Infinity

Unit 1: Different-Sized Infinities!

Set Theory and Proof by Contradiction

Sets

Counting Squares

1. Consider the set $S = \{1, 2, \ldots, n^2\}$, where n is a natural number. What fraction of the elements of S are perfect squares? (Hint: first figure it out for n = 1, n = 2, and so on, making a table.)

2. Consider the set $S = \{1, 2, \ldots, n\}$, where n is a natural number. Is the fraction of perfect squares increasing or decreasing as n increases? Does that ever change? (Same hint as #1.)

Finite Sets

3. Consider the set $T = \{a, b\}$. Make a list of all its subsets, including the empty set.

4. Repeat for
   a. $\{a, b, c\}$
   b. $\{a, b, c, d\}$
   c. $\{a\}$

5. If a set has n elements, how many subsets does it have? Explain why this must be true.

Infinite Sets

Sets are called *equivalent* if they can be put in a one-to-one correspondence with each other.

6. Show that the sets $\{0, 1, 2, 3, \ldots\}$ and $\{1, 2, 3, \ldots\}$ are equivalent.

7. Show that the set of perfect cubes is equivalent to the set of perfect squares.

Assume $a \leq b$. The *closed interval* $[a, b]$ is a subset of the real number line going from a to b, including the endpoints. The *open interval* (a, b) is a subset of the real number line going from a to b, excluding the endpoints. Note that intervals can be *half-open*, i.e. include exactly one endpoint.

8. Show that the interval $[0,1]$ is equivalent to the interval $[0,10]$.

9. Show that the interval $[5, 7]$ is equivalent to the interval $[12, 19]$. (Hint: one method is to start by putting one interval on the x-axis and the other on the y-axis.)

10. Show that the interval $[0,1)$ is equivalent to the interval $(0,1]$

11. Show that the set of points on a circle is equivalent to the set of points on another circle.

12. Show that the interval $[0,\infty)$ is equivalent to the interval $(0,\infty)$. (Hint: remember #6.)

Set Notation

Describe these sets in words:

1. $\{x \in \mathbf{N} \mid x/3 \in \mathbf{N}\}$

2. $\{x \in \mathbf{N} \mid (x+1)/3 \in \mathbf{N}\}$

3. $\{x \in \mathbf{Z} \mid x > 0\}$

4. $\{x \in \mathbf{Q} \mid 0 < x < 1\}$

5. $\{p/q \mid p \in \mathbf{N}, q \in \mathbf{N}, \text{and } p < q\}$

Use mathematical notation to describe:

6. The set of perfect cubes. (Assume we are talking about integers.)

7. The set of negative integers.

8. The set of real numbers greater than or equal to 17.

9. The set of rational numbers between –1 and 1, including the endpoints of the interval.

10. The set of integers between –1 and 1, not including these two numbers.

Prime Numbers

Definition: A natural number is prime if it has exactly two factors — itself and 1.

1. **The sieve of Eratosthenes**:
   a. On a list of the natural numbers, cross out the 1.
      Circle 2, cross out its multiples.
      Circle the first number that is not crossed out, cross out its multiples.
      Repeat, until all the numbers are either crossed out or circled.
   b. Explain what you accomplished.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90
91	92	93	94	95	96

2. **Making primes**: start with any set of small primes (for example {3, 5, 11}). Multiply them together. Add 1. Study the result. Is it prime? What are its prime factors? Do the prime factors include any from your starting set?

3. $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$
   a. Is $p_1 + 1$ prime? Is it in the set $\{p_1\}$?
   b. Is $p_1 \, p_2 + 1$ prime? Is it in the set $\{p_1, p_2\}$?
   c. Is $p_1 \, p_2 \, p_3 + 1$ prime? Is it in the set $\{p_1, p_2, p_3\}$?
   d. Is $p_1 \, p_2 \, p_3 \, p_4 + 1$ prime? Is it in the set $\{p_1, p_2, p_3, p_4\}$?
   e. Is $p_1 \, p_2 \, p_3 \, p_4 \, p_5 + 1$ prime? Is it in the set $\{p_1, p_2, p_3, p_4, p_5\}$?
   f. Show that $p_1 \, p_2 \, p_3 \, p_4 \, p_5 \, p_6 + 1$ is NOT prime. Call this number A.
   g. Is p_1 a factor of A? Is p_2? Is p_3? p_4? p_5? p_6?
   h. What are the prime factors of A? Are they in the set $\{p_1, p_2, p_3, p_4, p_5, p_6\}$?

4. Assume that there is a *finite* number n of prime numbers $\{p_1, p_2, \ldots, p_n\}$. Explain why this is impossible. Therefore:

Theorem: There is an infinite number of prime numbers.

Larger Primes

To check whether a number is prime, you need to check whether it has any factors in addition to itself and 1. It is not necessary to check all numbers.

5. For example, if you already know the number is not a multiple of 3, explain why it cannot be a multiple of 6.

6. Explain why you only need to check prime factors, and only up to the square root of the number.

Slumber Theory

Slumber theory is a silly branch of mathematics, which exists only on this page.

Any number can be **sliced** into a sequence of numbers.

Example: 365 can be sliced in four different ways: 3 | 6 | 5; 36 | 5; 3 | 65; or 365.

(Note that the slices are indicated by a vertical slash. Note also that in slumber theory, not slicing is considered a form of slicing.)

1. How many ways are there to slice a four-digit number?

A number is **slime** if it can be sliced into a sequence of primes.

Examples: 5 is slime, since it is already prime.
2027 is slime (2 | 02 | 7)
4,155,243,311 is slime (41 | 5 | 5 | 2 | 43 | 3 | 11)

2. Which one of the following numbers is slime? 12; 345; 6789
3. 2 is the only even prime. Find the first three even slimes.
4. There are no prime squares. Find the first two slime squares.
5. There are no prime cubes. Find the first two slime cubes.
6. 2 and 3 are the only consecutive numbers that are both prime. Find the first three pairs of consecutive numbers that are both slime.
7. There is no triple of consecutive numbers that are all prime. Find the first two triples of consecutive numbers that are all slime.
8. Prove that there are an infinite number of slime numbers.
9. Find the smallest number that is slime in more than one way. (In other words, it can be sliced into two different sequences of primes.)
10. Find the smallest number that is slime in more than two ways.

A number is a **super-slime** if you get a sequence of primes no matter how you slice it.

Example: 53 is a super-slime since 53 and 5 | 3 are both sequences of primes.

11. Prove that there are only a finite number of super-slimes.

Thinking About Infinity

Autobiography

It took Tristram Shandy two years to write about two days of his life.

1. Assuming he wants to write about every single day, and will keep going at the same rate, will he ever finish his autobiography?

2. Imagine he lives forever. Does that change your answer to the question?

Marbles

I have a bag with an infinite number of marbles in it.

- ◊ It is one minute to midnight. I take marbles 1-10 from the bag, and put them on the table. I put marble 1 in the trash.
- ◊ It is 1/2 minute to midnight. I take marbles 11-20 from the bag, and put them on the table. I put marble 2 in the trash.
- ◊ It is 1/3 minute to midnight. I take marbles 21-30 from the bag, and put them on the table. I put marble 3 in the trash.
- ◊ …

3. How many marbles are on the table at midnight? In the bag? In the trash? (Hint: where is marble 100? Can you name any marble that is in the bag? On the table?)

4. Change the way the marbles are selected to go into the trash, so that there are exactly ten marbles on the table at midnight.

5. Change the way the marbles are selected to go into the trash, so that there are an infinite number of marbles on the table at midnight.

On or off?

I have a lamp.

It is one minute to midnight. I turn it on.
It is 1/2 minute to midnight. I turn it off.
It is 1/3 minute to midnight. I turn it on.
…

6. At midnight, is the lamp on or off?

Hilbert's Hotel

You are in charge of Hotel Infinity. There are an infinite number of rooms, but all are taken.

7. Seventeen more customers show up. How can you offer them rooms without kicking anyone out of the hotel?

8. An infinite number of people show up. How can you offer them rooms without kicking anyone out of the hotel?

Countable Infinite Sets

An infinite set is said to be *countable* if it is equivalent to the natural numbers.

The Integers

1. Prove that the set of integers is countable.

Fraction Action

2. Write the fraction $\frac{0}{1}$ at the left side of the page. Write the “fraction” $\frac{1}{0}$ at the right side of the page. (Please ignore the fact that $\frac{1}{0}$ is a meaningless symbol.) Half way in between, write the fraction with numerator the sum of the numerators, and denominator the sum of the denominators: $\frac{1}{1}$ (since $\frac{0+1}{1+0} = \frac{1}{1}$.) Repeat the process, until there is no room left between the fractions. What do you notice about the list of fractions you have? Write down at least three observations.

3. Imagine all the fractions (lowest terms or not) were arranged on the coordinate plane, with the numerators along the y–axis, and the denominators along the x-axis. Draw a figure of that arrangement on graph paper. What do you notice about the arrangement? Write down at least three observations.

4. Compare the arrangements described in exercises 2 and 3. Write down at least three differences between them.

The Rational Numbers

5. Show that the set of rational numbers is countable.

The Real Numbers

Cantor proved that the set of real numbers is not countable.

Puzzle

6. Put the natural numbers in alphabetical order. (Just kidding. To make this a finite task, list the first eight numbers on such a list.)

Cardinal Numbers

Cardinal numbers are the numbers that denote the "size" of a set. The cardinal number for a finite set is a natural number, the number of elements in the set.

Georg Cantor called the cardinal number of the natural numbers $\aleph_0$ (aleph zero or aleph null -- aleph is the first letter in the Hebrew alphabet.)

1. Name several other sets that have $\aleph_0$ elements. What is another way to say the same thing?

Cantor proved that the set of natural numbers has *more than* $\aleph_0$ *subsets!* Another way to say this is that the cardinal number of the set of subsets of the natural numbers is greater than $\aleph_0$.

2. After listening to an explanation of that proof, write down a summary of it.

Cardinal Arithmetic

To define addition and multiplication for cardinal numbers, we need the following concepts from set theory:

Intersection of two sets A and B: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Two sets are *disjoint* if their intersection is empty

Union of two sets: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

An *ordered pair* (a, b) is an ordered list of two elements.
Note: in general, $(a, b) \neq (b, a)$, while for two-element sets $\{a, b\} = \{b, a\}$ always.

The *Cartesian product* of two sets A and B: $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

If #(A) stands for the cardinal number of set A, here are the definitions of addition and multiplication of ordinals. Then:

if A and B are disjoint: $\#(A) + \#(B) = \#(A \cup B)$

for any sets A, B: $\#(A) \cdot \#(B) = \#(A \times B)$

Subtraction and division are their inverse operations, respectively.

3. Show that these definitions work the usual way for natural numbers.

4. Calculate, and explain your answers. (Warning: some of these calculations are not defined, so be careful.)

a. $\aleph_0 + 2 =$	b. $\aleph_0 - 2 =$	c. $\aleph_0 \cdot 2 =$
d. $\aleph_0 / 2 =$	e. $\aleph_0 + \aleph_0 =$	f. $\aleph_0 - \aleph_0 =$
g. $\aleph_0 \cdot \aleph_0 =$	h. $\aleph_0 / \aleph_0 =$	i. $2^{\aleph_0} =$

“The Power of the Continuum”

Cantor proved that the number of real numbers between 0 and 1 is greater than $\aleph_0$. He called that number C, “the power of the continuum”.

1. After listening to an explanation of that proof, write down a summary of it.

Cantor was never able to prove that C = $\aleph_1$ (the next cardinal after $\aleph_0$.) This equation is called *the continuum hypothesis*. In 1938, Kurt Gödel proved that if the continuum hypothesis is assumed to be true, set theory remains consistent. In 1963, Paul Cohen proved that if the continuum hypothesis is assumed to be false, set theory remains consistent. Mathematicians say that the continuum hypothesis is *undecidable*.

2. Show that any two open intervals are equivalent.
   a. visually
   b. algebraically

3. Show that any two closed intervals are equivalent.

4. Show that (0,1] is equivalent to (0,1). (Not so easy!)

5. Show that [0,1) is equivalent to [0,1].

6. Show that (0,1) is equivalent to [0,1].

7. Show that any two intervals are equivalent, regardless of length or closed- or open-ness.

8. Show that the set of points on a circle is equivalent to the set of points on an interval.

9. Show that the set of real numbers is equivalent to (0,1).

Cantor proved that there are C points in the inside of a 1 by 1 square.

10. After listening to an explanation of that proof, write down a summary of it.

Infinite Sets Review

1. Imagine that you are managing $\aleph_0$ hotels with $\aleph_0$ rooms in each, and that all the rooms are filled. You decide it's too much work. Show that it is possible to move all the guests into just one of the hotels.

2. Imagine that the devil took you prisoner. He will let you go to heaven as soon as you guess the natural number he's thinking of. You get to make one guess a day. Can you find a strategy to make sure you'll get out in a finite time (and therefore spend an infinite amount of time in Heaven?)

3. Same question, but you must guess:
   a. the integer he's thinking of
   b. the pair of integers he's thinking of
   c. the rational number he's thinking of
   d. the real number he's thinking of

4. Choose a countable infinite set, and prove that it is indeed so.

5. Choose an uncountable infinite set, and prove that it is indeed so.

6. Give examples of proofs for showing that two sets are equivalent, of the following types:
   a. enumeration (making a list)
   b. visual / geometric
   c. algebraic (using a formula)
   d. using digits in decimal expansions

7. How does one prove that two sets are not equivalent?