APPROXIMATION OF QUADRATIC IRRATIONALS AND THEIR PIERCE EXPANSIONS

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1. INTRODUCTION

In the year 1937, E. B. Escott published his paper "Rapid Method for Extracting a Square Root" [4], where he presented an algorithm to find rational approximations for the square root of any real number. Escott's algorithm is based upon the algebraic identity

$$\sqrt{\frac{x_1+2}{x_1-2}}=\frac{x_1+1}{x_1-1}\cdot\frac{x_2+1}{x_2-1}\cdot\frac{x_3+1}{x_3-1}\cdot\cdots,$$

where the x_i are obtained through the following recurrence:

$$x_n = x_{n-1}(x_{n-1}^2-3).$$

It is obvious that, in order to calculate $\sqrt{N}$, Escott's algorithm must use rational x_i and thus the actual computation is considerably retarded.

More recently, in 1993, Y. Lacroix [6] referred to Escott's algorithm in the context of the representation of real numbers by generalized Cantor products and their metrical study.

In Section 2 of this paper, we present an algorithm similar to Escott's but improved in the sense that we only use positive integers in the recurrence leading to the computation of $\sqrt{N}$. Moreover, the approximating fractions obtained by our algorithm constitute best approximations (of the second kind).

In 1984, J. O. Shallit [15] published the recurrence relations followed by the coefficients in the Pierce series development of irrational quadratics of the form $(c-\sqrt{c^2-4})/2$. His method is based on Pierce's algorithm [12] applied to the polynomial x^2-cx+1.

In Section 3, we use the infinite product expansion provided by our square root algorithm to find the Pierce expansions corresponding to irrationals of the form $p-\sqrt{p^2-1}$, as an alternative way to the one used by Shallit [15]. The same method can also be used in the case of irrationals of the form $2(p-1)(p-\sqrt{p^2-1})$, as we show in Section 4.

2. THE EXPANSION OF AN IRRATIONAL QUADRATIC AS AN INFINITE PRODUCT

It is well known (see [9], [10]) that the convergents p_n / q_n of the *regular continued fraction* development of $\sqrt{r}$, with r a positive integer, verify, alternatively, Pell's equation $p_n^2 - rq_n^2 = \pm 1$, and we get the recurrence relationships,

$$p_n = 2p_1p_{n-1} - p_{n-2}, \quad q_n = 2p_1q_{n-1} - q_{n-2}, \tag{1}$$

that allow us to find all the solutions of Pell's equation from the first one (p_1, q_1); (we take $p_0 = 1$, $q_0 = 0$).

Lemma 1: Let (p_1, q_1) be a positive solution $(p_1 > 0,\ q_1 > 0)$ of Pell's equation $x^2 - ry^2 = 1$, where r is a positive integer free of squares. The sequence $\{(\overline{p}_n,\ \overline{q}_n)\}$, obtained recurrently in the following way,

$$\begin{cases} \overline{p}_n = \overline{p}_{n-1}(4\overline{p}_{n-1}^2 - 3), & \overline{p}_1 = p_1, \\ \overline{q}_n = \overline{q}_{n-1}(4\overline{p}_{n-1}^2 - 1), & \overline{q}_1 = q_1, \end{cases} \tag{2}$$

is a subsequence of the sequence $\{(p_n,\ q_n)\}$ of all solutions of the given Pell equation, with the peculiarity that each solution is an integer multiple of the preceding one.

Proof: We shall proceed by induction on n. Let us suppose that $\overline{p}_{n-1}^2 = r\overline{q}_{n-1}^2 + 1$ is verified. We must ascertain that

$$\overline{p}_n^2 = r\overline{q}_n^2 + 1. \tag{3}$$

We replace $\overline{p}_n$ and $\overline{q}_n$ using the recurrence (2):

$$\overline{p}_{n-1}^2(4\overline{p}_{n-1}^2 - 3)^2 = r\overline{q}_{n-1}^2(4\overline{p}_{n-1}^2 - 1)^2 + 1. \tag{4}$$

To simplify, let us denote by α the expression $\alpha = 4\overline{p}_{n-1}^2 - 2$. Equality (4) becomes

$$\overline{p}_{n-1}^2(\alpha - 1)^2 = r\overline{q}_{n-1}^2(\alpha + 1)^2 + 1,$$

which can be written as $\overline{p}_{n-1}^2(\alpha^2 - 2\alpha + 1) = r\overline{q}_{n-1}^2(\alpha^2 + 2\alpha + 1) + 1$. Grouping together the terms corresponding to $\alpha^2 + 1$, we obtain the equality

$$(\alpha^2 + 1)(\overline{p}_{n-1}^2 - r\overline{q}_{n-1}^2) - 2\alpha(\overline{p}_{n-1}^2 + r\overline{q}_{n-1}^2) = 1, \tag{5}$$

and, by the induction hypothesis, $\overline{p}_{n-1}^2 - r\overline{q}_{n-1}^2 = 1$, and also $\overline{p}_{n-1}^2 + r\overline{q}_{n-1}^2 = 2\overline{p}_{n-1}^2 - 1$. Therefore, (5) becomes

$$\alpha^2 - 2\alpha(2\overline{p}_{n-1}^2 - 1) = 0, \tag{6}$$

and, as we have $\alpha = 2(2\overline{p}_{n-1}^2 - 1)$, we deduce that (6) is, in point of fact, an algebraic identity. □

Theorem 1: $\sqrt{r}$ expands in an infinite product of the form

$$\sqrt{r} = \frac{p_1}{q_1}\prod_{n=1}^{\infty}\frac{\alpha_n^2 - 3}{\alpha_n^2 - 1} = \frac{p_1}{q_1}\prod_{n=1}^{\infty}\left(1 - \frac{2}{\alpha_n^2 - 1}\right),$$

where $(p_1,\ q_1)$ is a positive solution of Pell's equation, $x^2 - ry^2 = 1$; $\alpha_1 = 2p_1$, $\alpha_n = \alpha_{n-1}(\alpha_{n-1}^2 - 3)$.

Proof: With the same notation as in Theorem 1, we have, on the one hand,

$$\sqrt{r} = \lim_{n\to\infty} \frac{\overline{p}_n}{\overline{q}_n}, \tag{7}$$

and, on the other, we have the recurrence

$$\frac{\overline{p}_n}{\overline{q}_n} = \frac{\overline{p}_{n-1}}{\overline{q}_{n-1}} \cdot \frac{4\overline{p}_{n-1}^2 - 3}{4\overline{p}_{n-1}^2 - 1}. \tag{8}$$

Iterating, we obtain the expansion

$$\frac{\overline{p}_n}{\overline{q}_n} = \frac{\overline{p}_1}{\overline{q}_1} \cdot \frac{4\overline{p}_1^2 - 3}{4\overline{p}_1^2 - 1} \cdot \frac{4\overline{p}_2^2 - 3}{4\overline{p}_2^2 - 1} \cdot \cdots \cdot \frac{4\overline{p}_{n-1}^2 - 3}{4\overline{p}_{n-1}^2 - 1}, \tag{9}$$

or, if we prefer, we can simplify expression (9) by defining the new recurrence

$$\alpha_n = \alpha_{n-1}(\alpha_{n-1}^2 - 3), \qquad \alpha_1 = 2p_1, \tag{10}$$

which allows us to write

$$\frac{\overline{p}_n}{\overline{q}_n} = \frac{p_1}{q_1} \cdot \frac{\alpha_1^2 - 3}{\alpha_1^2 - 1} \cdot \cdots \cdot \frac{\alpha_{n-1}^2 - 3}{\alpha_{n-1}^2 - 1}. \tag{11}$$

Finally, taking limits as $n \to \infty$, the expansion of $\sqrt{r}$ in an infinite product is

$$\sqrt{r} = \frac{p_1}{q_1} \prod_{n=1}^{\infty} \frac{\alpha_n^2 - 3}{\alpha_n^2 - 1} = \frac{p_1}{q_1} \prod_{n=1}^{\infty} \left(1 - \frac{2}{\alpha_n^2 - 1}\right). \quad \square \tag{12}$$

Using the recurrence (10) in (11), we obtain

$$\frac{\overline{p}_n}{\overline{q}_n} = \frac{\alpha_n}{2q_1(\alpha_1^2 - 1) \cdots (\alpha_{n-1}^2 - 1)}. \tag{13}$$

The recurrence (10) is a fast way to compute the fractions of (13) which constitute best approximations of the second kind of any irrational quadratic of the form $\sqrt{r}$, where r is a positive integer; to start, we need only a positive solution of Pell's equation $x^2 - ry^2 = 1$. With ten iterations of the algorithm, we obtain a fraction whose approximation to the irrational is of the order $10^{-30,000}$. With 14 iterations, the approximation gives us a million correct decimal figures.

Expansion (12), among others, is the one considered by Y. Lacroix [6] in connection with Cantor's representation of real numbers by infinite products (see [1]).

3. THE PIERCE EXPANSION OF $p - (p^2 - 1)^{1/2}$

Any real number $\alpha \in (0, 1]$ has a unique Pierce expansion of the form

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + \frac{(-1)^{n+1}}{a_1 a_2 \cdots a_n} + \cdots, \tag{14}$$

where $\{a_n\}$ is a strictly increasing sequence of positive integers. These a_i will be called *coefficients* or *partial quotients* of the development.

Following Erdös and Shallit [3], we will denote the right-hand side of (14) by the special symbol $\langle a_1, a_2, \ldots, a_n, \ldots \rangle$. If expansion (14) is infinite, α is irrational. Otherwise, α is rational.

One of the first mathematicians to consider these developments was Lambert in [8]. Later, Lagrange refers to them in [7], but their numerical properties were not discussed until the independent studies of Sierpinski [18] and Ostyrogadsky [13] were published. Perron mentions them in [11] among other unusual series representations of real numbers. T. A. Pierce [12] used these series to approximate roots of algebraic equations and, in 1986, Shallit [16] studied their metrical properties using methods developed by Rényi [14] to study the metrical properties of Engel's series, i.e., series of type (14) but with all its signs positive (see [2], [11], [14]). In 1984, using Pierce's algorithm, Shallit [15] obtained the Pierce expansion of all irrational quadratics of the form

$$\frac{c-\sqrt{c^2-4}}{2} \text{ with integer } c,\ c \geq 3. \tag{15}$$

Quite recently, in 1994, Shallit [17] used Pierce expansions to propose a very nice method for determining leap years which generalizes those existent and, in 1995, Knopfmacher and Mays [5] related the expansions obtained in (15) to the Pierce expansions of some particular quotients of consecutive Fibonacci numbers.

If $c = 2k$, the irrational in (15) is directly of the form $k-\sqrt{k^2-1}$. If $c = 2k+1$, it can be seen that the irrational in (15) can be written as

$$\frac{1}{2k}-\frac{1}{2k(2k+2)}+\frac{1}{2k(2k+2)}\cdot(p-\sqrt{p^2-1})$$

with $p=(2k+1)(2k^2+2k-1)$. Thus, the Pierce expansion of irrationals of the form studied by Shallit are directly related to the irrationals of the form $p-\sqrt{p^2-1}$. The aim of this section is to find the Pierce expansion of all irrationals of this particular form using different methods than those in [15].

Now, if $\sqrt{p^2-1}=q\sqrt{r}$ with r free of squares, (p, q) is a solution of Pell's equation

$$x^2-ry^2=1.$$

Theorem 2: Given r, a positive integer free of squares, let (p, q) be a positive solution of Pell's equation $x^2-ry^2=1$. The Pierce expansion of the irrational $p-q\sqrt{r}$ is exactly

$$p-q\sqrt{r}=\langle \alpha_1-1,\ \alpha_1+1,\ \alpha_2-1,\ \alpha_2+1, \ldots\rangle, \tag{16}$$

where $\alpha_1 = 2p$ and $\alpha_{n+1}=\alpha_n(\alpha_n^2-3)$.

Proof: Using expression (13),

$$\frac{\overline{p}_n}{\overline{q}_n}=\frac{\alpha_n}{2q_1(\alpha_1^2-1)\cdots(\alpha_{n-1}^2-1)}.$$

We can write its right-hand side as

$$\frac{\alpha_{n-1}}{2q_1(\alpha_1^2-1)\cdots(\alpha_{n-2}^2-1)}\cdot\frac{\alpha_{n-1}^2-3}{\alpha_{n-1}^2-1}=\frac{\alpha_{n-1}}{2q_1(\alpha_1^2-1)\cdots(\alpha_{n-2}^2-1)}\left(1-\frac{2}{\alpha_{n-1}^2-1}\right).$$

Now, as we have the algebraic identity

$$\frac{\alpha_{n-1}}{b(\alpha_{n-1}^2-1)}=\frac{1}{b(\alpha_{n-1}-1)}-\frac{1}{b(\alpha_{n-1}-1)(\alpha_{n-1}+1)},$$

iterating the former process we eventually reach the expansion,

$$\frac{\overline{p}_n}{\overline{q}_n}=\frac{p_1}{q_1}-\frac{1}{q_1(\alpha_1-1)}+\frac{1}{q_1(\alpha_1-1)(\alpha_1+1)}+\cdots$$
$$+\frac{1}{q_1(\alpha_1^2-1)\cdots(\alpha_{n-2}^2-1)(\alpha_{n-1}-1)}-\frac{1}{q_1(\alpha_1^2-1)\cdots(\alpha_{n-1}^2-1)}.$$

In our case, $p_1=p$ and $q_1=q$; thus, we can write

$$\frac{p}{q}-\frac{p_n}{q_n}=\frac{1}{q(\alpha_1-1)}-\frac{1}{q(\alpha_1-1)(\alpha_1+1)}+\cdots$$
$$+\frac{1}{q(\alpha_1^2-1)\cdots(\alpha_{n-2}^2-1)(\alpha_{n-1}-1)}-\frac{1}{q(\alpha_1^2-1)\cdots(\alpha_{n-1}^2-1)}. \tag{17}$$

As $n\to\infty$ we obtain the infinite Pierce expansion,

$$\frac{p}{q}-\sqrt{r}=\sum_{i=1}^{\infty}\left(\frac{1}{q\prod_{k=1}^{i-1}(\alpha_k^2-1)\cdot(\alpha_i-1)}-\frac{1}{q\prod_{k=1}^{i}(\alpha_k^2-1)}\right), \tag{18}$$

which is equivalent to (16). □

4. THE PIERCE EXPANSION OF $2(p-1)[p-(p^2-1)^{1/2}]$

In this section we are going to see how the method we have just used can be extended to find the Pierce expansion of irrational quadratics of the form $2(p-1)(p-\sqrt{p^2-1})$.

As above, our starting point will be Pell's equation $x^2-ry^2=1$, and we will choose a subsequence of the sequence of its solutions. We will need the following result.

Lemma 2: Given a positive solution (p, q) of Pell's equation $x^2-ry^2=1$, with r free of squares, the recurrent sequence $\{(\overline{p}_n, \overline{q}_n)\}$, obtained in the form

$$\begin{cases}\overline{p}_n=2\overline{p}_{n-1}^2-1, & \overline{p}_1=p,\\ \overline{q}_n=2\overline{p}_{n-1}\overline{q}_{n-1}, & \overline{q}_1=q,\end{cases}$$

is a subsequence of the sequence $\{(p_n, q_n)\}$ of all the equation solutions.

Proof: The result is easily proved by induction. Let us suppose that $\overline{p}_{n-1}$ and $\overline{q}_{n-1}$ verify $\overline{p}_{n-1}^2-r\overline{q}_{n-1}^2=1$. For the next index we will have

$$\overline{p}_n^2=(2\overline{p}_{n-1}^2-1)^2=4\overline{p}_{n-1}^4-4\overline{p}_{n-1}^2+1,$$
$$r\overline{q}_n^2=r4\overline{p}_{n-1}^2\overline{q}_{n-1}^2,$$

and subtracting gives $\overline{p}_n^2-r\overline{q}_n^2=4\overline{p}_{n-1}^2(\overline{p}_{n-1}^2-r\overline{q}_{n-1}^2)-4\overline{p}_{n-1}^2+1=1$. □

Having proved that all pairs $(\overline{p}_n, \overline{q}_n)$ are solutions of the given Pell equation and using the fact that $\sqrt{r}=\lim_{n\to\infty}(\overline{p}_n/\overline{q}_n)$, we will try, as before, to expand the fraction $\overline{p}_n/\overline{q}_n$ as a finite

Pierce expansion and then, taking limits, obtain the infinite Pierce expansion corresponding to the irrational $\sqrt{r}$ or an equivalent one.

Let us start with the fraction $\overline{p}_n / \overline{q}_n$, and let us express its numerator and denominator in terms of the preceding pair of solutions, i.e.,

$$\frac{\overline{p}_n}{\overline{q}_n} = \frac{2\overline{p}_{n-1}^2 - 1}{2\overline{p}_{n-1}\overline{q}_{n-1}} = \frac{\overline{p}_{n-1}}{\overline{q}_{n-1}} - \frac{1}{2\overline{p}_{n-1}\overline{q}_{n-1}}. \tag{19}$$

Proceeding with the expansion of the equation above, we will eventually reach the first one, $\overline{p}_1 / \overline{q}_1$, and the chain of equalities

$$\begin{aligned}\frac{\overline{p}_n}{\overline{q}_n} &= \frac{\overline{p}_1}{\overline{q}_1} - \frac{1}{2\overline{p}_1\overline{q}_1} - \frac{1}{2\overline{p}_2\overline{q}_2} - \cdots - \frac{1}{2\overline{p}_{n-1}\overline{q}_{n-1}} \\ &= \frac{\overline{p}_1}{\overline{q}_1} - \frac{1}{\overline{q}_1 2\overline{p}_1} - \frac{1}{\overline{q}_1 2\overline{p}_1 2\overline{p}_2} - \cdots - \frac{1}{\overline{q}_1 2\overline{p}_1 2\overline{p}_2 \ldots 2\overline{p}_{n-1}} \\ &= \frac{\overline{p}_1}{\overline{q}_1} - \frac{1}{\overline{q}_1}\left(\frac{1}{2\overline{p}_1} + \frac{1}{2\overline{p}_1 2\overline{p}_2} + \cdots + \frac{1}{2\overline{p}_1 2\overline{p}_2 \ldots 2\overline{p}_{n-1}}\right).\end{aligned}$$

Taking limits in this last expression and remembering that $\overline{p}_1 = p$ and $\overline{q}_1 = q$, we obtain

$$\sqrt{r} = \frac{p}{q} - \frac{1}{q}\left(\frac{1}{2p} + \frac{1}{2p2\overline{p}_2} + \cdots + \frac{1}{2p2\overline{p}_2 \ldots 2\overline{p}_{n-1}} + \cdots\right), \tag{20}$$

where the $\overline{p}_i$ follow the recurrence $\overline{p}_n = 2\overline{p}_{n-1}^2 - 1$, $\overline{p}_1 = p$. The series within parentheses on the right-hand side of (20) is an Engel series.

Equality (20) can also be expressed in the form

$$p - q\sqrt{r} = \frac{1}{2p} + \frac{1}{2p\overline{p}_2} + \cdots + \frac{1}{2p2\overline{p}_2 \ldots 2\overline{p}_{n-1}} + \cdots, \tag{21}$$

or, if we prefer, we can state the following result.

Lemma 3: For all positive integers p, we have

$$p - \sqrt{p^2 - 1} = \frac{1}{2\overline{p}_1} + \frac{1}{2\overline{p}_1\overline{p}_2} + \cdots + \frac{1}{2\overline{p}_1 2\overline{p}_2 \ldots 2\overline{p}_{n-1}} + \cdots, \tag{22}$$

with $\overline{p}_i = 2\overline{p}_{i-1}^2 - 1$, $\overline{p}_1 = p$.

Expression (22) is known as *Stratemeyer's formula*, and can be obtained algebraically by the method described in Perron (see [11], Ch. IV). We mention in passing that the recurrence (22) verified by the $\overline{p}_i$ is exactly the recurrence verified by the denominators in the infinite product expansion presented by Cantor in [1]. We are now ready for the following result.

Theorem 3: If p is a positive integer greater than one, then $2(p-1)(p-\sqrt{p^2-1}) = \langle 1, p_1, p_2, p_3, \ldots \rangle$, where $p_1 = p$ and the p_i verify

$$\begin{cases} p_{2n} = 4(p_{2n-1} + 1), \\ p_{2n+1} = 2p_{2n-1}^2 - 1. \end{cases}$$

Proof: To prove Theorem 3, we just have to change the Engel series in (22) into a Pierce expansion. In order to do that, let us consider the Pierce expansion of Theorem 3,

$$\langle 1, p_1, p_2, p_3, \dots \rangle, \tag{23}$$

with the recurrence $p_{2n} = 4(p_{2n-1}+1)$, $p_{2n+1} = 2p_{2n-1}^2 - 1$, $p_1 = p$. If we denote by S the irrational number represented by (23), we have the following expansion:

$$S = 1 - \frac{1}{p_1} + \frac{1}{p_1 p_2} - \dots = \frac{p_1 - 1}{p_1} + \frac{p_3 - 1}{p_1 p_2 p_3} + \dots + \frac{p_{2n+1} - 1}{p_1 p_2 \cdots p_{2n+1}}.$$

We want to see that each fraction in the sum above is of the form

$$\frac{p_{2n+1} - 1}{p_1 p_2 \cdots p_{2n+1}} = \frac{p_1 - 1}{p_1 2 p_3 \dots 2 p_{2n+1}}. \tag{24}$$

We will proceed by induction on n. For $n = 0$, it is trivially true. Let us expand the left-hand side of (24) in the following way:

$$\frac{p_{2n+1} - 1}{p_1 \cdots p_{2n-1} p_{2n} p_{2n+1}} = \underbrace{\frac{p_{2n-1} - 1}{p_1 \cdots p_{2n-1}}}_{(*)} \cdot \underbrace{\left(\frac{p_{2n+1} - 1}{p_{2n+1} p_{2n}}\right) \cdot \frac{1}{p_{2n-1} - 1}}_{(**)}. \tag{25}$$

The term $(**)$ can be written as follows:

$$\frac{2p_{2n-1}^2 - 1 - 1}{p_{2n+1} 4(p_{2n-1} + 1)} \cdot \frac{1}{p_{2n-1} - 1} = \frac{2(p_{2n-1}^2 - 1)}{p_{2n+1} 4(p_{2n-1}^2 - 1)} = \frac{1}{2 p_{2n+1}}.$$

Finally, by the induction hypothesis applied to factor $(*)$ in (25), we obtain

$$\frac{p_{2n+1} - 1}{p_1 \cdots p_{2n-1} p_{2n} p_{2n+1}} = \frac{p_1 - 1}{p_1 2 p_3 \dots 2 p_{2n-1}} \cdot \frac{1}{2 p_{2n+1}}. \tag{26}$$

Thus, S can be written as

$$\begin{aligned} S = \langle 1, p_1, p_2, \dots \rangle &= \frac{p_1 - 1}{p_1} + \frac{p_1 - 1}{p_1 2 p_3} + \dots + \frac{p_1 - 1}{p_1 2 p_3 \dots 2 p_{2n+1}} + \cdots \\ &= \frac{p_1 - 1}{p_1}\left(1 + \frac{1}{2p_3} + \frac{1}{2p_3 2p_5} + \dots + \frac{1}{2p_3 2p_5 \dots 2p_{2n+1}} + \cdots\right) \\ &= 2(p_1 - 1)\underbrace{\left(\frac{1}{2p_1} + \frac{1}{2p_1 2p_3} + \dots + \frac{1}{2p_1 2p_3 \dots 2p_{2n+1}}\right)}_{(***)}. \end{aligned}$$

But, by Stratemeyer's formula (22), the term $(***)$ is precisely $p_1 - \sqrt{p_1^2 - 1}$. □

5. CONCLUSIONS

The algorithm presented in this article provides fast best approximations to any irrational of the form $\sqrt{r}$, where r is a positive integer. At the same time, the algorithm provides the necessary background to obtain the Pierce expansion of some quadratic irrationals whose partial quotients, a_i, grow as x^3. The procedure used proves also that the convergents in the Pierce expansions of these irrationals are best approximations of the second kind.

We also present the Pierce series development of irrationals of the form

$$2(p-1)(p-\sqrt{p^2-1}),$$

whose partial quotients grow as x^2.

However, there exist quadratic irrationals that escape the above laws, whose partial quotients obey the metrical behavior, $\lim_{n\to\infty}\sqrt[n]{a_n}=e$, found by Shallit in [16].

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