Commun. Math. Phys. 214, 57 – 89 (2000)

Communications in
Mathematical Physics

Random Matrix Theory and $\zeta(1/2 + it)$

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Received: 20 December 1999 / Accepted: 24 March 2000

Abstract: We study the characteristic polynomials $Z(U,\theta)$ of matrices U in the Circular Unitary Ensemble (CUE) of Random Matrix Theory. Exact expressions for any matrix size N are derived for the moments of $|Z|$ and Z/Z^*, and from these we obtain the asymptotics of the value distributions and cumulants of the real and imaginary parts of $\log Z$ as $N \to \infty$. In the limit, we show that these two distributions are independent and Gaussian. Costin and Lebowitz [15] previously found the Gaussian limit distribution for $\mathrm{Im}\log Z$ using a different approach, and our result for the cumulants proves a conjecture made by them in this case. We also calculate the leading order $N \to \infty$ asymptotics of the moments of $|Z|$ and Z/Z^*. These CUE results are then compared with what is known about the Riemann zeta function $\zeta(s)$ on its critical line $\mathrm{Re}s = 1/2$, assuming the Riemann hypothesis. Equating the mean density of the non-trivial zeros of the zeta function at a height T up the critical line with the mean density of the matrix eigenvalues gives a connection between N and T. Invoking this connection, our CUE results coincide with a theorem of Selberg for the value distribution of $\log\zeta(1/2 + iT)$ in the limit $T \to \infty$. They are also in close agreement with numerical data computed by Odlyzko [29] for large but finite T. This leads us to a conjecture for the moments of $|\zeta(1/2 + it)|$. Finally, we generalize our random matrix results to the Circular Orthogonal (COE) and Circular Symplectic (CSE) Ensembles.

1. Introduction

We investigate the distribution of values taken by the characteristic polynomials

$$Z(U,\theta) = \det(I - Ue^{-i\theta}) \tag{1}$$

of $N \times N$ unitary matrices U with respect to the circular unitary ensemble (CUE) of random matrix theory (RMT). Our motivation is that it has been conjectured that the limiting distribution of the non-trivial zeros of the Riemann zeta function (and other

L-functions), on the scale of their mean spacing, is the same as that of the eigenphases θ_n of matrices in the CUE in the limit as $N \to \infty$ [28,29,31]. Hence the distribution of values taken by the zeta function might be expected to be related to those of $Z(U,\theta)$, averaged over the CUE.

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{2}$$

for $\mathrm{Re}s > 1$, and then by analytic continuation to the rest of the complex plane. It has infinitely many *non-trivial* zeros in the *critical strip* $0 < \mathrm{Re}s < 1$. The Riemann Hypothesis (RH) states that all of these non-trivial zeros lie on the *critical line* $\mathrm{Re}s = 1/2$; that is, $\zeta(1/2 + it) = 0$ has non-trivial solutions only when $t = t_n \in \mathbb{R}$.

Montgomery [28] has conjectured that the two-point correlations between the heights t_n (assumed real), on the scale of the mean asymptotic spacing $2\pi/\log t_n$, in the limit $n \to \infty$, are the same as those which exist between the eigenvalues of random complex hermitian matrices in the limit as the matrix size tends to infinity. Such matrices form the Gaussian Unitary Ensemble (GUE) of RMT. The GUE correlations are in turn the same as those of the phases θ_n of the eigenvalues of $N \times N$ unitary matrices, on the scale of their mean separation $2\pi/N$, averaged over the CUE, in the limit $N \to \infty$. (For a review of the spectral statistics of random matrices, see [27]).

This conjecture is supported by a theorem, also due to Montgomery [28], which implies that, in the appropriate limits, the Fourier transform of the two-point correlation function of the Riemann zeros coincides over a restricted range with the corresponding CUE result. It is also supported by extensive numerical computations [29].

Both the conjecture and Montgomery's theorem (again for restricted ranges) extend to all n-point correlations [30]. There is also strong numerical evidence in support of this generalization; for example, the distribution of spacings between adjacent zeros, measured in units of the mean spacing, appears to have the same limit as for the CUE [29]. Furthermore, heuristic calculations based on a Hardy-Littlewood conjecture for the pair correlation of the primes imply the validity of the generalized conjecture for all n, without restriction on the correlation range [24,7,9].

Thus all available evidence suggests that, in the limit as $N \to \infty$, local (i.e. short-range) statistics of the scaled (to have unit mean spacing) zeros $w_n = t_n \frac{1}{2\pi} \log \frac{t_n}{2\pi}$, defined by averaging over the zeros up to the N^{th}, coincide with the corresponding statistics of the similarly scaled eigenphases $\phi_n = \theta_n \frac{N}{2\pi}$, defined by averaging over the CUE of $N \times N$ unitary matrices.

This then implies that locally-determined statistical properties of $\zeta(s)$, high up the critical line, might be modelled by the corresponding properties of $Z(\theta)$, averaged over the CUE. One of our aims here is to explore this link by comparing certain RMT calculations with the following theorem and conjecture concerning the value distribution of $\zeta(1/2 + it)$.

First, according to a theorem of Selberg [33,29], for any rectangle E in $\mathbb{R}^2$,

$$\lim_{T\to\infty} \frac{1}{T} \left| \left\{ t : T \le t \le 2T, \frac{\log \zeta(1/2+it)}{\sqrt{(1/2)\log\log T}} \in E \right\} \right| = \frac{1}{2\pi} \int\int_E e^{-(x^2+y^2)/2} dx\, dy; \tag{3}$$

that is, in the limit as T, the height up the critical line, tends to infinity, the value distributions of the real and imaginary parts of $\log\zeta(1/2+iT)/\sqrt{(1/2)\log\log T}$ each tend independently to a Gaussian with unit variance and zero mean. Interestingly, Odlyzko's computations for these distributions when $T \approx t_{10^{20}}$ show systematic deviations from this limiting form [29]. For example, increasing moments of both the real and imaginary parts diverge from the Gaussian values. We review this data in more detail in Sect. 3.

Second, it is a long-standing conjecture that $f(\lambda)$, defined by

$$\lim_{T\to\infty}\frac{1}{(\log T)^{\lambda^2}}\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2\lambda}dt = f(\lambda)a(\lambda), \tag{4}$$

where

$$a(\lambda)=\prod_p\left\{(1-1/p)^{\lambda^2}\left(\sum_{m=0}^{\infty}\left(\frac{\Gamma(\lambda+m)}{m!\Gamma(\lambda)}\right)^2 p^{-m}\right)\right\}, \tag{5}$$

exists, and a much-studied problem then to determine the values it takes, in particular for integer λ (see, for example, [33, 21]). Obviously $f(0)=1$. It is also known that $f(1)=1$ [17] and $f(2)=1/12$ [20]. Based on number-theoretical arguments, Conrey and Ghosh have conjectured that $f(3)=42/9!$ [13], and Conrey and Gonek that $f(4)=24024/16!$ [14]. Conrey and Ghosh have obtained a lower bound for f when $\lambda\geq 0$ [12], and Heath-Brown [18] has obtained an upper bound for $0<\lambda<2$.

We now state our main results, all of which hold for $\theta\in\mathbb{R}$.

(i) For $\mathrm{Re}s>-1$,

$$M_N(s)=\left\langle |Z(U,\theta)|^s\right\rangle_{U(N)}=\prod_{j=1}^{N}\frac{\Gamma(j)\Gamma(j+s)}{(\Gamma(j+s/2))^2}, \tag{6}$$

where the average is over the CUE of $N\times N$ unitary matrices, that is over the group $U(N)$ with respect to the normalized translation-invariant (Haar) measure [34, 27]. Clearly the result extends by analytic continuation to the rest of the complex s-plane. It follows from (6) that, for integers $k\geq 0$, $M_N(2k)$ is a polynomial in N of degree k^2.

(ii) For $s\in\mathbb{C}$,

$$L_N(s)=\left\langle\left(\frac{Z(U,\theta)}{Z^*(U,\theta)}\right)^{s/2}\right\rangle_{U(N)}=\prod_{j=1}^{N}\frac{(\Gamma(j))^2}{\Gamma(j+s/2)\Gamma(j-s/2)}, \tag{7}$$

where $\arg Z(U,\theta)$ is defined by continuous variation along $\theta-i\epsilon$, starting at $-i\epsilon$, in the limit $\epsilon\to 0$, assuming θ is not equal to any of the eigenphases θ_n, with $\log Z(U,\theta-i\epsilon)\to 0$ as $\epsilon\to\infty$. Thus $\mathrm{Im}\log Z(U,\theta)$ has a jump discontinuity of size π when $\theta=\theta_n$.

(iii) The value distributions of the real and imaginary parts of $\log Z(U,\theta)/\sqrt{(1/2)\log N}$ each tend independently to a Gaussian with zero mean and unit variance in the limit as $N\to\infty$. This corresponds directly to Selberg's theorem (3) for $\log\zeta(1/2+it)$ if we identify the mean density of the eigenangles θ_n, $N/2\pi$, with the mean density of the Riemann zeros at a height T up the critical line, $\frac{1}{2\pi}\log\frac{T}{2\pi}$; that is if

$$N=\log\frac{T}{2\pi}. \tag{8}$$

This is a natural connection to make between matrix size and position on the critical line, because the mean eigenvalue density is the only parameter in the theory of spectral statisitics for the circular and Gaussian ensembles of RMT.

The central limit theorem for $\mathrm{Im}\log Z$ was first proved by Costin and Lebowitz [15] for the characteristic polynomials of matrices in the GUE (see also [32] for a review of related results). Our proof is new, and goes further in that it allows us to compute the cumulants.

(iv) Let $Q_n(N)$ be the n^{th} cumulant of the distribution of values of $\mathrm{Re}\log Z$, defined with respect to the CUE, and let $R_n(N)$ be the corresponding cumulant for $\mathrm{Im}\log Z$. Then

$$Q_n(N) = \frac{2^{n-1}-1}{2^{n-1}} \sum_{j=1}^{N} \psi^{(n-1)}(j), \tag{9}$$

and

$$R_n(N) = \begin{cases} \frac{(-1)^{1+n/2}}{2^{n-1}} \sum_{j=1}^{N} \psi^{(n-1)}(j) & n \quad \text{even} \\ 0 & n \quad \text{odd} \end{cases}, \tag{10}$$

where ψ is a polygamma function. Thus $Q_1(N) = R_1(N) = 0$. It is straightforward to obtain a complete (large N) asymptotic expansion for these cumulants. For example,

$$Q_2(N) = \left\langle (\mathrm{Re}\log Z)^2 \right\rangle_{U(N)} = \frac{1}{2}\log N + \frac{1}{2}(\gamma+1) + \frac{1}{24N^2} + O(N^{-4}), \tag{11}$$

$$Q_n(N) = (-1)^n \frac{2^{n-1}-1}{2^{n-1}} \zeta(n-1)\Gamma(n) + O(N^{2-n}), \quad n \geq 3, \tag{12}$$

and

$$\begin{aligned} R_2(N) &= \left\langle (\mathrm{Im}\log Z)^2 \right\rangle_{U(N)} = Q_2(N) \\ &= \frac{1}{2}\log N + \frac{1}{2}(\gamma+1) + \frac{1}{24N^2} + O(N^{-4}), \end{aligned} \tag{13}$$

$$R_{2k}(N) = \frac{(-1)^{(k+1)}}{2^{2k-1}} \zeta(2k-1)\Gamma(2k) + O(N^{2-2k}), \quad k > 1. \tag{14}$$

The fact that when $k > 1$ $R_{2k}(N)$ tends to a constant as $N \to \infty$ proves a conjecture made by Costin and Lebowitz [15].

(v) It follows from (6) that

$$f_{\mathrm{CUE}}(\lambda) = \lim_{N\to\infty} \frac{1}{N^{\lambda^2}} \left\langle |Z(U,\theta)|^{2\lambda} \right\rangle_{U(N)} = \frac{G^2(1+\lambda)}{G(1+2\lambda)}, \tag{15}$$

where G denotes the Barnes G-function [3], and hence that $f_{\mathrm{CUE}}(0) = 1$ (trivial) and

$$f_{\mathrm{CUE}}(k) = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \tag{16}$$

for integers $k \geq 1$. Thus, for example, $f_{\mathrm{CUE}}(1) = 1$, $f_{\mathrm{CUE}}(2) = 1/12$, $f_{\mathrm{CUE}}(3) = 42/9!$ and $f_{\mathrm{CUE}}(4) = 24024/16!$. $f_{\mathrm{CUE}}(k)$ is the coefficient of N^{k^2} in $M_N(2k)$, which, as noted above, is a polynomial in N of degree k^2. The coefficients of the lower-order terms can also be calculated explicitly. Similarly,

$$\lim_{N\to\infty} N^{\lambda^2} L_N(2\lambda) = G(1-\lambda)G(1+\lambda). \tag{17}$$

The results listed above allow us to compute the value distributions of $\operatorname{Re}\log Z$, $\operatorname{Im}\log Z$, and $|Z|$, for any N, and to derive explicit asymptotics for these distributions when $N \to \infty$.

In comparing our random-matrix results with what is known about the zeta function, we find the following. First, the value distributions of $\operatorname{Re}\log Z$ and $\operatorname{Im}\log Z$ coincide with Odlyzko's numerical data for the corresponding distributions of the values of the zeta function at a height T up the critical line if we make the identification (8). This implies that, with respect to its local statistics, the zeta function behaves like a finite polynomial of degree N given by (8). The value distribution of $|Z|$ is similarly in agreement with our numerical data for that of $|\zeta(1/2+it)|$.

It is important at this stage to remark that Montgomery's conjecture (and its generalization) refers to the *short range* correlations (i.e. correlations on the scale of mean separation) between the Riemann zeros at a height T up the critical line, in the limit as $T \to \infty$. The finite-T correlations take the form of a sum of two contributions, one being the random-matrix limit and the other representing *long range* deviations which may be expressed as a sum over the primes [4,25,5]. This is also known to be the case for the second moment of $\operatorname{Im}\log\zeta(1/2+it)$. Specifically, Goldston [16] has proved, under the assumption of RH and Montgomery's conjecture, that as $T \to \infty$,

$$\frac{1}{T}\int_0^T (\operatorname{Im}\log\zeta(1/2+it))^2 dt$$
$$= \frac{1}{2}\log\log\frac{T}{2\pi} + \frac{1}{2}(\gamma+1) + \sum_{m=2}^{\infty}\sum_p \frac{(1-m)}{m^2}\frac{1}{p^m} + o(1). \tag{18}$$

Here the first two terms on the right-hand side agree with those in (13) if we again make the identification (8). The same general behaviour also holds for the higher moments of $\log\zeta$. It is plausible then that the moments of $|\zeta(1/2+it)|$ (which are determined by long-range correlations between the zeros) asymptotically split into a product of two terms, one coming from random matrix theory and the other from the primes. Taken together with the fact that $f_{\mathrm{CUE}}(k) = f(k)$ for $k = 1, 2$, and, conjecturally, for $k = 3, 4$, this leads us to conjecture that

$$f(\lambda) = f_{\mathrm{CUE}}(\lambda) \tag{19}$$

for all λ where the moments are defined. This is further supported by other heuristic arguments, and by the fact that the product of $a(\lambda)$ and our formula (6) for the moments of $|Z(U,\theta)|$ matches Odlyzko's numerical data for the moments of $|\zeta(1/2+it)|$ over the range $0 < \lambda \le 2$, where we can compare them, again making the identification (8).

These results were first announced in lectures at the Erwin Schrödinger Institute in Vienna, in September 1998 and at the Mathematical Sciences Research Institute in Berkeley in June 1999.

The structure of this paper is as follows. We derive the CUE results listed above in Sect. 2, and then compare them with numerical data (almost all taken from [29]) for the Riemann zeta-function in Sect. 3. Our conjecture (19) is also discussed in more detail in this section. In Sect. 4 we state the analogues of the CUE results for the other circular ensembles of RMT, namely the Circular Orthogonal (COE) and Circular Symplectic (CSE) Ensembles.

Numerical evidence suggests that the eigenvalues of the laplacian on certain compact (non-arithmetic) surfaces of constant negative curvature are asymptotically the same as those of matrices in the COE, and so our results might be expected to describe the associated Selberg zeta functions. More generally, it has been suggested that in the semiclassical ($\hbar \to 0$) limit the quantum eigenvalue statistics of all generic, classically chaotic systems are related to those of the RMT ensembles (COE for time-reversal symmetric integer-spin systems, CUE for non-time-reversal integer-spin systems, and CSE for half-integer-spin systems) [10], and our results might then be expected to apply to the corresponding quantum spectral determinants. It is worth noting in this respect that extensive numerical evidence supports the conclusion that for classically chaotic systems the value distribution of the fluctuating part of the spectral counting function (which is proportional to the imaginary part of the logarithm of the spectral determinant) tends to a Gaussian in the semiclassical limit [6, 2].

Finally, it is worth remarking that Montgomery's conjecture extends to many other classes of L-functions, and hence our results are expected to apply to them too, in the same way. However, Katz and Sarnak [22, 23] have conjectured that correlations between the zeros *low down* on the critical line, defined by averaging over L-functions within certain particular families, are described not by averages over the CUE, that is, over the unitary group $U(N)$, but by averages over other classical compact groups, for example the orthogonal group $O(N)$ or the unitary symplectic group $USp(2N)$. Thus the value distributions within these families close to the symmetry point $t = 0$ on the critical line will also be described by averages over the corresponding groups. We shall present our results in this case in a second paper [26].

2. CUE Random Matrix Polynomials

2.1. Generating functions. All of our CUE random-matrix results follow from the formulae (6) and (7) for the generating functions $M_N(s)$ and $L_N(s)$, and our goal in this section is to derive these expressions.

Consider first $M_N(s)$. We start with the representation of $Z(U, \theta)$ in terms of the eigenvalues $e^{i\theta_n}$ of U:

$$Z(U, \theta) = \prod_{n=1}^{N} \left(1 - e^{i(\theta_n - \theta)}\right). \tag{20}$$

The CUE average can then be performed using the joint probability density for the eigenphases θ_n, $((2\pi)^N N!)^{-1} \prod_{j<m} \left|e^{i\theta_j} - e^{i\theta_m}\right|^2$ [34, 27]. Thus

$$\langle |Z|^s \rangle_{U(N)} = \frac{1}{(2\pi)^N N!} \int_0^{2\pi} \cdots \int_0^{2\pi} d\theta_1 \cdots d\theta_N \times \prod_{1 \le j < m \le N} \left|e^{i\theta_j} - e^{i\theta_m}\right|^2 \left|\prod_{n=1}^{N} (1 - e^{i(\theta_n - \theta)})\right|^s. \tag{21}$$

This integral can be evaluated exactly using Selberg's formula (see, for example, Chapter 17 of [27]):

$$\begin{aligned} &J(a,b,\alpha,\beta,\gamma,N) \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\prod_{1\le j<\ell\le N}(x_j-x_\ell)\right|^{2\gamma}\prod_{j=1}^{N}(a+ix_j)^{-\alpha}(b-ix_j)^{-\beta}dx_j \qquad (22)\\ &= \frac{(2\pi)^N}{(a+b)^{(\alpha+\beta)N-\gamma N(N-1)-N}}\prod_{j=0}^{N-1}\frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+\beta-(N+j-1)\gamma-1)}{\Gamma(1+\gamma)\Gamma(\alpha-j\gamma)\Gamma(\beta-j\gamma)}, \end{aligned}$$

where a, b, α, β and γ are complex numbers, Re a, Re b, Re α and Re β are all greater than zero, Re $(\alpha+\beta)>1$ and

$$-\frac{1}{N} < \mathrm{Re}\ \gamma < \min\left(\frac{\mathrm{Re}\ \alpha}{N-1}, \frac{\mathrm{Re}\ \beta}{N-1}, \frac{\mathrm{Re}\ (\alpha+\beta-1)}{2(N-1)}\right). \qquad (23)$$

To see this, note that (21) can be written in the form

$$\begin{aligned} \langle |Z|^s\rangle_{U(N)} &= \frac{2^{N(N-1)}2^{sN}}{N!(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi} d\theta_1\cdots d\theta_N \qquad (24)\\ &\times \prod_{1\le j<m\le N}\left|\sin(\theta_j/2-\theta_m/2)\right|^2\prod_{n=1}^{N}|\sin(\theta_n/2-\theta/2)|^s . \end{aligned}$$

Clearly this integral is independent of θ (as it must be, since we are averaging over all unitary matrices) and so we set $\theta=0$. Using $\sin(\theta_j-\theta_m)=\sin\theta_j\cos\theta_m-\cos\theta_j\sin\theta_m$, we then have

$$\begin{aligned} \langle |Z|^s\rangle_{U(N)} &= \frac{2^{N^2+sN}}{N!(2\pi)^N}\int_0^{\pi}\cdots\int_0^{\pi} d\theta_1\cdots d\theta_N \prod_{1\le j<m\le N}\left|\cot\theta_m-\cot\theta_j\right|^2 \\ &\times\prod_{n=1}^{N}\left(\sin^2\theta_n\right)^{N-1}\prod_{k=1}^{N}|\sin\theta_k|^s . \qquad (25) \end{aligned}$$

Finally, the change of variables $x_n=\cot\theta_n$ gives

$$\begin{aligned} \langle |Z|^s\rangle_{U(N)} &= \frac{2^{N^2+sN}}{N!(2\pi)^N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dx_1\cdots dx_N \prod_{1\le j<m\le N}|x_m-x_j|^2 \\ &\times\prod_{n=1}^{N}((1+ix_n)(1-ix_n))^{-N-s/2} \\ &= \frac{2^{N^2+sN}}{N!(2\pi)^N}J(1,1,N+s/2,N+s/2,1,N) \\ &= \prod_{j=1}^{N}\frac{\Gamma(j)\Gamma(s+j)}{(\Gamma(j+s/2))^2}, \qquad (26) \end{aligned}$$

provided Re$s > -1$, which is just the result (6). Clearly the product (26) has an analytic continuation to the rest of the complex plane.

Consider next $L_N(s)$. Note first that, according to the definition given in the Introduction,

$$\left(\frac{Z}{Z^*}\right)^{\frac{1}{2}} = \exp\left(i \operatorname{Im} \log Z(U,\theta)\right)$$
$$= \exp\left(-i \sum_{n=1}^{N} \sum_{m=1}^{\infty} \frac{\sin[(\theta_n - \theta)m]}{m}\right), \tag{27}$$

where for each value of n, the sum of sines lies in $(-\pi, \pi]$. Hence, again using the joint probability density of the eigenphases θ_n,

$$\left\langle \left(\frac{Z}{Z^*}\right)^{\frac{s}{2}} \right\rangle_{U(N)} = \frac{1}{N!(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} d\theta_1 \cdots d\theta_N \prod_{1\le j<m\le N} \left|e^{i\theta_j} - e^{i\theta_m}\right|^2$$
$$\times \prod_{n=1}^{N} \exp\left(-is \sum_{k=1}^{\infty} \frac{\sin[(\theta_n - \theta)k]}{k}\right). \tag{28}$$

As before, this integral is independent of θ, and so we set $\theta = 0$.

The sum in (28) can be evaluated using

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2}, \quad \text{for} \quad 0 < x < 2\pi. \tag{29}$$

Note that this relation keeps the sine sum within the range $(-\pi, \pi]$ prescribed by the definition of the logarithm. Substituting (29) into (28) then yields

$$\left\langle \left(\frac{Z}{Z^*}\right)^{\frac{s}{2}} \right\rangle_{U(N)} = \frac{2^{N(N-1)}}{N!(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} d\theta_1 \cdots d\theta_N \prod_{1\le j<m\le N} \left|\sin(\theta_j/2 - \theta_m/2)\right|^2$$
$$\times \prod_{n=1}^{N} \exp\left(-\frac{is}{2}(\pi - \theta_n)\right). \tag{30}$$

Making the transformation $\phi_j = \theta_j/2 - \pi/2$ and using the identity $\sin(\phi_j - \phi_m) = (\tan\phi_j - \tan\phi_m) \times \cos\phi_j \cos\phi_m$ gives

$$\left\langle \left(\frac{Z}{Z^*}\right)^{\frac{s}{2}} \right\rangle_{U(N)} = \frac{2^{N^2}}{N!(2\pi)^N} \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} d\phi_1 \cdots d\phi_N \tag{31}$$
$$\times \prod_{1\le j<m\le N} \left|\tan\phi_j - \tan\phi_m\right|^2 \prod_{n=1}^{N} (\cos^2\phi_n)^{N-1}$$
$$\times \prod_{k=1}^{N} (\cos\phi_k + i\sin\phi_k)^s.$$

Finally, changing variables to $x_j = \tan\phi_j$, we have that

$$\left\langle \left(\frac{Z}{Z^*}\right)^{\frac{s}{2}} \right\rangle_{U(N)} = \frac{2^{N^2}}{N!(2\pi)^N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_N \prod_{1\le j<m\le N} \left|x_j - x_m\right|^2 \times \prod_{n=1}^{N} \left(\frac{1}{1+x_n^2}\right)^N \times \prod_{k=1}^{N} \left(\frac{1}{\sqrt{1+x_k^2}} + i\frac{x_k}{\sqrt{1+x_k^2}}\right)^s$$

$$= \frac{2^{N^2}}{N!(2\pi)^N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_N \prod_{1\le j<m\le N} \left|x_j - x_m\right|^2 \times \prod_{n=1}^{N} \left(\frac{1}{1+x_n^2}\right)^N \times \prod_{k=1}^{N} \left(\frac{\sqrt{1+ix_k}}{\sqrt{1-ix_k}}\right)^s. \tag{32}$$

This is in the form of Selberg's integral (22) with $a = b = 1$, $\alpha = N - s/2$, $\beta = N + s/2$ and $\gamma = 1$ (the condition (23) is satisfied when $|s| < 2$) and so we have

$$\left\langle \left(\frac{Z}{Z^*}\right)^{\frac{s}{2}} \right\rangle_{U(N)} = \prod_{j=1}^{N} \frac{(\Gamma(j))^2}{\Gamma(j+s/2)\Gamma(j-s/2)}, \tag{33}$$

as required.

2.2. Value distribution of Re log Z. All information about the value distribution of Re log Z can be obtained from the generating function $M_N(s)$: the moments may be obtained from the coefficients in the Taylor expansion of M_N about $s = 0$,

$$M_N(s) = \sum_{j=0}^{\infty} \frac{\langle(\log|Z|)^j\rangle_{U(N)}}{j!} s^j; \tag{34}$$

the corresponding cumulants $Q_j(N)$ are related to the Taylor coefficients of $\log M_N$,

$$\log M_N(s) = \sum_{j=1}^{\infty} \frac{Q_j(N)}{j!} s^j; \tag{35}$$

and the probability density for the values taken by Re log Z,

$$\rho_N(x) =< \delta(\log|Z| - x) >_{U(N)}, \tag{36}$$

is given by

$$\rho_N(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iyx} M_N(iy) dy. \tag{37}$$

We now analyse these general formulae using the explicit expression (6) for $M_N(s)$.

Differentiating $\log M_N(s)$, we have that

$$Q_n(N) = \frac{2^{n-1}-1}{2^{n-1}} \sum_{j=1}^{N} \psi^{(n-1)}(j), \tag{38}$$

where

$$\psi^{(n)}(z) = \frac{d^{n+1}\log\Gamma(z)}{dz^{n+1}} \tag{39}$$

are the polygamma functions. Thus it follows immediately that

$$Q_1(N) = \langle(\log|Z|)\rangle_{U(N)} = 0. \tag{40}$$

Furthermore, substituting the well-known integral representation for the polygamma functions [1], when $n \geq 2$,

$$\begin{aligned} Q_n(N) &= \frac{2^{n-1}-1}{2^{n-1}} \sum_{j=1}^{N} (-1)^n \int_0^\infty \frac{t^{n-1}e^{-jt}}{1-e^{-t}} dt \\ &= \frac{2^{n-1}-1}{2^{n-1}} (-1)^n \int_0^\infty \frac{t^{n-1}e^{-t}}{1-e^{-t}} \frac{1-e^{-Nt}}{1-e^{-t}} dt \\ &= \frac{2^{n-1}-1}{2^{n-1}} (-1)^n \int_0^\infty \frac{e^{-t}}{1-e^{-t}} \\ &\quad \times \left((n-1)t^{n-2} - (n-1)t^{n-2}e^{-Nt} + Nt^{n-1}e^{-Nt}\right) dt, \end{aligned} \tag{41}$$

where the last equality follows from an integration by parts.

Consider first the second cumulant $Q_2(N)$. Rearranging the integrand in the final equality of (41),

$$Q_2(N) = \frac{1}{2} \int_0^\infty \left(\frac{1-e^{-Nt}}{1-e^{-t}} e^{-t} + Nt \frac{e^{-(N+1)t}}{1-e^{-t}} \right) dt, \tag{42}$$

and so, re-expanding the terms written as fractions to give geometric series and integrating these term-by-term, we have that

$$Q_2(N) = \langle(\log|Z|)^2\rangle_{U(N)} = \frac{1}{2} \sum_{n=1}^{N} \frac{1}{n} + \frac{N}{2} \sum_{n=N+1}^{\infty} \frac{1}{n^2}. \tag{43}$$

The large-N asymptotics can then be obtained by substituting

$$\sum_{k=1}^{n} \frac{1}{k} = \gamma + \log n + \frac{1}{2n} - \sum_{k=2}^{\infty} \frac{A_k}{n(n+1)\cdots(n+k-1)}, \tag{44}$$

where $A_k = \frac{1}{k}\int_0^1 x(1-x)(2-x)(3-x)\cdots(k-1-x)dx$, into the first sum and applying the Euler–Maclaurin formula to the second. Any number of terms in the expansion in inverse powers of N can be calculated in this way; for example

$$Q_2(N) = \langle(\log|Z|)^2\rangle_{U(N)} = \frac{1}{2}\log N + \frac{1}{2}(\gamma+1) + \frac{1}{24N^2} - \frac{1}{80N^4} + O\left(\frac{1}{N^6}\right). \tag{45}$$

Consider next the cumulants $Q_n(N)$ when $n \geq 3$. We now write

$$Q_n(N) = \frac{2^{n-1}-1}{2^{n-1}}(-1)^n \int_0^\infty \left((n-1)\frac{t^{n-2}}{e^t-1} + (Nt^{n-1}-(n-1)t^{n-2})\frac{e^{-Nt}}{e^t-1} \right) dt. \tag{46}$$

The first term, which is independent of N, can be integrated explicitly using a well-known representation of the zeta-function [33]. Changing variables in the second to $y = tN$ then gives

$$Q_n(N) = \frac{2^{n-1}-1}{2^{n-1}}(-1)^n \times \left(\Gamma(n)\zeta(n-1) + \frac{1}{N^{n-1}} \int_0^\infty (y^{n-1}-(n-1)y^{n-2})e^{-y}\frac{1}{e^{y/N}-1} dy \right). \tag{47}$$

The N-dependent term in this equation clearly vanishes in the limit as $N \to \infty$. Its large-N asymptotics can be obtained by expanding $(e^{y/N}-1)^{-1}$ in powers of y/N and then integrating term-by-term; for example

$$Q_n(N) = \frac{2^{n-1}-1}{2^{n-1}}(-1)^n \left(\Gamma(n)\zeta(n-1) - \frac{(n-3)!}{N^{n-2}} \right) + O(N^{1-n}). \tag{48}$$

It follows immediately from the fact that $Q_n(N)/(Q_2(N))^{n/2} \to 0$ as $N \to \infty$ for all $n > 2$ that the value distribution of $\mathrm{Re}\log Z/\sqrt{Q_2(N)}$ tends to a Gaussian in this limit. Specifically, we have from (37) and the definition of the cumulants that if

$$\tilde{\rho}_N(x) = \sqrt{Q_2(N)}\rho_N(\sqrt{Q_2(N)}x) \tag{49}$$

then

$$\tilde{\rho}_N(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-iyx - \frac{y^2}{2} - \frac{iQ_3y^3}{3!Q_2^{3/2}} + \frac{Q_4y^4}{4!Q_2^2} + \cdots \right) dy. \tag{50}$$

Hence all terms in the exponential that involve higher powers of y than y^2 vanish in the limit as $N \to \infty$. Evaluating the resulting Gaussian integral then gives

$$\lim_{N\to\infty} \tilde{\rho}_N(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2} \right). \tag{51}$$

The large-N asymptotics describing the approach to this limit can be obtained by retaining more terms in (50). There are several ways to do this. One is to expand the exponential of all terms that involve higher powers of y than y^2 as a series in increasing powers of y, so that

$$\tilde{\rho}_N(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2} \right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyx}e^{-y^2/2} \left(\frac{Q_3(iy)^3}{3!Q_2^{3/2}} + \frac{Q_4(iy)^4}{4!Q_2^2} + \cdots \right. + \left(\frac{Q_3(iy)^3}{3!Q_2^{3/2}} + \frac{Q_4(iy)^4}{4!Q_2^2} + \cdots \right)^2 \Big/ 2!$$

$$+\left(\frac{Q_3(iy)^3}{3!Q_2^{3/2}}+\frac{Q_4(iy)^4}{4!Q_2^2}+\cdots\right)^3\Big/3!+\cdots\Bigg)dy$$

$$=\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)+\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iyx}e^{-y^2/2}\left(\frac{A_3(iy)^3}{Q_2^{3/2}}+\frac{A_4(iy)^4}{Q_2^2}\right.$$
$$\left.+\frac{A_5(iy)^5}{Q_2^{5/2}}+\cdots\right)dy, \tag{52}$$

where the coefficients $A_n(N)$ are defined in terms of combinations of the cumulants $Q_n(N)$ with $n \geq 3$ (for example, $A_3 = Q_3/3!$). Integrating term-by-term then gives

$$\tilde{\rho}_N(x)=\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)+\frac{1}{\sqrt{2\pi}}\sum_{m=3}^{\infty}\frac{A_m}{Q_2^{m/2}}e^{-x^2/2}$$
$$\times\sum_{p=0}^{m}\binom{m}{p}x^p\begin{cases}i^{m-p}(m-p-1)!!, & m-p \text{ even}\\ 0, & m-p \text{ odd}\end{cases} \tag{53}$$

from which it follows that the deviation from the Gaussian limit is of the order of $(\log N)^{-3/2}$ (because $A_n(N) \to$ constant as $N \to \infty$).

It may be seen from (53) that it is only in the limit as $N \to \infty$ that $\tilde{\rho}_N(x)$ becomes even in x: when N is finite it is asymmetric about $x = 0$. This can be traced back to the fact that the series in the exponential in (50) involves both even and odd powers of y. Indeed, the dominant $N \to \infty$ asymptotics can also be computed by retaining only the y^3 term in the exponential (and not expanding the exponential as a series itself). Thus

$$\tilde{\rho}_N(x)\sim\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-ixy-\frac{y^2}{2}-\frac{iQ_3y^3}{3!Q_2^{3/2}}\right)dy, \tag{54}$$

and this integral can then be computed exactly in terms of the Airy function $\mathrm{Ai}(z)$, giving

$$\tilde{\rho}_N(x)\sim\sqrt{Q_2}\left(\frac{-2}{Q_3}\right)^{1/3}\exp\left(\frac{Q_2^3}{3Q_3^2}+\frac{xQ_2^{3/2}}{Q_3}\right)\mathrm{Ai}\left(\frac{2^{1/3}x\sqrt{Q_2}}{Q_3^{1/3}}+\frac{Q_2^2}{2^{2/3}Q_3^{4/3}}\right), \tag{55}$$

which itself is manifestly asymmetric in x.

Finally, we note that the formulae derived above lead directly to corresponding expressions for the moments, since these may be related to the cumulants by taking the exponential of the right-hand side of (35), re-expanding as a Taylor series in powers of s, and equating the coefficients with those in (34). Thus, for example, it is straightforward to see that

$$\langle(\log|Z|)^n\rangle_{U(N)}=\frac{d^n}{ds^n}M_N(s)\ |_{s=0}$$
$$=\begin{cases}(2k-1)!!\langle(\log|Z|)^2\rangle_{U(N)}^k+O((\log N)^{k-2}) & \text{if } n=2k\\ O((\log N)^{k-1}) & \text{if } n=2k+1\end{cases}, \tag{56}$$

where the second moment is given by (45). This again implies that the limiting distribution is Gaussian.

2.3. Value distribution of Im log Z. In the same way as for the real part, all information about the value distribution of Im log Z is contained in the generating function $L_N(s)$. Thus,

$$L_N(-it) = \sum_{j=0}^{\infty} \frac{\langle(\mathrm{Im}\log Z)^j\rangle_{U(N)}}{j!} t^j, \tag{57}$$

and similarly for the corresponding cumulants R_j,

$$\log L_N(-it) = \sum_{j=1}^{\infty} \frac{R_j(N)}{j!} t^j, \tag{58}$$

where $L_N(s)$ is given by (7). Likewise, the probability density for the values taken by Im log Z,

$$\sigma_N(x) =< \delta(\mathrm{Im}\log Z - x) >_{U(N)}, \tag{59}$$

is given by

$$\sigma_N(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyx} L_N(y)dy. \tag{60}$$

All of the results of the previous section then extend immediately to Im log Z. Thus, taking the logarithm of (7) and differentiating,

$$\begin{aligned} R_n(N) &= \frac{(-i)^n}{2^n}\sum_{j=1}^{N}\left[-\psi^{(n-1)}(j) + (-1)^{n-1}\psi^{(n-1)}(j)\right] \\ &= \begin{cases} 0 & \text{if } n \text{ odd} \\ \frac{(-1)^{n/2+1}}{2^{n-1}}\sum_{j=1}^{N}\psi^{(n-1)}(j) & \text{if } n \text{ even} \end{cases}. \end{aligned} \tag{61}$$

The fact that all of the odd cumulants are zero implies that all of the odd moments are also zero. This is the main difference compared to the case of Re log Z. For the even cumulants we have

$$R_{2m}(N) = \frac{(-1)^{m+1}}{2^{2m-1}-1} Q_{2m}(N), \tag{62}$$

and so the asymptotics computed in the previous section apply immediately in this case too. Thus

$$\begin{aligned} R_2(N) &= \langle(\mathrm{Im}\log Z)^2\rangle_{U(N)} \\ &= Q_2(N) = \frac{1}{2}\log N + \frac{1}{2}(\gamma+1) + \frac{1}{24N^2} + O\left(\frac{1}{N^4}\right), \end{aligned} \tag{63}$$

and for $m > 1$,

$$R_{2m}(N) = \frac{(-1)^{m+1}}{2^{2m-1}}\left(\Gamma(2m)\zeta(2m-1) - \frac{(2m-3)!}{N^{2m-2}}\right) + O(N^{1-2m}). \tag{64}$$

The fact that, for $m \geq 2$, $R_{2m}/R_2^m \to 0$ as $N \to \infty$ implies that the value distribution of $\mathrm{Im}\log Z/\sqrt{R_2(N)}$ tends to a Gaussian in the limit. This was first proved by Costin and Lebowitz [15] for the GUE of random matrices. Specifically, they proved that the fluctuating part of the eigenvalue counting function has a limiting value distribution that is Gaussian. The connection comes because the two functions are the same, up to multiplication by π; specifically, if $n(U, a, b)$ denotes the number of eigenvalues of U with $a < \theta_n < b$, then

$$n(U, a, b) = \frac{(b-a)N}{2\pi} + \frac{1}{\pi}\mathrm{Im}\log\frac{Z(U, b)}{Z(U, a)}, \tag{65}$$

assuming that none of the eigenphases coincides with the end-points of the range. In addition, Costin and Lebowitz conjectured that, for $m \geq 2$, $R_{2m}(N) \to$ constant when $N \to \infty$. Our asymptotic formula (64) proves this for averages over the CUE, and provides the value of the constant. Wieand [35] has independently given a proof of the central limit theorem for $n(U, a, b) - (b-a)N/(2\pi)$ in the CUE case.

The asymptotics of the approach to the Gaussian can be calculated from (58) and (60). Defining

$$\tilde{\sigma}_N(x) = \sqrt{R_2(N)}\sigma_N(\sqrt{R_2(N)}x), \tag{66}$$

we have that

$$\begin{aligned}\tilde{\sigma}_N(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-iyx - \frac{y^2}{2}\right)\\ &\quad\times\exp\left(\frac{R_4 y^4}{R_2^2 4!} - \frac{R_6 y^6}{R_2^3 6!} + \cdots\right)dy \\ &= \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-iyx - \frac{y^2}{2}\right)\\ &\quad\times\left[\frac{C_4 y^4}{R_2^2} + \frac{C_6 y^6}{R_2^3} + \frac{C_8 y^8}{R_2^4} + \cdots\right]dy,\end{aligned} \tag{67}$$

where the coefficents $C_n(N)$ are defined in terms of the cumulants $R_{2m}(N)$ with $m > 1$; for example $C_4(N) = R_4(N)/4!$. Thus, integrating term-by-term,

$$\begin{aligned}\tilde{\sigma}_N(x) &= \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right) + \frac{1}{\sqrt{2\pi}}\sum_{m=2}^{\infty}\frac{C_{2m}}{R_2^m}e^{-x^2/2}\\ &\quad\times\sum_{p=0}^{2m}\binom{2m}{p}(-ix)^p\begin{cases}(2m-p-1)!! & \text{if } 2m-p \text{ is even}\\ 0 & \text{if } 2m-p \text{ is odd}\end{cases}.\end{aligned} \tag{68}$$

In this case $\tilde{\sigma}_N(x)$ is an even function of x for all N, and not just in the limit as $N \to \infty$. This is a consequence of the fact that all of the odd cumulants are identically zero. It follows from (68) that the deviation from the Gaussian limit is of the order of $(\log N)^{-2}$, and so is asymptotically smaller than in the case of $\mathrm{Re}\log Z$.

Finally, the expressions derived above for the cumulants may again be used to deduce information about the moments. We have already noted that the odd moments are identically zero. For the even moments we find the usual Gaussian relationship:

$$\langle(\operatorname{Im}\log Z)^{2k}\rangle_{U(N)} = (2k-1)!!\langle(\operatorname{Im}\log Z)^2\rangle_{U(N)}^k + O((\log N)^{k-2}), \tag{69}$$

where the asymptotics of the second moment are given by (63).

2.4. Independence. We have shown in Sects. 2.2 and 2.3 that the values of both Re log Z and Im log Z have a Gaussian limit distribution as $N\to\infty$. Our purpose in this section is to show that they are also independent in this limit.

The generating function for the joint distribution is

$$\begin{aligned}\langle|Z|^t e^{is(\operatorname{Im}\log Z)}\rangle_{U(N)} &= \frac{1}{N!(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi} d\theta_1\cdots d\theta_N \prod_{1\le j<k\le N}\left|e^{i\theta_j}-e^{i\theta_k}\right|^2\\ &\quad\times\prod_{n=1}^N\left|1-e^{i(\theta_n-\theta)}\right|^t\prod_{l=1}^N\exp\left(-is\sum_{m=1}^\infty\frac{\sin[(\theta_l-\theta)m]}{m}\right)\\ &= \frac{1}{N!(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi} d\theta_1\cdots d\theta_N \prod_{1\le j<k\le N}\left|e^{i\theta_j}-e^{i\theta_k}\right|^2\\ &\quad\times\prod_{n=1}^N\left|1-e^{i\theta_n}\right|^t\prod_{l=1}^N\exp\left(-is\sum_{m=1}^\infty\frac{\sin(\theta_l m)}{m}\right).\end{aligned} \tag{70}$$

Making the same transformations as in Sect. 2.1,

$$\begin{aligned}\langle|Z|^t e^{is(\operatorname{Im}\log Z)}\rangle_{U(N)} &= \frac{2^{N^2}2^{tN}}{N!(2\pi)^N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dx_1\cdots dx_N\prod_{1\le j<k\le N}|x_j-x_k|^2\\ &\quad\times\prod_{n=1}^N\left(\frac{1}{1+x_n^2}\right)^{N+t/2}\prod_{l=1}^N\left(\frac{1}{\sqrt{1+x_l^2}}+i\frac{x_l}{\sqrt{1+x_l^2}}\right)^s\\ &= \frac{2^{N^2}2^{tN}}{N!(2\pi)^N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dx_1\cdots dx_N\prod_{1\le j<k\le N}|x_j-x_k|^2\\ &\quad\times\prod_{n=1}^N(1+ix_n)^{-N-t/2+s/2}(1-ix_n)^{-N-t/2-s/2}\\ &= \frac{2^{N^2}2^{tN}}{N!(2\pi)^N}J(1,1,N+t/2-s/2,N+t/2+s/2,1,N)\\ &= \prod_{j=1}^N\frac{\Gamma(j)\Gamma(t+j)}{\Gamma(j+t/2+s/2)\Gamma(j+t/2-s/2)}.\end{aligned} \tag{71}$$

The conditions on the validity of Selberg's integral translate into the restrictions $t/2+s/2>-1$, $t/2-s/2>-1$ and $t>-1$.

Next we expand the logarithm of the generating function as a series in powers of s and t:

$$\sum_{j=1}^{N} \log\Gamma(j) + \log\Gamma(t+j) - \log\Gamma(j+t/2+s/2) - \log\Gamma(j+t/2-s/2)$$
$$= \alpha_{00} + \alpha_{10}t + \alpha_{01}s + \frac{\alpha_{20}}{2}t^2 + \alpha_{11}ts + \frac{\alpha_{02}}{2}s^2 + \frac{\alpha_{30}}{3!}t^3 + \frac{\alpha_{21}}{2!1!}t^2 s$$
$$+ \frac{\alpha_{12}}{2!1!}ts^2 + \frac{\alpha_{03}}{3!}s^3 + \cdots, \tag{72}$$

where

$$\alpha_{n0} = Q_n(N), \tag{73a}$$
$$\alpha_{0n} = i^n R_n(N), \tag{73b}$$

and for $n \neq 0$ and $m \neq 0$,

$$\alpha_{mn} = \frac{\partial^m}{\partial t^m}\left[\sum_{j=1}^{N} \frac{1}{2^n}\Big(-\psi^{(n-1)}(j+t/2+s/2)\right.$$
$$\left. + (-1)^{n-1}\psi^{(n-1)}(j+t/2-s/2)\Big)\right]_{(0,0)}$$
$$= \sum_{j=1}^{N} \frac{1}{2^n}\frac{1}{2^m}\Big[-\psi^{(n+m-1)}(j+t/2+s/2)$$
$$+ (-1)^{n-1}\psi^{(n+m-1)}(j+t/2-s/2)\Big]_{(0,0)}$$
$$= \begin{cases} 0 & \text{if } n \text{ odd} \\ \frac{-1}{2^{n+m-1}} \sum_{j=1}^{N} \psi^{(n+m-1)}(j) & \text{if } n \text{ even} \end{cases}. \tag{74}$$

The joint value distribution is then given by

$$\tau_N(x,y) = \langle \delta(\log|Z|-x)\delta(\operatorname{Im}\log Z - y)\rangle_{U(N)} \tag{75}$$
$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(itx+isy)}\left\langle e^{it\log|Z|}e^{is\operatorname{Im}\log Z}\right\rangle_{U(N)} dt\, ds$$
$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(itx+isy)}$$
$$\times \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(it+j)}{\Gamma(j+it/2+s/2)\Gamma(j+it/2-s/2)} dt\, ds$$
$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(itx+isy)} \exp\Big(\alpha_{10}it + \alpha_{01}s + \frac{\alpha_{20}}{2}(it)^2$$
$$+ \alpha_{11}its + \frac{\alpha_{02}}{2}s^2 + \frac{\alpha_{30}}{3!}(it)^3 + \frac{\alpha_{21}}{2!1!}(it)^2 s$$
$$+ \frac{\alpha_{12}}{2!1!}its^2 + \frac{\alpha_{03}}{3!}s^3 + \cdots\Big) dt\, ds.$$

Hence, using $\alpha_{10}=\alpha_{01}=\alpha_{11}=0$,

$$\alpha_{20}=-\alpha_{02}=\frac{1}{2}\log N+\frac{1}{2}(\gamma+1)+\frac{1}{24N^2}-\frac{1}{80N^4}+O\left(\frac{1}{N^6}\right), \tag{76}$$

and $\alpha_{mn}=O(1)$ for $m+n\geq 3$, which follows from a comparison with the cumulants of $\mathrm{Re}\log Z$, the scaled joint distribution

$$\tilde{\tau}_N(x,y)=\sqrt{Q_2(N)R_2(N)}\tau_N(\sqrt{Q_2(N)}x,\sqrt{R_2(N)}y) \tag{77}$$

satisfies

$$\begin{aligned}\tilde{\tau}_N(x,y)&=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\Bigg(-ivx-iwy-\frac{v^2}{2}-\frac{w^2}{2}\\&\quad+\frac{\alpha_{30}}{3!\alpha_{20}^{3/2}}(iv)^3+\frac{\alpha_{21}}{2!\alpha_{20}^{3/2}}(iv)^2w+\frac{\alpha_{12}}{2!\alpha_{20}^{3/2}}ivw^2\\&\quad+\frac{\alpha_{03}}{3!\alpha_{20}^{3/2}}w^3+\cdots\Bigg)dv\,dw\\&=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-ivx-iwy-\frac{v^2}{2}-\frac{w^2}{2}\right)\\&\quad\times\left(1+O\left(\frac{1}{(\log N)^{3/2}}\right)\right)dv\,dw.\end{aligned} \tag{78}$$

Thus

$$\begin{aligned}\lim_{N\to\infty}\tilde{\tau}_N(x,y)&=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}e^{-ivx-\frac{v^2}{2}}dv\int_{-\infty}^{\infty}e^{-iwy-\frac{w^2}{2}}dw\\&=\frac{1}{2\pi}\exp\left(\frac{-x^2}{2}\right)\exp\left(\frac{-y^2}{2}\right).\end{aligned} \tag{79}$$

Therefore, as claimed, the limiting value distributions of the real and imaginary parts of $\log Z$ are independent and Gaussian as $N\to\infty$.

2.5. Asymptotics of the generating functions. Our goal in this section is to derive the leading-order asymptotics of the generating functions $M_N(s)$ and $L_N(s)$ as $N\to\infty$. The results are most easily stated in terms of the Barnes G-function [3], defined by

$$G(1+z)=(2\pi)^{z/2}e^{-[(1+\gamma)z^2+z]/2}\prod_{n=1}^{\infty}\left[(1+z/n)^n e^{-z+z^2/(2n)}\right], \tag{80}$$

which has the following important properties:

$$\begin{aligned}G(1)&=1,\\G(z+1)&=\Gamma(z)\;G(z),\end{aligned} \tag{81}$$

and

$$\log G(1+z)=(\log(2\pi)-1)\frac{z}{2}-(1+\gamma)\frac{z^2}{2}+\sum_{n=3}^{\infty}(-1)^{n-1}\zeta(n-1)\frac{z^n}{n}, \tag{82}$$

where the sum converges for $|z| < 1$. It follows from the definition (80) that $G(z)$ is an entire function of order two and that $G(1+z)$ has zeros at the negative integers, $-n$, with multiplicity n ($n = 1, 2, 3 \ldots$).

Consider first $M_N(s)$. Define

$$f_{\mathrm{CUE}}(s/2) = \lim_{N\to\infty} \frac{M_N(s)}{N^{(s/2)^2}} = \lim_{N\to\infty} \frac{1}{N^{(s/2)^2}} \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+s)}{(\Gamma(j+s/2))^2}. \tag{83}$$

We claim that

$$f_{\mathrm{CUE}}(s/2) = \frac{(G(1+s/2))^2}{G(1+s)}. \tag{84}$$

To prove this, we use the fact that for $|s| < 1$,

$$f_{\mathrm{CUE}}(s/2) = \exp\left(\left(\frac{s}{2}\right)^2 (\gamma+1) + \sum_{j=3}^{\infty} (-s)^j \left(\frac{2^{j-1}-1}{2^{j-1}}\right) \frac{\zeta(j-1)}{j} \right), \tag{85}$$

which follows from (35), (40), (45), and (48). Comparing this to

$$\begin{aligned}
\log\left(\frac{(G(1+s/2))^2}{G(1+s)}\right) &= 2\log G(1+s/2) - \log G(1+s) \\
&= 2(\log(2\pi)-1)\frac{s}{4} - 2(1+\gamma)\frac{s^2}{8} + 2\sum_{n=3}^{\infty} (-1)^{n-1}\zeta(n-1)\frac{s^n}{2^n n} \\
&\quad -(\log(2\pi)-1)\frac{s}{2} + (1+\gamma)\frac{s^2}{2} - \sum_{n=3}^{\infty}(-1)^{n-1}\zeta(n-1)\frac{s^n}{n} \\
&= (1+\gamma)\frac{s^2}{4} + \sum_{n=3}^{\infty} \frac{2^{n-1}-1}{2^{n-1}} \zeta(n-1)\frac{(-s)^n}{n},
\end{aligned} \tag{86}$$

which also holds for $|s| < 1$, we see that (84) holds when $|s| < 1$, and hence, by analytic continuation, in the rest of the complex s-plane. It follows that $f_{\mathrm{CUE}}(s/2)$ is a meromorphic function of order two with a pole of order $2k-1$ at each odd negative integer $s = -(2k-1)$, for $k = 1, 2, 3, \ldots$.

The value of $f_{\mathrm{CUE}}(n)$, where n is an integer, can be calculated directly from (84), since we have from (81) that

$$G(n) = \prod_{j=1}^{n-1} \Gamma(j), \quad n = 2, 3, 4 \ldots. \tag{87}$$

Thus

$$
\begin{aligned}
f_{\mathrm{CUE}}(n) &= \frac{(G(1+n))^2}{G(1+2n)} \\
&= \frac{\prod_{j=1}^{n}\Gamma(j)^2}{\prod_{m=1}^{2n}\Gamma(m)} \\
&= \frac{\prod_{j=1}^{n}\Gamma(j)}{\prod_{m=n+1}^{2n}\Gamma(m)} \\
&= \prod_{j=0}^{n-1}\frac{j!}{(j+n)!},
\end{aligned}
\tag{88}
$$

for $n = 1, 2, \ldots$. Inspired by a talk by one of us (JPK) at the Mathematical Sciences Research Institute, Berkeley, in June 1999, in which this result was discussed, Brézin and Hikami have since checked that the same formula holds for the integer moments of a wider class of random-matrix characteristic polynomials, including the GUE [11].

The leading order asymptotics of $L_N(s)$ can be obtained in the same way. In this case we claim that

$$
\lim_{N\to\infty} L_N(s)N^{s^2/4} = \mathrm{G}(1-s/2)\mathrm{G}(1+s/2). \tag{89}
$$

To prove this we note that

$$
\begin{aligned}
\lim_{N\to\infty} L_N(s)N^{s^2/4} &= \lim_{N\to\infty} N^{s^2/4}\prod_{j=1}^{N}\frac{(\Gamma(j))^2}{\Gamma(j+s/2)\Gamma(j-s/2)} \\
&= \exp\left(-(\gamma+1)\left(\frac{s}{2}\right)^2 - \sum_{j=2}^{\infty}\frac{\zeta(2j-1)s^{2j}}{2^{2j}j}\right),
\end{aligned}
\tag{90}
$$

where the second equality follows from (58), (61), (63), and (64). We also have that

$$
\begin{aligned}
\log(G\ (1-s/2)G(1+s/2)) &= \log G(1-s/2) + \log G(1+s/2) \\
&= -(\log(2\pi)-1)\frac{s}{4} - (1+\gamma)\frac{s^2}{8} + \sum_{n=3}^{\infty}(-1)^{n-1}\zeta(n-1)\frac{(-s)^n}{2^n n} \\
&\quad + (\log(2\pi)-1)\frac{s}{4} - (1+\gamma)\frac{s^2}{8} + \sum_{n=3}^{\infty}(-1)^{n-1}\zeta(n-1)\frac{s^n}{2^n n} \\
&= -(1+\gamma)\frac{s^2}{4} + 2\sum_{n=2}^{\infty}(-1)^{2n-1}\zeta(2n-1)\frac{(-s)^{2n}}{2^{2n}(2n)} \\
&= -(1+\gamma)\frac{s^2}{4} - \sum_{n=2}^{\infty}\frac{\zeta(2n-1)s^{2n}}{2^{2n}n},
\end{aligned}
\tag{91}
$$

when $|s/2| < 1$. Thus (89) holds for $|s/2| < 1$, and hence, by analytic continuation, in the rest of the complex s-plane. It follows that $\lim_{N\to\infty} L_N(s)N^{s^2/4}$ has zeros of order n at $s = \pm 2n$ for $n = 1, 2, \ldots$.

3. $\zeta(1/2 + it)$

Our aim now is to compare the CUE results for $Z(U, \theta)$ derived in the previous sections with the behaviour of the Riemann zeta function on its critical line. First, we have to identify the analogue of the matrix size N, which is the one parameter that appears in the CUE formulae. With this in mind we note that under the identification

$$N = \log\left(\frac{T}{2\pi}\right), \tag{92}$$

the fact that value distributions of $\mathrm{Re}\log Z/\sqrt{\frac{1}{2}\log N}$ and $\mathrm{Im}\log Z/\sqrt{\frac{1}{2}\log N}$ tend independently to Gaussians with zero mean and unit variance in the limit as $N \to \infty$ coincides precisely with Selberg's theorem (3). (Of course, the fact that $\log Z$ has zero mean is a consequence of its definition: we could multiply the determinant in (1) by any function with no zeros, for example a constant, but this would correspond to a trivial shift of the mean.) In the random matrix theory of spectral statistics, the natural parameter is the mean eigenvalue separation. For the eigenphases θ_n of U, this is $2\pi/N$. In the same way, the mean spacing between the Riemann zeros t_n at a height T up the critical line, $2\pi/\log(T/2\pi)$, is the only property of the zeta function that appears in Montgomery's conjecture and its generalizations. Equation (92) corresponds to equating these two parameters.

As already mentioned in the Introduction, Odlyzko's computations of the value distributions of both the real and imaginary parts of $\log\zeta(1/2 + it)$, for ranges of t near to the $10^{20\,\text{th}}$ zero (that is, $t \approx 1.5202 \times 10^{19}$), exhibit striking deviations from the Gaussian limit guaranteed by Selberg's theorem [29]. In Figs. 1 and 2 we show some of Odlyzko's data, for the real and imaginary parts respectively, normalized as in (3), together with the Gaussian. It is apparent that the deviations are larger for $\mathrm{Re}\log\zeta$, and that in this case the value distribution is not symmetric about zero.

This deviation can be quantified by comparing the moments of these distributions with the corresponding Gaussian values. These moments are listed in Tables 1 ($\mathrm{Re}\log\zeta$) and 2 ($\mathrm{Im}\log\zeta$). Again, it is clear from the size of the odd moments that the distribution is not symmetric about zero in the case of $\mathrm{Re}\log\zeta$.

We begin by comparing these data with the CUE results derived in Sects. 2.2 and 2.3. The matrix size N corresponding, via (92), to the height of the $10^{20\text{th}}$ zero is about 42 (the results we now present are not sensitive to the precise value). In Figs. 1 and 2 we also plot the CUE value distributions for $\mathrm{Re}\log Z$ and $\mathrm{Im}\log Z$ corresponding to $N = 42$, computed by direct numerical evaluation of the Fourier integrals in (37), using (6), and (60), using (7). The $N = 42$ random matrix curves are clearly much closer to the data than the limiting Gaussians ($N = \infty$). This is even more apparent in Fig. 3, where we show minus the logarithm of the value distributions plotted in Fig. 1. Similarly, we also give in Tables 1 and 2 the values of the CUE moments, normalized in the same way (so that the second moment takes the value one). These confirm the improved agreement. In this context we recall two relevant facts about the deviations of the CUE value-distributions from their Gaussian limiting forms: first, these deviations are larger for $\mathrm{Re}\log Z$ than for $\mathrm{Im}\log Z$; and second, in the case of $\mathrm{Re}\log Z$ they are not symmetric (even) about zero for N finite, whereas for $\mathrm{Im}\log Z$ they are.

As was already pointed out in the Introduction, random matrix theory cannot give a complete description of the finite-T distribution of values of $\log\zeta(1/2 + it)$, because it contains no information about the long-range zero-correlations associated with the primes. These can be computed separately, using the methods of [4]. For the moments of

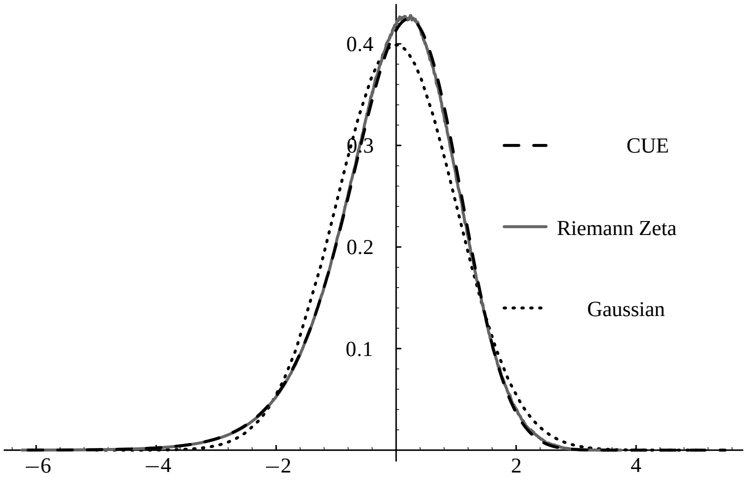

Fig. 1. The CUE value distribution for $\mathrm{Re}\log Z$ with $N = 42$, Odlyzko's data for the value distribution of $\mathrm{Re}\log\zeta(1/2+it)$ near the $10^{20\text{th}}$ zero (taken from [29]), and the standard Gaussian, all scaled to have unit variance

Table 1. Moments of $\mathrm{Re}\log\zeta(1/2+it)$, calculated over two ranges (labelled a and b) near the $10^{20\text{th}}$ zero ($T \simeq 1.520 \times 10^{19}$) (taken from [29]), compared with the CUE moments of $\mathrm{Re}\log Z$ with $N = 42$ and the Gaussian moments, all scaled to have unit variance

Moment	ζ a)	ζ b)	CUE	Normal
1	0.0	0.0	0.0	0
2	1.0	1.0	1.0	1
3	−0.53625	−0.55069	−0.56544	0
4	3.9233	3.9647	3.89354	3
5	−7.6238	−7.8839	−7.76965	0
6	38.434	39.393	38.0233	15
7	−144.78	−148.77	−145.043	0
8	758.57	765.54	758.036	105
9	−4002.5	−3934.7	−4086.92	0
10	24060.5	22722.9	25347.77	945

Table 2. Moments of $\mathrm{Im}\log\zeta(1/2+it)$ near the $10^{2\text{th}}$ zero ($T = 1.520 \times 10^{19}$) (taken from [29]) compared with the CUE moments for $\mathrm{Im}\log Z$ when $N = 42$ and the Gaussian moments, all scaled to have unit variance

Moment	ζ	CUE	Normal
1	-6.3×10^{-6}	0.0	0
2	1.0	1.0	1
3	-4.7×10^{-4}	0.0	0
4	2.831	2.87235	3
5	-9.1×10^{-3}	0.0	0
6	12.71	13.29246	15
7	−0.140	0.0	0
8	76.57	83.76939	105

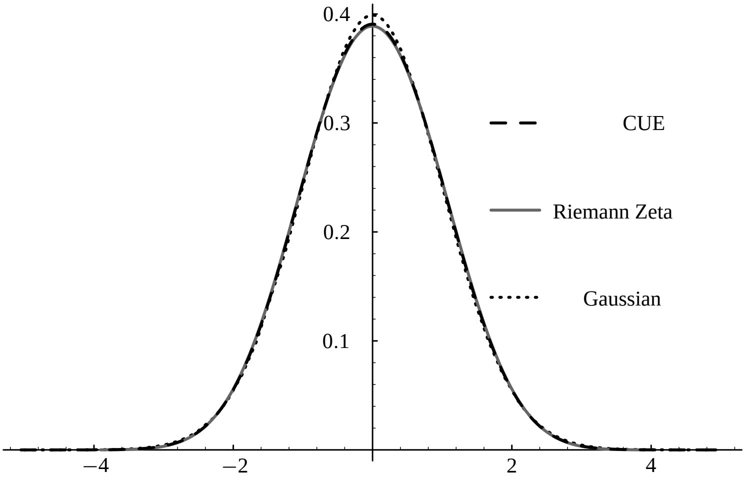

Fig. 2. The CUE value distribution for $\mathrm{Im}\log Z$ with N=42, Odlyzko's data for $\mathrm{Im}\log\zeta(1/2+it)$ near the $10^{20\text{th}}$ zero (taken from [29]), and the standard Gaussian, all scaled to have unit variance

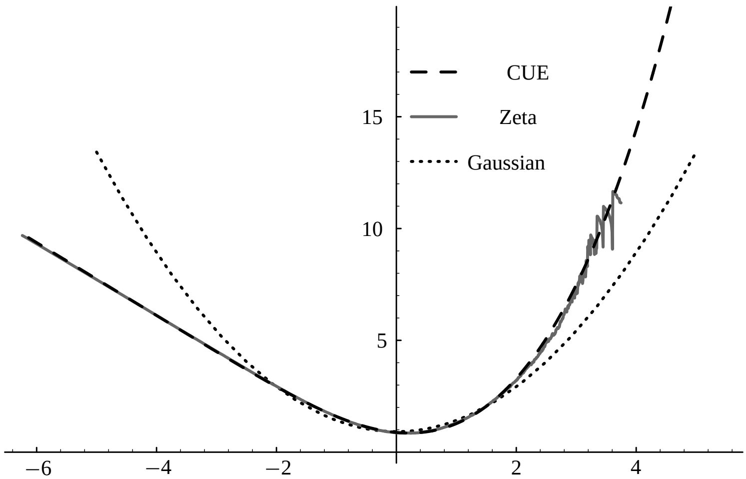

Fig. 3. Minus the logarithm of the value distributions plotted in Fig. 1

$\log\zeta(1/2+it)$, the results take the same form as Goldston's formula (18): the long-range contributions may be expressed as convergent sums over the primes. These prime-sums all have the property that, if each prime p is replaced by p^{γ}, they vanish in the limit $\gamma\to\infty$. We give explicit formulae below, but first turn to the moments of $|\zeta(1/2+it)|$.

We expect a relationship between the moments of $|\zeta(1/2+it)|$, defined by averaging over t, and those of $|Z(U,\theta)|$, averaged over the CUE; but clearly the moments of $|\zeta(1/2+it)|$ are related to those of $\mathrm{Re}\log\zeta(1/2+it)$ by exponentiation, and so it is natural to anticipate a long-range contribution in the form of a multiplicative factor given by a convergent product over the primes. We are thus led to a connection resembling the conjecture (4). The precise form of the prime product in (4) can, in fact, be recovered using heuristic arguments similar to those of [8] and [25] (essentially by substituting for $\zeta(1/2+it)$ the prime product (2), truncated to include only primes with $p < T/2\pi$, and treating these prime-contributions as being independent). However, our main focus here is on the CUE component, and so we merely observe that if each prime p in (5) is replaced by p^γ, then $a(\lambda) \to 1$ in the limit as $\gamma \to \infty$. This leads us to conjecture, again invoking (92), that $f(\lambda)$, defined by (4), is equal to $f_{\mathrm{CUE}}(\lambda)$, defined by (15). Based on the results of Section 2.5, we thus conjecture that

$$f(\lambda) = \frac{(G(1+\lambda))^2}{G(1+2\lambda)}, \tag{93}$$

and

$$f(n) = \prod_{j=0}^{n-1} \frac{j!}{(j+n)!}. \tag{94}$$

The main evidence in support of this conjecture is, as already noted in the Introduction, that (94) coincides with the known values $f(1) = 1$ [17] and $f(2) = 1/12$ [20], and agrees with other conjectures (based on number-theoretical calculations) that $f(3) = 42/9!$ [13] and $f(4) = 24024/16!$ [14] (this last conjecture and ours were announced independently at the Erwin Schrödinger Institute in Vienna, in September 1998). In addition, we can compare with numerical data. Odlyzko has computed

$$r(\lambda, H) = \frac{1}{H(\log T)^{\lambda^2}} \int_T^{T+H} |\zeta(1/2+it)|^{2\lambda}\, dt \tag{95}$$

for T close to $t_{10^{20}}$ [29]. It is obviously natural to compare this to

$$r_{\mathrm{CUE}}(\lambda) = \frac{1}{N^{\lambda^2}} M_N(2\lambda) a(\lambda), \tag{96}$$

with N satisfying (92). The results, shown in Table 3, would appear to support the conjecture.

We can also test our conjecture by returning to the moments of $\mathrm{Re}\log\zeta(1/2+it)$. Based on the arguments of the previous paragraph, we expect that as $T \to \infty$,

$$\frac{1}{T}\int_0^T (\mathrm{Re}\log\zeta(1/2+it))^k\, dt \sim \frac{d^k}{ds^k}\left[M_N(s)a(s/2)\right]_{s=0}, \tag{97}$$

where N is related to T via (92). The resulting expressions incorporate both the random matrix and the prime contributions. A comparison with Odlyzko's data may be made by computing the moments using (97) with $N = 42$. These values are listed in Table 4 (in this case, unlike in Table 1, the moments have not been normalized, in order to focus on the subdominant role played by the primes). They clearly match the data more closely than the CUE values.

The moments of $\mathrm{Im}\log\zeta(1/2+it)$ can be treated in the same way. These are obviously related to derivatives of the generating function $L_N(s)$. Applying the same heuristic method which underpins (4) leads us to conjecture that

$$\frac{1}{T}\int_0^T (\mathrm{Im}\log\zeta(1/2+it))^k\, dt \sim (-i)^k \frac{d^k}{ds^k}\left[L_N(s)b(s/2)\right]_{s=0}, \tag{98}$$

where

$$b(\lambda) = \prod_p \left[(1 - 1/p)^{-\lambda^2} \sum_{n=0}^{\infty} \frac{\Gamma(1+\lambda)\Gamma(1-\lambda)}{\Gamma(1+\lambda-n)\Gamma(1-\lambda-n)n!n!} p^{-n} \right]. \tag{99}$$

Moments calculated using (98) with $N = 42$ are listed in Table 5 (again, unlike in Table 2, these have not been scaled), together with Odlyzko's data. In this case too, the prime contribution leads to a noticeable improvement compared to the CUE values. It is also simple to check that for $k = 2$ (98) coincides with (18), and that for $k = 3$ and $k = 4$ it agrees with heuristic calculations based on the methods of [4, 7] and [9]. The conjecture corresponding to (4) is then that

$$\lim_{T\to\infty} (\log T)^{\lambda^2} \frac{1}{T} \int_0^T \left(\frac{\zeta(1/2+it)}{\zeta(1/2-it)} \right)^{\lambda} dt = \mathrm{G}(1-\lambda)\mathrm{G}(1+\lambda)b(\lambda), \tag{100}$$

where we have used (89).

Finally, it is also instructive to examine the distribution of values of $|Z|$,

$$P_N(w) = \langle \delta(w - |Z|) \rangle_{U(N)} . \tag{101}$$

Table 3. Comparison of $r(\lambda, H)$, calculated numerically for the Riemann zeta function near the $10^{20\text{th}}$ zero (taken from [29]), the corresponding CUE quantity ($N = 42$), with and without the prime product $a(\lambda)$, and the lower bound on the leading order coefficient [12], $C_1(\lambda)$

λ	CUE with prime product	$r(\lambda, H)$	$C_1(\lambda)$ (lower bound)	CUE	% error CUE with primes	% error CUE
0.1	1.011	1.004	1.0042	1.0129	0.741	0.886
0.2	1.038	1.034	1.0172	1.0430	0.395	0.870
0.3	1.071	1.067	1.0381	1.0803	0.423	1.25
0.4	1.105	1.098	1.064	1.1171	0.649	1.74
0.5	1.133	1.123	1.0904	1.1466	0.914	2.10
0.6	1.151	1.135	1.1113	1.1631	1.37	2.25
0.7	1.152	1.132	1.1195	1.1616	1.77	2.26
0.8	1.133	1.107	1.1076	1.1386	2.38	2.85
0.9	1.091	1.06	1.069	1.0925	2.92	3.07
1 .	1.024	0.989	1.	1.0238	3.52	3.52
1.1	0.933	0.896	0.901	0.9350	4.16	4.35
1.2	0.822	0.787	0.776	0.8307	4.48	5.55
1.3	0.699	0.667	0.637	0.7167	4.89	7.45
1.4	0.571	0.544	0.494	0.5996	4.99	10.2
1.5	0.446	0.426	0.36	0.4858	4.65	14.0
1.6	0.333	0.319	0.246	0.3806	4.27	19.3
1.7	0.237	0.229	0.157	0.2880	3.37	25.8
1.8	0.158	0.156	0.092	0.2103	1.41	34.8
1.9	0.100	0.101	0.05	0.1480	0.542	46.5
2.	0.0602	0.0624	0.025	0.1003	3.53	60.7

Table 4. Moments of $\mathrm{Re}\log\zeta(1/2+it)$ near the $10^{20\text{th}}$ zero ($T \simeq 1.520\times 10^{19}$) (averages in a) and b) taken over different intervals) compared with $\mathrm{Re}\log Z$ when $N=42$ with and without the prime contributions

Moment	ζ a)	ζ b)	CUE + primes	CUE
1	−0.001595	0.000549	0.0	0.0
2	2.5736	2.51778	2.56939	2.65747
3	−2.2263	−2.19591	−2.21609	−2.44955
4	25.998	25.1283	26.017	27.4967
5	−81.2144	−79.2332	−81.2922	−89.4481
6	655.921	628.48	663.493	713.597
7	−3966.46	−3765.29	−4052.98	−4437.47
8	33328.6	30385.5	34808.2	37806
9	−282163	−250744	−304267	−332278
10	2.271×10^6	2.298×10^6	3.082×10^6	3.359×10^6

Table 5. Moments of $\mathrm{Im}\log\zeta(1/2+it)$ near the $10^{20\text{th}}$ zero ($T \simeq 1.520\times 10^{19}$) compared with $\mathrm{Im}\log Z$ when $N=42$, with and without the prime contributions

Moment	ζ	CUE + primes	CUE
1	-1.0×10^{-5}	0.0	0.0
2	2.573	2.569	2.657
3	-1.9×10^{-3}	0.0	0.0
4	18.74	18.69	20.28
5	−0.097	0.0	0.0
6	216.5	215.6	249.5
7	−3.8	0.0	0.0
8	3355	3321	4178

Obviously

$$P_N(w) = \frac{1}{2\pi w}\int_{-\infty}^{\infty} e^{-is\log w} M_N(is)ds. \tag{102}$$

We can approximate this for large N in the same manner as for $\mathrm{Re}\log Z$:

$$\begin{aligned} P_N(w) &= \frac{1}{2\pi w}\int_{-\infty}^{\infty}\exp\left(-is\log w - Q_2s^2/2! - iQ_3s^3/3! + Q_4s^4/4! + \cdots\right)ds \\ &= \frac{1}{2\pi w\sqrt{Q_2}}\int_{-\infty}^{\infty}\exp\left(\frac{-is\log w}{\sqrt{Q_2}} - \frac{s^2}{2} - \frac{iQ_3s^3}{Q_2^{3/2}3!} + \cdots\right)ds \\ &\sim \frac{1}{2\pi w\sqrt{Q_2}}\int_{-\infty}^{\infty}\exp\left(\frac{-is\log w}{\sqrt{Q_2}} - \frac{s^2}{2}\right)ds \qquad (103) \\ &= \frac{1}{w\sqrt{2\pi Q_2}}\exp\left(\frac{-\log^2 w}{2Q_2}\right), \qquad (104) \end{aligned}$$

which is valid when w is fixed and $N\to\infty$, and more generally if $\log w >> -\frac{1}{2}\log N$, the lower bound being determined by the first pole of $M_N(s)$.

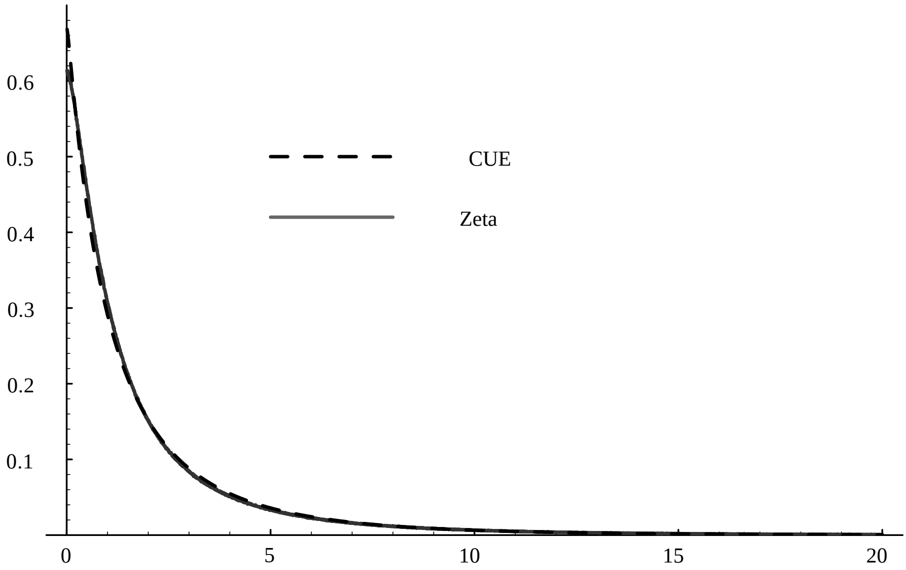

Fig. 4. The CUE value distribution of $|Z|$, corresponding to $N = 12$, with numerical data for the value distribution of $|\zeta(1/2+it)|$ near $t = 10^6$

For any finite N we can plot $P_N(w)$ numerically by direct evaluation of (102). This is done in Fig. 4 together with data for the value distribution of $|\zeta(1/2+it)|$ when $t \approx 10^6$, which corresponds via (92) to $N = 12$.

As $w \to 0$, $P_N(w)$ tends to a constant for a given N, the value of which can be calculated by noting that the contribution to the integral is dominated by the pole of $M_N(is)$ (at $s = i$) closest to the real axis. Hence

$$\lim_{w\to 0} P_N(w) = \frac{1}{\Gamma(N)} \prod_{j=1}^{N} \left(\frac{\Gamma(j)}{\Gamma(j-1/2)} \right)^2 . \qquad (105)$$

If N is large, this is asymptotic to [19]

$$N^{1/4}\,(G(1/2))^2 = \exp\left(\frac{1}{12}\log 2 + 3\zeta'(-1) - \frac{1}{2}\log\pi \right) N^{1/4}. \qquad (106)$$

Based on the previous discussion of its moments, it is natural to expect that as $t \to \infty$ the way in which the primes contribute to the value distribution of $|\zeta(1/2+it)|$ is given by

$$\tilde{P}_N(w) = \frac{1}{2\pi w} \int_{-\infty}^{\infty} e^{-is\log w} a(is/2) M_N(is) ds. \qquad (107)$$

Consequently,

$$\tilde{P}_N(0) = a(-1/2) P_N(0). \qquad (108)$$

We find that $a(-1/2) \approx 0.919$, $P_{12}(0) \approx 0.671$, and so $a(-1/2)P_{12}(0) \approx 0.617$, which is indeed close to the numerically computed value of the probability density at zero, 0.613.

Away from $w = 0$, in the region where (104) is valid, the stationary point of (107) is at $s^* = -i \log w/Q_2$, so $a(is^*/2) = a(\log w/(2Q_2))$. Since $a(0) = 1$, when $|\log w| << Q_2$, $a(\log w/(2Q_2))$ is close to 1 and so the contribution from the prime product recedes to the extremes of the distribution when N is large.

4. COE and CSE Results

Our main focus in this paper has been on the CUE of random matrix theory. However, the methods and results of Sect. 2 extend immediately to the other circular ensembles – the Circular Orthogonal Ensemble (COE) and the Circular Symplectic Ensemble (CSE) [27] – and for completeness we outline the form these generalizations take.

Let Z now represent the characteristic polynomial of an $N \times N$ matrix U in either the CUE ($\beta = 2$), the COE ($\beta = 1$), or the CSE ($\beta = 4$). The generalization of (21) is that

$$\langle |Z|^s \rangle_{RMT} = \frac{(\beta/2)!^N}{(N\beta/2)!(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} d\theta_1 \cdots d\theta_N \prod_{1\le j<m\le N} \left| e^{i\theta_j} - e^{i\theta_m} \right|^\beta \times \left| \prod_{p=1}^{N} \left(1 - e^{i(\theta_p - \theta)}\right) \right|^s, \tag{109}$$

where the average is over the appropriate ensemble. Exactly the same method as was applied in Sect. 2.1 leads to

$$M_N(\beta, s) = \langle |Z|^s \rangle_{RMT} = \prod_{j=0}^{N-1} \frac{\Gamma(1 + j\beta/2)\Gamma(1 + s + j\beta/2)}{(\Gamma(1 + s/2 + j\beta/2))^2}. \tag{110}$$

It follows from expanding $\log M_N(\beta, s)$ as a series in powers of s that the cumulants of the distribution of values taken by $\mathrm{Re} \log Z$ are given by

$$Q_n^\beta(N) = \frac{2^{n-1} - 1}{2^{n-1}} \sum_{j=0}^{N-1} \psi^{(n-1)}(1 + j\beta/2). \tag{111}$$

As in the CUE case, $Q_1^\beta(N) = 0$. Replacing the polygamma functions by their integral representations and interchanging the integral and the sum in (111) provides the leading-order asymptotics

$$\begin{aligned} Q_2^\beta(N) &= \frac{1}{2} \sum_{j=0}^{N-1} \frac{1}{1 + j\beta/2} + O(1) \\ &= \frac{1}{\beta} \log N + O(1). \end{aligned} \tag{112}$$

For $Q_n^\beta(N)$, $n \geq 3$, the sum in (111) converges as $N \to \infty$. Its value in the limit is

$$\begin{aligned} Q_n^\beta(\infty) \equiv \lim_{N\to\infty} Q_n^\beta(N) &= \frac{2^{n-1}-1}{2^{n-1}}(-1)^n \int_0^\infty \frac{e^{-t}t^{n-1}}{(1-e^{-t})(1-e^{-\beta t/2})} dt \\ &= \frac{2^{n-1}-1}{2^{n-1}}(-1)^n \sum_{r=0}^\infty \sum_{s=0}^\infty \int_0^\infty e^{-(1+s+\beta r/2)t} t^{n-1} dt \\ &= \frac{2^{n-1}-1}{2^{n-1}}(-1)^n \sum_{r=0}^\infty \sum_{s=1}^\infty \Gamma(n)(s+\beta r/2)^{-n}. \end{aligned} \tag{113}$$

When $\beta = 4$, the number of ways in which $s + 2r = k$ is $k/2$ if k is even, and $(k+1)/2$ if k is odd. Thus

$$\begin{aligned} Q_n^4(\infty) &= \frac{2^{n-1}-1}{2^{n-1}}(-1)^n \Gamma(n) \left(\sum_{k=1}^\infty \frac{k}{(2k-1)^n} + \sum_{k=1}^\infty \frac{k}{(2k)^n} \right) \\ &= \frac{2^{n-1}-1}{2^{n-1}}(-1)^n \Gamma(n) \frac{1}{2} \left(\sum_{k=1}^\infty \frac{2k-1}{(2k-1)^n} + \sum_{k=1}^\infty \frac{1}{(2k-1)^n} + \sum_{k=1}^\infty \frac{2k}{(2k)^n} \right) \\ &= \frac{2^{n-1}-1}{2^n}(-1)^n \Gamma(n) \left(\zeta(n-1) + \left(1 - \frac{1}{2^n}\right) \zeta(n) \right). \end{aligned} \tag{114}$$

Similarly,

$$Q_n^1(\infty) = (2^{n-1}-1)(-1)^n \Gamma(n) \left(\zeta(n-1) - \left(1 - \frac{1}{2^n}\right) \zeta(n) \right). \tag{115}$$

The asymptotic convergence of these cumulants ensures that the distribution of values taken by $\operatorname{Re}\log Z$ is Gaussian in the limit $N \to \infty$ (with unit variance if normalized with respect to $Q_2^\beta(N)$) when the zeros are distributed with COE or CSE statistics, just as it was for the CUE. All of the calculations carried out for the CUE transfer immediately to the other two ensembles by replacing Q_n with Q_n^β.

A similar equivalence holds for $\operatorname{Im}\log Z$. We have

$$L_N(\beta, s) = \langle e^{is(\operatorname{Im}\log Z)} \rangle_{RMT} = \prod_{j=0}^{N-1} \frac{(\Gamma(1+j\beta/2))^2}{\Gamma(j\beta/2+1+s/2)\Gamma(j\beta/2+1-s/2)}, \tag{116}$$

from which it follows that

$$R_n^\beta(N) = \begin{cases} 0 & \text{if } n \text{ odd} \\ \frac{(-1)^{n/2+1}}{2^{n-1}} \sum_{j=0}^{N-1} \psi^{n-1}(1+j\beta/2) & \text{if } n \text{ even} \end{cases}. \tag{117}$$

Comparison with (111) then shows that $R_2^\beta(N) = Q_2^\beta(N)$, and that the value distribution of $\operatorname{Im}\log Z$ has a Gaussian limit in all three cases.

In order to generalize the results of Sect. 2.5, we need the next term in the asymptotic expansion (112) for $Q_2^\beta(N)$. Applying the recurrence formula for the polygamma function,

$$\psi^{(1)}(z+1) = \psi^{(1)}(z) - \frac{1}{z^2}, \tag{118}$$

we have in the CSE case that

$$\begin{aligned} Q_2^4(N) &= \frac{1}{2}\sum_{j=0}^{N-1}\psi^{(1)}(1+2j) \\ &= \frac{1}{2}\left(\psi^{(1)}(1) + \sum_{j=1}^{N-1}\left(\psi^{(1)}(1) - \sum_{m=1}^{2j}\frac{1}{m^2}\right)\right) \\ &= \frac{N}{2}\psi^{(1)}(1) - \frac{1}{2}\sum_{k=1}^{N-1}\frac{N-k}{(2k-1)^2} - \frac{1}{2}\sum_{k=1}^{N-1}\frac{N-k}{(2k)^2} \\ &= \frac{1}{4}\log N + \frac{1}{4}(1+\gamma) + \frac{1}{4}\log 2 + \frac{3}{16}\zeta(2) + O(N^{-1}). \end{aligned} \tag{119}$$

As $Q_2^4(N) = R_2^4(N)$, this also gives us the second cumulant for $\operatorname{Im}\log Z$.

In the COE case we follow a very similar procedure, except that as we now have polygamma functions of half-integers, we need to consider the case of even and odd N seperately. We start with N even, relating the polygamma functions of integers back to $\psi^{(1)}(1)$ and those with half-integer argument to $\psi^{(1)}(1/2)$, and find that

$$\begin{aligned} Q_2^1(N) &= \frac{1}{2}\sum_{j=0}^{N-1}\psi^{(1)}(1+j/2) \\ &= \frac{1}{2}\left(\frac{N}{2}\psi^{(1)}(1) + \frac{N}{2}\psi^{(1)}(1/2) - \sum_{k=1}^{N/2}\frac{4(N/2-k+1)}{(2k-1)^2} - \sum_{k=1}^{(N-2)/2}\frac{N/2-k}{k^2}\right) \\ &= \log N + 1 + \gamma - \frac{3}{4}\zeta(2) + O(N^{-1}). \end{aligned} \tag{120}$$

The calculation for odd N is very similar and the result is the same. Once again $Q_2^1(N) = R_2^1(N)$.

The procedure for calculating the leading-order coefficient of $\langle |Z|^s\rangle$ or $\langle (Z/Z^*)^{s/2}\rangle$ for averages over the CSE and COE ensembles is also very similar to that already detailed for the CUE. In these cases we need

$$\log\Gamma(1+z) = -z\gamma + \sum_{n=2}^{\infty}(-1)^n\zeta(n)\frac{z^n}{n}, \tag{121}$$

valid for $|z| < 1$, as well as the expansion (82) for the Barnes G-function.

Using (114) and (119), we have that

$$
\begin{aligned}
f_{\mathrm{CSE}}(s) &= \lim_{N\to\infty} \frac{M_N(4,s)}{N^{s^2/8}} \\
&= \exp\left(\left(\frac{1}{4}(1+\gamma)+\frac{1}{4}\log 2+\frac{3}{16}\zeta(2)\right)\frac{s^2}{2}+\sum_{n=3}^{\infty}(-1)^n\left(\frac{1}{2}\zeta(n-1)\right.\right. \\
&\quad \left.\left.-\frac{1}{2^n}\zeta(n-1)+\frac{1}{2}\zeta(n)-\frac{1}{2^n}\zeta(n)-\frac{1}{2^{n+1}}\zeta(n)+\frac{1}{4^n}\zeta(n)\right)\frac{s^n}{n}\right), \quad (122)
\end{aligned}
$$

and from (121) and (82) we see that

$$
\begin{aligned}
&\log G(1+s/2)+\frac{1}{2}\log\Gamma(1+s)+\log\Gamma(1+s/4) \\
&\quad -\frac{1}{2}\log G(1+s)-\frac{1}{2}\log\Gamma(1+s/2)-\log\Gamma(1+s/2) \\
&= \left(\frac{1}{4}(1+\gamma)+\frac{3}{16}\zeta(2)\right)\frac{s^2}{2}+\sum_{n=3}^{\infty}(-1)^n \\
&\quad \times\left(\frac{1}{2}\zeta(n-1)-\frac{1}{2^n}\zeta(n-1)+\frac{1}{2}\zeta(n)-\frac{1}{2^n}\zeta(n)\right. \\
&\quad \left.-\frac{1}{2^{n+1}}\zeta(n)+\frac{1}{4^n}\zeta(n)\right)\frac{s^n}{n}. \qquad (123)
\end{aligned}
$$

Thus

$$
f_{\mathrm{CSE}}(s) = 2^{s^2/8}\frac{G(1+s/2)\Gamma(1+s/4)\sqrt{\Gamma(1+s)}}{\sqrt{G(1+s)\Gamma(1+s/2)}\Gamma(1+s/2)}, \qquad (124)
$$

for $|s|<1$. It then follows by analytic continuation that the equality holds for all s.

The above combination of gamma and G-functions also has the correct poles and zeros, namely a pole of order k at negative integers of the form $-(2k-1)$ and a zero of order 1 at $-(4k-2)$, where $k=1,2,3\ldots.$

The coefficients which reduce to rational numbers, as for the $2k^{\text{th}}$ moments in the CUE case, are those where $s=4k$ for positive integers k. With the help of (87) we see that

$$
f_{\mathrm{CSE}}(4k) = \frac{2^k}{\left(\prod_{j=1}^{2k-1}(2j-1)!!\right)(2k-1)!!}. \qquad (125)
$$

This can also been checked directly by writing $M_N(4,s)$ as a polynomial of order $2k^2$ in N.

For the imaginary part of the log of Z we have, in the CSE case, that

$$
\begin{aligned}
\lim_{N\to\infty} L_N(4,s)\times N^{s^2/8} &= \exp\left(-\left(\frac{1}{4}(1+\gamma)+\frac{1}{4}\log 2+\frac{3}{16}\zeta(2)\right)\frac{s^2}{2}\right. \qquad (126) \\
&\quad \left.-\sum_{n=2}^{\infty}\left(\frac{1}{2^{2n}}\zeta(2n-1)+\frac{1}{2^{2n}}\zeta(2n)-\frac{1}{4^{2n}}\zeta(2n)\right)\frac{s^{2n}}{2n}\right),
\end{aligned}
$$

and the expansions of the gamma and G-functions allow us to show that

$$\lim_{N\to\infty} L_N(4,s)\times N^{s^2/8} = 2^{-s^2/8}\sqrt{\frac{G(1+s/2)G(1-s/2)\Gamma(1+s/4)\Gamma(1-s/4)}{\Gamma(1+s/2)\Gamma(1-s/2)}}, \qquad (127)$$

which has zeros of order k at $\pm(4k-2)$ and also k^{th} order zeros at $\pm 4k$, $k=1,2,3\ldots$, just as an examination of $L_N(4,s)$ indicates it should.

Moving on to the COE, we have in this case that

$$\lim_{N\to\infty}\frac{M_N(1,s)}{N^{s^2/2}} = \exp\left(\left(1+\gamma-\frac{3}{4}\zeta(2)\right)\frac{s^2}{2}+\sum_{n=3}^{\infty}(-1)^n\left(2^{n-1}\zeta(n-1) - \zeta(n-1)-2^{n-1}\zeta(n)+\frac{3}{2}\zeta(n)-\frac{1}{2^n}\zeta(n)\right)\frac{s^n}{n}\right). \qquad (128)$$

Comparing this to

$$\log G(1+s)+\frac{3}{2}\log\Gamma(1+s) - \frac{1}{2}\log G(1+2s)-\frac{1}{2}\log\Gamma(1+2s)-\log\Gamma(1+s/2) = \left(1+\gamma-\frac{3}{4}\zeta(2)\right)\frac{s^2}{2}+\sum_{n=3}^{\infty}(-1)^n\left(2^{n-1}\zeta(n-1)-\zeta(n-1)-2^{n-1}\zeta(n) + \frac{3}{2}\zeta(n)-\frac{1}{2^n}\zeta(n)\right)\frac{s^n}{n} \qquad (129)$$

when $|s|<1/2$, it follows that

$$f_{\mathrm{COE}}(s)=\lim_{N\to\infty}\frac{M_N(1,s)}{N^{s^2/2}}=\frac{G(1+s)\Gamma(1+s)\sqrt{\Gamma(1+s)}}{\Gamma(1+s/2)\sqrt{G(1+2s)\Gamma(1+2s)}}, \qquad (130)$$

in this range, and hence, by analytic continuation, for all s. This expression has a simple poles at $s=-(2k-1)$ and a pole of order k at $s=-(2k+1)/2$, with $k=1,2,3,\ldots$.

We find rational values of this coefficient when $s=2k$:

$$f_{\mathrm{COE}}(2k)=\prod_{j=1}^{k}\frac{(2j-1)!}{(2k+2j-1)!}. \qquad (131)$$

Again, this can be verified by computing the leading order term of $M_N(1,2k)$, which turns out to be a polynomial of order $2k^2$ in N.

Finally,

$$\lim_{N\to\infty} L_N(1,s)\times N^{s^2/2} = \exp\left(-\left(1+\gamma-\frac{3}{4}\zeta(2)\right)\frac{s^2}{2} - \sum_{n=2}^{\infty}\left(\zeta(2n-1)-\zeta(2n) + \frac{1}{2^{2n}}\zeta(2n)\right)\frac{s^{2n}}{2n}\right)$$
$$= \sqrt{\frac{G(1+s)G(1-s)\Gamma(1+s)\Gamma(1-s)}{\Gamma(1+s/2)\Gamma(1-s/2)}}, \quad (132)$$

with the correct zeros of order k at $\pm 2k$ and order k at $\pm(2k+1)$, where $k = 1, 2, 3, \ldots$.

Acknowledgements. We are grateful to Peter Sarnak, for a number of very helpful suggestions, to Brian Conrey, for several stimulating discussions, and, in particular, to Andrew Odlyzko, for giving us access to his numerical data. NCS also wishes to acknowledge generous funding from NSERC and the CFUW in Canada.

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Communicated by P. Sarnak