PELL IDENTITIES

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1. INTRODUCTION

Recent issues of this Journal have contained several interesting special results involving Pell numbers. Allowing for extension to the usual Pell numbers to negative subscripts, we define the Pell numbers by the Pell sequence $\{P_n\}$ thus:

$$(1) \qquad \{P_n\}: \begin{matrix} \cdots & P_{-4} & P_{-3} & P_{-2} & P_{-1} & P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & \cdots \\ \cdots & -12 & 5 & -2 & 1 & 0 & 1 & 2 & 5 & 12 & 29 & \cdots \end{matrix}$$

in which

$$(2) \qquad P_0 = 0, \quad P_1 = 1, \quad P_{n+2} = 2P_{n+1} + P_n$$

and

$$(2') \qquad P_{-n} = (-1)^{n+1} P_n .$$

The purpose of this article is to urge a greater use of the properties of the generalized recurrence sequence $\{W_n(a,b;\ p,q)\}$, discussed by the author in a series of papers [2], [3], and [4]. The Pell sequence is but a special case of the generalized sequence.

2. THE SEQUENCE $\{W_n(a,b;\ p,q)\}$

Our generalized sequence $\{W_n(a,b;\ p,q)\}$ is defined [2] as

$$(3) \qquad \{W_n\}: \begin{matrix} \cdots & W_{-1}, & W_0, & W_1, & W_2, & W_3, & W_4 & \cdots \\ \cdots & \frac{pa-b}{q}, & a, & b, & pb-qa, & p^2b-pqa-qb, & \cdots & \cdots \end{matrix}$$

in which

$$W_0 = a, \qquad W_1 = b, \qquad W_{n+2} = pW_{n+1} - qW_n , \tag{4}$$

where a, b, p, q are arbitrary integers at our disposal.

The Pell sequence is the special case for which

$$a = 0, \qquad b = 1, \qquad p = 2, \qquad q = -1, \tag{5}$$

i.e., $P_n = W_n(0, 1; 2 - 1)$.

From the general term W_n [2], namely,

$$W_n = \frac{b - \alpha\beta}{\alpha - \beta}\alpha^n + \frac{a\alpha - b}{\alpha - \beta}\beta^n , \tag{6}$$

where

$$\begin{cases} \alpha = (p + d)/2, \quad \beta = (p - d)/2 \\ d = (p^2 - 4q)^{1/2} \end{cases} , \tag{7}$$

We have, for the Pell sequence, using (5),

$$\begin{cases} d = 2^{3/2} \\ \alpha = 1 + \sqrt{2} \\ \beta = 1 - \sqrt{2} \end{cases} \tag{8}$$

so that, from (5), (6) and (8), the n^{th} term of the Pell sequence is

$$P_n = \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2^{3/2}} . \tag{9}$$

A generating function for $\{W_n\}$, namely [4],

(10) $$\frac{a + (b - pa)x}{1 - pz + qx^2} = \sum_{n=0}^{\infty} W_n x^n$$

becomes, using (5) for $\{P_n\}$,

(11) $$\frac{x}{1 - 2x - x^2} = \sum_{n=0}^{\infty} W_n x^n$$

(11') $$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} P_{n+1} x^n .$$

Associated with $\{W_n\}$ is [2] the characteristic number

(12) $$e = pab - qa^2 - b^2$$

with Pell value

(13) $$ep = -1$$

by (5).

Another special case of subsequent interest to us in (32) is the sequence $\{U_n(p,q)\}$ defined by

(14) $$U_0 = 0, \qquad U_1 = 1, \qquad U_{n+2} = pU_{n+1} - qU_n ,$$

i.e.,

$$U_n(p,q) = W_n(0, 1; p,q) ,$$

for which

(15) $$e_U = -1$$

and

$$U_{-n} = -q^{-n} U_n \ . \tag{16}$$

Result (16) was noted long ago by Lucas [6], p. 308, to whom much of the knowledge of sequences like $\{U_n(p,q)\}$ is due. Obviously, by (5) and (14),

$$P_n = U_n(2, -1) \ . \tag{17}$$

3. PELL IDENTITIES

Specific Pell identities to which we refer are:

$$P_k = \sum_{r=0}^{[(k-1)/2]} \binom{k}{2r+1} 2^r \tag{18}$$

$$P_{2k} = \sum_{r=1}^{k} \binom{k}{r} 2^r P_r \tag{19}$$

$$P_{2n+1} = P_n^2 + P_{n+1}^2 \tag{20}$$

$$P_{2n+1} + P_{2n} = 2P_{n+1}^2 - 2P_n^2 - (-1)^n \tag{21}$$

$$(-1)^n P_a P_b = P_{n+a} P_{n+b} - P_n P_{n+a+b} \ . \tag{22}$$

These identities occur as Problems B-161 [5], B-161 [5], B-136 [7], B-137 [7], and B-155 [8], respectively.

Identity (18) follows readily from formula (3.20) of [2]:

$$2^n W_n = a \sum_{j=0}^{[n/2]} p^{n-2j} d^{2j} \binom{n}{2j} + (2b - pa) \sum_{j=0}^{[(n-1)/2]} \binom{n}{2j+1} p^{n-2j-1} d^{2j} \tag{23}$$

on using (5) and (8).

Identity (19) follows from formula (3.19) of [2]:

$$W_{2n} = (-q)^n \sum_{j=0}^{n} \binom{n}{j}\left(\frac{-p}{q}\right)^{n-j} W_{n-j} \tag{24}$$

on using (5) and recognizing that

$$\sum_{r=0}^{k} \binom{k}{r} 2^{k-r} P_{k-r} = \sum_{r=1}^{k} \binom{k}{r} 2^r P_r \ . \tag{25}$$

Employing the formula (3.14) of [2] and replacing U_n therein (and subsequently as required) by U_{n+1} in accordance with (14) to get

$$W_{n+r} = W_r U_{n+1} - qW_{r-1} U_n \ , \tag{26}$$

we put $r = n + 1$, and identity (20) follows immediately with the aid of (5) and (17).

Furthermore, (20) may simply be obtained from formula (4.5) of [2]:

$$W_{n+r} W_{n-r} = W_n^2 + e q^{n-r} U_r^2 \tag{27}$$

on choosing $r = n + 1$ and utilizing (1), (5), (13) and (17). ($P_{-1} = P_1 = 1.$)

An immediate consequence of (26) is, by (5) and (17), the result

$$P_{n+r} = P_r P_{n+1} + P_{r-1} P_n \ . \tag{28}$$

Setting $r = n$ in (28), we deduce that

$$P_{2n} = P_n (P_{n+1} + P_{n-1}) \ . \tag{29}$$

From (27), with $r = 1$ and using (5), (13) and (17) ($P_1 = 1$), we have

$$P_{n+1} P_{n-1} - P_n^2 = (-1)^n \ . \tag{30}$$

Now, to prove identity (21), merely add (20) and (29). Then

$$\begin{aligned} P_{2n+1} + P_{2n} &= P_{n+1}^2 + (P_{n+1}P_{n-1} - (-1)^n) + P_n(2P_{n+1} - 2P_n) \\ &= P_{n+1}^2 + P_{n+1}(P_{n+1} - 2P_n) - (-1)^n + P_n(2P_{n+1} - 2P_n) \\ &= 2(P_{n+1}^2 - P_n^2) - (-1)^n \end{aligned}$$

on using (2) twice, and (30).

Next, consider formula (4.18) of [2]:

(31) $$W_{n-r}W_{n+r+t} - W_nW_{n+t} = eq^{n-r}U_rU_{r+t} .$$

Put $r = -a$, $b = r + t$, $t = a + b$ in (31). Using (2'), (5), (13), and (17), we observe that identity (22) evolves without difficulty.

4. CONCLUDING COMMENTS

I. Problem B-174, proposed by Zeitlin [10] from the solution to Problem B-155 [8], namely, to show that

(32) $$U_{n+a}U_{n+b} - U_nU_{n+a+b} = q^nU_aU_b ,$$

is proved for identity (22) from (31) on using (14), (15), and (16).

II. Discussing briefly the sequence $\{T_n\}$ for which

(33) $$T_n = Ar^n + Bs^n ,$$

where

(34) $$r = \frac{1 + \sqrt{5}}{2}, \qquad s = \frac{1 - \sqrt{5}}{2}$$

and A,B depend on initial conditions, Bro. Brousseau [1] asks, and answers, the questions:

(i) Which sequences have a limiting ratio T_n/T_{n-1}?

(ii) Which sequences do not have a limiting ratio?

(iii) On what does the limiting ratio depend?

He finds that

(35) $$\lim_{n \to \infty} \left[\frac{T_n}{T_{n-1}} \right] = r \; .$$

This accords with our more general result (3.1) of [2]:

(36) $$\lim_{n \to \infty} \left[\frac{W_n}{W_{n-1}} \right] = \begin{matrix} \alpha & \text{if} & |\beta| \le 1 \\ \beta & \text{if} & |\alpha| \le 1 \end{matrix} \; ,$$

where α, β are defined in (7). Result (36) probably answers Bro. Brousseau's queries (i), (ii), (iii) from a slightly different point of view.

Clearly, the particular sequence he quotes, namely, the one defined by

(37) $$T_1 = 5, \quad T_2 = 9, \quad T_{n+2} = 3T_{n+1} - 4T_n \; ,$$

i.e., our $\{W_n(5,9;\ 3,4)\}$, cannot converge to a real limit, since by (7),

(38) $$\begin{cases} \alpha = (3 + i\sqrt{7})/2 \\ \beta = (3 - i\sqrt{7})/2 \end{cases}$$

which are both complex numbers.

III. Corresponding to the specifically stated Pell identities (18)-(22), and to the incidental Pell identities (28)-(30), one may write down identities for the

(39) $$\begin{cases} \text{Fibonacci sequence} & \{F_n\} & = \{W_n(0,\ 1;\ 1,\ -1)\} \\ \text{Lucas sequence} & \{L_n\} & = \{W_n(2,\ 1;\ 1,\ -1)\} \; . \\ \text{Generalized sequence} & \{H_n(s,r)\} & = \{W_n(s,\ r;\ 1,\ -1)\} \end{cases}$$

Readers are invited to explore these pleasant mathematical pastures. Reversing our previous procedure of using the $\{W_n\}$ sequence to obtain special

(Pell) identities, one could be motivated to discover generalized W_n identities commencing with only a simple recurrence-relation result.

Consider, for example, the relationship

$$F_n^2 + F_{n+3}^2 = 2(F_{n+1}^2 + F_{n+2}^2) . \tag{40}$$

an aesthetically attractive result known in embryonic form, at least, in 1929 when it was described in a philosophical article by D'Arcy Thompson [9] as "another of the many curious properties" of $\{F_n\}$. Readily, we have

$$L_n^2 + L_{n+3}^2 = 2(L_{n+1}^2 + L_{n+2}^2) \tag{41}$$

$$H_n^2 + H_{n+3}^2 = 2(H_{n+1}^2 + H_{n+2}^2) . \tag{42}$$

Not unexpectedly, the results (40)-(42) are alike simply because we have $p = 1$, $q = -1$ for each of the sequences concerned. But what, we ask, will happen in the case of the Pell sequence, for which $p = 2$, $q = -1$?

Proceeding to the generalized situation, we find

$$W_n^2 + W_{n+3}^2 = q^{-2}\ (p^2q^2 + 1)W_{n+2}^2 + (p^2 + q^4)W_{n+1}^2 - 2p(q^3 + 1)W_{n+1}W_{n+2} . \tag{43}$$

Pell's sequence reduces (43) to

$$P_n^2 + P_{n+3}^2 = 5(P_{n+1}^2 + P_{n+2}^2) . \tag{44}$$

IV. By now, the message of this article should be evident. Simply, it is this:

While the discovery of individual properties of a particular sequence, elegant though they may be, is a satisfying experience, I believe that a more fruitful mathematical enterprise is an investigation of the properties of the generalized sequence $\{W_n\}$. In this way, otherwise hidden relationships are brought to light. To this objective, I commend the reader.

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