Area-perimeter generating functions of lattice walks: q-series and their asymptotics

(A lattice model of vesicles attached to a skewed surface)

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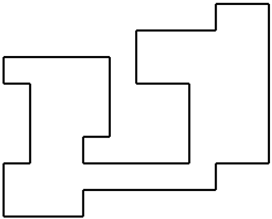
Topic Outline

Outline

1. Motivation
   - Vesicle Generating Function
   - Singularity Diagram
   - Scaling Function
2. From Lattice Walks to Basic Hypergeometric Series
   - q-Deformed Algebraic Equations
   - q-Difference Equations
   - Basic Hypergeometric Series
3. Asymptotic Analysis
   - Contour Integral Representation
   - Saddle Point Analysis
   - Uniform Asymptotics
4. Outlook

Vesicle Generating Function

- 3-dim vesicle (bubble) with surface and volume
- 2-dim lattice model: polygons on the square lattice



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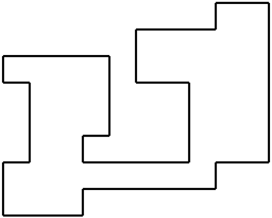


$c_{m,n}$ number of polygons with area m and perimeter $2n$

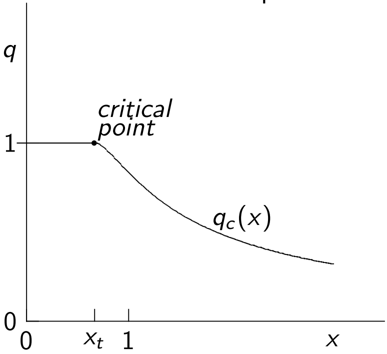
$$G(x,q) = \sum_{n,m} c_{m,n} x^n q^m \qquad \text{generating function}$$

Wanted:

- an explicit formula for $G(x,q)$
- singularity structure, e.g. $q_c(x)$

Singularity Diagram

Folklore: universal behaviour near a "critical point"



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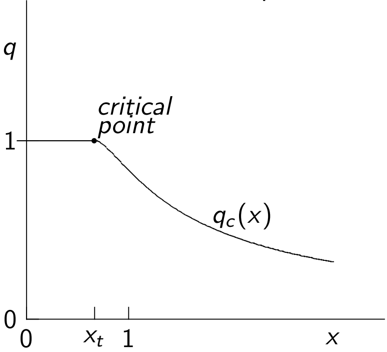

- scaling function f with crossover exponent ϕ:

$$G^{sing}(x,q) \sim (1-q)^{-\gamma_t} f\left([1-q]^{-\phi}[x_t - x]\right)$$

as $q \to 1$ and $x \to x_t$ with $z = [1-q]^{-\phi}[x_t - x]$ fixed

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  - Rigorous derivation (Prellberg)

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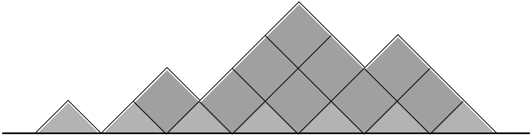
- q-Analogue of the Painlevé II equation (Witte)

Outline

Example 1: Dyck Paths



$2n = 14$ steps enclosing an area of size $m = 9$

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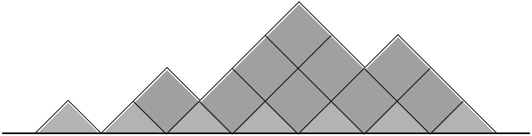


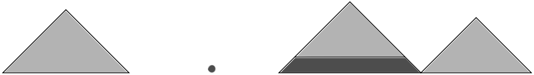
$2n = 14$ steps enclosing an area of size $m = 9$

$$G(t, q) = \sum_{m,n} c_{m,n} t^n q^m$$

t counts pairs of up/down steps, q counts enclosed area

Example 1: Dyck Paths

- A functional equation



$$G(t,q) \quad = \quad 1 \quad + \quad tG(qt,q)G(t,q)$$

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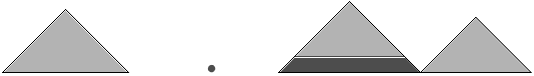


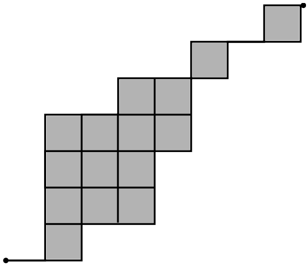
$$G(t,q) \quad = \quad 1 \quad + \quad tG(qt,q)G(t,q)$$

- $C(t) = G(t,1)$ satisfies $C(t) = 1 + tC(t)^2$

$$C(t) = \frac{1-\sqrt{1-4t}}{2t} = \sum_{n=0}^{\infty} \frac{t^n}{n+1}\binom{2n}{n}$$

Generating function of Catalan numbers

Example 2: A Pair of Directed Walks



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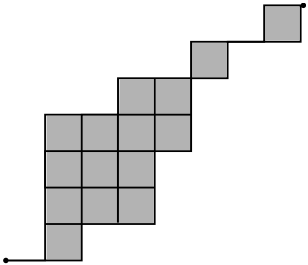

Two directed walks not allowed to cross

$$G(x,y,q) = \sum_{m,n_x,n_y} c_{m,n_x,n_y} x^{n_x} y^{n_y} q^m$$

x and y count pairs of east and north steps, q counts enclosed area

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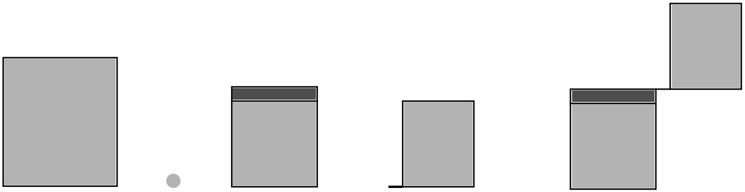

$$G(x,y,q) = 1 + yG(qx,y,q) + xG(x,y,q) + yG(qx,y,q)xG(x,y,q)$$

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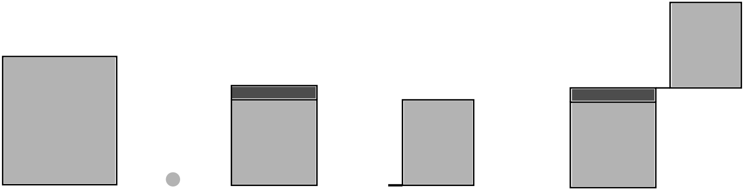

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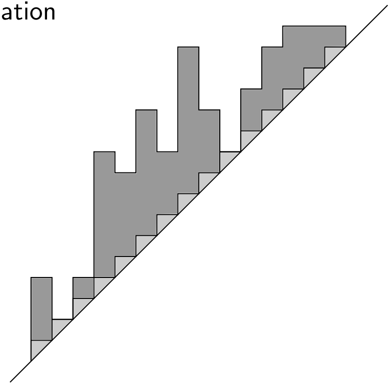
- $G(t,t,1) = 1 + tC(t)$ Catalan generating function

Example 3: Partially Directed Walks Above $y = x$

- A functional equation



$$G(x, y, q) = 1 + yG(qx, y, q)xG(x, y, q) + y(G(qx, y, q) - 1)y$$

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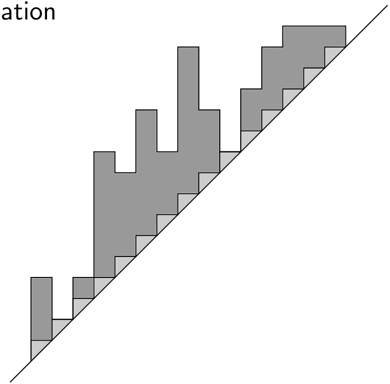

$$G(x, y, q) = 1 + yG(qx, y, q)xG(x, y, q) + y(G(qx, y, q) - 1)y$$

- $G(x, y, 1) = C\left(\frac{xy}{1-y^2}\right)$ Catalan generating function

Summary of the Examples

Different q-deformations of Catalan-type generating functions:

- Dyck paths

$$G(t) = 1 + tG(t)G(qt)$$

- Pair of directed walks

$$G(x) = (1 + xG(x))(1 + yG(qx))$$

- Partially directed walks above the diagonal

$$G(x) = 1 + xyG(x)G(qx) + y^2(G(qx) - 1)$$

Example 1: Solving $G(t) = 1 + tG(t)G(qt)$

An aside:

- $G(t)$ admits a nice continued fraction expansion

$$G(t) = \cfrac{1}{1 - \cfrac{t}{1 - \cfrac{qt}{1 - \cfrac{q^2 t}{1 - \dots}}}}$$

- Connections with orthogonal polynomials, combinatorics of weighted lattice paths, ...

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- Connections with orthogonal polynomials, combinatorics of weighted lattice paths, ...
- However, useless for finer asymptotic analysis of $q \to 1$.

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Better:

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$$H(t) = \sum_{n=0}^{\infty} \frac{q^{n^2-n}(-t)^n}{(q;q)_n} = {}_0\phi_1(-;0;q,-t)$$

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[${}_0\phi_1(-;0;q,-qt)$ a q-Airy function (Ismail)]

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Better:

- Linearise the functional equation using

$$G(x) = \frac{1}{x}\left(\frac{H(qx)}{H(x)} - 1\right)$$

- Obtain a linear q-difference equation

$$q(H(qx) - H(x)) = qxH(qx) + y(H(q^2x) - H(qx))$$

- Explicit solution

$$H(t) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-x)^n}{(y;q)_n(q;q)_n} = {}_1\phi_1(0;y;q,x)$$

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$$H(t) = \sum_{n=0}^{\infty} \frac{(-x(1-y^2)/y)^n}{(y^2;q)_n(q;q)_n} = {}_2\phi_1(0,0;y^2;q,-x(1-y^2)/y)$$

Summary:

Different q-deformations of Catalan-type generating functions:

- Dyck paths

$$G(t,q) = \frac{{}_0\phi_1(-;0;q,-qt)}{{}_0\phi_1(-;0;q,-t)}$$

- Pair of directed walks

$$G(x,y,q) = \frac{1}{x}\left(\frac{{}_1\phi_1(0;y;q,qx)}{{}_1\phi_1(0;y;q,x)} - 1\right)$$

- Partially directed walks above the diagonal

$$G(x,y,q) = \frac{y}{x}\left(\frac{{}_2\phi_1(0,0;y^2;q,qx(y-1/y))}{{}_2\phi_1(0,0;y^2;q,x(y-1/y))} - 1\right)$$

Outline

A Puzzle

- The full generating function is a quotient of q-series, e.g.

$$G(t,q) = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2}(-t)^n}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2-n}(-t)^n}{(q;q)_n}}$$

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How can one understand the limit $q \to 1$?

A Standard Trick For Evaluating Alternating Series

- Write an alternating series as a contour integral

$$\sum_{n=0}^{\infty}(-x)^n c_n = \frac{1}{2\pi i}\oint_{\mathcal{C}} x^s c(s) \frac{\pi}{\sin(\pi s)} ds$$

$\mathcal{C}$ runs counterclockwise around the zeros of $\sin(\pi s)$

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- For example,

$$\exp(-x) = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!} = \frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty} x^s\Gamma(-s)ds$$

where $c > 0$ (here, we have used $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$)

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Find suitable q-version for this trick

Contour Integral Representation

Use that

$$\text{Res}\left[(z;q)_\infty^{-1}; z=q^{-n}\right] = -\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n (q;q)_\infty} \qquad n=0,1,2,\ldots$$

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to prove that

Lemma

For complex t with $|\arg(x)| < \pi$*, non-negative integer n, and* $0 < q < 1$ *we have for* $0 < \rho < 1$

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}(-t)^n}{(q;q)_n} = \frac{(q;q)_\infty}{2\pi i} \int_{\rho-i\infty}^{\rho+i\infty} \frac{z^{\frac{1}{2}\log_q z - \log_q t}}{(z;q)_\infty} \sqrt{z}\, dz$$

Some Asymptotics

Approximate $\log(z;q)_\infty \sim \frac{1}{\log q}\mathrm{Li}_2(z) + \frac{1}{2}\log(1-z)$ to get

Lemma

For $0 < t < 1$ *and with* $\varepsilon = -\log q$

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where $t < \rho < 1$

We find a Laplace-type integral, where the saddles are given by

$$0 = \frac{d}{dz}\left[-\frac{1}{2}(\log z)^2 + \log(z)\log(t) + \mathrm{Li}_2(z)\right]$$

Saddle Point Analysis

- The asymptotics of

$$\int_{\mathcal{C}} e^{\frac{1}{\varepsilon}g(z)} f(z) dz$$

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$$z(1-z) = t \quad \Rightarrow \quad z = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4t}$$

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As t approaches $t_t = 1/4$, the saddles coalesce

Standard procedure: reparametrise locally by a cubic and compute a uniform asymptotic expansion (involving Airy functions)...

Saddle Point Summary:

Saddle Point coalescence occurs in all three cases:

- Dyck paths, ${}_0\phi_1(-;0;q,-t)$:

$$g(z) = -\frac{1}{2}(\log z)^2 + \log(z)\log(t) + \mathrm{Li}_2(z) \quad \Rightarrow \quad (z-1)z + t = 0$$

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- Pair of directed walks, ${}_1\phi_1(0;y;q,x)$:

$$g(z) = -\mathrm{Li}_2(y/z)+\log(z)\log(x)+\mathrm{Li}_2(z) \quad \Rightarrow \quad (z-1)(z-y)+zx = 0$$

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- Part. directed walks above the diagonal, ${}_2\phi_1(0,0;y^2;q,x(y-1/y))$:

$$g(z) = \dots \qquad \Rightarrow \qquad (z-1)(z-y^2) + z^2x(1/y-y) = 0$$

Uniform Asymptotics

Theorem

Let $0 < t < 1$ *and* $\varepsilon = -\log q$*. Then, as* $\varepsilon \to 0^+$,

$$G(t,q) \sim \frac{1}{2}\left(1 - \sqrt{1-4t}\left[-\frac{\mathrm{Ai}'(\alpha\varepsilon^{-2/3})}{\alpha^{1/2}\varepsilon^{-1/3}\mathrm{Ai}(\alpha\varepsilon^{-2/3})}\right]\right)$$

where $\alpha = \alpha(t)$ *is an explicitly given function of* t*. In particular,*

$$\alpha(t) \sim 1 - 4t \qquad \text{as } t \to 1/4$$

Uniform Asymptotics

Theorem

Let $0 < t < 1$ and $\varepsilon = -\log q$. Then, as $\varepsilon \to 0^+$,

$$G(t,q) \sim \frac{1}{2}\left(1 - \sqrt{1-4t}\left[-\frac{\mathrm{Ai}'(\alpha\varepsilon^{-2/3})}{\alpha^{1/2}\varepsilon^{-1/3}\mathrm{Ai}(\alpha\varepsilon^{-2/3})}\right]\right)$$

where $\alpha = \alpha(t)$ is an explicitly given function of t. In particular,

$$\alpha(t) \sim 1 - 4t \qquad \text{as } t \to 1/4$$

Some remarks:

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The result is completely analogous for the other examples.

Outline

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