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On The Negative Pell Equation

$$y^2 = 72x^2 - 23$$

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Abstract – The binary quadratic equation represented by the negative Pellian $y^2 = 72x^2 - 23$ is analyzed for its distinct integer solutions .A few interesting relations among the solutions are also given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.

Index Terms – Binary quadratic, hyperbola, parabola, integral solutions, Pell equation.

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1. INTRODUCTION

Diophantine equation of the form $y^2 = Dx^2 + 1,$ where D is a given positive square-free integer is known as Pell equation and is one of the oldest Diophantine equation that has interested mathematicians all over the world, since antiquity, J.L.Lagrange proved that the positive Pell equation $y^2 = Dx^2 + 1$ has infinitely many distinct integer solutions whereas the negative Pell equation $y^2 = Dx^2 - 1$ does not always have a solution. In [1], an elementary proof of a criterion for the solvability of the Pell equation $x^2 - Dy^2 = -1$ where D is any positive non-square integer has been presented. For examples the equations $y^2 = 3x^2 - 1, y^2 = 7x^2 - 4$ have no integer solution whereas $y^2 = 65x^2 - 1, y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer [2- 9]. More specifically, one may refer " The On-line Encyclopedia of integer sequences " (A031396,A130226,A031398) for values of D for which the negative Pell equation $y^2 = Dx^2 - 1$ is solvable or not. In this communication, the negative Pell equation given by $y^2 = 72x^2 - 23$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solutions are presented.

2. METHOD OF ANALYSIS

The negative Pell equation representing hyperbola under consideration is

$$y^2 = 72x^2 - 23 \qquad (1)$$

whose smallest positive integer solution is $x_0 = 1, y_0 = 7$

To obtain the other solutions of (1), consider the Pell equation $y^2 = 72x^2 + 1$ whose solution is given by

$$\tilde{x}_n = \frac{1}{2} f_n, \tilde{y}_n = \frac{1}{2\sqrt{72}} g_n$$

where,

$$f_n = \left(17 + 2\sqrt{72}\right)^{n+1} + \left(17 - 2\sqrt{72}\right)^{n+1}$$

$$g_n = \left(17 + 2\sqrt{72}\right)^{n+1} - \left(17 - 2\sqrt{72}\right)^{n+1}$$

Applying Brahmagupta Lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{7}{2\sqrt{72}} g_n$$

$$y_{n+1} = \frac{7}{2} f_n + \frac{36}{\sqrt{72}} g_n$$

The recurrence relations satisfied by the solutions $x \,\&\, y$ are given by

$$x_{n+1} - 34x_{n+2} + x_{n+3} = 0, x_0 = 1, x_1 = 31$$

$$y_{n+1} - 34y_{n+2} + y_{n+3} = 0, y_0 = 7, y_1 = 263$$

Some numerical examples of $x \,\&\, y$ satisfying (1) are given in the table below

n	x_n	y_n
0	1	7
1	31	263
2	1053	8935
3	35771	303527
4	1215161	10310983

From the above table, we observe some interesting relations among the solutions which are presented below

1) x_n is always odd
2) y_n is always even
3) Each of the following expressions is a nasty number
   i) $\frac{144x_{2n+2} - 14y_{2n+2} + 46}{23}$
   ii) $\frac{17870x_{2n+2} - 14x_{2n+4} + 3128}{1564}$
   iii) $\frac{17870x_{2n+3} - 526x_{2n+4} + 92}{46}$
   iv) $\frac{2y_{2n+4} - 2106y_{2n+2} + 3128}{1564}$
   v) $\frac{2y_{2n+3} - 62y_{2n+2} + 92}{46}$
   vi) $\frac{62y_{2n+4} - 2106y_{2n+3} + 92}{46}$
   vii) $\frac{4464x_{2n+2} - 14y_{2n+3} + 782}{391}$
   viii) $\frac{14x_{2n+3} - 526y_{2n+2} + 782}{391}$
4) $\frac{10x_{3n+3} - 2y_{3n+3} + 342}{3}$ is a cubical integer.
5) $2x_{n+3} = 68x_{n+2} - 2x_{n+1}$
6) $2y_{n+1} = x_{n+2} - 17x_{n+1}$
7) $2y_{n+2} = 17x_{n+2} - x_{n+1}$
8) $2y_{n+3} = 577x_{n+2} - 17x_{n+1}$
9) $1154x_{n+1} = 2x_{n+3} - 136y_{n+1}$
10) $288x_{n+2} = 2y_{n+1} - 34y_{n+2}$
11) $288x_{n+2} = 2y_{n+3} - 34y_{n+2}$
12) $2x_{n+2} = 34x_{n+3} - 4y_{n+3}$
13) $576x_{n+2} = 2y_{n+3} - 2y_{n+1}$
14) $34x_{n+2} = 2x_{n+3} - 4y_{n+2}$
15) $2x_{n+1} = 2x_{n+3} - 8y_{n+2}$
16) $1154x_{n+2} = 34x_{n+3} - 4y_{n+1}$
17) $288x_{n+1} = 2y_{n+2} - 34y_{n+1}$
18) $288x_{n+1} = 34y_{n+3} - 1154y_{n+2}$
19) $1154x^2{}_{n+2} = 34x_{n+3}x_{n+2} - 4y_{n+1}x_{n+2}$
20) $288x^2{}_{n+2} = 34y_{n+2}x_{n+2} - 2y_{n+1}x_{n+2}$
21) $1154x^2{}_{n+1} = 2x_{n+3}x_{n+1} - 136y_{n+1}x_{n+1}$
22) $288x^2{}_{n+1} = 2y_{n+2}x_{n+1} - 34y_{n+1}x_{n+1}$
23) $9792x^2{}_{n+1} = 2y_{n+3}x_{n+1} - 1154y_{n+1}x_{n+3}$
24) $9792x^2{}_{n+1} = 2y_{n+3}x_{n+1} - 1154y_{n+1}x_{n+1}$

3. REMARKABLE OBSERVATIONS

I: Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the table 1 below

II: Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the table 2 below

III:Consider $m = x_{n+1} + y_{n+1}, n = x_{n+1}$, Observe that $m > n > 0$.Treat m, n as the generators of the Pythagorean triangle $T(\alpha, \beta, \gamma)$,where

$$\alpha = 2mn, \beta = m^2 - n^2, \gamma = m^2 + n^2$$

Then the following interesting relations are observed.

a) $\alpha - 36\beta + 35\gamma = 23$

b) $37\beta - 36\gamma - \frac{4A}{P} = -23$

c) c) $\gamma - 37\alpha + \frac{144A}{P} = -23$

TABLE: 1

a	HYPERBOLA	(X,Y)
1	$X^2 - Y^2 = 2116$	$(\frac{144x_{n+1} - 14y_{n+1}}{23}, \frac{2\sqrt{72}y_{n+1} - 14\sqrt{72}x_{n+1}}{23})$
2	$X^2 - Y^2 = 9784384$	$(\frac{17870x_{n+1} - 14x_{n+3}}{1564}, \frac{2\sqrt{72}x_{n+3} - 2106\sqrt{72}x_{n+1}}{1564})$
3	$X^2 - Y^2 = 8464$	$(\frac{17870x_{n+2} - 526x_{n+3}}{46}, \frac{62\sqrt{72}x_{n+3} - 2106\sqrt{72}x_{n+2}}{18})$
4	$X^2 - Y^2 = 9784384$	$(\frac{2y_{n+3} - 2106y_{n+1}}{1564}, \frac{17870\sqrt{72}y_{n+1} - 14\sqrt{72}y_{n+3}}{1564})$
5	$X^2 - Y^2 = 8464$	$(\frac{2y_{n+2} - 62y_{n+1}}{46}, \frac{526\sqrt{72}y_{n+1} - 14\sqrt{72}y_{n+2}}{46})$
6	$X^2 - Y^2 = 8464$	$(\frac{62y_{n+3} - 2106y_{n+2}}{46}, \frac{17870y_{n+2} - 526y_{n+3}}{46\sqrt{72}})$
7	$X^2 - Y^2 = 611524$	$(\frac{4464x_{n+1} - 14y_{n+2}}{391}, \frac{2\sqrt{72}y_{n+2} - 526\sqrt{72}x_{n+1}}{391})$
8	$X^2 - Y^2 = 611524$	$(\frac{144x_{n+2} - 526y_{n+1}}{391}, \frac{62\sqrt{72}y_{n+1} - 14\sqrt{72}x_{n+2}}{391})$

TABLE: 2

S.No	PARABOLA	(X,Y)
1	$Y^2 = 23X - 2116$	$(\frac{144x_{2n+2} - 14y_{2n+2}}{23}, \frac{2\sqrt{72}y_{n+1} - 14\sqrt{72}x_{n+1}}{23})$
2	$Y^2 = 1564X - 9784384$	$(\frac{17870x_{2n+2} - 14x_{2n+4}}{1564}, \frac{2\sqrt{72}x_{n+3} - 2106\sqrt{72}x_{n+1}}{1564})$
3	$Y^2 = 46X - 8464$	$(\frac{17870x_{2n+3} - 526x_{2n+4}}{46}, \frac{62\sqrt{72}x_{n+3} - 2106\sqrt{72}x_{n+2}}{46})$
4	$Y^2 = 1564X - 9784384$	$(\frac{2y_{2n+4} - 2106y_{2n+2}}{1564}, \frac{17870\sqrt{72}y_{n+1} - 14\sqrt{72}y_{n+3}}{1564})$
5	$Y^2 = 46X - 8464$	$(\frac{2y_{2n+3} - 62y_{2n+2}}{46}, \frac{526\sqrt{72}y_{n+1} - 14\sqrt{72}y_{n+2}}{46})$

6	$Y^2 = 46X - 8464$	$(\frac{62y_{2n+4} - 2106y_{2n+3}}{46}, \frac{17870y_{n+2} - 526x_{n+1}}{46\sqrt{72}})$
7	$Y^2 = 391X - 611524$	$(\frac{4464x_{2n+2} - 14y_{2n+3}}{391}, \frac{2\sqrt{72}y_{n+2} - 526\sqrt{72}x_{n+2}}{391})$
8	$Y^2 = 391X - 611524$	$(\frac{144x_{2n+3} - 526y_{2n+2}}{391}, \frac{62\sqrt{72}y_{n+1} - 14\sqrt{72}x_{n+2}}{391})$

4. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the negative Pell equation $y^2 = 72x^2 - 23$. As the binary quadratic Diophantine equation is rich in variety, one may search for the other choices of negative Pell equations and determine their integer solutions along with suitable properties.

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