# ON A GENERAL METHOD IN DYNAMICS

**By**

**William Rowan Hamilton**

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# NOTE ON THE TEXT

This edition is based on the original publication in the *Philosophical Transactions of the Royal Society*, part II for 1834.

The following errors in the original published text have been corrected:

a term $w^{(n)}$ in the last summand on the right hand side of equation ($S^5$.) has been corrected to $w^{(n-1)}$;

a minus sign ($-$) missing from equation ($K^6$.) has been inserted.

The paper *On a General Method in Dynamics* has also been republished in *The Mathematical Papers of Sir William Rowan Hamilton, Volume II: Dynamics*, edited for the Royal Irish Academy by A. W. Conway and A. J. McConnell, and published by Cambridge University Press in 1940.

David R. Wilkins
Dublin, February 2000

*On a General Method in Dynamics; by which the Study of the Motions of all free Systems of attracting or repelling Points is reduced to the Search and Differentiation of one central Relation, or characteristic Function. By* WILLIAM ROWAN HAMILTON, *Member of several scientific Societies in the British Dominions, and of the American Academy of Arts and Sciences, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland. Communicated by Captain* BEAUFORT, *R.N. F.R.S.*

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*Introductory Remarks.*

The theoretical development of the laws of motion of bodies is a problem of such interest and importance, that it has engaged the attention of all the most eminent mathematicians, since the invention of dynamics as a mathematical science by GALILEO, and especially since the wonderful extension which was given to that science by NEWTON. Among the successors of those illustrious men, LAGRANGE has perhaps done more than any other analyst, to give extent and harmony to such deductive researches, by showing that the most varied consequences respecting the motions of systems of bodies may be derived from one radical formula; the beauty of the method so suiting the dignity of the results, as to make of his great work a kind of scientific poem. But the science of force, or of power acting by law in space and time, has undergone already another revolution, and has become already more dynamic, by having almost dismissed the conceptions of solidity and cohesion, and those other material ties, or geometrically imaginably conditions, which LAGRANGE so happily reasoned on, and by tending more and more to resolve all connexions and actions of bodies into attractions and repulsions of points: and while the science is advancing thus in one direction by the improvement of physical views, it may advance in another direction also by the invention of mathematical methods. And the method proposed in the present essay, for the deductive study of the motions of attracting or repelling systems, will perhaps be received with indulgence, as an attempt to assist in carrying forward so high an inquiry.

In the methods commonly employed, the determination of the motion of a free point in space, under the influence of accelerating forces, depends on the integration of three equations in ordinary differentials of the second order; and the determination of the motions of a system of free points, attracting or repelling one another, depends on the integration of a system of such equations, in number threefold the number of the attracting or repelling points, unless we previously diminish by unity this latter number, by considering only relative motions. Thus, in the solar system, when we consider only the mutual attractions of the sun and the ten known planets, the determination of the motions of the latter about the former is reduced, by the usual methods, to the integration of a system of thirty ordinary differential equations of the second order, between the coordinates and the time; or, by a transformation of LAGRANGE, to the integration of a system of sixty ordinary differential equations of the first order, between the time and the elliptic elements: by which integrations, the thirty varying coordinates, or the sixty varying elements, are to be found as functions of the time. In the method of the present essay, this problem is reduced to the search and differentiation of a single function, which satisfies two partial differential equations of the first order and of the second degree: and every other dynamical problem, respecting the motions of any system, however numerous, of attracting or repelling points, (even if we suppose those points restricted by any conditions of connexion consistent with the law of living force,) is reduced, in like manner, to the study of one central function, of which the form marks out and characterizes the properties of the moving system, and is to be determined by a pair of partial differential equations of the first order, combined with some simple considerations. The difficulty is therefore at least transferred from the integration of many equations of one class to the integration of two of another: and even if it should be thought that no practical facility is gained, yet an intellectual pleasure may result from the reduction of the most complex and, probably, of all researches respecting the forces and motions of body, to the study of one characteristic function,* the unfolding of one central relation.

The present essay does not pretend to treat fully of this extensive subject,—a task which may require the labours of many years and many minds; but only to suggest the thought and propose the path to others. Although, therefore, the method may be used in the most varied dynamical researches, it is at present only applied to the orbits and perturbations of a system with any laws of attraction or repulsion, and with one predominant mass or centre of predominant energy; and only so far, even in this one research, as appears sufficient to make the principle itself understood. It may be mentioned here, that this dynamical principle is only another form of that idea which has already been applied to optics in the *Theory of systems of rays*, and that an intention of applying it to the motion of systems of bodies was announced† at the publication of that theory. And besides the idea itself, the manner of calculation also, which has been thus exemplified in the sciences of optics and dynamics, seems not confined to those two sciences, but capable of other applications; and the peculiar combination which it involves, of the principles of variations with those of partial differentials, for the determination and use of an important class of integrals, may constitute, when it shall be matured by the future labours of mathematicians, a separate branch of analysis.

WILLIAM R. HAMILTON.

*Observatory, Dublin, March* 1834.

* LAGRANGE and, after him, LAPLACE and others, have employed a single function to express the different forces of a system, and so to form in an elegant manner the differential equations of its motion. By this conception, great simplicity has been given to the statement of the problem of dynamics; but the solution of that problem, or the expression of the motions themselves, and of their integrals, depends on a very different and hitherto unimagined function, as it is the purpose of this essay to show.

† Transactions of the Royal Irish Academy, Vol. XV, page 80. A notice of this dynamical principle was also lately given in an article "On a general Method of expressing the Paths of Light and of the Planets," published in the Dublin University Review for October 1833.

*Integration of the Equations of Motion of a System, characteristic Function of such Motion, and Law of varying Action.*

1. The known differential equations of motion of a system of free points, repelling or attracting one another according to any functions of their distances, and not disturbed by any foreign force, may be comprised in the following formula:

$$\Sigma\,.m(x''\,\delta x + y''\,\delta y + z''\,\delta z) = \delta U. \tag{1.}$$

In this formula the sign of summation $\Sigma$ extends to all the points of the system; $m$ is, for any one such point, the constant called its mass; $x''$, $y''$, $z''$, are its component accelerations, or the second differential coefficients of its rectangular coordinates $x$, $y$, $z$, taken with respect to the time; $\delta x$, $\delta y$, $\delta z$, are any arbitrary infinitesimal displacements which the point can be imagined to receive in the same three rectangular directions; and $\delta U$ is the infinitesimal variation corresponding, of a function $U$ of the masses and mutual distances of the several points of the system, of which the form depends on the laws of their mutual actions, by the equation

$$U = \Sigma\,.mm_{\prime}f(r), \tag{2.}$$

$r$ being the distance between any two points $m$, $m_{\prime}$, and the function $f(r)$ being such that the derivative or differential coefficient $f'(r)$ expresses the law of their repulsion, being negative in the case of attraction. The function which has been here called $U$ may be named the *force-function* of a system: it is of great utility in theoretical mechanics, into which it was introduced by LAGRANGE, and it furnishes the following elegant forms for the differential equations of motion, included in the formula (1.):

$$\left.\begin{array}{llll} m_1x_1'' = \dfrac{\delta U}{\delta x_1}; & m_2x_2'' = \dfrac{\delta U}{\delta x_2}; & \dots & m_nx_n'' = \dfrac{\delta U}{\delta x_n}; \\ m_1y_1'' = \dfrac{\delta U}{\delta y_1}; & m_2y_2'' = \dfrac{\delta U}{\delta y_2}; & \dots & m_ny_n'' = \dfrac{\delta U}{\delta y_n}; \\ m_1z_1'' = \dfrac{\delta U}{\delta z_1}; & m_2z_2'' = \dfrac{\delta U}{\delta z_2}; & \dots & m_nz_n'' = \dfrac{\delta U}{\delta z_n}; \end{array}\right\} \tag{3.}$$

the second members of these equations being the partial differential coefficients of the first order of the function $U$. But notwithstanding the elegance and simplicity of this known manner of stating the principal problem of dynamics, the difficulty of solving that problem, or even of expressing its solution, has hitherto appeared insuperable; so that only seven intermediate integrals, or integrals of the first order, with as many arbitrary constants, have hitherto been found for these general equations of motion of a system of $n$ points, instead of $3n$ intermediate and $3n$ final integrals, involving ultimately $6n$ constants; nor has any integral been found which does not need to be integrated again. No general solution has been obtained assigning (as a complete solution ought to do) $3n$ relations between the $n$ masses $m_1, m_2, \dots\ m_n$, the $3n$ varying coordinates $x_1, y_1, z_1, \dots\ x_n, y_n, z_n$, the varying time $t$, and the $6n$ initial data of the problem, namely, the initial coordinates $a_1, b_1, c_1, \dots\ a_n, b_n, c_n$, and their initial rates of increase $a_1', b_1', c_1', \dots\ a_n', b_n', c_n'$; the quantities called here initial being those which correspond to the arbitrary origin of time. It is, however, possible (as we shall see) to express these long-sought relations by the partial differential coefficients of a new central or radical function, to the search and employment of which the difficulty of mathematical dynamics becomes henceforth reduced.

2. If we put for abridgement

$$T = \tfrac{1}{2}\,\Sigma\,.m(x'^2 + y'^2 + z'^2), \tag{4.}$$

so that $2T$ denotes, as in the Mécanique Analytique, the whole living force of the system; ($x'$, $y'$, $z'$, being here, according to the analogy of our foregoing notation, the rectangular components of velocity of the point $m$, or the first differential coefficients of its coordinates taken with respect to the time;) an easy and well known combination of the differential equations of motion, obtained by changing in the formula (1.) the variations to the differentials of the coordinates, may be expressed in the following manner,

$$dT = dU, \tag{5.}$$

and gives, by integration, the celebrated law of living force, under the form

$$T = U + H. \tag{6.}$$

In this expression, which is one of the seven known integrals already mentioned, the quantity $H$ is independent of the time, and does not alter in the passage of the points of the system from one set of positions to another. We have, for example, an initial equation of the same form, corresponding to the origin of time, which may be written thus,

$$T_0 = U_0 + H. \tag{7.}$$

The quantity $H$ may, however, receive any arbitrary increment whatever, when we pass in thought from a system moving in one way, to the same system moving in another, with the same dynamical relations between the accelerations and positions of its points, but with different initial data; but the increment of $H$, thus obtained, is evidently connected with the analogous increments of the functions $T$ and $U$, by the relation

$$\Delta T = \Delta U + \Delta H, \tag{8.}$$

which, for the case of infinitesimal variations, may be conveniently be written thus,

$$\delta T = \delta U + \delta H; \tag{9.}$$

and this last relation, when multiplied by $dt$, and integrated, conducts to an important result. For it thus becomes, by (4.) and (1.),

$$\int \Sigma\,.m(dx\,.\,\delta x' + dy\,.\,\delta y' + dz\,.\,\delta z') = \int \Sigma\,.m(dx'\,.\,\delta x + dy'\,.\,\delta y + dz'\,.\,\delta z) + \int \delta H\,.\,dt, \tag{10.}$$

that is, by the principles of the calculus of variations,

$$\delta V = \Sigma\,.m(x'\,\delta x + y'\,\delta y + z'\,\delta z) - \Sigma\,.m(a'\,\delta a + b'\,\delta b + c'\,\delta c) + t\,\delta H, \tag{A.}$$

if we denote by $V$ the integral

$$V = \int \Sigma\,.m(x'\,dx + y'\,dy + z'\,dz) = \int_0^t 2T\,dt, \tag{B.}$$

namely, the accumulated living force, called often the action of the system, from its initial to its final position.

If, then, we consider (as it is easy to see that we may) the action $V$ as a function of the initial and final coordinates, and of the quantity $H$, we shall have, by (A.), the following groups of equations; first, the group,

$$\left.\begin{array}{llll} \dfrac{\delta V}{\delta x_1} = m_1 x_1'; & \dfrac{\delta V}{\delta x_2} = m_2 x_2'; & \dots & \dfrac{\delta V}{\delta x_n} = m_n x_n'; \\ \dfrac{\delta V}{\delta y_1} = m_1 y_1'; & \dfrac{\delta V}{\delta y_2} = m_2 y_2'; & \dots & \dfrac{\delta V}{\delta y_n} = m_n y_n'; \\ \dfrac{\delta V}{\delta z_1} = m_1 z_1'; & \dfrac{\delta V}{\delta z_2} = m_2 z_2'; & \dots & \dfrac{\delta V}{\delta z_n} = m_n z_n'. \end{array}\right\} \tag{C.}$$

Secondly, the group,

$$\left.\begin{array}{llll} \dfrac{\delta V}{\delta a_1} = -m_1 a_1'; & \dfrac{\delta V}{\delta a_2} = -m_2 a_2'; & \dots & \dfrac{\delta V}{\delta a_n} = -m_n a_n'; \\ \dfrac{\delta V}{\delta b_1} = -m_1 b_1'; & \dfrac{\delta V}{\delta b_2} = -m_2 b_2'; & \dots & \dfrac{\delta V}{\delta b_n} = -m_n b_n'; \\ \dfrac{\delta V}{\delta c_1} = -m_1 c_1'; & \dfrac{\delta V}{\delta c_2} = -m_2 c_2'; & \dots & \dfrac{\delta V}{\delta c_n} = -m_n c_n'; \end{array}\right\} \tag{D.}$$

and finally, the equation,

$$\frac{\delta V}{\delta H} = t. \tag{E.}$$

So that if this function $V$ were known, it would only remain to eliminate $H$ between the $3n+1$ equations (C.) and (E.), in order to obtain all the $3n$ intermediate integrals, or between (D.) and (E.) to obtain all the $3n$ final integrals of the differential equations of motion; that is, ultimately, to obtain the $3n$ sought relations between the $3n$ varying coordinates and the time, involving also the masses and the $6n$ initial data above mentioned; the discovery of which relations would be (as we have said) the general solution of the general problem of dynamics. We have, therefore, at least reduced that general problem to the search and differentiation of a single function $V$, which we shall call on this account the CHARACTERISTIC FUNCTION of motion of a system; and the equation (A.), expressing the fundamental law of its variation, we shall call the *equation of the characteristic function*, or the LAW OF VARYING ACTION.

3. To show more clearly that the action or accumulated living force of a system, or in other words, the integral of the product of the living force by the element of the time, may be regarded as a function of the $6n + 1$ quantities already mentioned, namely, of the initial and final coordinates, and of the quantity $H$, we may observe, that whatever depends on the manner and time of motion of the system may be considered as such a function; because the initial form of the law of living force, when combined with the $3n$ known or unknown relations between the time, the initial data, and the varying coordinates, will always furnish $3n + 1$ relations, known or unknown, to connect the time and the initial components of velocities with the initial and final coordinates, and with $H$. Yet from not having formed the conception of the action as a *function* of this kind, the consequences that have been here deduced from the formula (A.) for the variation of that definite integral appear to have escaped the notice of Lagrange, and of the other illustrious analysts who have written on theoretical mechanics; although they were in possession of a formula for the variation of this integral not greatly differing from ours. For although Lagrange and others, in treating of the motion of a system, have shown that the variation of this definite integral vanishes when the extreme coordinates and the constant $H$ are given, they appear to have deduced from this result only the well known law of *least action*; namely, that if the points or bodies of a system be imagined to move from a given set of initial to a given set of final positions, not as they do nor even as they could move consistently with the general dynamical laws or differential equations of motion, but so as not to violate any supposed geometrical connexions, nor that one dynamical relation between velocities and configurations which constitutes the law of living force; and if, besides, this geometrically imaginable, but dynamically impossible motion, be made to differ infinitely *little* from the actual manner of motion of the system, between the given extreme positions; then the varied value of the definite integral called action, or the accumulated living force of the system in the motion thus imagined, will differ infinitely *less* from the actual value of that integral. But when this well known law of least, or as it might be better called, of *stationary action*, is applied to the determination of the actual motion of the system, it serves only to form, by the rules of the calculus of variations, the differential equations of motion of the second order, which can always be otherwise found. It seems, therefore, to be with reason that Lagrange, Laplace, and Poisson have spoken lightly of the utility of this principle in the present state of dynamics. A different estimate, perhaps, will be formed of that other principle which has been introduced in the present paper, under the name of the *law of varying action*, in which we pass from an actual motion to another motion dynamically possible, by varying the extreme positions of the system, and (in general) the quantity $H$, and which serves to express, by means of a single function, not the mere differential equations of motion, but their intermediate and their final integrals.

*Verification of the foregoing Integrals.*

4. A verification, which ought not to be neglected, and at the same time an illustration of this new principle, may be obtained by deducing the known differential equations of motion from our system of intermediate integrals, and by showing the consistence of these again with our final integral system. As preliminary to such verification, it is useful to observe that the final equation (6.) of living force, when combined with the system (C.), takes this new form,

$$\tfrac{1}{2}\,\Sigma\,.\,\frac{1}{m}\left\{\left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2+\left(\frac{\delta V}{\delta z}\right)^2\right\}=U+H; \tag{F.}$$

and that the initial equation (7.) of living force becomes by (D.)

$$\tfrac{1}{2}\,\Sigma\,.\,\frac{1}{m}\left\{\left(\frac{\delta V}{\delta a}\right)^2+\left(\frac{\delta V}{\delta b}\right)^2+\left(\frac{\delta V}{\delta c}\right)^2\right\}=U_0+H. \tag{G.}$$

These two partial differential equations, inital and final, of the first order and the second degree, must both be identically satisfied by the characteristic function $V$: they furnish (as we shall find) the principal means of discovering the form of that function, and are of essential importance in its theory. If the form of this function were known, we might eliminate $3n-1$ of the $3n$ initial coordinates between the $3n$ equations (C.); and although we cannot yet perform the actual process of this elimination, we are entitled to assert that it would remove along with the others the remaining initial coordinate, and would conduct to the equation (6.) of final living force, which might then be transformed into the equation (F.). In like manner we may conclude that all the $3n$ final coordinates could be eliminated together from the $3n$ equations (D.), and that the result would be the initial equation (7.) of living force, or the transformed equation (G.). We may therefore consider the law of living force, which assisted us in discovering the properties of our characteristic function $V$, as included reciprocally in those properties, and as resulting by elimination, in every particular case, from the systems (C.) and (D.); and in treating of either of these systems, or in conducting any other dynamical investigation by the method of this characteristic function, we are at liberty to employ the partial differential equations (F.) and (G.) which that function must necessarily satisfy.

It will now be easy to deduce, as we proposed, the known equations of motion (3.) of the second order, by differentiation and elimination of constants, from our intermediate integral system (C.), (E.), or even from a part of that system, namely, from the group (C.), when combined with the equation (F.). For we thus obtain

$$\left.\begin{aligned}
m_1x_1'' = \frac{d}{dt}\frac{\delta V}{\delta x_1} &= x_1'\frac{\delta^2 V}{\delta x_1^2}+x_2'\frac{\delta^2 V}{\delta x_1\,\delta x_2}+\cdots+x_n'\frac{\delta^2 V}{\delta x_1\,\delta x_n}\\
&\quad+y_1'\frac{\delta^2 V}{\delta x_1\,\delta y_1}+y_2'\frac{\delta^2 V}{\delta x_1\,\delta y_2}+\cdots+y_n'\frac{\delta^2 V}{\delta x_1\,\delta y_n}\\
&\quad+z_1'\frac{\delta^2 V}{\delta x_1\,\delta z_1}+z_2'\frac{\delta^2 V}{\delta x_1\,\delta z_2}+\cdots+z_n'\frac{\delta^2 V}{\delta x_1\,\delta z_n}\\
&= \frac{1}{m_1}\frac{\delta V}{\delta x_1}\frac{\delta^2 V}{\delta x_1^2}+\frac{1}{m_2}\frac{\delta V}{\delta x_2}\frac{\delta^2 V}{\delta x_1\,\delta x_2}+\cdots+\frac{1}{m_n}\frac{\delta V}{\delta x_n}\frac{\delta^2 V}{\delta x_1\,\delta x_n}\\
&\quad+\frac{1}{m_1}\frac{\delta V}{\delta y_1}\frac{\delta^2 V}{\delta x_1\,\delta y_1}+\frac{1}{m_2}\frac{\delta V}{\delta y_2}\frac{\delta^2 V}{\delta x_1\,\delta y_2}+\cdots+\frac{1}{m_n}\frac{\delta V}{\delta y_n}\frac{\delta^2 V}{\delta x_1\,\delta y_n}\\
&\quad+\frac{1}{m_1}\frac{\delta V}{\delta z_1}\frac{\delta^2 V}{\delta x_1\,\delta z_1}+\frac{1}{m_2}\frac{\delta V}{\delta z_2}\frac{\delta^2 V}{\delta x_1\,\delta z_2}+\cdots+\frac{1}{m_n}\frac{\delta V}{\delta z_n}\frac{\delta^2 V}{\delta x_1\,\delta z_n}\\
&= \frac{\delta}{\delta x_1}\,\Sigma\,.\,\frac{1}{2m}\left\{\left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2+\left(\frac{\delta V}{\delta z}\right)^2\right\}=\frac{\delta}{\delta x_1}(U+H);
\end{aligned}\right\} \tag{11.}$$

that is, we obtain

$$m_1 x_1'' = \frac{\delta U}{\delta x_1}. \tag{12.}$$

And in like manner we might deduce, by differentiation, from the integrals (C.) and from (F.) all the other known differential equations of motion, of the second order, contained in the set marked (3.); or, more concisely, we may deduce at once the formula (1.), which contains all those known equations, by observing that the intermediate integrals (C.), when combined with the relation (F.), give

$$\left.\begin{aligned}
&\Sigma\,.m(x''\,\delta x + y''\,\delta y + z''\,\delta z)\\
&\quad= \Sigma\left(\frac{d}{dt}\frac{\delta V}{\delta x}\,.\,\delta x + \frac{d}{dt}\frac{\delta V}{\delta y}\,.\,\delta y + \frac{d}{dt}\frac{\delta V}{\delta z}\,.\,\delta z\right)\\
&\quad= \Sigma\,.\frac{1}{m}\left(\frac{\delta V}{\delta x}\frac{\delta}{\delta x} + \frac{\delta V}{\delta y}\frac{\delta}{\delta y} + \frac{\delta V}{\delta z}\frac{\delta}{\delta z}\right)\Sigma\left(\frac{\delta V}{\delta x}\delta x + \frac{\delta V}{\delta y}\delta y + \frac{\delta V}{\delta z}\delta z\right)\\
&\quad= \Sigma\left(\delta x\,\frac{\delta}{\delta x} + \delta y\,\frac{\delta}{\delta y} + \delta z\,\frac{\delta}{\delta z}\right)\Sigma\,.\frac{1}{2m}\left\{\left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2\right\}\\
&\quad= \Sigma\left(\delta x\,\frac{\delta}{\delta x} + \delta y\,\frac{\delta}{\delta y} + \delta z\,\frac{\delta}{\delta z}\right)(U + H)\\
&\quad= \delta U.
\end{aligned}\right\} \tag{13.}$$

5. Again, we were to show that our intermediate integral system, composed of the equations (C.) and (E.), with the $3n$ arbitrary constants $a_1, b_1, c_1, \dots\ a_n, b_n, c_n$, (and involving also the auxiliary constant $H$,) is consistent with our final integral system of equations (D.) and (E.), which contain $3n$ other arbitrary constants, namely $a_1', b_1', c_1', \dots\ a_n', b_n', c_n'$. The immediate differentials of the equations (C.), (D.), (E.), taken with respect to the time, are, for the first group,

$$\left.\begin{aligned}
&\frac{d}{dt}\frac{\delta V}{\delta x_1} = m_1 x_1''; \quad \frac{d}{dt}\frac{\delta V}{\delta x_2} = m_2 x_2''; \quad \dots \quad \frac{d}{dt}\frac{\delta V}{\delta x_n} = m_n x_n'';\\
&\frac{d}{dt}\frac{\delta V}{\delta y_1} = m_1 y_1''; \quad \frac{d}{dt}\frac{\delta V}{\delta y_2} = m_2 y_2''; \quad \dots \quad \frac{d}{dt}\frac{\delta V}{\delta y_n} = m_n y_n'';\\
&\frac{d}{dt}\frac{\delta V}{\delta z_1} = m_1 z_1''; \quad \frac{d}{dt}\frac{\delta V}{\delta z_2} = m_2 z_2''; \quad \dots \quad \frac{d}{dt}\frac{\delta V}{\delta z_n} = m_n z_n'';
\end{aligned}\right\} \tag{H.}$$

for the second group,

$$\left.\begin{aligned}
&\frac{d}{dt}\frac{\delta V}{\delta a_1} = 0; \quad \frac{d}{dt}\frac{\delta V}{\delta a_2} = 0; \quad \dots \quad \frac{d}{dt}\frac{\delta V}{\delta a_n} = 0;\\
&\frac{d}{dt}\frac{\delta V}{\delta b_1} = 0; \quad \frac{d}{dt}\frac{\delta V}{\delta b_2} = 0; \quad \dots \quad \frac{d}{dt}\frac{\delta V}{\delta b_n} = 0;\\
&\frac{d}{dt}\frac{\delta V}{\delta c_1} = 0; \quad \frac{d}{dt}\frac{\delta V}{\delta c_2} = 0; \quad \dots \quad \frac{d}{dt}\frac{\delta V}{\delta c_n} = 0;
\end{aligned}\right\} \tag{I.}$$

and finally, for the last equation,

$$\frac{d}{dt}\frac{\delta V}{\delta H} = 1. \tag{K.}$$

By combining the equations (C.) with their differentials (H.), and with the relation (F.), we deduced, in the foregoing number, the known equations of motion (3.); and we are now to show the consistence of the same intermediate integrals (C.) with the group of differentials (I.) which have been obtained from the final integrals.

The first equation of the group (I.) may be developed thus:

$$\left.\begin{aligned} 0 = x'_1 \frac{\delta^2 V}{\delta a_1\, \delta x_1} + x'_2 \frac{\delta^2 V}{\delta a_1\, \delta x_2} + \cdots + x'_n \frac{\delta^2 V}{\delta a_1\, \delta x_n} \\ + y'_1 \frac{\delta^2 V}{\delta a_1\, \delta y_1} + y'_2 \frac{\delta^2 V}{\delta a_1\, \delta y_2} + \cdots + y'_n \frac{\delta^2 V}{\delta a_1\, \delta y_n} \\ + z'_1 \frac{\delta^2 V}{\delta a_1\, \delta z_1} + z'_2 \frac{\delta^2 V}{\delta a_1\, \delta z_2} + \cdots + z'_n \frac{\delta^2 V}{\delta a_1\, \delta z_n} \end{aligned}\right\} \tag{14.}$$

and the others may be similarly developed. In order, therefore, to show that they are satisfied by the group (C.), it is sufficient to prove that the following equations are true,

$$\left.\begin{aligned} 0 &= \frac{\delta}{\delta a_i} \Sigma . \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2 \right\}, \\ 0 &= \frac{\delta}{\delta b_i} \Sigma . \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2 \right\}, \\ 0 &= \frac{\delta}{\delta c_i} \Sigma . \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2 \right\}, \end{aligned}\right\} \tag{L.}$$

the integer $i$ receiving any value from 1 to $n$ inclusive; which may be shown at once, and the required verification thereby be obtained, if we merely take the variation of the relation (F.) with respect to the initial coordinates, as in the former verification we took its variation with respect to the final coordinates, and so obtained results which agreed with the known equations of motion, and which may be thus collected,

$$\left.\begin{aligned} \frac{\delta}{\delta x_i} \Sigma . \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2 \right\} &= \frac{\delta U}{\delta x_i}; \\ \frac{\delta}{\delta y_i} \Sigma . \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2 \right\} &= \frac{\delta U}{\delta y_i}; \\ \frac{\delta}{\delta z_i} \Sigma . \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2 \right\} &= \frac{\delta U}{\delta z_i}. \end{aligned}\right\} \tag{M.}$$

The same relation (F.), by being varied with respect to the quantity $H$, conducts to the expression

$$\frac{\delta}{\delta H} \Sigma . \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2 \right\} = 1; \tag{N.}$$

and this, when developed, agrees with the equation (K.), which is a new verification of the consistence of our foregoing results. Nor would it have been much more difficult, by the help of the foregoing principles, to have integrated directly our integrals of the first order, and so to have deduced in a different way our final integral system.

6. It may be considered as still another verification of our own general integral equations, to show that they include not only the known law of living force, or the integral expressing that law, but also the six other known integrals of the first order, which contain the law of motion of the centre of gravity, and the law of description of areas. For this purpose, it is only necessary to observe that it evidently follows from the conception of our characteristic function $V$, that the function depends on the initial and final positions of the attracting or repelling points of a system, not as referred to any foreign standard, but only as compared with one another; and therefore that this function will not vary, if without making any real change in either initial or final configuration, or in the relation of these to each other, we alter at once all the initial and all the final positions of the points of the system, by any common motion, whether of translation or of rotation. Now by considering these coordinate translations, we obtain the three following partial differential equations of the first order, which the function $V$ must satisfy,

$$\left.\begin{aligned} \Sigma\frac{\delta V}{\delta x} + \Sigma\frac{\delta V}{\delta a} &= 0; \\ \Sigma\frac{\delta V}{\delta y} + \Sigma\frac{\delta V}{\delta b} &= 0; \\ \Sigma\frac{\delta V}{\delta z} + \Sigma\frac{\delta V}{\delta c} &= 0; \end{aligned}\right\} \qquad \text{(O.)}$$

and by considering three coordinate rotations, we obtain these three other relations between the partial differential coefficients of the same order of the same characteristic function,

$$\left.\begin{aligned} \Sigma\left(x\frac{\delta V}{\delta y} - y\frac{\delta V}{\delta x}\right) + \Sigma\left(a\frac{\delta V}{\delta b} - b\frac{\delta V}{\delta a}\right) &= 0; \\ \Sigma\left(y\frac{\delta V}{\delta z} - z\frac{\delta V}{\delta y}\right) + \Sigma\left(b\frac{\delta V}{\delta c} - c\frac{\delta V}{\delta b}\right) &= 0; \\ \Sigma\left(z\frac{\delta V}{\delta x} - x\frac{\delta V}{\delta z}\right) + \Sigma\left(c\frac{\delta V}{\delta a} - a\frac{\delta V}{\delta c}\right) &= 0; \end{aligned}\right\} \qquad \text{(P.)}$$

and if we change the final coefficients of $V$ to the final components of momentum, and the initial coefficients to the initial components taken negatively, according to the dynamical properties of this function expressed by the integrals (C.) and (D.), we shall change these partial differential equations (O.) (P.), to the following,

$$\Sigma\,.mx' = \Sigma\,.ma'; \quad \Sigma\,.my' = \Sigma\,.mb'; \quad \Sigma\,.mz' = \Sigma\,.mc'; \qquad (15.)$$

and

$$\left.\begin{aligned} \Sigma\,.m(xy' - yx') &= \Sigma\,.m(ab' - ba'); \\ \Sigma\,.m(yz' - zy') &= \Sigma\,.m(bc' - cb'); \\ \Sigma\,.m(zx' - xz') &= \Sigma\,.m(ca' - ac'). \end{aligned}\right\} \qquad (16.)$$

In this manner, therefore, we can deduce from the properties of our characteristic function the six other known integrals above mentioned, in addition to that seventh which contains the law of living force, and which assisted in the discovery of our method.

*Introduction of relative or polar Coordinates, or other marks of position of a System.*

7. The property of our characteristic function, by which it depends only on the internal or mutual relations between the positions initial and final of the points of an attracting or repelling system, suggests an advantage in employing internal or relative coordinates; and from the analogy of other applications of algebraical methods to researches of a geometrical kind, it may be expected that polar and other marks of position will also often be found useful. Supposing, therefore, that the $3n$ final coordinates $x_1\, y_1\, z_1 \,\ldots\, x_n\, y_n\, z_n$ have been expressed as functions of $3n$ other variables $\eta_1\, \eta_2 \,\ldots\, \eta_{3n}$, and that the $3n$ initial coordinates have in like manner been expressed as functions of $3n$ similar quantities, which we shall call $e_1\, e_2 \,\ldots\, e_{3n}$, we shall proceed to assign a general method for introducing these new marks of position into the expressions of our fundamental relations.

For this purpose we have only to transform the law of varying action, or the fundamental formula (A.), by transforming the two sums,

$$\Sigma\,.m(x'\,\delta x + y'\,\delta y + z'\,\delta z), \quad \text{and} \quad \Sigma\,.m(a'\,\delta a + b'\,\delta b + c'\,\delta c),$$

which it involves, and which are respectively equivalent to the following more developed expressions,

$$\left.\begin{aligned}\Sigma\,.m(x'\,\delta x + y'\,\delta y + z'\,\delta z) = m_1(x'_1\,\delta x_1 + y'_1\,\delta y_1 + z'_1\,\delta z_1)\\ + m_2(x'_2\,\delta x_2 + y'_2\,\delta y_2 + z'_2\,\delta z_2)\\ + \&\text{c.} + m_n(x'_n\,\delta x_n + y'_n\,\delta y_n + z'_n\,\delta z_n);\end{aligned}\right\} \tag{17.}$$

$$\left.\begin{aligned}\Sigma\,.m(a'\,\delta a + b'\,\delta b + c'\,\delta c) = m_1(a'_1\,\delta a_1 + b'_1\,\delta b_1 + c'_1\,\delta c_1)\\ + m_2(a'_2\,\delta a_2 + b'_2\,\delta b_2 + c'_2\,\delta c_2)\\ + \&\text{c.} + m_n(a'_n\,\delta a_n + b'_n\,\delta b_n + c'_n\,\delta c_n).\end{aligned}\right\} \tag{18.}$$

Now $x_i$ being by supposition a function of the $3n$ new marks of position $\eta_1 \,\ldots\, \eta_{3n}$, its variation $\delta x_i$, and its differential coefficient $x'_i$ may be thus expressed:

$$\delta x_i = \frac{\delta x_i}{\delta \eta_1}\delta\eta_1 + \frac{\delta x_i}{\delta \eta_2}\delta\eta_2 + \cdots + \frac{\delta x_i}{\delta \eta_{3n}}\delta\eta_{3n}; \tag{19.}$$

$$x'_i = \frac{\delta x_i}{\delta \eta_1}\eta'_1 + \frac{\delta x_i}{\delta \eta_2}\eta'_2 + \cdots + \frac{\delta x_i}{\delta \eta_{3n}}\eta'_{3n}; \tag{20.}$$

and similarly for $y_i$ and $z_i$. If, then, we consider $x'_i$ as a function, by (20.), of $\eta'_1 \,\ldots\, \eta'_{3n}$, involving also in general $\eta_1 \,\ldots\, \eta_{3n}$, and if we take its partial differential coefficients of the first order with respect to $\eta'_1 \,\ldots\, \eta'_{3n}$, we find the relations,

$$\frac{\delta x'_i}{\delta \eta'_1} = \frac{\delta x_i}{\delta \eta_1}; \quad \frac{\delta x'_i}{\delta \eta'_2} = \frac{\delta x_i}{\delta \eta_2}; \quad \ldots \quad \frac{\delta x'_i}{\delta \eta'_{3n}} = \frac{\delta x_i}{\delta \eta_{3n}}; \tag{21.}$$

and therefore we obtain these new expressions for the variations $\delta x_i$, $\delta y_i$, $\delta z_i$,

$$\left.\begin{aligned}
\delta x_i &= \frac{\delta x'_i}{\delta \eta'_1}\delta\eta_1 + \frac{\delta x'_i}{\delta \eta'_2}\delta\eta_2 + \cdots + \frac{\delta x'_i}{\delta \eta'_{3n}}\delta\eta_{3n},\\
\delta y_i &= \frac{\delta y'_i}{\delta \eta'_1}\delta\eta_1 + \frac{\delta y'_i}{\delta \eta'_2}\delta\eta_2 + \cdots + \frac{\delta y'_i}{\delta \eta'_{3n}}\delta\eta_{3n},\\
\delta z_i &= \frac{\delta z'_i}{\delta \eta'_1}\delta\eta_1 + \frac{\delta z'_i}{\delta \eta'_2}\delta\eta_2 + \cdots + \frac{\delta z'_i}{\delta \eta'_{3n}}\delta\eta_{3n}.
\end{aligned}\right\} \tag{22.}$$

Substituting these expressions (22.) for the variations in the sum (17.), we easily transform it into the following,

$$\left.\begin{aligned}
\Sigma\,.m(x'\,\delta x + y'\,\delta y + z'\,\delta z) &= \Sigma\,.m\left(x'\frac{\delta x'}{\delta\eta'_1} + y'\frac{\delta y'}{\delta\eta'_1} + z'\frac{\delta z'}{\delta\eta'_1}\right).\,\delta\eta_1\\
&\quad + \Sigma\,.m\left(x'\frac{\delta x'}{\delta\eta'_2} + y'\frac{\delta y'}{\delta\eta'_2} + z'\frac{\delta z'}{\delta\eta'_2}\right).\,\delta\eta_2\\
&\quad + \&\text{c.} + \Sigma\,.m\left(x'\frac{\delta x'}{\delta\eta'_{3n}} + y'\frac{\delta y'}{\delta\eta'_{3n}} + z'\frac{\delta z'}{\delta\eta'_{3n}}\right).\,\delta\eta_{3n}\\
&= \frac{\delta T}{\delta\eta'_1}\delta\eta_1 + \frac{\delta T}{\delta\eta'_2}\delta\eta_2 + \cdots + \frac{\delta T}{\delta\eta'_{3n}}\delta\eta_{3n};
\end{aligned}\right\} \tag{23.}$$

$T$ being the same quantity as before, namely, the half of the final living force of system, but being now considered as a function of $\eta'_1 \ldots \eta'_{3n}$, involving also the masses, and in general $\eta_1 \ldots \eta_{3n}$, and obtained by substituting for the quantities $x'$ $y'$ $z'$ their values of the form (20.) in the equation of definition

$$T = \tfrac{1}{2}\,\Sigma\,.m(x'^2 + y'^2 + z'^2). \tag{4.}$$

In like manner we find this transformation for the sum (18.),

$$\Sigma\,.m(a'\,\delta a + b'\,\delta b + c'\,\delta c) = \frac{\delta T_0}{\delta e'_1}\delta e_1 + \frac{\delta T_0}{\delta e'_2}\delta e_2 + \cdots + \frac{\delta T_0}{\delta e'_{3n}}\delta e_{3n}. \tag{24.}$$

The law of varying action, or the formula (A.), becomes therefore, when expressed by the present more general coordinates or marks of position,

$$\delta V = \Sigma\,.\frac{\delta T}{\delta\eta'}\delta\eta - \Sigma\,.\frac{\delta T}{\delta e'}\delta e + t\,\delta H; \tag{Q.}$$

and instead of the groups (C.) and (D.), into which, along with the equation (E.), this law resolved itself before, it gives now these other groups,

$$\frac{\delta V}{\delta\eta_1} = \frac{\delta T}{\delta\eta'_1};\quad \frac{\delta V}{\delta\eta_2} = \frac{\delta T}{\delta\eta'_2};\quad \cdots\quad \frac{\delta V}{\delta\eta_{3n}} = \frac{\delta T}{\delta\eta'_{3n}}; \tag{R.}$$

and

$$\frac{\delta V}{\delta e_1} = -\frac{\delta T_0}{\delta e'_1}; \quad \frac{\delta V}{\delta e_2} = -\frac{\delta T_0}{\delta e'_2}; \quad \cdots \quad \frac{\delta V}{\delta e_{3n}} = -\frac{\delta T_0}{\delta e'_{3n}}. \tag{S.}$$

The quantities $e_1\, e_2\, \ldots\, e_{3n}$, and $e'_1\, e'_2\, \ldots\, e'_{3n}$, are now the initial data respecting the manner of motion of the system; and the $3n$ final integrals, connecting these $6n$ initial data, and the $n$ masses, with the time $t$, and with the $3n$ final or varying quantities $\eta_1\, \eta_2\, \ldots\, \eta_{3n}$, which mark the varying positions of the $n$ moving points of the system, are now to be obtained by eliminating the auxiliary constant $H$ between the $3n+1$ equations (S.) and (E.); while the $3n$ intermediate integrals, or integrals of the first order, which connect the same varying marks of position and their first differential coefficients with the time, the masses, and the initial marks of position, are the result of elimination of the same auxiliary constant $H$ between the equations (R.) and (E.). Our fundamental formula, and intermediate and final integrals, can therefore be very simply expressed with any new sets of coordinates; and the partial differential equations (F.) (G.), which our characteristic function $V$ must satisfy, and which are, as we have said, essential in the theory of that function, can also easily be expressed with any such transformed coordinates, by merely combining the final and initial expressions of the law of living force,

$$T = U + H, \tag{6.}$$

$$T_0 = U_0 + H, \tag{7.}$$

with the new groups (R.) and (S.). For this purpose we must now consider the function $U$, of the masses and mutual distances of the several points of the system, as depending on the new marks of position $\eta_1\, \eta_2\, \ldots\, \eta_{3n}$; and the analogous function $U_0$, as depending similarly on the initial quantities $e_1\, e_2\, \ldots\, e_{3n}$; we must also suppose that $T$ is expressed (as it may) as a function of its own coefficients, $\frac{\delta T}{\delta \eta'_1}, \frac{\delta T}{\delta \eta'_2}, \ldots \frac{\delta T}{\delta \eta'_{3n}}$, which will always be, with respect to these, homogeneous of the second dimension, and may also involve explicitly the quantities $\eta_1\, \eta_2\, \ldots\, \eta_{3n}$; and that $T_0$ is expressed as a similar function of its coefficients $\frac{\delta T_0}{\delta e'_1}, \frac{\delta T_0}{\delta e'_2}, \ldots \frac{\delta T_0}{\delta e'_{3n}}$; so that

$$\left.\begin{aligned} T &= F\left(\frac{\delta T}{\delta \eta'_1}, \frac{\delta T}{\delta \eta'_2}, \ldots \frac{\delta T}{\delta \eta'_{3n}}\right), \\ T_0 &= F\left(\frac{\delta T_0}{\delta e'_1}, \frac{\delta T_0}{\delta e'_2}, \ldots \frac{\delta T_0}{\delta e'_{3n}}\right); \end{aligned}\right\} \tag{25.}$$

and that then these coefficients of $T$ and $T_0$ are changed to their values (R.) and (S.), so as to give, instead of (F.) and (G.), two other transformed equations, namely,

$$F\left(\frac{\delta V}{\delta \eta_1}, \frac{\delta V}{\delta \eta_2}, \ldots \frac{\delta V}{\delta \eta_{3n}}\right) = U + H, \tag{T.}$$

and, on account of the homogeneity and dimension of $T_0$,

$$F\left(\frac{\delta V}{\delta e_1}, \frac{\delta V}{\delta e_2}, \ldots \frac{\delta V}{\delta e_{3n}}\right) = U_0 + H. \tag{U.}$$

8. Nor is there any difficulty in deducing analogous transformations for the known differential equations of motion of the second order, of any system of free points, by taking the variation of the new form (T.) of the law of living force, and by attending to the dynamical meanings of the coefficients of our characteristic function. For if we observe that the final living force $2T$, when considered as a function of $\eta_1\,\eta_2\,\ldots\,\eta_{3n}$, and of $\eta'_1\,\eta'_2\,\ldots\,\eta'_{3n}$, is necessarily homogeneous of the second dimension with respect to the latter set of variables, and must therefore satisfy the condition

$$2T = \eta'_1 \frac{\delta T}{\delta \eta'_1} + \eta'_2 \frac{\delta T}{\delta \eta'_2} + \cdots + \eta'_{3n} \frac{\delta T}{\delta \eta'_{3n}}, \tag{26.}$$

we shall perceive that its total variation,

$$\left.\begin{aligned} \delta T = \frac{\delta T}{\delta \eta_1}\delta\eta_1 + \frac{\delta T}{\delta \eta_2}\delta\eta_2 + \cdots + \frac{\delta T}{\delta \eta_{3n}}\delta\eta_{3n} \\ + \frac{\delta T}{\delta \eta'_1}\delta\eta'_1 + \frac{\delta T}{\delta \eta'_2}\delta\eta'_2 + \cdots + \frac{\delta T}{\delta \eta'_{3n}}\delta\eta'_{3n}, \end{aligned}\right\} \tag{27.}$$

may be put under the form

$$\left.\begin{aligned} \delta T &= \eta'_1\,\delta\frac{\delta T}{\delta \eta'_1} + \eta'_2\,\delta\frac{\delta T}{\delta \eta'_2} + \cdots + \eta'_{3n}\,\delta\frac{\delta T}{\delta \eta'_{3n}} \\ &\quad - \frac{\delta T}{\delta \eta_1}\delta\eta_1 - \frac{\delta T}{\delta \eta_2}\delta\eta_2 - \cdots - \frac{\delta T}{\delta \eta_{3n}}\delta\eta_{3n} \\ &= \Sigma\,.\eta'\,\delta\frac{\delta T}{\delta \eta'} - \Sigma\,.\frac{\delta T}{\delta \eta}\delta\eta \\ &= \Sigma\,.\left(\eta'\,\delta\frac{\delta V}{\delta \eta} - \frac{\delta T}{\delta \eta}\delta\eta\right), \end{aligned}\right\} \tag{28.}$$

and therefore that the total variation of the new partial differential equation (T.) may be thus written,

$$\Sigma\,.\left(\eta'\,\delta\frac{\delta V}{\delta \eta} - \frac{\delta T}{\delta \eta}\delta\eta\right) = \Sigma\,.\frac{\delta U}{\delta \eta}\delta\eta + \delta H: \tag{V.}$$

in which, if we observe that $\eta' = \dfrac{d\eta}{dt}$, and that the quantities of the form $\eta$ are the only ones which vary with the time, we shall see that

$$\Sigma\,.\eta'\delta\frac{\delta V}{\delta \eta} = \Sigma\left(\frac{d}{dt}\frac{\delta V}{\delta \eta}\,.\,\delta\eta + \frac{d}{dt}\frac{\delta V}{\delta e}\,.\,\delta e\right) + \frac{d}{dt}\frac{\delta V}{\delta H}\,.\,\delta H, \tag{29.}$$

because the identical equation $\delta dV = d\delta V$ gives, when developed,

$$\Sigma\left(\delta\frac{\delta V}{\delta \eta}\,.\,d\eta + \delta\frac{\delta V}{\delta e}\,.\,de\right) + \delta\frac{\delta V}{\delta H}\,.\,dH = \Sigma\left(d\frac{\delta V}{\delta \eta}\,.\,\delta\eta + d\frac{\delta V}{\delta e}\,.\,\delta e\right) + d\frac{\delta V}{\delta H}\,.\,\delta H. \tag{30.}$$

Decomposing, therefore, the expression (V.), for the variation of half the living force, into as many separate equations as it contains independent variations, we obtain, not only the equation

$$\frac{d}{dt}\frac{\delta V}{\delta H} = 1, \tag{K.}$$

which had already presented itself, and the group

$$\frac{d}{dt}\frac{\delta V}{\delta e_1} = 0, \quad \frac{d}{dt}\frac{\delta V}{\delta e_2} = 0, \quad \cdots \quad \frac{d}{dt}\frac{\delta V}{\delta e_{3n}} = 0, \tag{W.}$$

which might have been at once obtained by differentiation from the final integrals (S.), but also a group of $3n$ other equations of the form

$$\frac{d}{dt}\frac{\delta V}{\delta \eta} - \frac{\delta T}{\delta \eta} = \frac{\delta U}{\delta \eta}, \tag{X.}$$

which give, by the intermediate integrals (R.),

$$\frac{d}{dt}\frac{\delta T}{\delta \eta'} - \frac{\delta T}{\delta \eta} = \frac{\delta U}{\delta \eta}: \tag{Y.}$$

that is, more fully,

$$\left.\begin{array}{c} \dfrac{d}{dt}\dfrac{\delta T}{\delta \eta'_1} - \dfrac{\delta T}{\delta \eta_1} = \dfrac{\delta U}{\delta \eta_1}; \\ \dfrac{d}{dt}\dfrac{\delta T}{\delta \eta'_2} - \dfrac{\delta T}{\delta \eta_2} = \dfrac{\delta U}{\delta \eta_2}; \\ \cdots\cdots \\ \dfrac{d}{dt}\dfrac{\delta T}{\delta \eta'_{3n}} - \dfrac{\delta T}{\delta \eta_{3n}} = \dfrac{\delta U}{\delta \eta_{3n}}. \end{array}\right\} \tag{Z.}$$

These last transformations of the differential equations of motion of the second order, of an attracting or repelling system, coincide in all respects (a slight difference of notation excepted,) with the elegant canonical forms in the *Mécanique Analytique* of Lagrange; but it seemed worth while to deduce them here anew, from the properties of our characteristic function. And if we were to suppose (as it has often been thought convenient and even necessary to do,) that the $n$ points of a system are not entirely free, nor subject only to their own mutual attractions or repulsions, but connected by any geometrical conditions, and influenced by any foreign agencies, consistent with the law of conservation of living force; so that the number of independent marks of position would be now less numerous, and the force-function $U$ less simple than before; it might still be proved, by a reasoning very similar to the foregoing, that on these suppositions also (which however, the dynamical spirit is tending more and more to exclude,) the accumulated living force or action $V$ of the system is a *characteristic motion-function* of the kind already explained; having the same law and formula of variation, which are susceptible of the same transformations; obliged to satisfy in the same way a final and an initial relation between its partial differential coefficients of the first order; conducting, by the variation of one of these two relations, to the same canonical forms assigned by LAGRANGE for the differential equations of motion; and furnishing, on the same principles as before, their intermediate and their final integrals. To those imaginable cases, indeed, in which the law of living force no longer holds, our method also would not apply; but it appears to be the growing conviction of the persons who have meditated the most profoundly on the mathematical dynamics of the universe, that these are cases suggested by insufficient views of the mutual actions of body.

9. It results from the foregoing remarks, that in order to apply our method of the characteristic function to any problem of dynamics respecting any moving system, the known law of living force is to be combined with our law of varying action; and that the general expression of this latter law is to be obtained in the following manner. We are first to express the quantity $T$, namely, the half of the living force of the system, as a function (which will always be homogeneous of the second dimension,) of the differential coefficients or rates of increase $\eta'_1$, $\eta'_2$ &c., of any rectangular coordinates, or other marks of position of the system: we are next to take the variation of this homogeneous function with respect to those rates of increase, and to change the variations of those rates $\delta\eta'_1$, $\delta\eta'_2$, &c., to the variations $\delta\eta_1$, $\delta\eta_2$, &c., of the marks of position themselves; and then to subtract the initial from the final value of the result, and to equate the remainder to $\delta V - t\,\delta H$. A slight consideration will show that this general rule or process for obtaining the variation of the characteristic function $V$, is applicable even when the marks of position $\eta_1$, $\eta_2$, &c. are not all independent of each other; which will happen when they have been made, from any motive of convenience, more numerous than the rectangular coordinates of the several points of the system. For if we suppose that the $3n$ rectangular coordinates $x_1\,y_1\,z_1\,\ldots\,x_n\,y_n\,z_n$ have been expressed by any transformation as functions of $3n+k$ other marks of position, $\eta_1\,\eta_2\,\ldots\,\eta_{3n+k}$, which must therefore be connected by $k$ equations of condition,

$$\left.\begin{array}{l} 0 = \phi_1(\eta_1, \eta_2, \ldots\ \eta_{3n+k}), \\ 0 = \phi_2(\eta_1, \eta_2, \ldots\ \eta_{3n+k}), \\ \cdots\cdots \\ 0 = \phi_k(\eta_1, \eta_2, \ldots\ \eta_{3n+k}), \end{array}\right\} \tag{31.}$$

giving $k$ of the new marks of position as functions of the remaining $3n$,

$$\left.\begin{array}{l} \eta_{3n+1} = \psi_1(\eta_1, \eta_2, \ldots\ \eta_{3n}), \\ \eta_{3n+2} = \psi_2(\eta_1, \eta_2, \ldots\ \eta_{3n}), \\ \cdots\cdots \\ \eta_{3n+k} = \psi_k(\eta_1, \eta_2, \ldots\ \eta_{3n}), \end{array}\right\} \tag{32.}$$

the expression

$$T = \tfrac{1}{2}\,\Sigma\,.m(x'^2 + y'^2 + z'^2), \tag{4.}$$

will become, by the introduction of these new variables, a homogeneous function of the second dimension of the $3n+k$ rates of increase $\eta'_1, \eta'_2, \ldots\ \eta'_{3n+k}$, involving also in general

$\eta_1, \eta_2, \ldots \eta_{3n+k}$, and having a variation which may be thus expressed:

$$\left.\begin{aligned}\delta T = \left(\frac{\delta T}{\delta \eta'_1}\right)\delta\eta'_1 + \left(\frac{\delta T}{\delta \eta'_2}\right)\delta\eta'_2 + \cdots + \left(\frac{\delta T}{\delta \eta'_{3n+k}}\right)\delta\eta'_{3n+k} \\ + \left(\frac{\delta T}{\delta \eta_1}\right)\delta\eta_1 + \left(\frac{\delta T}{\delta \eta_2}\right)\delta\eta_2 + \cdots + \left(\frac{\delta T}{\delta \eta_{3n+k}}\right)\delta\eta_{3n+k};\end{aligned}\right\} \tag{33.}$$

or in this other way,

$$\left.\begin{aligned}\delta T = & \frac{\delta T}{\delta \eta'_1}\delta\eta'_1 + \frac{\delta T}{\delta \eta'_2}\delta\eta'_2 + \cdots + \frac{\delta T}{\delta \eta'_{3n}}\delta\eta'_{3n} \\ & + \frac{\delta T}{\delta \eta_1}\delta\eta_1 + \frac{\delta T}{\delta \eta_2}\delta\eta_2 + \cdots + \frac{\delta T}{\delta \eta_{3n}}\delta\eta_{3n},\end{aligned}\right\} \tag{34.}$$

on account of the relations (32.) which give, when differentiated with respect to the time,

$$\left.\begin{aligned}&\eta'_{3n+1} = \eta'_1\frac{\delta\psi_1}{\delta\eta_1} + \eta'_2\frac{\delta\psi_1}{\delta\eta_2} + \cdots + \eta'_{3n}\frac{\delta\psi_1}{\delta\eta_{3n}}, \\ &\eta'_{3n+2} = \eta'_1\frac{\delta\psi_2}{\delta\eta_1} + \eta'_2\frac{\delta\psi_2}{\delta\eta_2} + \cdots + \eta'_{3n}\frac{\delta\psi_2}{\delta\eta_{3n}}, \\ &\cdots\cdots \\ &\eta'_{3n+k} = \eta'_1\frac{\delta\psi_k}{\delta\eta_1} + \eta'_2\frac{\delta\psi_k}{\delta\eta_2} + \cdots + \eta'_{3n}\frac{\delta\psi_k}{\delta\eta_{3n}},\end{aligned}\right\} \tag{35.}$$

and therefore, attending only to the variations of quantities of the form $\eta'$,

$$\left.\begin{aligned}&\delta\eta'_{3n+1} = \frac{\delta\psi_1}{\delta\eta_1}\delta\eta'_1 + \frac{\delta\psi_1}{\delta\eta_2}\delta\eta'_2 + \cdots + \frac{\delta\psi_1}{\delta\eta_{3n}}\delta\eta'_{3n}, \\ &\delta\eta'_{3n+2} = \frac{\delta\psi_2}{\delta\eta_1}\delta\eta'_1 + \frac{\delta\psi_2}{\delta\eta_2}\delta\eta'_2 + \cdots + \frac{\delta\psi_2}{\delta\eta_{3n}}\delta\eta'_{3n}, \\ &\cdots\cdots \\ &\delta\eta'_{3n+k} = \frac{\delta\psi_k}{\delta\eta_1}\delta\eta'_1 + \frac{\delta\psi_k}{\delta\eta_2}\delta\eta'_2 + \cdots + \frac{\delta\psi_k}{\delta\eta_{3n}}\delta\eta'_{3n}.\end{aligned}\right\} \tag{36.}$$

Comparing the two expressions (33.) and (34.), we find by (36.) the relations

$$\left.\begin{aligned}&\frac{\delta T}{\delta\eta'_1} = \left(\frac{\delta T}{\delta\eta'_1}\right) + \left(\frac{\delta T}{\delta\eta'_{3n+1}}\right)\frac{\delta\psi_1}{\delta\eta_1} + \left(\frac{\delta T}{\delta\eta'_{3n+2}}\right)\frac{\delta\psi_2}{\delta\eta_1} + \cdots + \left(\frac{\delta T}{\delta\eta'_{3n+k}}\right)\frac{\delta\psi_k}{\delta\eta_1}, \\ &\frac{\delta T}{\delta\eta'_2} = \left(\frac{\delta T}{\delta\eta'_2}\right) + \left(\frac{\delta T}{\delta\eta'_{3n+1}}\right)\frac{\delta\psi_1}{\delta\eta_2} + \left(\frac{\delta T}{\delta\eta'_{3n+2}}\right)\frac{\delta\psi_2}{\delta\eta_2} + \cdots + \left(\frac{\delta T}{\delta\eta'_{3n+k}}\right)\frac{\delta\psi_k}{\delta\eta_2}, \\ &\cdots\cdots \\ &\frac{\delta T}{\delta\eta'_{3n}} = \left(\frac{\delta T}{\delta\eta'_{3n}}\right) + \left(\frac{\delta T}{\delta\eta'_{3n+1}}\right)\frac{\delta\psi_1}{\delta\eta_{3n}} + \left(\frac{\delta T}{\delta\eta'_{3n+2}}\right)\frac{\delta\psi_2}{\delta\eta_{3n}} + \cdots + \left(\frac{\delta T}{\delta\eta'_{3n+k}}\right)\frac{\delta\psi_k}{\delta\eta_{3n}};\end{aligned}\right\} \tag{37.}$$

which give, by (32.),

$$\frac{\delta T}{\delta \eta_1'} \delta\eta_1 + \frac{\delta T}{\delta \eta_2'} \delta\eta_2 + \cdots + \frac{\delta T}{\delta \eta_{3n}'} \delta\eta_{3n} = \left(\frac{\delta T}{\delta \eta_1'}\right) \delta\eta_1 + \left(\frac{\delta T}{\delta \eta_2'}\right) \delta\eta_2 + \cdots + \left(\frac{\delta T}{\delta \eta_{3n+k}'}\right) \delta\eta_{3n+k}; \quad (38.)$$

we may therefore put the expression (Q.) under the following more general form,

$$\delta V = \Sigma \,.\left(\frac{\delta T}{\delta \eta'}\right) \delta\eta - \Sigma \,.\left(\frac{\delta T_0}{\delta e'}\right) \delta e + t\,\delta H, \qquad (\mathrm{A}^1.)$$

the coefficients $\left(\frac{\delta T}{\delta \eta'}\right)$ being formed by treating all the $3n+k$ quantities $\eta_1', \eta_2', \ldots, \eta_{3n+k}'$, as independent; which was the extension above announced, of the rule for forming the variation of the characteristic function $V$.

We cannot, however, immediately decompose this new expression (A$^1$.) for $\delta V$, as we did the expression (Q.), by treating all the variations $\delta\eta$, $\delta e$, as independent; but we may decompose it so, if we previously combine it with the final equations of condition (31.), and with the analogous initial equations of condition, namely,

$$\left.\begin{array}{l} 0 = \Phi_1(e_1, e_2, \ldots\, e_{3n+k}), \\ 0 = \Phi_2(e_1, e_2, \ldots\, e_{3n+k}), \\ \cdots\cdots \\ 0 = \Phi_k(e_1, e_2, \ldots\, e_{3n+k}), \end{array}\right\} \qquad (39.)$$

which we may do by adding the variations of the connecting functions $\phi_1, \ldots, \phi_k$, $\Phi_1, \ldots\, \Phi_k$ multiplied respectively by the factors to be determined, $\lambda_1, \ldots\, \lambda_k$, $\Lambda_1, \ldots\, \Lambda_k$. In this manner the law of varying action takes this new form,

$$\delta V = \Sigma \,.\left(\frac{\delta T}{\delta \eta'}\right) \delta\eta - \Sigma \,.\left(\frac{\delta T_0}{\delta e'}\right) \delta e + t\,\delta H + \Sigma \,.\lambda\, \delta\phi + \Sigma \,.\Lambda\, \delta\Phi; \qquad (\mathrm{B}^1.)$$

and decomposes itself into $6n + 2k + 1$ separate expressions, for the partial differential coefficients of the first order of the characteristic function $V$, namely, into the following,

$$\left.\begin{array}{l} \frac{\delta V}{\delta \eta_1} = \left(\frac{\delta T}{\delta \eta_1'}\right) + \lambda_1 \frac{\delta \phi_1}{\delta \eta_1} + \lambda_2 \frac{\delta \phi_2}{\delta \eta_1} + \cdots + \lambda_k \frac{\delta \phi_k}{\delta \eta_1}, \\ \frac{\delta V}{\delta \eta_2} = \left(\frac{\delta T}{\delta \eta_2'}\right) + \lambda_1 \frac{\delta \phi_1}{\delta \eta_2} + \lambda_2 \frac{\delta \phi_2}{\delta \eta_2} + \cdots + \lambda_k \frac{\delta \phi_k}{\delta \eta_2}, \\ \cdots\cdots \\ \frac{\delta V}{\delta \eta_{3n+k}} = \left(\frac{\delta T}{\delta \eta_{3n+k}'}\right) + \lambda_1 \frac{\delta \phi_1}{\delta \eta_{3n+k}} + \cdots + \lambda_k \frac{\delta \phi_k}{\delta \eta_{3n+k}}, \end{array}\right\} \qquad (\mathrm{C}^1.)$$

and

$$\left.\begin{array}{l} \frac{\delta V}{\delta e_1} = -\left(\frac{\delta T}{\delta e_1'}\right) + \Lambda_1 \frac{\delta \Phi_1}{\delta e_1} + \Lambda_2 \frac{\delta \Phi_2}{\delta e_1} + \cdots + \Lambda_k \frac{\delta \Phi_k}{\delta e_1}, \\ \frac{\delta V}{\delta e_2} = -\left(\frac{\delta T}{\delta e_2'}\right) + \Lambda_1 \frac{\delta \Phi_1}{\delta e_2} + \Lambda_2 \frac{\delta \Phi_2}{\delta e_2} + \cdots + \Lambda_k \frac{\delta \Phi_k}{\delta e_2}, \\ \cdots\cdots \\ \frac{\delta V}{\delta e_{3n+k}} = -\left(\frac{\delta T}{\delta e_{3n+k}'}\right) + \Lambda_1 \frac{\delta \Phi_1}{\delta e_{3n+k}} + \cdots + \Lambda_k \frac{\delta \Phi_k}{\delta e_{3n+k}}, \end{array}\right\} \qquad (\mathrm{D}^1.)$$

besides the old equation (E.). The analogous introduction of multipliers in the canonical forms of LAGRANGE, for the differential equations of motion of the second order, by which a sum such as $\Sigma\,.\lambda\dfrac{\delta\phi}{\delta\eta}$ is added to $\dfrac{\delta U}{\delta\eta}$ in the second member of the formula (Y.), is also easily justified on the principles of the present essay.

*Separation of the relative motion of a system from the motion of its centre of gravity; characteristic function for such motion, and law of its variation.*

10. As an example of the foregoing transformations, and at the same time as an important application, we shall now introduce relative coordinates, $x_{\prime}\ y_{\prime}\ z_{\prime}$, referred to an internal origin $x_{\prime\prime}\ y_{\prime\prime}\ z_{\prime\prime}$; that is, we shall put

$$x_i = x_{\prime i} + x_{\prime\prime}, \quad y_i = y_{\prime i} + y_{\prime\prime}, \quad z_i = z_{\prime i} + z_{\prime\prime}, \tag{40.}$$

and in like manner

$$a_i = a_{\prime i} + a_{\prime\prime}, \quad b_i = b_{\prime i} + b_{\prime\prime}, \quad c_i = c_{\prime i} + c_{\prime\prime}; \tag{41.}$$

together with the differentiated expressions

$$x'_i = x'_{\prime i} + x'_{\prime\prime}, \quad y'_i = y'_{\prime i} + y'_{\prime\prime}, \quad z'_i = z'_{\prime i} + z'_{\prime\prime}, \tag{42.}$$

and

$$a'_i = a'_{\prime i} + a'_{\prime\prime}, \quad b'_i = b'_{\prime i} + b'_{\prime\prime}, \quad c'_i = c'_{\prime i} + c'_{\prime\prime}. \tag{43.}$$

Introducing the expressions (42.) for the rectangular components of velocity, we find that the value given by (4.) for the living force $2T$ decomposes itself into the three following parts,

$$\left.\begin{aligned} 2T &= \Sigma\,.m(x'^2 + y'^2 + z'^2) \\ &= \Sigma\,.m(x'^2_{\prime} + y'^2_{\prime} + z'^2_{\prime}) + 2(x'_{\prime\prime}\,\Sigma\,.mx'_{\prime} + y'_{\prime\prime}\,\Sigma\,.my'_{\prime} + z'_{\prime\prime}\,\Sigma\,.mz'_{\prime}) \\ &\quad + (x'^2_{\prime\prime} + y'^2_{\prime\prime} + z'^2_{\prime\prime})\,\Sigma\,m; \end{aligned}\right\} \tag{44.}$$

if then we establish, as we may, the three equations of condition,

$$\Sigma\,.mx_{\prime} = 0, \quad \Sigma\,.my_{\prime} = 0, \quad \Sigma\,.mz_{\prime} = 0, \tag{45.}$$

which give by (40.),

$$x_{\prime\prime} = \frac{\Sigma\,.mx}{\Sigma\,m}, \quad y_{\prime\prime} = \frac{\Sigma\,.my}{\Sigma\,m}, \quad z_{\prime\prime} = \frac{\Sigma\,.mz}{\Sigma\,m}, \tag{46.}$$

so that $x_{\prime\prime}\ y_{\prime\prime}\ z_{\prime\prime}$ are now the coordinates of the point which is called the centre of gravity of the system, we may reduce the function $T$ to the form

$$T = T_{\prime} + T_{\prime\prime}, \tag{47.}$$

in which

$$T_{\prime} = \tfrac{1}{2}\,\Sigma\,.m(x'^2_{\prime} + y'^2_{\prime} + z'^2_{\prime}), \tag{48.}$$

and

$$T_{\prime\prime} = \tfrac{1}{2}(x_{\prime\prime}'^2 + y_{\prime\prime}'^2 + z_{\prime\prime}'^2)\,\Sigma\, m. \tag{49.}$$

By this known decomposition, the whole living force $2T$ of the system is resolved into the two parts $2T_{\prime}$ and $2T_{\prime\prime}$, of which the former, $2T_{\prime}$, may be called the *relative living force*, being that which results solely from the relative velocities of the points of the system, in their motions about their common centre of gravity $x_{\prime\prime}$ $y_{\prime\prime}$ $z_{\prime\prime}$; while the latter part, $2T_{\prime\prime}$, results only from the absolute motion of that centre of gravity in space, and is the same as if all the masses of the system were united in that common centre. At the same time, the law of living force, $T = U + H$, (6.), resolves itself by the law of motion of the centre of gravity into the two following separate equations,

$$T_{\prime} = U + H_{\prime}, \tag{50.}$$

and

$$T_{\prime\prime} = H_{\prime\prime}; \tag{51.}$$

$H_{\prime}$ and $H_{\prime\prime}$ being two new constants independent of the time $t$, and such that their sum

$$H_{\prime} + H_{\prime\prime} = H. \tag{52.}$$

And we may in like manner decompose the action, or accumulated living force $V$, which is equal to the definite integral $\int_0^t 2T\,dt$, into the two following analogous parts,

$$V = V_{\prime} + V_{\prime\prime}, \tag{E$^1$.}$$

determined by the two equations,

$$V_{\prime} = \int_0^t 2T_{\prime}\,dt, \tag{F$^1$.}$$

and

$$V_{\prime\prime} = \int_0^t 2T_{\prime\prime}\,dt. \tag{G$^1$.}$$

The last equation gives by (51.),

$$V_{\prime\prime} = 2H_{\prime\prime}t; \tag{53.}$$

a result which, by the law of motion of the centre of gravity, may be thus expressed,

$$V_{\prime\prime} = \sqrt{(x_{\prime\prime} - a_{\prime\prime})^2 + (y_{\prime\prime} - b_{\prime\prime})^2 + (z_{\prime\prime} - c_{\prime\prime})^2}\,.\,\sqrt{2H_{\prime\prime}\,\Sigma\, m}\,: \tag{H$^1$.}$$

$a_{\prime\prime}$ $b_{\prime\prime}$ $c_{\prime\prime}$ being the initial coordinates of the centre of gravity, so that

$$a_{\prime\prime} = \frac{\Sigma\,.ma}{\Sigma\, m}, \quad b_{\prime\prime} = \frac{\Sigma\,.mb}{\Sigma\, m}, \quad c_{\prime\prime} = \frac{\Sigma\,.mc}{\Sigma\, m}. \tag{54.}$$

And for the variation $\delta V$ of the whole function $V$, the rule of the last number gives

$$\left.\begin{aligned} \delta V = {} & \Sigma\,.m(x_{\prime}'\,\delta x_{\prime} - a_{\prime}'\,\delta a_{\prime} + y_{\prime}'\,\delta y_{\prime} - b_{\prime}'\,\delta b_{\prime} + z_{\prime}'\,\delta z_{\prime} - c_{\prime}'\,\delta c_{\prime}) \\ & + (x_{\prime\prime}'\,\delta x_{\prime\prime} - a_{\prime\prime}'\,\delta a_{\prime\prime} + y_{\prime\prime}'\,\delta y_{\prime\prime} - b_{\prime\prime}'\,\delta b_{\prime\prime} + z_{\prime\prime}'\,\delta z_{\prime\prime} - c_{\prime\prime}'\,\delta c_{\prime\prime})\,\Sigma\, m \\ & + t\,\delta H + \lambda_1\,\Sigma\,.m\,\delta x_{\prime} + \lambda_2\,\Sigma\,.m\,\delta y_{\prime} + \lambda_3\,\Sigma\,.m\,\delta z_{\prime} \\ & + \Lambda_1\,\Sigma\,.m\,\delta a_{\prime} + \Lambda_2\,\Sigma\,.m\,\delta b_{\prime} + \Lambda_3\,\Sigma\,.m\,\delta c_{\prime}; \end{aligned}\right\} \tag{I$^1$.}$$

while the variation of the part $V_{\prime\prime}$, determined by the equation (H¹.), is easily shown to be equivalent to the part

$$\delta V_{\prime\prime} = (x'_{\prime\prime}\,\delta x_{\prime\prime} - a'_{\prime\prime}\,\delta a_{\prime\prime} + y'_{\prime\prime}\,\delta y_{\prime\prime} - b'_{\prime\prime}\,\delta b_{\prime\prime} + z'_{\prime\prime}\,\delta z_{\prime\prime} - c'_{\prime\prime}\,\delta c_{\prime\prime})\,\Sigma\, m + t\,\delta H_{\prime\prime}; \qquad (\mathrm{K}^1.)$$

the variation of the other part $V_{\prime}$ may therefore be thus expressed,

$$\left.\begin{aligned} \delta V_{\prime} = \Sigma\,.m(x'_{\prime}\,\delta x_{\prime} - a'_{\prime}\,\delta a_{\prime} + y'_{\prime}\,\delta y_{\prime} - b'_{\prime}\,\delta b_{\prime} + z'_{\prime}\,\delta z_{\prime} - c'_{\prime}\,\delta c_{\prime}) \\ + t\,\delta H_{\prime} + \lambda_1\,\Sigma\,.m\,\delta x_{\prime} + \lambda_2\,\Sigma\,.m\,\delta y_{\prime} + \lambda_3\,\Sigma\,.m\,\delta z_{\prime} \\ + \Lambda_1\,\Sigma\,.m\,\delta a_{\prime} + \Lambda_2\,\Sigma\,.m\,\delta b_{\prime} + \Lambda_3\,\Sigma\,.m\,\delta c_{\prime} : \end{aligned}\right\} \qquad (\mathrm{L}^1.)$$

and it resolves itself into the following separate expressions, in which the part $V_{\prime}$ is considered as a function of the $6n+1$ quantities $x_{\prime i}\ y_{\prime i}\ z_{\prime i}\ a_{\prime i}\ b_{\prime i}\ c_{\prime i}\ H_{\prime}$, of which, however, only $6n-5$ are really independent:
first group,

$$\left.\begin{aligned} \frac{\delta V_{\prime}}{\delta x_{\prime 1}} &= m_1 x'_{\prime 1} + \lambda_1 m_1; \quad \cdots \quad \frac{\delta V_{\prime}}{\delta x_{\prime n}} = m_n x'_{\prime n} + \lambda_1 m_n; \\ \frac{\delta V_{\prime}}{\delta y_{\prime 1}} &= m_1 y'_{\prime 1} + \lambda_2 m_1; \quad \cdots \quad \frac{\delta V_{\prime}}{\delta y_{\prime n}} = m_n y'_{\prime n} + \lambda_2 m_n; \\ \frac{\delta V_{\prime}}{\delta z_{\prime 1}} &= m_1 z'_{\prime 1} + \lambda_3 m_1; \quad \cdots \quad \frac{\delta V_{\prime}}{\delta z_{\prime n}} = m_n z'_{\prime n} + \lambda_3 m_n; \end{aligned}\right\} \qquad (\mathrm{M}^1.)$$

second group,

$$\left.\begin{aligned} \frac{\delta V_{\prime}}{\delta a_{\prime 1}} &= -m_1 a'_{\prime 1} + \Lambda_1 m_1; \quad \cdots \quad \frac{\delta V_{\prime}}{\delta a_{\prime n}} = -m_n a'_{\prime n} + \Lambda_1 m_n; \\ \frac{\delta V_{\prime}}{\delta b_{\prime 1}} &= -m_1 b'_{\prime 1} + \Lambda_2 m_1; \quad \cdots \quad \frac{\delta V_{\prime}}{\delta b_{\prime n}} = -m_n b'_{\prime n} + \Lambda_2 m_n; \\ \frac{\delta V_{\prime}}{\delta c_{\prime 1}} &= -m_1 c'_{\prime 1} + \Lambda_3 m_1; \quad \cdots \quad \frac{\delta V_{\prime}}{\delta c_{\prime n}} = -m_n c'_{\prime n} + \Lambda_3 m_n; \end{aligned}\right\} \qquad (\mathrm{N}^1.)$$

and finally,

$$\frac{\delta V_{\prime}}{\delta H_{\prime}} = t. \qquad (\mathrm{O}^1.)$$

With respect to the six multipliers $\lambda_1\ \lambda_2\ \lambda_3\ \Lambda_1\ \Lambda_2\ \Lambda_3$ which were introduced by the 3 final equations of condition (45.), and by the 3 analogous initial equations of condition,

$$\Sigma\,.ma_{\prime} = 0, \quad \Sigma\,.mb_{\prime} = 0, \quad \Sigma\,.mc_{\prime} = 0; \qquad (55.)$$

we have, by differentiating these conditions,

$$\Sigma\,.mx'_{\prime} = 0, \quad \Sigma\,.my'_{\prime} = 0, \quad \Sigma\,.mz'_{\prime} = 0, \qquad (56.)$$

and

$$\Sigma\,.ma'_{\prime} = 0, \quad \Sigma\,.mb'_{\prime} = 0, \quad \Sigma\,.mc'_{\prime} = 0; \qquad (57.)$$

and therefore

$$\lambda_1 = \frac{\Sigma \dfrac{\delta V_\prime}{\delta x_\prime}}{\Sigma m}, \quad \lambda_2 = \frac{\Sigma \dfrac{\delta V_\prime}{\delta y_\prime}}{\Sigma m}, \quad \lambda_3 = \frac{\Sigma \dfrac{\delta V_\prime}{\delta z_\prime}}{\Sigma m}, \qquad (58.)$$

and

$$\Lambda_1 = \frac{\Sigma \dfrac{\delta V_\prime}{\delta a_\prime}}{\Sigma m}, \quad \Lambda_2 = \frac{\Sigma \dfrac{\delta V_\prime}{\delta b_\prime}}{\Sigma m}, \quad \Lambda_3 = \frac{\Sigma \dfrac{\delta V_\prime}{\delta c_\prime}}{\Sigma m}. \qquad (59.)$$

11. As an example of the determination of these multipliers, we may suppose that the part $V_\prime$, of the whole action $V$, has been expressed, before differentiation, as a function of $H_\prime$, and of these other $6n - 6$ independent quantities

$$\left.\begin{array}{llll} x_{\prime 1} - x_{\prime n} = \xi_1, & x_{\prime 2} - x_{\prime n} = \xi_2, & \ldots & x_{\prime n-1} - x_{\prime n} = \xi_{n-1}, \\ y_{\prime 1} - y_{\prime n} = \eta_1, & y_{\prime 2} - y_{\prime n} = \eta_2, & \ldots & y_{\prime n-1} - y_{\prime n} = \eta_{n-1}, \\ z_{\prime 1} - z_{\prime n} = \zeta_1, & z_{\prime 2} - z_{\prime n} = \zeta_2, & \ldots & z_{\prime n-1} - z_{\prime n} = \zeta_{n-1}, \end{array}\right\} \qquad (60.)$$

and

$$\left.\begin{array}{llll} a_{\prime 1} - a_{\prime n} = \alpha_1, & a_{\prime 2} - a_{\prime n} = \alpha_2, & \ldots & a_{\prime n-1} - a_{\prime n} = \alpha_{n-1}, \\ b_{\prime 1} - b_{\prime n} = \beta_1, & b_{\prime 2} - b_{\prime n} = \beta_2, & \ldots & b_{\prime n-1} - b_{\prime n} = \beta_{n-1}, \\ c_{\prime 1} - c_{\prime n} = \gamma_1, & c_{\prime 2} - c_{\prime n} = \gamma_2, & \ldots & c_{\prime n-1} - c_{\prime n} = \gamma_{n-1}; \end{array}\right\} \qquad (61.)$$

that is, of the *differences* only of the *centrobaric* coordinates; or, in other words, as a function of the coordinates (initial and final) of $n - 1$ points of the system, referred to the $n^{\text{th}}$ point, as an internal or moveable origin: because the centrobaric coordinates $x_{\prime i}$, $y_{\prime i}$, $z_{\prime i}$, $a_{\prime i}$, $b_{\prime i}$, $c_{\prime i}$, may themselves, by the equations of condition, be expressed as a function of these, namely,

$$x_{\prime i} = \xi_i - \frac{\Sigma\,.m\xi}{\Sigma m}, \quad y_{\prime i} = \eta_i - \frac{\Sigma\,.m\eta}{\Sigma m}, \quad z_{\prime i} = \zeta_i - \frac{\Sigma\,.m\zeta}{\Sigma m}, \qquad (62.)$$

and in like manner,

$$a_{\prime i} = \alpha_i - \frac{\Sigma\,.m\alpha}{\Sigma m}, \quad b_{\prime i} = \beta_i - \frac{\Sigma\,.m\beta}{\Sigma m}, \quad c_{\prime i} = \gamma_i - \frac{\Sigma\,.m\gamma}{\Sigma m}; \qquad (63.)$$

in which we are to observe, that the six quantities $\xi_n\ \eta_n\ \zeta_n\ \alpha_n\ \beta_n\ \gamma_n$ must be considered as separately vanishing. When $V_\prime$ has been thus expressed as a function of the centrobaric coordinates, involving their differences only, it will evidently satisfy the six partial differential equations,

$$\left.\begin{array}{lll} \Sigma \dfrac{\delta V_\prime}{\delta x_\prime} = 0, & \Sigma \dfrac{\delta V_\prime}{\delta y_\prime} = 0, & \Sigma \dfrac{\delta V_\prime}{\delta z_\prime} = 0, \\ \Sigma \dfrac{\delta V_\prime}{\delta a_\prime} = 0, & \Sigma \dfrac{\delta V_\prime}{\delta b_\prime} = 0, & \Sigma \dfrac{\delta V_\prime}{\delta c_\prime} = 0; \end{array}\right\} \qquad (\mathrm{P}^1.)$$

after this preparation, therefore, of the function $V_\prime$, the six multipliers determined by (58.) and (59.) will vanish, so that we shall have

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0, \quad \Lambda_1 = 0, \quad \Lambda_2 = 0, \quad \Lambda_3 = 0, \qquad (64.)$$

and the groups ($M^1$.) and ($N^1$.) will reduce themselves to the two following:

$$\left.\begin{array}{llll}
\dfrac{\delta V_\prime}{\delta x_{\prime 1}} = m_1 x'_{\prime 1}; & \dfrac{\delta V_\prime}{\delta x_{\prime 2}} = m_2 x'_{\prime 2}; & \cdots & \dfrac{\delta V_\prime}{\delta x_{\prime n}} = m_n x'_{\prime n};\\
\dfrac{\delta V_\prime}{\delta y_{\prime 1}} = m_1 y'_{\prime 1}; & \dfrac{\delta V_\prime}{\delta y_{\prime 2}} = m_2 y'_{\prime 2}; & \cdots & \dfrac{\delta V_\prime}{\delta y_{\prime n}} = m_n y'_{\prime n};\\
\dfrac{\delta V_\prime}{\delta z_{\prime 1}} = m_1 z'_{\prime 1}; \quad \cdots & \dfrac{\delta V_\prime}{\delta z_{\prime 2}} = m_2 z'_{\prime 2}; & \cdots & \dfrac{\delta V_\prime}{\delta z_{\prime n}} = m_n z'_{\prime n};
\end{array}\right\} \qquad (\mathrm{Q}^1.)$$

and

$$\left.\begin{array}{llll}
\dfrac{\delta V_\prime}{\delta a_{\prime 1}} = -m_1 a'_{\prime 1}; & \dfrac{\delta V_\prime}{\delta a_{\prime 2}} = -m_2 a'_{\prime 2}; & \cdots & \dfrac{\delta V_\prime}{\delta a_{\prime n}} = -m_n a'_{\prime n};\\
\dfrac{\delta V_\prime}{\delta b_{\prime 1}} = -m_1 b'_{\prime 1}; & \dfrac{\delta V_\prime}{\delta b_{\prime 2}} = -m_2 b'_{\prime 2}; & \cdots & \dfrac{\delta V_\prime}{\delta b_{\prime n}} = -m_n b'_{\prime n};\\
\dfrac{\delta V_\prime}{\delta c_{\prime 1}} = -m_1 c'_{\prime 1}; \quad \cdots & \dfrac{\delta V_\prime}{\delta c_{\prime 2}} = -m_2 c'_{\prime 2}; & \cdots & \dfrac{\delta V_\prime}{\delta c_{\prime n}} = -m_n c'_{\prime n};
\end{array}\right\} \qquad (\mathrm{R}^1.)$$

analogous in all respects to the groups (C.) and (D.). We find, therefore, for the relative motion of a system about its own centre of gravity, equations of the same form as those which we had obtained before for the absolute motion of the same system of points in space, And we see that in investigating such relative motion only, it is useful to confine ourselves to the part $V_\prime$ of our whole characteristic function, that is, to the *relative action* of the system, or accumulated living force of the motion about the centre of gravity; and to consider this part as the *characteristic function* of such relative motion, in a sense analogous to that which has been already explained.

This relative action, or part $V_\prime$, may, however, be otherwise expressed, and even in an infinite variety of ways, on account of the six equations of condition which connect the $6n$ centrobaric coordinates; and every different preparation of its form will give a different set of values for the six multipliers $\lambda_1$ $\lambda_2$ $\lambda_3$ $\Lambda_1$ $\Lambda_2$ $\Lambda_3$. For example, we might eliminate, by a previous preparation, the six centrobaric coordinates of the point $m_n$ from the expression of $V_\prime$, so as to make this expression involve only the centrobaric coordinates of the other $n-1$ points of the system, and then we should have

$$\frac{\delta V_\prime}{\delta x_{\prime n}} = 0, \quad \frac{\delta V_\prime}{\delta y_{\prime n}} = 0, \quad \frac{\delta V_\prime}{\delta z_{\prime n}} = 0, \quad \frac{\delta V_\prime}{\delta a_{\prime n}} = 0, \quad \frac{\delta V_\prime}{\delta b_{\prime n}} = 0, \quad \frac{\delta V_\prime}{\delta c_{\prime n}} = 0, \qquad (\mathrm{S}^1.)$$

and therefore, by the six last equations of the groups ($M^1$.) and ($N^1$.), the multipliers would take the values

$$\lambda_1 = -x'_{\prime n}, \quad \lambda_2 = -y'_{\prime n}, \quad \lambda_3 = -z'_{\prime n}, \quad \Lambda_1 = a'_{\prime n}, \quad \Lambda_2 = b'_{\prime n}, \quad \Lambda_3 = c'_{\prime n}, \qquad (65.)$$

and would reduce, by (60.) and (61.), the preceding $6n-6$ equations of the same groups ($M^1$.) and ($N^1$.), to the forms

$$\left.\begin{array}{llll}
\dfrac{\delta V_\prime}{\delta x_{\prime 1}} = m_1 \xi'_1, & \dfrac{\delta V_\prime}{\delta x_{\prime 2}} = m_2 \xi'_2, & \cdots & \dfrac{\delta V_\prime}{\delta x_{\prime n-1}} = m_{n-1} \xi'_{n-1},\\
\dfrac{\delta V_\prime}{\delta y_{\prime 1}} = m_1 \eta'_1, & \dfrac{\delta V_\prime}{\delta y_{\prime 2}} = m_2 \eta'_2, & \cdots & \dfrac{\delta V_\prime}{\delta y_{\prime n-1}} = m_{n-1} \eta'_{n-1},\\
\dfrac{\delta V_\prime}{\delta z_{\prime 1}} = m_1 \zeta'_1, \quad \cdots & \dfrac{\delta V_\prime}{\delta z_{\prime 2}} = m_2 \zeta'_2, & \cdots & \dfrac{\delta V_\prime}{\delta z_{\prime n-1}} = m_{n-1} \zeta'_{n-1},
\end{array}\right\} \qquad (\mathrm{T}^1.)$$

and

$$\left.\begin{array}{llll}
\dfrac{\delta V_{\prime}}{\delta a_{\prime 1}} = -m_1 \alpha'_1, & \dfrac{\delta V_{\prime}}{\delta a_{\prime 2}} = -m_2 \alpha'_2, & \cdots & \dfrac{\delta V_{\prime}}{\delta a_{\prime n-1}} = -m_{n-1} \alpha'_{n-1}, \\
\dfrac{\delta V_{\prime}}{\delta b_{\prime 1}} = -m_1 \beta'_1, & \dfrac{\delta V_{\prime}}{\delta b_{\prime 2}} = -m_2 \beta'_2, & \cdots & \dfrac{\delta V_{\prime}}{\delta b_{\prime n-1}} = -m_{n-1} \beta'_{n-1}, \\
\dfrac{\delta V_{\prime}}{\delta c_{\prime 1}} = -m_1 \gamma'_1, \quad \cdots & \dfrac{\delta V_{\prime}}{\delta c_{\prime 2}} = -m_2 \gamma'_2, & \cdots & \dfrac{\delta V_{\prime}}{\delta c_{\prime n-1}} = -m_{n-1} \gamma'_{n-1}.
\end{array}\right\} \qquad (\mathrm{U}^1.)$$

12. We might also express the relative action $V_{\prime}$, not as a function of the centrobaric, but of some other internal coordinates, or marks of relative position. We might, for instance, express it and its variation as functions of the $6n - 6$ independent internal coordinates $\xi$ $\eta$ $\zeta$ $\alpha$ $\beta$ $\gamma$ already mentioned, and of their variations, defining these without any reference to the centre of gravity, by the equations

$$\left.\begin{array}{lll}
\xi_i = x_i - x_n, & \eta_i = y_i - y_n, & \zeta_i = z_i - z_n, \\
\alpha_i = a_i - a_n, & \beta_i = b_i - b_n, & \gamma_i = c_i - c_n.
\end{array}\right\} \qquad (66.)$$

For all such transformations of $\delta V_{\prime}$ it is easy to establish a rule or law, which may be called the *law of varying relative action* (exactly analogous to the rule (B[1].)), namely, the following:

$$\delta V_{\prime} = \Sigma \,.\left(\frac{\delta T_{\prime}}{\delta \eta'_{\prime}}\right) \delta \eta_{\prime} - \Sigma \,.\left(\frac{\delta T_{\prime 0}}{\delta e'_{\prime}}\right) \delta e_{\prime} + t\, \delta H_{\prime} + \Sigma \,.\lambda_{\prime}\, \delta \phi_{\prime} + \Sigma \,.\Lambda_{\prime}\, \delta \Phi_{\prime}; \qquad (\mathrm{V}^1.)$$

which implies that we are to express the half $T_{\prime}$ of the relative living force of the system as a function of the rates of increase $\eta'_{\prime}$ of any marks of relative position; and after taking its variation with respect to these rates, to change their variations to the variations of the marks of position themselves; then to subtract the initial from the final value of the result, and to add the variations of the final and initial functions $\phi_{\prime}$ $\Phi_{\prime}$, which enter into the equations of condition, if any, of the form $\phi_{\prime} = 0$, $\Phi_{\prime} = 0$, (connecting the final and initial marks of relative position,) multiplied respectively by undetermined factors $\lambda_{\prime}$ $\Lambda_{\prime}$; and lastly, to equate the whole result to $\delta V_{\prime} - t\, \delta H_{\prime}$, $H_{\prime}$ being the quantity independent of the time in the equation (50.) of relative living force, and $V_{\prime}$ being the relative action, of which we desired to express the variation. It is not necessary to dwell here on the demonstration of this new rule (V[1].), which may easily be deduced from the principles already laid down; or by the calculus of variations from the law of relative living force, combined with the differential equations of second order of relative motion.

But to give an example of its application, let us resume the problem already mentioned, namely to express $\delta V_{\prime}$ by means of the $6n - 5$ independent variations $\delta\xi_i$ $\delta\eta_i$ $\delta\zeta_i$ $\delta\alpha_i$ $\delta\beta_i$ $\delta\gamma_i$ $\delta H_{\prime}$. For this purpose we shall employ a known transformation of the relative living force $2T_{\prime}$, multiplied by the sum of the masses of the system, namely the following:

$$2T_{\prime}\, \Sigma m = \Sigma \,.m_i m_k \{(x'_i - x'_k)^2 + (y'_i - y'_k)^2 + (z'_i - z'_k)^2\} : \qquad (67.)$$

the sign of summation $\Sigma$ extending, in the second member, to all the combinations of points two by two, which can be formed without repetition. This transformation gives, by (66.),

$$\left.\begin{array}{l}
2T_{\prime}\, \Sigma m = m_n\, \Sigma_{\prime} \,.m(\xi'^2 + \eta'^2 + \zeta'^2) \\
\qquad + \Sigma_{\prime} \,.m_i m_k \{(\xi'_i - \xi'_k)^2 + (\eta'_i - \eta'_k)^2 + (\zeta'_i - \zeta'_k)^2\};
\end{array}\right\} \qquad (68.)$$

the sign of summation $\Sigma_{\prime}$ extending only to the first $n-1$ points of the system. Applying, therefore, our general rule or law of varying relative action, and observing that the $6n-6$ internal coordinates $\xi\ \eta\ \zeta\ \alpha\ \beta\ \gamma$ are independent, we find the following new expression:

$$\left.\begin{aligned} \delta V_{\prime} = t\,\delta H_{\prime} &+ \frac{m_n}{\Sigma\, m} \,.\, \Sigma_{\prime}\,.m(\xi'\,\delta\xi - \alpha'\,\delta\alpha + \eta'\,\delta\eta - \beta'\,\delta\beta + \zeta'\,\delta\zeta - \gamma'\,\delta\gamma) \\ &+ \frac{1}{\Sigma\, m} \,.\, \Sigma_{\prime}\,.m_i m_k\{(\xi'_i - \xi'_k)(\delta\xi_i - \delta\xi_k) + (\eta'_i - \eta'_k)(\delta\eta_i - \delta\eta_k) \\ &\qquad\qquad + (\zeta'_i - \zeta'_k)(\delta\zeta_i - \delta\zeta_k)\} \\ &- \frac{1}{\Sigma\, m} \,.\, \Sigma_{\prime}\,.m_i m_k\{(\alpha'_i - \alpha'_k)(\delta\alpha_i - \delta\alpha_k) + (\beta'_i - \beta'_k)(\delta\beta_i - \delta\beta_k) \\ &\qquad\qquad + (\gamma'_i - \gamma'_k)(\delta\gamma_i - \delta\gamma_k)\} : \end{aligned}\right\} \quad (\mathrm{W}^1.)$$

which gives, besides the equation (O$^1$.), the following groups:

$$\left.\begin{aligned} \frac{\delta V_{\prime}}{\delta\xi_i} &= \frac{m_i}{\Sigma\, m} \,.\, \Sigma\,.m(\xi'_i - \xi') = m_i \left(\xi'_i - \frac{\Sigma_{\prime}\, m\xi'}{\Sigma\, m}\right), \\ \frac{\delta V_{\prime}}{\delta\eta_i} &= \frac{m_i}{\Sigma\, m} \,.\, \Sigma\,.m(\eta'_i - \eta') = m_i \left(\eta'_i - \frac{\Sigma_{\prime}\, m\eta'}{\Sigma\, m}\right), \\ \frac{\delta V_{\prime}}{\delta\zeta_i} &= \frac{m_i}{\Sigma\, m} \,.\, \Sigma\,.m(\zeta'_i - \zeta') = m_i \left(\zeta'_i - \frac{\Sigma_{\prime}\, m\zeta'}{\Sigma\, m}\right), \end{aligned}\right\} \quad (\mathrm{X}^1.)$$

and

$$\left.\begin{aligned} \frac{\delta V_{\prime}}{\delta\alpha_i} &= \frac{-m_i}{\Sigma\, m} \,.\, \Sigma\,.m(\alpha'_i - \alpha') = -m_i \left(\alpha'_i - \frac{\Sigma_{\prime}\, m\alpha'}{\Sigma\, m}\right), \\ \frac{\delta V_{\prime}}{\delta\beta_i} &= \frac{-m_i}{\Sigma\, m} \,.\, \Sigma\,.m(\beta'_i - \beta') = -m_i \left(\beta'_i - \frac{\Sigma_{\prime}\, m\beta'}{\Sigma\, m}\right), \\ \frac{\delta V_{\prime}}{\delta\gamma_i} &= \frac{-m_i}{\Sigma\, m} \,.\, \Sigma\,.m(\gamma'_i - \gamma') = -m_i \left(\gamma'_i - \frac{\Sigma_{\prime}\, m\gamma'}{\Sigma\, m}\right); \end{aligned}\right\} \quad (\mathrm{Y}^1.)$$

results which may be thus summed up:

$$\left.\begin{aligned} \delta V_{\prime} = t\,\delta H_{\prime} &+ \Sigma_{\prime}\,.m(\xi'\,\delta\xi - \alpha'\,\delta\alpha + \eta'\,\delta\eta - \beta'\,\delta\beta + \zeta'\,\delta\zeta - \gamma'\,\delta\gamma) \\ &- \frac{1}{\Sigma\, m}(\Sigma_{\prime}\, m\xi' \,.\, \Sigma_{\prime}\, m\,\delta\xi + \Sigma_{\prime}\, m\eta' \,.\, \Sigma_{\prime}\, m\,\delta\eta + \Sigma_{\prime}\, m\zeta' \,.\, \Sigma_{\prime}\, m\,\delta\zeta) \\ &+ \frac{1}{\Sigma\, m}(\Sigma_{\prime}\, m\alpha' \,.\, \Sigma_{\prime}\, m\,\delta\alpha + \Sigma_{\prime}\, m\beta' \,.\, \Sigma_{\prime}\, m\,\delta\beta + \Sigma_{\prime}\, m\gamma' \,.\, \Sigma_{\prime}\, m\,\delta\gamma), \end{aligned}\right\} \quad (\mathrm{Z}^1.)$$

and might have been otherwise deduced by our rule, from this other known transformation of $T_{\prime}$,

$$T_{\prime} = \tfrac{1}{2}\,\Sigma_{\prime}\,.m(\xi'^2 + \eta'^2 + \zeta'^2) - \frac{(\Sigma_{\prime}\, m\xi')^2 + (\Sigma_{\prime}\, m\eta')^2 + (\Sigma_{\prime}\, m\zeta')^2}{2\,\Sigma\, m}. \quad (69.)$$

And to obtain, with any set of internal or relative marks of position, the two partial differential equations which the characteristic function $V_{\prime}$ of relative motion must satisfy, and which offer (as we shall find) the chief means of discovering its form, namely, the equations analogous to those marked (F.) and (G.), we have only to eliminate the rates of increase of the marks of position of the system, which determine the final and initial components of the relative velocities of its points, by the law of varying relative action, from the final and initial expressions of the law of relative living force; namely, from the following equations:

$$T_{\prime} = U + H_{\prime}, \tag{50.}$$

and

$$T_{\prime 0} = U_0 + H_{\prime}. \tag{70.}$$

The law of areas, or the property respecting rotation which was expressed by the partial differential equations (P.), will also always admit of being expressed in relative coordinates, and will assist in discovering the form of the characteristic function $V_{\prime}$; by showing that this function involves only such internal coordinates (in number $6n - 9$) as do not alter by any common rotation of all points final and initial, round the centre of gravity, or round any other internal origin; that origin being treated as fixed, and the quantity $H_{\prime}$ as constant, in determining the effects of this rotation. The general problem of dynamics, respecting the motions of a free system of $n$ points attracting or repelling one another, is therefore reduced, in the last analysis, by the method of the present essay, to the research and differentiation of a function $V_{\prime}$, depending on $6n - 9$ internal or relative coordinates, and on the quantity $H_{\prime}$, and satisfying a pair of partial differential equations of the first order and second degree; in integrating which equations, we are to observe, that at the assumed origin of the motion, namely at the moment when $t = 0$, the final or variable coordinates are equal to their initial values, and the partial differential coefficient $\dfrac{\delta V_{\prime}}{\delta H_{\prime}}$ vanishes; and, that at a moment infinitely little distant, the differential alterations of the coordinates have ratios connected with the other partial differential coefficients of the characteristic function $V_{\prime}$, by the law of varying relative action. It may be here observed, that, although the consideration of the point, called usually the centre of gravity, is very simply suggested by the process of the tenth number, yet this internal centre is even more simply indicated by our early corollaries from the law of varying action; which show that the components of relative final velocities, in any system of attracting or repelling points, may be expressed by the differences of quantities of the form $\dfrac{1}{m}\dfrac{\delta V}{\delta x}$, $\dfrac{1}{m}\dfrac{\delta V}{\delta y}$, $\dfrac{1}{m}\dfrac{\delta V}{\delta z}$: and that therefore in calculating these relative velocities, it is advantageous to introduce the final sums $\Sigma\, mx$, $\Sigma\, my$, $\Sigma\, mz$, and, for an analogous reason, the initial sums $\Sigma\, ma$, $\Sigma\, mb$, $\Sigma\, mc$, among the marks of the extreme positions of the system, in the expression of the characteristic function $V$; because, in differentiating that expression for the calculation of relative velocities, those sums may be treated as constant.

*On Systems of two Points, in general; Characteristic Function of the motion of any Binary System.*

13. To illustrate the foregoing principles, which extend to any free system of points, however numerous, attracting or repelling one another, let us now consider, in particular, a system of two such points. For such a system, the known *force-function U* becomes, by (2.)

$$U = m_1 m_2 f(r), \tag{71.}$$

$r$ being the mutual distance

$$r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \tag{72.}$$

between the two points $m_1$, $m_2$, and $f(r)$ being a function of this distance such that its derivative or differential coefficient $f'(r)$ expresses the law of their repulsion or attraction, according as it is positive or negative. The known differential equations of motion, of the second order, are now, by (1.), comprised in the following formula:

$$m_1(x_1''\,\delta x_1 + y_1''\,\delta y_1 + z''\,\delta z_1) + m_2(x_2''\,\delta x_2 + y_2''\,\delta y_2 + z''\,\delta z_2) = m_1 m_2\,\delta f(r); \tag{73.}$$

they are therefore, separately,

$$\left.\begin{array}{lll} x_1'' = m_2 \dfrac{\delta f(r)}{\delta x_1}, & y_1'' = m_2 \dfrac{\delta f(r)}{\delta y_1}, & z_1'' = m_2 \dfrac{\delta f(r)}{\delta z_1}, \\ x_2'' = m_1 \dfrac{\delta f(r)}{\delta x_2}, & y_2'' = m_1 \dfrac{\delta f(r)}{\delta y_2}, & z_2'' = m_1 \dfrac{\delta f(r)}{\delta z_2}. \end{array}\right\} \tag{74.}$$

The problem of integrating these equations consists in proposing to assign, by their means, six relations between the time $t$, the masses $m_1$ $m_2$, the six varying coordinates $x_1$ $y_1$ $z_1$ $x_2$ $y_2$ $z_2$, and their initial values and initial rates of increase $a_1$ $b_1$ $c_1$ $a_2$ $b_2$ $c_2$ $a_1'$ $b_1'$ $c_1'$ $a_2'$ $b_2'$ $c_2'$. If we knew these six final integrals, and combined them with the initial form of the law of living force, or of the known intermediate integral

$$\tfrac{1}{2} m_1(x_1'^2 + y_1'^2 + z_1'^2) + \tfrac{1}{2} m_2(x_2'^2 + y_2'^2 + z_2'^2) = m_1 m_2 f(r) + H; \tag{75.}$$

that is, with the following formula,

$$\tfrac{1}{2} m_1(a_1'^2 + b_1'^2 + c_1'^2) + \tfrac{1}{2} m_2(a_2'^2 + b_2'^2 + c_2'^2) = m_1 m_2 f(r_0) + H, \tag{76.}$$

in which $r_0$ is the initial distance

$$r_0 = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2}, \tag{77.}$$

and $H$ is a constant quantity, introduced by integration; we could, by the combination of these seven relations, determine the time $t$, and the six initial components of velocity $a_1'$ $b_1'$ $c_1'$ $a_2'$ $b_2'$ $c_2'$, as functions of the twelve final and initial coordinates $x_1$ $y_1$ $z_1$ $x_2$ $y_2$ $z_2$ $a_1$ $b_1$ $c_1$ $a_2$ $b_2$ $c_2$, and of the quantity $H$, (involving also the masses:) we could therefore determine whatever else depends on the manner and time of motion of this system of two points, as a function of the same extreme coordinates and of the same quantity $H$. In particular, we could determine the action, or accumulated living force of the system, namely,

$$V = m_1 \int_0^t (x_1'^2 + y_1'^2 + z_1'^2)\,dt + m_2 \int_0^t (x_2'^2 + y_2'^2 + z_2'^2)\,dt, \tag{A$^2$.}$$

as a function of those thirteen quantities $x_1$ $y_1$ $z_1$ $x_2$ $y_2$ $z_2$ $a_1$ $b_1$ $c_1$ $a_2$ $b_2$ $c_2$ $H$: and might then calculate the variation of this function,

$$\left.\begin{aligned}\delta V = {} & \frac{\delta V}{\delta x_1}\,\delta x_1 + \frac{\delta V}{\delta y_1}\,\delta y_1 + \frac{\delta V}{\delta z_1}\,\delta z_1 + \frac{\delta V}{\delta x_2}\,\delta x_2 + \frac{\delta V}{\delta y_2}\,\delta y_2 + \frac{\delta V}{\delta z_2}\,\delta z_2 \\ & + \frac{\delta V}{\delta a_1}\,\delta a_1 + \frac{\delta V}{\delta b_1}\,\delta b_1 + \frac{\delta V}{\delta c_1}\,\delta c_1 + \frac{\delta V}{\delta a_2}\,\delta a_2 + \frac{\delta V}{\delta b_2}\,\delta b_2 + \frac{\delta V}{\delta c_2}\,\delta c_2 \\ & + \frac{\delta V}{\delta H}\,\delta H.\end{aligned}\right\} \qquad (\mathrm{B}^2.)$$

But the essence of our method consists in *forming previously the expression of this variation by our law of varying action*, namely,

$$\left.\begin{aligned}\delta V = {} & m_1(x_1'\,\delta x_1 - a_1'\,\delta a_1 + y_1'\,\delta y_1 - b_1'\,\delta b_1 + z_1'\,\delta z_1 - c_1'\,\delta c_1) \\ & + m_2(x_2'\,\delta x_2 - a_2'\,\delta a_2 + y_2'\,\delta y_2 - b_2'\,\delta b_2 + z_2'\,\delta z_2 - c_2'\,\delta c_2) \\ & + t\,\delta H;\end{aligned}\right\} \qquad (\mathrm{C}^2.)$$

and in *considering $V$ as a characteristic function of the motion*, from the form of which may be deduced all the intermediate and all the final integrals of the known differential equations, by resolving the expression (C$^2$.) into the following separate groups, (included in (C.) and (D.),)

$$\left.\begin{aligned}\frac{\delta V}{\delta x_1} = m_1 x_1', \quad \frac{\delta V}{\delta y_1} = m_1 y_1', \quad \frac{\delta V}{\delta z_1} = m_1 z_1', \\ \frac{\delta V}{\delta x_2} = m_2 x_2', \quad \frac{\delta V}{\delta y_2} = m_2 y_2', \quad \frac{\delta V}{\delta z_2} = m_2 z_2';\end{aligned}\right\} \qquad (\mathrm{D}^2.)$$

and

$$\left.\begin{aligned}\frac{\delta V}{\delta a_1} = -m_1 a_1', \quad \frac{\delta V}{\delta b_1} = -m_1 b_1', \quad \frac{\delta V}{\delta c_1} = -m_1 c_1', \\ \frac{\delta V}{\delta a_2} = -m_2 a_2', \quad \frac{\delta V}{\delta b_2} = -m_2 b_2', \quad \frac{\delta V}{\delta c_2} = -m_2 c_2';\end{aligned}\right\} \qquad (\mathrm{E}^2.)$$

besides this other equation, which had occurred before,

$$\frac{\delta V}{\delta H} = t. \qquad (\mathrm{E}.)$$

By this new method, the difficulty of integrating the six known equations of motion of the second order (74.) is reduced to the search and differentiation of a single function $V$; and to find the form of this function, we are to employ the following pair of partial differential equations of the first order:

$$\left.\begin{aligned}& \frac{1}{2m_1}\left\{\left(\frac{\delta V}{\delta x_1}\right)^2 + \left(\frac{\delta V}{\delta y_1}\right)^2 + \left(\frac{\delta V}{\delta z_1}\right)^2\right\} + \frac{1}{2m_2}\left\{\left(\frac{\delta V}{\delta x_2}\right)^2 + \left(\frac{\delta V}{\delta y_2}\right)^2 + \left(\frac{\delta V}{\delta z_2}\right)^2\right\} \\ & \qquad = m_1 m_2 f(r) + H,\end{aligned}\right\} \qquad (\mathrm{F}^2.)$$

$$\left.\begin{aligned}&\frac{1}{2m_1}\left\{\left(\frac{\delta V}{\delta a_1}\right)^2+\left(\frac{\delta V}{\delta b_1}\right)^2+\left(\frac{\delta V}{\delta c_1}\right)^2\right\}+\frac{1}{2m_2}\left\{\left(\frac{\delta V}{\delta a_2}\right)^2+\left(\frac{\delta V}{\delta b_2}\right)^2+\left(\frac{\delta V}{\delta c_2}\right)^2\right\}\\&\qquad=m_1m_2f(r_0)+H,\end{aligned}\right\}\quad(\mathrm{G}^2.)$$

combined with some simple considerations. And it easily results from the principles already laid down, that the integral of this pair of equations, adapted to the present question, is

$$\left.\begin{aligned}V&=\sqrt{(x_{\prime\prime}-a_{\prime\prime})^2+(y_{\prime\prime}-b_{\prime\prime})^2+(z_{\prime\prime}-c_{\prime\prime})^2}\,.\,\sqrt{2H_{\prime\prime}(m_1+m_2)}\\&\quad+\frac{m_1m_2}{m_1+m_2}\left(h\vartheta+\int_{r_0}^{r}\rho\,dr\right);\end{aligned}\right\}\quad(\mathrm{H}^2.)$$

in which $x_{\prime\prime}$ $y_{\prime\prime}$ $z_{\prime\prime}$ $a_{\prime\prime}$ $b_{\prime\prime}$ $c_{\prime\prime}$ denote the coordinates, final and initial, of the centre of gravity of the system,

$$\left.\begin{aligned}x_{\prime\prime}&=\frac{m_1x_1+m_2x_2}{m_1+m_2},\quad y_{\prime\prime}=\frac{m_1y_1+m_2y_2}{m_1+m_2},\quad z_{\prime\prime}=\frac{m_1z_1+m_2z_2}{m_1+m_2},\\a_{\prime\prime}&=\frac{m_1a_1+m_2a_2}{m_1+m_2},\quad b_{\prime\prime}=\frac{m_1b_1+m_2b_2}{m_1+m_2},\quad c_{\prime\prime}=\frac{m_1c_1+m_2c_2}{m_1+m_2},\end{aligned}\right\}\quad(78.)$$

and $\vartheta$ is the angle between the final and initial distances $r$, $r_0$: we have also put for abridgement

$$\rho=\pm\sqrt{2(m_1+m_2)\left(f(r)+\frac{H_\prime}{m_1m_2}\right)-\frac{h^2}{r^2}},\quad(79.)$$

the upper or the lower sign to be used, according as the distance $r$ is increasing or decreasing, and have introduced three auxiliary quantities $h$, $H_\prime$, $H_{\prime\prime}$, to be determined by this condition,

$$0=\vartheta+\int_{r_0}^{r}\frac{\delta\rho}{\delta h}dr,\quad(\mathrm{I}^2.)$$

combined with the two following,

$$\left.\begin{aligned}&\frac{m_1m_2}{m_1+m_2}\int_{r_0}^{r}\frac{\delta\rho}{\delta H_\prime}dr=\sqrt{(x_{\prime\prime}-a_{\prime\prime})^2+(y_{\prime\prime}-b_{\prime\prime})^2+(z_{\prime\prime}-c_{\prime\prime})^2}\,.\,\sqrt{\frac{m_1+m_2}{2H_{\prime\prime}}},\\&\qquad H_\prime+H_{\prime\prime}=H;\end{aligned}\right\}\quad(\mathrm{K}^2.)$$

which auxiliary quantities, although in one view they are functions of the twelve extreme coordinates, are yet to be treated as constant in calulating the three definite integrals, or limits of sums of numerous small elements,

$$\int_{r_0}^{r}\rho\,dr,\quad\int_{r_0}^{r}\frac{\delta\rho}{\delta h}dr,\quad\int_{r_0}^{r}\frac{\delta\rho}{\delta H_\prime}dr.$$

The form (H$^2$.), for the *characteristic function of a binary system*, may be regarded as a central or radical relation, which includes the whole theory of the motion of such a system; so that all the details of this motion may be deduced from it by the application of our general method. But because the theory of binary systems has been brought to great perfection already, by the labours of former writers, it may suffice to give briefly here a few instances of such deduction.

14. The form (H².), for the characteristic function of a binary system, involves explicitly, when $\rho$ is changed to its value (79.), the twelve quantities $x_{\prime\prime}\ y_{\prime\prime}\ z_{\prime\prime}\ a_{\prime\prime}\ b_{\prime\prime}\ c_{\prime\prime}\ r\ r_0\ \vartheta\ h\ H_{\prime}\ H_{\prime\prime}$, (besides the masses $m_1\ m_2$ which are always considered as given;) its variation may therefore be thus expressed:

$$\left.\begin{aligned}\delta V = {} & \frac{\delta V}{\delta x_{\prime\prime}}\delta x_{\prime\prime} + \frac{\delta V}{\delta y_{\prime\prime}}\delta y_{\prime\prime} + \frac{\delta V}{\delta z_{\prime\prime}}\delta z_{\prime\prime} + \frac{\delta V}{\delta a_{\prime\prime}}\delta a_{\prime\prime} + \frac{\delta V}{\delta b_{\prime\prime}}\delta b_{\prime\prime} + \frac{\delta V}{\delta c_{\prime\prime}}\delta c_{\prime\prime} \\ & + \frac{\delta V}{\delta r}\delta r + \frac{\delta V}{\delta r_0}\delta r_0 + \frac{\delta V}{\delta \vartheta}\delta \vartheta + \frac{\delta V}{\delta H_{\prime}}\delta H_{\prime} + \frac{\delta V}{\delta H_{\prime\prime}}\delta H_{\prime\prime}.\end{aligned}\right\} \tag{L².}$$

In this expression, if we put for abridgement

$$\lambda = \sqrt{\frac{2H_{\prime\prime}(m_1+m_2)}{(x_{\prime\prime}-a_{\prime\prime})^2+(y_{\prime\prime}-b_{\prime\prime})^2+(z_{\prime\prime}-c_{\prime\prime})^2}}, \tag{80.}$$

we shall have

$$\left.\begin{aligned}\frac{\delta V}{\delta x_{\prime\prime}} &= \lambda(x_{\prime\prime}-a_{\prime\prime}), & \frac{\delta V}{\delta y_{\prime\prime}} &= \lambda(y_{\prime\prime}-b_{\prime\prime}), & \frac{\delta V}{\delta z_{\prime\prime}} &= \lambda(z_{\prime\prime}-c_{\prime\prime}), \\ \frac{\delta V}{\delta a_{\prime\prime}} &= \lambda(a_{\prime\prime}-x_{\prime\prime}), & \frac{\delta V}{\delta b_{\prime\prime}} &= \lambda(b_{\prime\prime}-y_{\prime\prime}), & \frac{\delta V}{\delta c_{\prime\prime}} &= \lambda(c_{\prime\prime}-z_{\prime\prime});\end{aligned}\right\} \tag{M².}$$

and if we put

$$\rho_0 = \pm\sqrt{2(m_1+m_2)\left(f(r_0)+\frac{H_{\prime}}{m_1m_2}\right)-\frac{h_2}{r_0^2}}, \tag{81.}$$

the sign of the radical being determined by the same rule as that of $\rho$, we shall have

$$\frac{\delta V}{\delta r} = \frac{m_1m_2\rho}{m_1+m_2}, \quad \frac{\delta V}{\delta r_0} = \frac{-m_1m_2\rho_0}{m_1+m_2}, \quad \frac{\delta V}{\delta \vartheta} = \frac{m_1m_2h}{m_1+m_2}; \tag{N².}$$

besides, by the equations of condition (I².), (K².), we have

$$\frac{\delta V}{\delta h} = 0, \tag{O².}$$

and

$$\frac{\delta V}{\delta H_{\prime\prime}} = \frac{\delta V}{\delta H_{\prime}} = \int_{r_0}^{r}\frac{dr}{\rho}, \quad \delta H_{\prime} + \delta H_{\prime\prime} = \delta H. \tag{P².}$$

The expression (L².) may therefore be thus transformed:

$$\left.\begin{aligned}\delta V = {} & \lambda\{(x_{\prime\prime}-a_{\prime\prime})(\delta x_{\prime\prime}-\delta a_{\prime\prime})+(y_{\prime\prime}-b_{\prime\prime})(\delta y_{\prime\prime}-\delta b_{\prime\prime})+(z_{\prime\prime}-c_{\prime\prime})(\delta z_{\prime\prime}-\delta c_{\prime\prime})\} \\ & + \frac{m_1m_2}{m_1+m_2}(\rho\,\delta r - \rho_0\,\delta r_0 + h\,\delta\vartheta) + \int_{r_0}^{r}\frac{\delta r}{\rho}\,.\,\delta H;\end{aligned}\right\} \tag{Q².}$$

and may be resolved by our general method into twelve separate expressions for the final and initial components of velocities, namely,

$$\left.\begin{aligned}
x_1' &= \frac{1}{m_1}\frac{\delta V}{\delta x_1} = \frac{\lambda}{m_1+m_2}(x_{\prime\prime} - a_{\prime\prime}) + \frac{m_2}{m_1+m_2}\left(\rho\frac{\delta r}{\delta x_1} + h\frac{\delta\vartheta}{\delta x_1}\right),\\
y_1' &= \frac{1}{m_1}\frac{\delta V}{\delta y_1} = \frac{\lambda}{m_1+m_2}(y_{\prime\prime} - b_{\prime\prime}) + \frac{m_2}{m_1+m_2}\left(\rho\frac{\delta r}{\delta y_1} + h\frac{\delta\vartheta}{\delta y_1}\right),\\
z_1' &= \frac{1}{m_1}\frac{\delta V}{\delta z_1} = \frac{\lambda}{m_1+m_2}(z_{\prime\prime} - c_{\prime\prime}) + \frac{m_2}{m_1+m_2}\left(\rho\frac{\delta r}{\delta z_1} + h\frac{\delta\vartheta}{\delta z_1}\right),\\
x_2' &= \frac{1}{m_2}\frac{\delta V}{\delta x_2} = \frac{\lambda}{m_1+m_2}(x_{\prime\prime} - a_{\prime\prime}) + \frac{m_1}{m_1+m_2}\left(\rho\frac{\delta r}{\delta x_2} + h\frac{\delta\vartheta}{\delta x_2}\right),\\
y_2' &= \frac{1}{m_2}\frac{\delta V}{\delta y_2} = \frac{\lambda}{m_1+m_2}(y_{\prime\prime} - b_{\prime\prime}) + \frac{m_1}{m_1+m_2}\left(\rho\frac{\delta r}{\delta y_2} + h\frac{\delta\vartheta}{\delta y_2}\right),\\
z_2' &= \frac{1}{m_2}\frac{\delta V}{\delta z_2} = \frac{\lambda}{m_1+m_2}(z_{\prime\prime} - c_{\prime\prime}) + \frac{m_1}{m_1+m_2}\left(\rho\frac{\delta r}{\delta z_2} + h\frac{\delta\vartheta}{\delta z_2}\right),
\end{aligned}\right\} \qquad (\mathrm{R}^2.)$$

and

$$\left.\begin{aligned}
a_1' &= \frac{-1}{m_1}\frac{\delta V}{\delta a_1} = \frac{\lambda}{m_1+m_2}(x_{\prime\prime} - a_{\prime\prime}) + \frac{m_2}{m_1+m_2}\left(\rho_0\frac{\delta r_0}{\delta a_1} - h\frac{\delta\vartheta}{\delta a_1}\right),\\
b_1' &= \frac{-1}{m_1}\frac{\delta V}{\delta b_1} = \frac{\lambda}{m_1+m_2}(y_{\prime\prime} - b_{\prime\prime}) + \frac{m_2}{m_1+m_2}\left(\rho_0\frac{\delta r_0}{\delta b_1} - h\frac{\delta\vartheta}{\delta b_1}\right),\\
c_1' &= \frac{-1}{m_1}\frac{\delta V}{\delta c_1} = \frac{\lambda}{m_1+m_2}(z_{\prime\prime} - c_{\prime\prime}) + \frac{m_2}{m_1+m_2}\left(\rho_0\frac{\delta r_0}{\delta c_1} - h\frac{\delta\vartheta}{\delta c_1}\right),\\
a_2' &= \frac{-1}{m_2}\frac{\delta V}{\delta a_2} = \frac{\lambda}{m_1+m_2}(x_{\prime\prime} - a_{\prime\prime}) + \frac{m_1}{m_1+m_2}\left(\rho_0\frac{\delta r_0}{\delta a_2} - h\frac{\delta\vartheta}{\delta a_2}\right),\\
b_2' &= \frac{-1}{m_2}\frac{\delta V}{\delta b_2} = \frac{\lambda}{m_1+m_2}(y_{\prime\prime} - b_{\prime\prime}) + \frac{m_1}{m_1+m_2}\left(\rho_0\frac{\delta r_0}{\delta b_2} - h\frac{\delta\vartheta}{\delta b_2}\right),\\
c_2' &= \frac{-1}{m_2}\frac{\delta V}{\delta c_2} = \frac{\lambda}{m_1+m_2}(z_{\prime\prime} - c_{\prime\prime}) + \frac{m_1}{m_1+m_2}\left(\rho_0\frac{\delta r_0}{\delta c_2} - h\frac{\delta\vartheta}{\delta c_2}\right);
\end{aligned}\right\} \qquad (\mathrm{S}^2.)$$

besides the following expression for the time of motion of the system:

$$t = \frac{\delta V}{\delta H} = \int_{r_0}^{r} \frac{dr}{\rho}, \qquad (\mathrm{T}^2.)$$

which gives by (K$^2$.), and by (79.), (80.),

$$t = \frac{m_1 + m_2}{\lambda}. \qquad (\mathrm{U}^2.)$$

The six equations (R$^2$.) give the six intermediate integrals, and the six equations (S$^2$.) give the six final integrals of the six known differential equations of motion (74.) for any binary system, if we eliminate or determine the three auxiliary quantities $h$, $H_{\prime}$, $H_{\prime\prime}$, by the three conditions (I$^2$.) (T$^2$.) (U$^2$.). Thus, if we observe that the distances $r$, $r_0$, and the included angle $\vartheta$, depend only on relative coordinates, which may be thus denoted,

$$\left.\begin{aligned} x_1 - x_2 = \xi, \quad y_1 - y_2 = \eta, \quad z_1 - z_2 = \zeta, \\ a_1 - a_2 = \alpha, \quad b_1 - b_2 = \beta, \quad c_1 - c_2 = \gamma, \end{aligned}\right\} \tag{82.}$$

we obtain by easy combinations the three following intermediate integrals for the centre of gravity of the system:

$$x'_{\prime\prime}t = x_{\prime\prime} - a_{\prime\prime}, \quad y'_{\prime\prime}t = y_{\prime\prime} - b_{\prime\prime}, \quad z'_{\prime\prime}t = z_{\prime\prime} - c_{\prime\prime}, \tag{83.}$$

and the three following final integrals,

$$a'_{\prime\prime}t = x_{\prime\prime} - a_{\prime\prime}, \quad b'_{\prime\prime}t = y_{\prime\prime} - b_{\prime\prime}, \quad c'_{\prime\prime}t = z_{\prime\prime} - c_{\prime\prime}, \tag{84.}$$

expressing the well-known law of the rectilinear and uniform motion of that centre. We obtain also the three following intermediate integrals for the relative motion of one point of the system about the other:

$$\left.\begin{aligned} \xi' &= \rho\frac{\delta r}{\delta\xi} + h\frac{\delta\vartheta}{\delta\xi}, \\ \eta' &= \rho\frac{\delta r}{\delta\eta} + h\frac{\delta\vartheta}{\delta\eta}, \\ \zeta' &= \rho\frac{\delta r}{\delta\zeta} + h\frac{\delta\vartheta}{\delta\zeta}, \end{aligned}\right\} \tag{85.}$$

and the three following final integrals,

$$\left.\begin{aligned} \alpha' &= \rho_0\frac{\delta r_0}{\delta\alpha} - h\frac{\delta\vartheta}{\delta\alpha}, \\ \beta' &= \rho_0\frac{\delta r_0}{\delta\beta} - h\frac{\delta\vartheta}{\delta\beta}, \\ \gamma' &= \rho_0\frac{\delta r_0}{\delta\gamma} - h\frac{\delta\vartheta}{\delta\gamma}; \end{aligned}\right\} \tag{86.}$$

in which the auxiliary quantities $h$, $H_{\prime}$ are to be determined by (I$^2$.), (T$^2$.), and in which the dependence of $r$, $r_0$, $\vartheta$, on $\xi$, $\eta$, $\zeta$, $\alpha$, $\beta$, $\gamma$, is expressed by the following equations:

$$\left.\begin{aligned} & r = \sqrt{\xi^2 + \eta^2 + \zeta^2}, \quad r_0 = \sqrt{\alpha^2 + \beta^2 + \gamma^2}, \\ & r r_0 \cos\vartheta = \xi\alpha + \eta\beta + \zeta\gamma. \end{aligned}\right\} \tag{87.}$$

If then we put, for abridgement,

$$A = \frac{\rho}{r} + \frac{h}{r^2 \tan\vartheta}, \quad B = \frac{h}{r r_0 \sin\vartheta}, \quad C = \frac{-\rho_0}{r_0} + \frac{h}{r_0^2 \tan\vartheta}, \tag{88.}$$

we shall have these three intermediate integrals,

$$\xi' = A\xi - B\alpha, \quad \eta' = A\eta - B\beta, \quad \zeta' = A\zeta - B\gamma, \tag{89.}$$

and these three final integrals,

$$\alpha' = B\xi - C\alpha, \quad \beta' = B\eta - C\beta, \quad \gamma' = B\zeta - C\gamma, \tag{90.}$$

of the equations of relative motion. These integrals give,

$$\left.\begin{aligned} \xi\eta' - \eta\xi' = \alpha\beta' - \beta\alpha' = B(\alpha\eta - \beta\xi), \\ \eta\zeta' - \zeta\eta' = \beta\gamma' - \gamma\beta' = B(\beta\zeta - \gamma\eta), \\ \zeta\xi' - \xi\zeta' = \gamma\alpha' - \alpha\gamma' = B(\gamma\xi - \alpha\zeta), \end{aligned}\right\} \tag{91.}$$

and

$$\zeta(\alpha\beta' - \beta\alpha') + \xi(\beta\gamma' - \gamma\beta') + \eta(\gamma\alpha' - \alpha\gamma') = 0; \tag{92.}$$

they contain therefore the known law of equable description of areas, and the law of a plane relative orbit. If we take for simplicity this plane for the plane $\xi$ $\eta$, the quantities $\zeta$ $\zeta'$ $\gamma$ $\gamma'$ will vanish; and we may put,

$$\left.\begin{aligned} \xi = r\cos\theta, \quad & \eta = r\sin\theta, \quad & \zeta = 0, \\ \alpha = r_0\cos\theta_0, \quad & \beta = r_0\sin\theta_0, \quad & \gamma = 0, \end{aligned}\right\} \tag{93.}$$

and

$$\left.\begin{aligned} \xi' = r'\cos\theta - \theta' r\sin\theta, \quad & \eta' = r'\sin\theta + \theta' r\cos\theta, \quad & \zeta' = 0, \\ \alpha' = r_0'\cos\theta_0 - \theta_0' r_0\sin\theta_0, \quad & \beta' = r_0'\sin\theta_0 + \theta_0' r_0\cos\theta_0, \quad & \gamma' = 0, \end{aligned}\right\} \tag{94.}$$

the angles $\theta$ $\theta_0$ being counted from some fixed line in the plane, and being such that their difference

$$\theta - \theta_0 = \vartheta. \tag{95.}$$

These values give

$$\xi\eta' - \eta\xi' = r^2\theta', \quad \alpha\beta' - \beta\alpha' = r_0^2\theta_0', \quad \alpha\eta - \beta\xi = rr_0\sin\vartheta, \tag{96.}$$

and therefore, by (88.) and (91.),

$$r^2\theta' = r_0^2\theta_0' = h; \tag{97.}$$

the quantity $\frac{1}{2}h$ is therefore the constant areal velocity in the relative motion of the system; a result which is easily seen to be independent of the directions of the three rectangular coordinates. The same values (93.), (94.), give

$$\left.\begin{aligned} \xi\cos\theta + \eta\sin\theta = r, \quad & \xi'\cos\theta + \eta'\sin\theta = r', \quad & \alpha\cos\theta + \beta\sin\theta = r_0\cos\vartheta, \\ \alpha\cos\theta_0 + \beta\sin\theta_0 = r_0, \quad & \alpha'\cos\theta_0 + \beta'\sin\theta_0 = r_0', \quad & \xi\cos\theta_0 + \eta\sin\theta_0 = r\cos\vartheta, \end{aligned}\right\} \tag{98.}$$

and therefore, by the intermediate and final integrals, (89.), (90.),

$$r' = \rho, \quad r'_0 = \rho_0; \tag{99.}$$

results which evidently agree with the condition (T².), and which give by (79.) and (81.), for all directions of coordinates,

$$r'^2 + \frac{h^2}{r^2} - 2(m_1 + m_2)f(r) = r_0'^2 + \frac{h^2}{r_0^2} - 2(m_1 + m_2)f(r_0) = 2H_\prime \left(\frac{1}{m_1} + \frac{1}{m_2}\right); \tag{100.}$$

the other auxiliary quantity $H_\prime$ is therefore also a constant, independent of the time, and enters as such into the constant part in the expression for $\left(r'^2 + \dfrac{h^2}{r^2}\right)$ the square of the relative velocity. The equation of condition (I².), connecting these two constants $h$, $H_\prime$, with the extreme lengths of the radius vector $r$, and with the angle $\vartheta$ described by this radius in revolving from its initial to its final direction, is the equation of the plane relative orbit; and the other equation of condition (T².), connecting the same two constants with the same extreme distances and with the time, gives the law of the velocity of mutual approach or recess.

We may remark that the part $V_\prime$ of the whole characteristic function $V$, which represents the relative action and determines the relative motion in the system, namely,

$$V_\prime = \frac{m_1 m_2}{m_1 + m_2} \left(h\vartheta + \int_{r_0}^{r} \rho \, dr\right), \tag{V².}$$

may be put, by (I².), under the form

$$V_\prime = \frac{m_1 m_2}{m_1 + m_2} \int_{r_0}^{r} \left(\rho - h\frac{\delta\rho}{\delta h}\right) dr, \tag{W².}$$

or finally, by (79.)

$$V_\prime = 2\int_{r_0}^{r} \frac{m_1 m_2 f(r) + H_\prime}{\rho} \, dr; \tag{X².}$$

the condition (I².) may also itself be transformed, by (79.), as follows:

$$\vartheta = h\int_{r_0}^{r} \frac{dr}{r^2 \rho} : \tag{Y².}$$

results which all admit of easy verifications. The partial differential equations connected with the law of relative living force, which the characteristic function $V_\prime$ of relative motion must satisfy, may be put under the following forms:

$$\left.\begin{aligned} \left(\frac{\delta V_\prime}{\delta r}\right)^2 + \frac{1}{r^2}\left(\frac{\delta V_\prime}{\delta \vartheta}\right)^2 &= \frac{2m_1 m_2}{m_1 + m_2}(U + H_\prime), \\ \left(\frac{\delta V_\prime}{\delta r_0}\right)^2 + \frac{1}{r_0^2}\left(\frac{\delta V_\prime}{\delta \vartheta}\right)^2 &= \frac{2m_1 m_2}{m_1 + m_2}(U_0 + H_\prime); \end{aligned}\right\} \tag{Z².}$$

and if the first of the equations of this pair have its variation taken with respect to $r$ and $\vartheta$, attention being paid to the dynamical meanings of the coefficients of the characteristic function, it will conduct (as in former instances) to the known differential equations of motion of the second order.

*On the undisturbed Motion of a Planet or Comet about the Sun: Dependence of the Characteristic Function of such Motion, on the chord and the sum of the Radii.*

15. To particularize still further, let

$$f(r) = \frac{1}{r}, \tag{101.}$$

that is, let us consider a binary system, such as a planet or comet and the sun, with the Newtonian law of attraction; and let us put, for abridgement,

$$m_1 + m_2 = \mu, \quad \frac{h^2}{\mu} = p, \quad \frac{-m_1 m_2}{2H_\prime} = \mathrm{a}. \tag{102.}$$

The characteristic function $V_\prime$ of relative motion may now be expressed as follows

$$V_\prime = \frac{m_1 m_2}{\sqrt{\mu}} \left( \vartheta \sqrt{p} + \int_{r_0}^{r} \pm \sqrt{\frac{2}{r} - \frac{1}{\mathrm{a}} - \frac{p}{r^2}} . dr \right); \tag{A$^3$.}$$

in which $p$ is to be considered as a function of the extreme radii vectores $r$, $r_0$, and of their included angle $\vartheta$, involving also the quantity a, or the connected quantity $H_\prime$, and determined by the condition

$$\vartheta = \int_{r_0}^{r} \frac{\pm dr}{r^2 \sqrt{\frac{2}{rp} - \frac{1}{\mathrm{a}p} - \frac{1}{r^2}}} \tag{B$^3$.}$$

that is, by the derivative of the formula (A$^3$.), taken with respect to $p$; the upper sign being taken in each expression when the distance $r$ is increasing, and the lower sign when that distance is diminishing, and the quantity $p$ being treated as constant in calculating the two definite integrals. It results from the foregoing remarks, that this quantity $p$ is constant also in the sense of being independent of the time, so as not to vary in the course of the motion; and that the condition (B$^3$.), connecting this constant with $r$ $r_0$ $\vartheta$ a, is the equation of the plane relative orbit; which is therefore (as it has long been known to be) an ellipse, hyperbola, or parabola, according as the constant a is positive, negative, or zero, the origin of $r$ being always a focus of the curve, and $p$ being the semiparameter. It results also, that the time of motion may be thus expressed:

$$t = \frac{\delta V_\prime}{\delta H_\prime} = \frac{2\mathrm{a}^2}{m_1 m_2} \frac{\delta V}{\delta \mathrm{a}}, \tag{C$^3$.}$$

and therefore thus:

$$t = \int_{r_0}^{r} \frac{\pm dr}{\sqrt{\frac{2\mu}{r} - \frac{\mu}{\mathrm{a}} - \frac{\mu p}{r^2}}}; \tag{D$^3$.}$$

which latter is a known expression. Confining ourselves at present to the case a $> 0$, and introducing the known auxiliary quantities called excentricity and excentric anomaly, namely,

$$e = \sqrt{1 - \frac{p}{\mathrm{a}}}, \tag{103.}$$

and

$$v = \cos^{-1}\left(\frac{\mathrm{a} - r}{\mathrm{a}e}\right), \tag{104.}$$

which give

$$\pm\sqrt{2\mathrm{a}r - r^2 - p\mathrm{a}} = \mathrm{a}e \sin v, \tag{105.}$$

$v$ being considered as continually increasing with the time; and therefore, as is well known,

$$\left.\begin{array}{l} r = \mathrm{a}(1 - e\cos v), \quad r_0 = \mathrm{a}(1 - e\cos v_0), \\ \vartheta = 2\tan^{-1}\left\{\sqrt{\dfrac{1+e}{1-e}}\tan\dfrac{v}{2}\right\} - 2\tan^{-1}\left\{\sqrt{\dfrac{1+e}{1-e}}\tan\dfrac{v_0}{2}\right\}, \end{array}\right\} \tag{106.}$$

and

$$t = \sqrt{\frac{\mathrm{a}^3}{\mu}} \,.\, (v - v_0 - e\sin v + e\sin v_0); \tag{107.}$$

we find that this expression for the characteristic function of relative motion,

$$V_{\prime} = \frac{m_1 m_2}{\sqrt{\mu}} \int_{r_0}^{r} \frac{\pm\left(\dfrac{2}{r} - \dfrac{1}{\mathrm{a}}\right) dr}{\sqrt{\dfrac{2}{r} - \dfrac{1}{\mathrm{a}} - \dfrac{p}{r^2}}}, \tag{E$^3$.}$$

deduced from (A$^3$.) and (B$^3$.), may be transformed as follows:

$$V_{\prime} = m_1 m_2 \sqrt{\frac{\mathrm{a}}{\mu}} (v - v_0 + e\sin v - e\sin v_0) : \tag{F$^3$.}$$

in which the excentricity $e$, and the final and initial excentric anomalies $v$, $v_0$, are to be considered as functions of the final and initial radii $r$, $r_0$, and of the included angle $\vartheta$, determined by the equations (106.). The expression (F$^3$.) may be thus written:

$$V_{\prime} = 2m_1 m_2 \sqrt{\frac{\mathrm{a}}{\mu}} (v_{\prime} + e_{\prime} \sin v_{\prime}), \tag{G$^3$.}$$

if we put, for abridgement,

$$v_{\prime} = \frac{v - v_0}{2}, \quad e_{\prime} = e\cos\frac{v + v_0}{2}; \tag{108.}$$

for the complete determination of the characteristic function of the present relative motion, it remains therefore to determine the two variables $v_{\prime}$ and $e_{\prime}$, as functions of $r$ $r_0$ $\vartheta$, or of some other set of quantities which mark the shape and size of the plane triangle bounded by the final and initial elliptic radii vectores and by the elliptic chord.

For this purpose it is convenient to introduce this elliptic chord itself, which we shall call $\pm\tau$, so that

$$\tau^2 = r^2 + r_0^2 - 2rr_0 \cos\vartheta; \tag{109.}$$

because this chord may be expressed as a function of the two variables $\upsilon_{,}$, $e_{,}$, (involving also the mean distance $a_{,}$,) as follows. The value (106.) for the angle $\vartheta$, that is, by (95.), for $\theta - \theta_0$, gives

$$\theta - 2\tan^{-1}\left\{\sqrt{\frac{1+e}{1-e}}\tan\frac{\upsilon}{2}\right\} = \theta_0 - 2\tan^{-1}\left\{\sqrt{\frac{1+e}{1-e}}\tan\frac{\upsilon_0}{2}\right\} = \varpi, \tag{110.}$$

$\varpi$ being a new constant independent of the time, namely, one of the values of the polar angle $\theta$, which correspond to the minimum of radius vector; and therefore, by (106.),

$$\left.\begin{aligned} r\cos(\theta-\varpi) &= \mathrm{a}(\cos\upsilon - e), & r\sin(\theta-\varpi) &= \mathrm{a}\sqrt{1-e^2}\sin\upsilon, \\ r_0\cos(\theta_0-\varpi) &= \mathrm{a}(\cos\upsilon_0 - e), & r_0\sin(\theta_0-\varpi) &= \mathrm{a}\sqrt{1-e^2}\sin\upsilon_0; \end{aligned}\right\} \tag{111.}$$

expressions which give the following value for the square of the elliptic chord:

$$\left.\begin{aligned} \tau^2 &= \{r\cos(\theta-\varpi) - r_0\cos(\theta_0-\varpi)\}^2 + \{r\sin(\theta-\varpi) - r_0\sin(\theta_0-\varpi)\}^2 \\ &= \mathrm{a}^2\{(\cos\upsilon - \cos\upsilon_0)^2 + (1-e^2)(\sin\upsilon - \sin\upsilon_0)^2\} \\ &= 4\mathrm{a}^2\sin\upsilon_{,}^2\left\{\left(\sin\frac{\upsilon+\upsilon_0}{2}\right)^2 + (1-e^2)\left(\cos\frac{\upsilon+\upsilon_0}{2}\right)^2\right\} \\ &= 4\mathrm{a}^2(1-e_{,}^2)\sin\upsilon_{,}^2 : \end{aligned}\right\} \tag{112.}$$

we may also consider $\tau$ as having the same sign with $\sin\upsilon_{,}$, if we consider it as alternately positive and negative, in the successive elliptic periods or revolutions, beginning with the initial position.

Besides, if we denote by $\sigma$ the sum of the two elliptic radii vectores, final and initial, so that

$$\sigma = r + r_0, \tag{113.}$$

we shall have, with our present abridgements,

$$\sigma = 2\mathrm{a}(1 - e_{,}\cos\upsilon_{,}); \tag{114.}$$

the variables $\upsilon_{,}$ $e_{,}$ are therefore functions of $\sigma$, $\tau$, a, and consequently the characteristic function $V_{,}$ is itself a function of those three quantities. We may therefore put

$$V_{,} = \frac{m_1 m_2 w}{m_1 + m_2}, \tag{H$^3$.}$$

$w$ being a function of $\sigma$, $\tau$, a, of which the form is to be determined by eliminating $\upsilon_{,}$ $e_{,}$ between the three equations,

$$\left.\begin{aligned} w &= 2\sqrt{\mu\mathrm{a}}(\upsilon_{,} + e_{,}\sin\upsilon_{,}), \\ \sigma &= 2\mathrm{a}(1 - e_{,}\cos\upsilon_{,}), \\ \tau &= 2\mathrm{a}(1-e_{,}^2)^{\frac{1}{2}}\sin\upsilon_{,}; \end{aligned}\right\} \tag{I$^3$.}$$

and we may consider this new function $w$ as itself a characteristic function of elliptic motion; the law of its variation being expressed as follows, in the notation of the present essay:

$$\delta w = \xi' \,\delta\xi - \alpha' \,\delta\alpha + \eta' \,\delta\eta - \beta' \,\delta\beta + \zeta' \,\delta\zeta - \gamma' \,\delta\gamma + \frac{t\mu \,\delta \mathrm{a}}{2\mathrm{a}^2}. \qquad (\mathrm{K}^3.)$$

In this expression $\xi\ \eta\ \zeta$ are the relative coordinates of the point $m_1$, at the time $t$, referred to the other attracting point $m_2$ as an origin, and to any three rectangular axes; $\xi'\ \eta'\ \zeta'$ are their rates of increase, or the three rectangular components of final relative velocity; $\alpha\ \beta\ \gamma$ $\alpha'\ \beta'\ \gamma'$ are the initial values, or values at the time zero, of these relative coordinates and components of relative velocity; a is a quantity independent of the time, namely, the mean distance of the two points $m_1$, $m_2$; and $\mu$ is the sum of their masses. And all the properties of the undisturbed elliptic motion of a planet or comet about the sun may be deduced in a new way, from the simplified characteristic function $w$, by comparing its variation (K$^3$.) with the following other form,

$$\delta w = \frac{\delta w}{\delta\sigma}\delta\sigma + \frac{\delta w}{\delta\tau}\delta\tau + \frac{\delta w}{\delta \mathrm{a}}\delta \mathrm{a}; \qquad (\mathrm{L}^3.)$$

in which we are to observe that

$$\left.\begin{aligned} \sigma &= \sqrt{\xi^2+\eta^2+\zeta^2} + \sqrt{\alpha^2+\beta^2+\gamma^2}, \\ \tau &= \pm\sqrt{(\xi-\alpha)^2+(\eta-\beta)^2+(\zeta-\gamma)^2}. \end{aligned}\right\} \qquad (\mathrm{M}^3.)$$

By this comparison we are brought back to the general integral equations of the relative motion of a binary system, (89.) and (90.); but we have now the following particular values for the coefficients $A$, $B$, $C$:

$$A = \frac{1}{r}\frac{\delta w}{\delta\sigma} + \frac{1}{\tau}\frac{\delta w}{\delta\tau}, \quad B = \frac{1}{\tau}\frac{\delta w}{\delta\tau}, \quad C = \frac{1}{r_0}\frac{\delta w}{\delta\sigma} + \frac{1}{\tau}\frac{\delta w}{\delta\tau}; \qquad (\mathrm{N}^3.)$$

and with respect to the three partial differential coefficients, $\frac{\delta w}{\delta\sigma}$, $\frac{\delta w}{\delta\tau}$, $\frac{\delta w}{\delta \mathrm{a}}$, we have the following relation between them:

$$\mathrm{a}\frac{\delta w}{\delta \mathrm{a}} + \sigma\frac{\delta w}{\delta\sigma} + \tau\frac{\delta w}{\delta\tau} = \frac{w}{2}, \qquad (\mathrm{O}^3.)$$

the function $w$ being homogeneous of the dimension $\frac{1}{2}$ with respect to the three quantities a, $\sigma$, $\tau$; we have also, by (I$^3$.),

$$\frac{\delta w}{\delta\sigma} = \sqrt{\frac{\mu}{\mathrm{a}}} \cdot \frac{\sin\upsilon_{\prime}}{e_{\prime} - \cos\upsilon_{\prime}}, \qquad \frac{\delta w}{\delta\tau} = \sqrt{\frac{\mu}{\mathrm{a}}} \cdot \frac{\sqrt{1-e_{\prime}^2}}{\cos\upsilon_{\prime} - e_{\prime}}, \qquad (\mathrm{P}^3.)$$

and therefore

$$\frac{\delta w}{\delta\sigma}\frac{\delta w}{\delta\tau} = \frac{-2\mu\tau}{\sigma^2-\tau^2}, \quad \left(\frac{\delta w}{\delta\sigma}\right)^2 + \left(\frac{\delta w}{\delta\tau}\right)^2 + \frac{\mu}{\mathrm{a}} = \frac{4\mu\sigma}{\sigma^2-\tau^2}, \qquad (\mathrm{Q}^3.)$$

from which may be deduced the following remarkable expressions:

$$\left.\begin{aligned} \left(\frac{\delta w}{\delta\sigma} + \frac{\delta w}{\delta\tau}\right)^2 &= \frac{4\mu}{\sigma+\tau} - \frac{\mu}{\mathrm{a}}, \\ \left(\frac{\delta w}{\delta\tau} - \frac{\delta w}{\delta\sigma}\right)^2 &= \frac{4\mu}{\sigma-\tau} - \frac{\mu}{\mathrm{a}}. \end{aligned}\right\} \qquad (\mathrm{R}^3.)$$

These expressions will be found to be important in the application of the present method to the theory of elliptic motion.

16. We shall not enter, on this occasion, into any details of such application; but we may remark, that the circumstance of the characteristic function involving only the elliptic chord and the sum of the extreme radii, (besides the mean distance and the sum of the masses,) affords, by our general method, a new proof of the well-known theorem that the elliptic time also depends on the same chord and sum of radii; and gives a new expression for the law of this dependence, namely,

$$t = \frac{2\mathrm{a}^2}{\mu} \frac{\delta w}{\delta \mathrm{a}}. \tag{S$^3$.}$$

We may remark also, that the same form of the characteristic function of elliptic motion conducts, by our general method, to the following curious, but not novel property, of the ellipse, that if any two tangents be drawn to such a curve, from any common point outside, these tangents subtend equal angles at one focus; they subtend also equal angles at the other. Reciprocally, if any plane curve possess this property, when referred to a fixed point in its own plane, which may be taken as the origin of polar coordinates $r$, $\theta$, the curve must satisfy the following equation in mixed differences:

$$\operatorname{cotan}\left(\frac{\Delta\theta}{2}\right) . \Delta\frac{1}{r} = (\Delta + 2)\frac{d}{d\theta}\frac{1}{r}, \tag{115.}$$

which may be brought to the following form,

$$\left(\frac{d}{d\theta} + \frac{d^3}{d\theta^3}\right)\frac{1}{r} = 0, \tag{116.}$$

and therefore gives, by integration,

$$r = \frac{p}{1 + e\cos(\theta - \varpi)}; \tag{117.}$$

the curve is, consequently, a conic section, and the fixed point is one of its foci.

The properties of parabolic are included as limiting cases in those of elliptic motion, and may be deduced from them by making

$$H_{\prime} = 0, \quad \text{or} \quad \mathrm{a} = \infty; \tag{118.}$$

and therefore the characteristic function $w$ and the time $t$, in parabolic as well as in elliptic motion, are functions of the chord and of the sum of the radii. By thus making a infinite in the foregoing expressions, we find, for parabolic motion, the partial differential equations

$$\left(\frac{\delta w}{\delta\sigma} + \frac{\delta w}{\delta\tau}\right)^2 = \frac{4\mu}{\sigma + \tau}, \quad \left(\frac{\delta w}{\delta\sigma} - \frac{\delta w}{\delta\tau}\right)^2 = \frac{4\mu}{\sigma - \tau}; \tag{T$^3$.}$$

and in fact the parabolic form of the simplified characteristic function $w$ may easily be shown to be

$$w = 2\sqrt{\mu}(\sqrt{\sigma + \tau} \mp \sqrt{\sigma - \tau}), \tag{U$^3$.}$$

$\tau$ being, as before, the chord, and $\sigma$ the sum of the radii; while the analogous limit of the expression ($S^3$.), for the time, is

$$t = \frac{1}{6\sqrt{\mu}} \{ (\sigma + \tau)^{\frac{3}{2}} \mp (\sigma - \tau)^{\frac{3}{2}} \} : \qquad (V^3.)$$

which latter is a known expression.

The formulæ ($K^3$.) and ($L^3$.), to the comparison of which we have reduced the study of elliptic motion, extend to hyperbolic motion also; and in any binary system, with NEWTON's law of attraction, the simplified characteristic function $w$ may be expressed by the definite integral

$$w = \int_{-\tau}^{\tau} \sqrt{\frac{\mu}{\sigma + \tau} - \frac{\mu}{4\mathrm{a}}} \,.\, d\tau, \qquad (W^3.)$$

this function $w$ being still connected with the relative action $V_{\prime}$ by the equation ($H^3$.); while the time $t$, which may always be deduced from this function, by the law of varying action, is represented by this other connected integral,

$$t = \frac{1}{4} \int_{-\tau}^{\tau} \left( \frac{\mu}{\sigma + \tau} - \frac{\mu}{4\mathrm{a}} \right)^{-\frac{1}{2}} d\tau : \qquad (X^3.)$$

provided that, within the extent of these integrations, the radical does not vanish nor become infinite. When this condition is not satisfied, we may still express the simplified characteristic function $w$, and the time $t$, by the following analogous integrals:

$$w = \int_{\tau_{\prime}}^{\sigma_{\prime}} \pm \sqrt{\frac{2\mu}{\sigma_{\prime}} - \frac{\mu}{\mathrm{a}}} \, d\sigma_{\prime}, \qquad (Y^3.)$$

and

$$t = \int_{\tau_{\prime}}^{\sigma_{\prime}} \pm \left( \frac{2\mu}{\sigma_{\prime}} - \frac{\mu}{\mathrm{a}} \right)^{-\frac{1}{2}} d\sigma_{\prime}, \qquad (Z^3.)$$

in which we have put for abridgement

$$\sigma_{\prime} = \frac{\sigma + \tau}{2}, \quad \tau_{\prime} = \frac{\sigma - \tau}{2}, \qquad (119.)$$

and in which it is easy to determine the signs of the radicals. But to treat fully of these various transformations would carry us too far at present, for it is time to consider the properties of systems with more points than two.

*On Systems of three Points, in general; and on their Characteristic Functions.*

17. For any system of three points, the known differential equations of motion of the 2nd order are included in the following formula:

$$\left.\begin{array}{l} m_1(x_1'' \,\delta x_1 + y_1'' \,\delta y_1 + z_1'' \,\delta z_1) + m_2(x_2'' \,\delta x_2 + y_2'' \,\delta y_2 + z_2'' \,\delta z_2) \\ + m_3(x_3'' \,\delta x_3 + y_3'' \,\delta y_3 + z_3'' \,\delta z_3) = \delta U, \end{array}\right\} \qquad (120.)$$

the known force-function $U$ having the form

$$U = m_1 m_2 f^{(1,2)} + m_1 m_3 f^{(1,3)} + m_2 m_3 f^{(2,3)}, \tag{121.}$$

in which $f^{(1,2)}$, $f^{(1,3)}$, $f^{(2,3)}$, are functions respectively of the three following mutual distances of the points of the system:

$$\left.\begin{aligned} r^{(1,2)} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (z_1 - z_2)^2}, \\ r^{(1,3)} &= \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2 - (z_1 - z_3)^2}, \\ r^{(2,3)} &= \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2 - (z_2 - z_3)^2}: \end{aligned}\right\} \tag{122.}$$

the known differential equations of motion are therefore, separately, for the point $m_1$,

$$\left.\begin{aligned} x_1'' &= m_2 \frac{\delta f^{(1,2)}}{\delta x_1} + m_3 \frac{\delta f^{(1,3)}}{\delta x_1}, \\ y_1'' &= m_2 \frac{\delta f^{(1,2)}}{\delta y_1} + m_3 \frac{\delta f^{(1,3)}}{\delta y_1}, \\ z_1'' &= m_2 \frac{\delta f^{(1,2)}}{\delta z_1} + m_3 \frac{\delta f^{(1,3)}}{\delta z_1}, \end{aligned}\right\} \tag{123.}$$

with six other analogous equations for the points $m_2$ and $m_3$: $x_1''$, &c., denoting the component accelerations of the three points $m_1$ $m_2$ $m_3$, or the second differential coefficients of their coordinates, taken with respect to the time. To integrate these equation is to assign, by their means, nine relations between the time $t$, the three masses $m_1$ $m_2$ $m_3$, the nine varying coordinates $x_1$ $y_1$ $z_1$ $x_2$ $y_2$ $z_2$ $x_3$ $y_3$ $z_3$, and their nine initial values and nine initial rates of increase, which may be thus denoted, $a_1$ $b_1$ $c_1$ $a_2$ $b_2$ $c_2$ $a_3$ $b_3$ $c_3$ $a_1'$ $b_1'$ $c_1'$ $a_2'$ $b_2'$ $c_2'$ $a_3'$ $b_3'$ $c_3'$. The known intermediate integral containing the law of living force, namely,

$$\left.\begin{aligned} &\tfrac{1}{2} m_1 (x_1'^2 + y_1'^2 + z_1'^2) + \tfrac{1}{2} m_2 (x_2'^2 + y_2'^2 + z_2'^2) + \tfrac{1}{2} m_3 (x_3'^2 + y_3'^2 + z_3'^2) \\ &\qquad = m_1 m_2 f^{(1,2)} + m_1 m_3 f^{(1,3)} + m_2 m_3 f^{(2,3)} + H, \end{aligned}\right\} \tag{124.}$$

gives the following initial relation:

$$\left.\begin{aligned} &\tfrac{1}{2} m_1 (a_1'^2 + b_1'^2 + c_1'^2) + \tfrac{1}{2} m_2 (a_2'^2 + b_2'^2 + c_2'^2) + \tfrac{1}{2} m_3 (a_3'^2 + b_3'^2 + c_3'^2) \\ &\qquad = m_1 m_2 f_0^{(1,2)} + m_1 m_3 f_0^{(1,3)} + m_2 m_3 f_0^{(2,3)} + H, \end{aligned}\right\} \tag{125.}$$

in which $f_0^{(1,2)}$, $f_0^{(1,3)}$, $f_0^{(2,3)}$ are composed of the initial coordinates, in the same manner as $f^{(1,2)}$ $f^{(1,3)}$ $f^{(2,3)}$ are composed of the final coordinates. If then we knew the nine final integrals of the equations of motion of this ternary system, and combined them with the initial form (125.) of the law of living force, we should have ten relations to determine the ten quantities $t$ $a_1'$ $b_1'$ $c_1'$ $a_2'$ $b_2'$ $c_2'$ $a_3'$ $b_3'$ $c_3'$, namely, the time and the nine initial components of the velocities of the three points, as functions of the nine final and nine initial coordinates, and of the quantity $H$, involving also the masses; we could therefore determine whatever else depends on the manner and time of motion of the system, from its initial to its final position, as a function of the same extreme coordinates, and of $H$. In particular, we could determine the action $V$, or the accumulated living force of the system, namely,

$$V = m_1 \int_0^t (x_1'^2 + y_1'^2 + z_1'^2)\, dt + m_2 \int_0^t (x_2'^2 + y_2'^2 + z_2'^2)\, dt + m_3 \int_0^t (x_3'^2 + y_3'^2 + z_3'^2)\, dt, \quad (\mathrm{A}^4.)$$

as a function of these nineteen quantities, $x_1\ y_1\ z_1\ x_2\ y_2\ z_2\ x_3\ y_3\ z_3\ a_1\ b_1\ c_1\ a_2\ b_2\ c_2\ a_3\ b_3\ c_3\ H$; and might then calculate the variation of this function,

$$\left.\begin{aligned} \delta V = {} & \frac{\delta V}{\delta x_1}\,\delta x_1 + \frac{\delta V}{\delta y_1}\,\delta y_1 + \frac{\delta V}{\delta z_1}\,\delta z_1 + \frac{\delta V}{\delta a_1}\,\delta a_1 + \frac{\delta V}{\delta b_1}\,\delta b_1 + \frac{\delta V}{\delta c_1}\,\delta c_1 \\ & + \frac{\delta V}{\delta x_2}\,\delta x_2 + \frac{\delta V}{\delta y_2}\,\delta y_2 + \frac{\delta V}{\delta z_2}\,\delta z_2 + \frac{\delta V}{\delta a_2}\,\delta a_2 + \frac{\delta V}{\delta b_2}\,\delta b_2 + \frac{\delta V}{\delta c_2}\,\delta c_2 \\ & + \frac{\delta V}{\delta x_3}\,\delta x_3 + \frac{\delta V}{\delta y_3}\,\delta y_3 + \frac{\delta V}{\delta z_3}\,\delta z_3 + \frac{\delta V}{\delta a_3}\,\delta a_3 + \frac{\delta V}{\delta b_3}\,\delta b_3 + \frac{\delta V}{\delta c_3}\,\delta c_3 \\ & + \frac{\delta V}{\delta H}\,\delta H. \end{aligned}\right\} \quad (\mathrm{B}^4.)$$

But the law of varying action gives, *previously*, the following expression for this variation:

$$\left.\begin{aligned} \delta V = {} & m_1(x_1'\,\delta x_1 - a_1'\,\delta a_1 + y_1'\,\delta y_1 - b_1'\,\delta b_1 + z_1'\,\delta z_1 - c_1'\,\delta c_1) \\ & + m_2(x_2'\,\delta x_2 - a_2'\,\delta a_2 + y_2'\,\delta y_2 - b_2'\,\delta b_2 + z_2'\,\delta z_2 - c_2'\,\delta c_2) \\ & + m_3(x_3'\,\delta x_3 - a_3'\,\delta a_3 + y_3'\,\delta y_3 - b_3'\,\delta b_3 + z_3'\,\delta z_3 - c_3'\,\delta c_3) \\ & + t\,\delta H; \end{aligned}\right\} \quad (\mathrm{C}^4.)$$

and shows, therefore, that the research of all the intermediate and all the final integral equations, of motion of the system, may be reduced, reciprocally, to the search and differentiation of this one characteristic function $V$; because if we knew this one function, we should have the nine intermediate integrals of the known differential equations, under the forms

$$\left.\begin{aligned} \frac{\delta V}{\delta x_1} &= m_1 x_1', & \frac{\delta V}{\delta y_1} &= m_1 y_1', & \frac{\delta V}{\delta z_1} &= m_1 z_1', \\ \frac{\delta V}{\delta x_2} &= m_2 x_2', & \frac{\delta V}{\delta y_2} &= m_2 y_2', & \frac{\delta V}{\delta z_2} &= m_2 z_2', \\ \frac{\delta V}{\delta x_3} &= m_3 x_3', & \frac{\delta V}{\delta y_3} &= m_3 y_3', & \frac{\delta V}{\delta z_3} &= m_3 z_3', \end{aligned}\right\} \quad (\mathrm{D}^4.)$$

and the nine final integrals under the forms

$$\left.\begin{aligned} \frac{\delta V}{\delta a_1} &= -m_1 a_1', & \frac{\delta V}{\delta b_1} &= -m_1 b_1', & \frac{\delta V}{\delta c_1} &= -m_1 c_1', \\ \frac{\delta V}{\delta a_2} &= -m_2 a_2', & \frac{\delta V}{\delta b_2} &= -m_2 b_2', & \frac{\delta V}{\delta c_2} &= -m_2 c_2', \\ \frac{\delta V}{\delta a_3} &= -m_3 a_3', & \frac{\delta V}{\delta b_3} &= -m_3 b_3', & \frac{\delta V}{\delta c_3} &= -m_3 c_3', \end{aligned}\right\} \quad (\mathrm{E}^4.)$$

the auxiliary constant $H$ being to be eliminated, and the time $t$ introduced, by this other equation, which has often occurred in this essay,

$$t = \frac{\delta V}{\delta H}. \tag{E.}$$

The same law of varying action suggests also a method of investigating the form of this characteristic function $V$, not requiring the previous integration of the known equations of motion; namely, the integration of a pair of partial differential equations connected with the law of living force; which are,

$$\left.\begin{aligned}
&\frac{1}{2m_1}\left\{\left(\frac{\delta V}{\delta x_1}\right)^2 + \left(\frac{\delta V}{\delta y_1}\right)^2 + \left(\frac{\delta V}{\delta z_1}\right)^2\right\} + \frac{1}{2m_2}\left\{\left(\frac{\delta V}{\delta x_2}\right)^2 + \left(\frac{\delta V}{\delta y_2}\right)^2 + \left(\frac{\delta V}{\delta z_2}\right)^2\right\} \\
&\quad + \frac{1}{2m_3}\left\{\left(\frac{\delta V}{\delta x_3}\right)^2 + \left(\frac{\delta V}{\delta y_3}\right)^2 + \left(\frac{\delta V}{\delta z_3}\right)^2\right\} \\
&\qquad = m_1 m_2 f^{(1,2)} + m_1 m_3 f^{(1,3)} + m_2 m_3 f^{(2,3)} + H,
\end{aligned}\right\} \tag{F$^4$.}$$

and

$$\left.\begin{aligned}
&\frac{1}{2m_1}\left\{\left(\frac{\delta V}{\delta a_1}\right)^2 + \left(\frac{\delta V}{\delta b_1}\right)^2 + \left(\frac{\delta V}{\delta c_1}\right)^2\right\} + \frac{1}{2m_2}\left\{\left(\frac{\delta V}{\delta a_2}\right)^2 + \left(\frac{\delta V}{\delta b_2}\right)^2 + \left(\frac{\delta V}{\delta c_2}\right)^2\right\} \\
&\quad + \frac{1}{2m_3}\left\{\left(\frac{\delta V}{\delta a_3}\right)^2 + \left(\frac{\delta V}{\delta b_3}\right)^2 + \left(\frac{\delta V}{\delta c_3}\right)^2\right\} \\
&\qquad = m_1 m_2 f_0^{(1,2)} + m_1 m_3 f_0^{(1,3)} + m_2 m_3 f_0^{(2,3)} + H.
\end{aligned}\right\} \tag{G$^4$.}$$

And to diminish the difficulty of thus determining the function $V$, which depends on 18 coordinates, we may separate it, by principles already explained, into a part $V_{\prime\prime}$ depending only on the motion of the centre of gravity of the system, and determined by the formula (H$^1$.), and another part $V_\prime$, depending only on the relative motions of the points of the system about this internal centre, and equal to the accumulated living force, connected with this relative motion only. In this manner the difficulty is reduced to determining the relative action $V_\prime$; and if we introduce the relative coordinates

$$\left.\begin{aligned}
\xi_1 &= x_1 - x_3, \quad \eta_1 = y_1 - y_3, \quad \zeta_1 = z_1 - z_3, \\
\xi_2 &= x_2 - x_3, \quad \eta_2 = y_2 - y_3, \quad \zeta_2 = z_2 - z_3,
\end{aligned}\right\} \tag{126.}$$

and

$$\left.\begin{aligned}
\alpha_1 &= a_1 - a_3, \quad \beta_1 = b_1 - b_3, \quad \gamma_1 = c_1 - c_3, \\
\alpha_2 &= a_2 - a_3, \quad \beta_2 = b_2 - b_3, \quad \gamma_2 = c_2 - c_3,
\end{aligned}\right\} \tag{127.}$$

we easily find, by the principles of the tenth and following numbers, that the function $V_\prime$ may be considered as depending only on these relative coordinates and on a quantity $H_\prime$

analogous to $H$ (besides the masses of the system); and that it must satisfy two partial differential equations, analogous to ($F^4$.) and ($G^4$.), namely

$$\left.\begin{array}{l}\dfrac{1}{2m_1}\left\{\left(\dfrac{\delta V_{\prime}}{\delta\xi_1}\right)^2+\left(\dfrac{\delta V_{\prime}}{\delta\eta_1}\right)^2+\left(\dfrac{\delta V_{\prime}}{\delta\zeta_1}\right)^2\right\}+\dfrac{1}{2m_2}\left\{\left(\dfrac{\delta V_{\prime}}{\delta\xi_2}\right)^2+\left(\dfrac{\delta V_{\prime}}{\delta\eta_2}\right)^2+\left(\dfrac{\delta V_{\prime}}{\delta\zeta_2}\right)^2\right\}\\ \qquad +\dfrac{1}{2m_3}\left\{\left(\dfrac{\delta V_{\prime}}{\delta\xi_1}+\dfrac{\delta V_{\prime}}{\delta\xi_2}\right)^2+\left(\dfrac{\delta V_{\prime}}{\delta\eta_1}+\dfrac{\delta V_{\prime}}{\delta\eta_2}\right)^2+\left(\dfrac{\delta V_{\prime}}{\delta\zeta_1}+\dfrac{\delta V_{\prime}}{\delta\zeta_2}\right)^2\right\}\\ \quad = m_1m_2f^{(1,2)}+m_1m_3f^{(1,3)}+m_2m_3f^{(2,3)}+H_{\prime};\end{array}\right\} \quad (\mathrm{H}^4.)$$

and

$$\left.\begin{array}{l}\dfrac{1}{2m_1}\left\{\left(\dfrac{\delta V_{\prime}}{\delta\alpha_1}\right)^2+\left(\dfrac{\delta V_{\prime}}{\delta\beta_1}\right)^2+\left(\dfrac{\delta V_{\prime}}{\delta\gamma_1}\right)^2\right\}+\dfrac{1}{2m_2}\left\{\left(\dfrac{\delta V_{\prime}}{\delta\alpha_2}\right)^2+\left(\dfrac{\delta V_{\prime}}{\delta\beta_2}\right)^2+\left(\dfrac{\delta V_{\prime}}{\delta\gamma_2}\right)^2\right\}\\ \qquad +\dfrac{1}{2m_3}\left\{\left(\dfrac{\delta V_{\prime}}{\delta\alpha_1}+\dfrac{\delta V_{\prime}}{\delta\alpha_2}\right)^2+\left(\dfrac{\delta V_{\prime}}{\delta\beta_1}+\dfrac{\delta V_{\prime}}{\delta\beta_2}\right)^2+\left(\dfrac{\delta V_{\prime}}{\delta\gamma_1}+\dfrac{\delta V_{\prime}}{\delta\gamma_2}\right)^2\right\}\\ \quad = m_1m_2f_0^{(1,2)}+m_1m_3f_0^{(1,3)}+m_2m_3f_0^{(2,3)}+H_{\prime}:\end{array}\right\} \quad (\mathrm{I}^4.)$$

the law of the variation of this function being, by ($Z^1$.),

$$\left.\begin{array}{l}\delta V_{\prime} = t\,\delta H_{\prime} + m_1(\xi_1'\,\delta\xi_1-\alpha_1'\,\delta\alpha_1+\eta_1'\,\delta\eta_1-\beta_1'\,\delta\beta_1+\zeta_1'\,\delta\zeta_1-\gamma_1'\,\delta\gamma_1)\\ \qquad + m_2(\xi_2'\,\delta\xi_2-\alpha_2'\,\delta\alpha_2+\eta_2'\,\delta\eta_2-\beta_2'\,\delta\beta_2+\zeta_2'\,\delta\zeta_2-\gamma_2'\,\delta\gamma_2)\\ \qquad -\dfrac{1}{m_1+m_2+m_3}\left\{\begin{array}{r}(m_1\xi_1'+m_2\xi_2')(m_1\,\delta\xi_1+m_2\,\delta\xi_2)\\ -(m_1\alpha_1'+m_2\alpha_2')(m_1\,\delta\alpha_1+m_2\,\delta\alpha_2)\\ +(m_1\eta_1'+m_2\eta_2')(m_1\,\delta\eta_1+m_2\,\delta\eta_2)\\ -(m_1\beta_1'+m_2\beta_2')(m_1\,\delta\beta_1+m_2\,\delta\beta_2)\\ +(m_1\zeta_1'+m_2\zeta_2')(m_1\,\delta\zeta_1+m_2\,\delta\zeta_2)\\ -(m_1\gamma_1'+m_2\gamma_2')(m_1\,\delta\gamma_1+m_2\,\delta\gamma_2)\end{array}\right\}\end{array}\right\} \quad (\mathrm{K}^4.)$$

which resolves itself in the same manner as before into the six intermediate and six final integrals of relative motion, namely, into the following equations:

$$\left.\begin{array}{ll}\dfrac{1}{m_1}\dfrac{\delta V_{\prime}}{\delta\xi_1}=\xi_1'-\dfrac{m_1\xi_1'+m_2\xi_2'}{m_1+m_2+m_3}; & \dfrac{1}{m_2}\dfrac{\delta V_{\prime}}{\delta\xi_2}=\xi_2'-\dfrac{m_1\xi_1'+m_2\xi_2'}{m_1+m_2+m_3};\\ \dfrac{1}{m_1}\dfrac{\delta V_{\prime}}{\delta\eta_1}=\eta_1'-\dfrac{m_1\eta_1'+m_2\eta_2'}{m_1+m_2+m_3}; & \dfrac{1}{m_2}\dfrac{\delta V_{\prime}}{\delta\eta_2}=\eta_2'-\dfrac{m_1\eta_1'+m_2\eta_2'}{m_1+m_2+m_3};\\ \dfrac{1}{m_1}\dfrac{\delta V_{\prime}}{\delta\zeta_1}=\zeta_1'-\dfrac{m_1\zeta_1'+m_2\zeta_2'}{m_1+m_2+m_3}; & \dfrac{1}{m_2}\dfrac{\delta V_{\prime}}{\delta\zeta_2}=\zeta_2'-\dfrac{m_1\zeta_1'+m_2\zeta_2'}{m_1+m_2+m_3};\end{array}\right\} \quad (\mathrm{L}^4.)$$

and

$$\left.\begin{array}{ll}\dfrac{-1}{m_1}\dfrac{\delta V_{\prime}}{\delta \alpha_1} = \alpha'_1 - \dfrac{m_1\alpha'_1 + m_2\alpha'_2}{m_1+m_2+m_3}; & \dfrac{-1}{m_2}\dfrac{\delta V_{\prime}}{\delta \alpha_2} = \alpha'_2 - \dfrac{m_1\alpha'_1 + m_2\alpha'_2}{m_1+m_2+m_3};\\ \dfrac{-1}{m_1}\dfrac{\delta V_{\prime}}{\delta \beta_1} = \beta'_1 - \dfrac{m_1\beta'_1 + m_2\beta'_2}{m_1+m_2+m_3}; & \dfrac{-1}{m_2}\dfrac{\delta V_{\prime}}{\delta \beta_2} = \beta'_2 - \dfrac{m_1\beta'_1 + m_2\beta'_2}{m_1+m_2+m_3};\\ \dfrac{-1}{m_1}\dfrac{\delta V_{\prime}}{\delta \gamma_1} = \gamma'_1 - \dfrac{m_1\gamma'_1 + m_2\gamma'_2}{m_1+m_2+m_3}; & \dfrac{-1}{m_2}\dfrac{\delta V_{\prime}}{\delta \gamma_2} = \gamma'_2 - \dfrac{m_1\gamma'_1 + m_2\gamma'_2}{m_1+m_2+m_3};\end{array}\right\} \qquad (\mathrm{M}^4.)$$

which must be combined with our old formula,

$$\frac{\delta V_{\prime}}{\delta H_{\prime}} = t. \qquad (\mathrm{O}^1.)$$

18. The quantity $H_{\prime}$ in $V_{\prime}$, and the analogous quantity $H_{\prime\prime}$ in $V_{\prime\prime}$, are indeed independent of the time, and do not vary in the course of the motion; but it is required by the spirit of our method, that in deducing the absolute action or original characteristic function $V$ from the two parts $V_{\prime}$ and $V_{\prime\prime}$, we should consider these two parts $H_{\prime}$ and $H_{\prime\prime}$ of the original quantity $H_{\prime}$ as functions involving each the nine initial and nine final coordinates of the points of the ternary system; the forms of these two functions, of the eighteen coordinates and of $H$, being determined by the two conditions,

$$\frac{\delta V_{\prime}}{\delta H_{\prime}} = \frac{\delta V_{\prime\prime}}{\delta H_{\prime\prime}}, \quad H_{\prime} + H_{\prime\prime} = H. \qquad (\mathrm{N}^4.)$$

However it results from these conditions, that in taking the variation of the whole original function $V$, of the first order, with respect to the eighteen coordinates, we may treat the two auxiliary quantities $H_{\prime}$ and $H_{\prime\prime}$ as constant; and therefore that we have the following expressions for the partial differential coefficients of the first order of $V$, taken with respect to the coordinates parallel to $x$,

$$\left.\begin{array}{ll}\dfrac{\delta V}{\delta x_1} = \dfrac{\delta V_{\prime}}{\delta \xi_1} + \dfrac{m_1}{m_1+m_2+m_3}\dfrac{\delta V_{\prime\prime}}{\delta x_{\prime\prime}}, & \dfrac{\delta V}{\delta a_1} = \dfrac{\delta V_{\prime}}{\delta \alpha_1} + \dfrac{m_1}{m_1+m_2+m_3}\dfrac{\delta V_{\prime\prime}}{\delta a_{\prime\prime}},\\ \dfrac{\delta V}{\delta x_2} = \dfrac{\delta V_{\prime}}{\delta \xi_2} + \dfrac{m_2}{m_1+m_2+m_3}\dfrac{\delta V_{\prime\prime}}{\delta x_{\prime\prime}}, & \dfrac{\delta V}{\delta a_2} = \dfrac{\delta V_{\prime}}{\delta \alpha_2} + \dfrac{m_2}{m_1+m_2+m_3}\dfrac{\delta V_{\prime\prime}}{\delta a_{\prime\prime}},\\ \dfrac{\delta V}{\delta x_3} = -\dfrac{\delta V_{\prime}}{\delta \xi_1} - \dfrac{\delta V_{\prime}}{\delta \xi_2} + \dfrac{m_3}{m_1+m_2+m_3}\dfrac{\delta V_{\prime\prime}}{\delta x_{\prime\prime}}, & \dfrac{\delta V}{\delta a_3} = -\dfrac{\delta V_{\prime}}{\delta \alpha_1} - \dfrac{\delta V_{\prime}}{\delta \alpha_2} + \dfrac{m_3}{m_1+m_2+m_3}\dfrac{\delta V_{\prime\prime}}{\delta a_{\prime\prime}},\end{array}\right\} \qquad (\mathrm{O}^4.)$$

together with analogous expressions for the partial differential coefficients of the same order taken with respect to the other coordinates. Substituting these expressions in the equations of the form (O.), namely, in the following,

$$\left.\begin{array}{l}\dfrac{\delta V}{\delta x_1} + \dfrac{\delta V}{\delta x_2} + \dfrac{\delta V}{\delta x_3} + \dfrac{\delta V}{\delta a_1} + \dfrac{\delta V}{\delta a_2} + \dfrac{\delta V}{\delta a_3} = 0,\\ \dfrac{\delta V}{\delta y_1} + \dfrac{\delta V}{\delta y_2} + \dfrac{\delta V}{\delta y_3} + \dfrac{\delta V}{\delta b_1} + \dfrac{\delta V}{\delta b_2} + \dfrac{\delta V}{\delta b_3} = 0,\\ \dfrac{\delta V}{\delta z_1} + \dfrac{\delta V}{\delta z_2} + \dfrac{\delta V}{\delta z_3} + \dfrac{\delta V}{\delta c_1} + \dfrac{\delta V}{\delta c_2} + \dfrac{\delta V}{\delta c_3} = 0,\end{array}\right\} \qquad (\mathrm{P}^4.)$$

we find that these equations become identical, because

$$\frac{\delta V_{\prime\prime}}{\delta x_{\prime\prime}} + \frac{\delta V_{\prime\prime}}{\delta a_{\prime\prime}} = 0, \quad \frac{\delta V_{\prime\prime}}{\delta y_{\prime\prime}} + \frac{\delta V_{\prime\prime}}{\delta b_{\prime\prime}} = 0, \quad \frac{\delta V_{\prime\prime}}{\delta z_{\prime\prime}} + \frac{\delta V_{\prime\prime}}{\delta c_{\prime\prime}} = 0, \qquad (\mathrm{Q}^4.)$$

But substituting, in like manner, the expressions (O$^4$.) in the equations of the form (P.), of which the first is, for a ternary system,

$$\left.\begin{array}{l} x_1 \frac{\delta V}{\delta y_1} - y_1 \frac{\delta V}{\delta x_1} + x_2 \frac{\delta V}{\delta y_2} - y_2 \frac{\delta V}{\delta x_2} + x_3 \frac{\delta V}{\delta y_3} - y_3 \frac{\delta V}{\delta x_3} \\ + a_1 \frac{\delta V}{\delta b_1} - b_1 \frac{\delta V}{\delta a_1} + a_2 \frac{\delta V}{\delta b_2} - b_2 \frac{\delta V}{\delta a_2} + a_3 \frac{\delta V}{\delta b_3} - b_3 \frac{\delta V}{\delta a_3}; \end{array}\right\} \qquad (\mathrm{R}^4.)$$

and observing that we have

$$x_{\prime\prime} \frac{\delta V_{\prime\prime}}{\delta y_{\prime\prime}} - y_{\prime\prime} \frac{\delta V_{\prime\prime}}{\delta x_{\prime\prime}} + a_{\prime\prime} \frac{\delta V_{\prime\prime}}{\delta b_{\prime\prime}} - b_{\prime\prime} \frac{\delta V_{\prime\prime}}{\delta a_{\prime\prime}} = 0, \qquad (\mathrm{S}^4.)$$

along with two other analogous conditions, we find that the part $V_{\prime}$, or the characteristic function of relative motion of the ternary system, must satisfy the three following conditions, involving its partial differential coefficients of the first order and in the first degree,

$$\left.\begin{array}{l} 0 = \xi_1 \frac{\delta V_{\prime}}{\delta \eta_1} - \eta_1 \frac{\delta V_{\prime}}{\delta \xi_1} + \xi_2 \frac{\delta V_{\prime}}{\delta \eta_2} - \eta_2 \frac{\delta V_{\prime}}{\delta \xi_2} + \alpha_1 \frac{\delta V_{\prime}}{\delta \beta_1} - \beta_1 \frac{\delta V_{\prime}}{\delta \alpha_1} + \alpha_2 \frac{\delta V_{\prime}}{\delta \beta_2} - \beta_2 \frac{\delta V_{\prime}}{\delta \alpha_2}, \\ 0 = \eta_1 \frac{\delta V_{\prime}}{\delta \zeta_1} - \zeta_1 \frac{\delta V_{\prime}}{\delta \eta_1} + \eta_2 \frac{\delta V_{\prime}}{\delta \zeta_2} - \zeta_2 \frac{\delta V_{\prime}}{\delta \eta_2} + \beta_1 \frac{\delta V_{\prime}}{\delta \gamma_1} - \gamma_1 \frac{\delta V_{\prime}}{\delta \beta_1} + \beta_2 \frac{\delta V_{\prime}}{\delta \gamma_2} - \gamma_2 \frac{\delta V_{\prime}}{\delta \beta_2}, \\ 0 = \zeta_1 \frac{\delta V_{\prime}}{\delta \xi_1} - \xi_1 \frac{\delta V_{\prime}}{\delta \zeta_1} + \zeta_2 \frac{\delta V_{\prime}}{\delta \xi_2} - \xi_2 \frac{\delta V_{\prime}}{\delta \zeta_2} + \gamma_1 \frac{\delta V_{\prime}}{\delta \alpha_1} - \alpha_1 \frac{\delta V_{\prime}}{\delta \gamma_1} + \gamma_2 \frac{\delta V_{\prime}}{\delta \alpha_2} - \alpha_2 \frac{\delta V_{\prime}}{\delta \gamma_2}, \end{array}\right\} \qquad (\mathrm{T}^4.)$$

which show that this function can depend only on the shape and size of a pentagon, not generally plane, formed by the point $m_3$ considered as fixed, and by the initial and final positions of the other two points $m_1$ and $m_2$; for example, the pentagon, of which the corners are, in order, $m_3$ $(m_1)$ $(m_2)$ $m_2$ $m_1$; $(m_1)$ and $(m_2)$ denoting the initial positions of the points $m_1$ and $m_2$, referred to $m_3$ as a fixed origin. The shape and size of this pentagon may be determined by the ten mutual distances of its five points, that is, by the five sides and five diagonals, which may be thus denoted:

$$\left.\begin{array}{lllll} m_3(m_1) = \sqrt{s_1}, & (m_1)(m_2) = \sqrt{s_2}, & (m_2)m_2 = \sqrt{s_3}, & m_2 m_1 = \sqrt{s_4}, & m_1 m_3 = \sqrt{s_5}, \\ m_3(m_2) = \sqrt{d_1}, & (m_1)m_2 = \sqrt{d_2}, & (m_2)m_1 = \sqrt{d_3}, & m_2 m_3 = \sqrt{d_4}, & m_1(m_1) = \sqrt{d_5}; \end{array}\right\} \qquad (128.)$$

the values of $s_1 \dots d_5$ as functions of the twelve relative coordinates being

$$\left.\begin{array}{ll} s_1 = \alpha_1^2 + \beta_1^2 + \gamma_1^2, & s_2 = (\alpha_2 - \alpha_1)^2 + (\beta_2 - \beta_1)^2 + (\gamma_2 - \gamma_1)^2, \\ & s_3 = (\xi_2 - \alpha_2)^2 + (\eta_2 - \beta_2)^2 + (\zeta_2 - \gamma_2)^2, \\ s_5 = \xi_1^2 + \eta_1^2 + \zeta_1^2, & s_4 = (\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2, \\ d_1 = \alpha_2^2 + \beta_2^2 + \gamma_2^2, & d_2 = (\xi_2 - \alpha_1)^2 + (\eta_2 - \beta_1)^2 + (\zeta_2 - \gamma_1)^2, \\ & d_3 = (\xi_1 - \alpha_2)^2 + (\eta_1 - \beta_2)^2 + (\zeta_1 - \gamma_2)^2, \\ d_4 = \xi_2^2 + \eta_2^2 + \zeta_2^2, & d_5 = (\xi_1 - \alpha_1)^2 + (\eta_1 - \beta_1)^2 + (\zeta_1 - \gamma_1)^2. \end{array}\right\} \qquad (129.)$$

These ten distances $\sqrt{s_1}$, &c., are not, however, all independent, but are connected by one equation of condition, namely,

$$
\left.
\begin{array}{llllll}
0 = & s_1^2 s_3^2 & +s_2^2 s_4^2 & +s_3^2 s_5^2 & +s_4^2 s_1^2 & +s_5^2 s_2^2 \\
& +s_1^2 d_3^2 & +s_2^2 d_4^2 & +s_3^2 d_5^2 & +s_4^2 d_1^2 & +s_5^2 d_2^2 \\
& +d_1^2 d_2^2 & +d_2^2 d_3^2 & +d_3^2 d_4^2 & +d_4^2 d_5^2 & +d_5^2 d_1^2 \\
& -2s_1^2 s_3 s_4 & -2s_2^2 s_4 s_5 & -2s_3^2 s_5 s_1 & -2s_4^2 s_1 s_2 & -2s_5^2 s_2 s_3 \\
& -2s_1^2 s_3 d_3 & -2s_2^2 s_4 d_4 & -2s_3^2 s_5 d_5 & -2s_4^2 s_1 d_1 & -2s_5^2 s_2 d_2 \\
& -2s_1^2 s_4 d_3 & -2s_1^2 s_5 d_4 & -2s_1^2 s_1 d_5 & -2s_1^2 s_2 d_1 & -2s_1^2 s_3 d_2 \\
& -2s_1 d_2 d_3^2 & -2s_2 d_3 d_4^2 & -2s_3 d_4 d_5^2 & -2s_4 d_5 d_1^2 & -2s_5 d_1 d_2^2 \\
& -2s_1 d_3^2 d_4 & -2s_2 d_4^2 d_5 & -2s_3 d_5^2 d_1 & -2s_4 d_1^2 d_2 & -2s_5 d_2^2 d_3 \\
& -2d_1 d_2^2 d_3 & -2d_2 d_3^2 d_4 & -2d_3 d_4^2 d_5 & -2d_4 d_5^2 d_1 & -2d_5 d_1^2 d_2 \\
& -4s_1 s_3 s_4 d_3 & -4s_2 s_4 s_5 d_4 & -4s_3 s_5 s_1 d_5 & -4s_4 s_1 s_2 d_1 & -4s_5 s_2 s_3 d_2 \\
& -4s_1 d_2 d_3 d_4 & -4s_2 d_3 d_4 d_5 & -4s_3 d_4 d_5 d_1 & -4s_4 d_5 d_1 d_2 & -4s_5 d_1 d_2 d_3 \\
& -2s_1 s_2 s_3 d_4 & -2s_2 s_3 s_4 d_5 & -2s_3 s_4 s_5 d_1 & -2s_4 s_5 s_1 d_2 & -2s_5 s_1 s_2 d_3 \\
& -2s_1 s_3 d_1 d_2 & -2s_2 s_4 d_2 d_3 & -2s_3 s_5 d_3 d_4 & -2s_4 s_1 d_4 d_5 & -2s_5 s_2 d_5 d_1 \\
& -2s_1 d_1 d_3 d_5 & -2s_2 d_2 d_4 d_1 & -2s_3 d_3 d_5 d_2 & -2s_4 d_4 d_1 d_3 & -2s_5 d_5 d_2 d_4 \\
& +2s_1 s_2 s_3 s_4 & +2s_2 s_3 s_4 s_5 & +2s_3 s_4 s_5 s_1 & +2s_4 s_5 s_1 s_2 & +2s_5 s_1 s_2 s_3 \\
& +2s_1 s_2 s_4 d_3 & +2s_2 s_3 s_5 d_4 & +2s_3 s_4 s_1 d_5 & +2s_4 s_5 s_2 d_1 & +2s_5 s_1 s_3 d_2 \\
& +2s_1 s_3 s_4 d_1 & +2s_2 s_4 s_5 d_2 & +2s_3 s_5 s_1 d_3 & +2s_4 s_1 s_2 d_4 & +2s_5 s_2 s_3 d_5 \\
& +2s_1 s_2 d_3 d_4 & +2s_2 s_3 d_4 d_5 & +2s_3 s_4 d_5 d_1 & +2s_4 s_5 d_1 d_2 & +2s_5 s_1 d_2 d_3 \\
& +2s_1 s_3 d_2 d_3 & +2s_2 s_4 d_3 d_4 & +2s_3 s_5 d_4 d_5 & +2s_4 s_1 d_5 d_1 & +2s_5 s_2 d_1 d_2 \\
& +2s_1 s_4 d_1 d_2 & +2s_2 s_5 d_2 d_3 & +2s_3 s_1 d_3 d_4 & +2s_4 s_2 d_4 d_5 & +2s_5 s_3 d_5 d_1 \\
& +2s_1 s_4 d_1 d_3 & +2s_2 s_5 d_2 d_4 & +2s_3 s_1 d_3 d_5 & +2s_4 s_2 d_4 d_1 & +2s_5 s_3 d_5 d_2 \\
& +2s_1 s_4 d_2 d_3 & +2s_2 s_5 d_3 d_4 & +2s_3 s_1 d_4 d_5 & +2s_4 s_2 d_5 d_1 & +2s_5 s_3 d_1 d_2 \\
& +2s_1 s_4 d_3 d_4 & +2s_2 s_5 d_4 d_5 & +2s_3 s_1 d_5 d_1 & +2s_4 s_2 d_1 d_2 & +2s_5 s_3 d_2 d_3 \\
& +2s_1 d_1 d_2 d_3 & +2s_2 d_2 d_3 d_4 & +2s_3 d_3 d_4 d_5 & +2s_4 d_4 d_5 d_1 & +2s_5 d_5 d_1 d_2 \\
& +2s_1 d_3 d_4 d_5 & +2s_2 d_4 d_5 d_1 & +2s_3 d_5 d_1 d_2 & +2s_4 d_1 d_2 d_3 & +2s_5 d_2 d_3 d_4 \\
& +2d_1 d_2 d_3 d_4 & +2d_2 d_3 d_4 d_5 & +2d_3 d_4 d_5 d_1 & +2d_4 d_5 d_1 d_2 & +2d_5 d_1 d_2 d_3;
\end{array}
\right\} \qquad (130.)
$$

they may therefore be expressed as functions of nine independent quantities; for example, of four lines and five angles, $r^{(1)}\ r_0^{(1)}\ r^{(2)}\ r_0^{(2)}$, $\theta^{(1)}\ \theta_0^{(1)}\ \theta^{(2)}\ \theta_0^{(2)}\ \iota$, on which they depend as follows:

$$
\left.
\begin{array}{l}
s_1 = r_0^{(1)2}, \\
s_2 = r_0^{(1)2} + r_0^{(2)2} - 2r_0^{(1)} r_0^{(2)} (\cos\theta_0^{(1)} \cos\theta_0^{(2)} + \sin\theta_0^{(1)} \sin\theta_0^{(2)} \cos\iota), \\
s_3 = r^{(2)2} + r_0^{(2)2} - 2r^{(2)} r_0^{(2)} \cos(\theta^{(2)} - \theta_0^{(2)}), \\
s_4 = r^{(2)2} + r^{(1)2} - 2r^{(2)} r^{(1)} (\cos\theta^{(1)} \cos\theta^{(2)} + \sin\theta^{(1)} \sin\theta^{(2)} \cos\iota), \\
s_5 = r^{(1)2}, \\
d_1 = r_0^{(2)2}, \\
d_2 = r^{(2)2} + r_0^{(1)2} - 2r^{(2)} r_0^{(1)} (\cos\theta^{(2)} \cos\theta_0^{(1)} + \sin\theta^{(2)} \sin\theta_0^{(1)} \cos\iota), \\
d_3 = r_0^{(2)2} + r^{(1)2} - 2r_0^{(2)} r^{(1)} (\cos\theta_0^{(2)} \cos\theta^{(1)} + \sin\theta_0^{(2)} \sin\theta^{(1)} \cos\iota), \\
d_4 = r^{(2)2}, \\
d_5 = r^{(1)2} + r_0^{(1)2} - 2r^{(1)} r_0^{(1)} \cos(\theta^{(1)} - \theta_0^{(1)}),
\end{array}
\right\} \qquad (131.)
$$

the two line-symbols $r^{(1)}$ $r^{(2)}$ denoting, for abridgement, the same two final radii vectores which were before denoted by $r^{(1,3)}$ $r^{(2,3)}$, and $r_0^{(1)}$ $r_0^{(2)}$ representing the initial values of these radii; while $\theta^{(1)}$ $\theta^{(2)}$ $\theta_0^{(1)}$ $\theta_0^{(2)}$ are angles made by these four radii, with the line of intersection of the two planes $r_0^{(1)} r^{(1)}$, $r_0^{(2)} r^{(2)}$; and $\iota$ is the inclination of these two planes to each other. We may therefore consider the characteristic function $V_\prime$ of relative motion, for any ternary system, as depending only on these latter lines and angles, along with the quantity $H_\prime$.

The reasoning which it has been thought useful to develope here, for any system of three points, attracting or repelling one another according to any functions of their distances, was alluded to, under a more general form, in the twelth number of this essay; and shows, for example, that the characteristic function of relative motion in a system of four such points, depends on the shape and size of a heptagon, and therefore only on the mutual distances of its seven corners, which are in number $\left(\frac{7 \times 6}{2} =\right)$ 21, but are connected by six equations of condition, leaving only fifteen independent. It is easy to extend these remarks to any multiple system.

*General method of improving an approximate expression for the Characteristic Function of motion of a System in any problem of Dynamics.*

19. The partial differential equation (F.), which the characteristic function $V$ must satisfy, in every dynamical question, may receive some useful general transformations, by the separation of this function $V$ into any two parts

$$V_1 + V_2 = V. \qquad (\mathrm{U}^4.)$$

For if we establish, for abridgement, the two following equations of definition,

$$\left.\begin{aligned} T_1 &= \Sigma \,.\, \frac{1}{2m}\left(\left(\frac{\delta V_1}{\delta x}\right)^2 + \left(\frac{\delta V_1}{\delta y}\right)^2 + \left(\frac{\delta V_1}{\delta z}\right)^2\right), \\ T_2 &= \Sigma \,.\, \frac{1}{2m}\left(\left(\frac{\delta V_2}{\delta x}\right)^2 + \left(\frac{\delta V_2}{\delta y}\right)^2 + \left(\frac{\delta V_2}{\delta z}\right)^2\right), \end{aligned}\right\} \qquad (\mathrm{V}^4.)$$

analogous to the relation

$$T = \Sigma \,.\, \frac{1}{2m}\left(\left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2\right), \qquad (\mathrm{W}^4.)$$

which served to transform the law of living force into the partial differential equation (F.); we shall have, by ($\mathrm{U}^4$.),

$$T = T_1 + T_2 + \Sigma \,.\, \frac{1}{m}\left(\frac{\delta V_1}{\delta x}\frac{\delta V_2}{\delta x} + \frac{\delta V_1}{\delta y}\frac{\delta V_2}{\delta y} + \frac{\delta V_1}{\delta z}\frac{\delta V_2}{\delta z}\right); \qquad (\mathrm{X}^4.)$$

and this expression may be further transformed by the help of the formula (C.), or by the law of varying action. For that law gives the following symbolic equation,

$$\Sigma \cdot \frac{1}{m} \left( \frac{\delta V}{\delta x} \frac{\delta}{\delta x} + \frac{\delta V}{\delta x} \frac{\delta}{\delta y} + \frac{\delta V}{\delta x} \frac{\delta}{\delta z} \right) = \frac{d}{dt}, \tag{Y$^4$.}$$

the symbols in both members being prefixed to any one function of the varying coordinates of a system, not expressly involving the time; it gives therefore by (U$^4$.), (V$^4$.),

$$\Sigma \cdot \frac{1}{m} \left( \frac{\delta V_1}{\delta x} \frac{\delta V_2}{\delta x} + \frac{\delta V_1}{\delta y} \frac{\delta V_2}{\delta y} + \frac{\delta V_1}{\delta z} \frac{\delta V_2}{\delta z} \right) = \frac{dV_2}{dt} - 2T_2. \tag{Z$^4$.}$$

In this manner we find the following general and rigorous transformation of the equation (F.),

$$\frac{dV_2}{dt} = T - T_1 + T_2; \tag{A$^5$.}$$

$T$ being here retained for the sake of symmetry and conciseness, instead of the equal expression $U + H$. And if we suppose, as we may, that the part $V_1$, like the whole function $V$, is chosen so as to vanish with the time, then the other part $V_2$ will also have that property, and may be expressed by the definite integral,

$$V_2 = \int_0^t (T - T_1 + T_2)\, dt. \tag{B$^5$.}$$

More generally, if we employ the principles of the seventh number, and introduce any $3n$ marks $\eta_1, \eta_2, \dots \eta_{3n}$, of the varying positions of the $n$ points of any system, (whether they be the rectangular coordinates themselves, or any functions of them,) we shall have

$$T = F \left( \frac{\delta V}{\delta \eta_1}, \frac{\delta V}{\delta \eta_2}, \dots \frac{\delta V}{\delta \eta_{3n}} \right), \tag{C$^5$.}$$

and may establish by analogy the two following equations of definition,

$$\left. \begin{aligned} T_1 &= F \left( \frac{\delta V_1}{\delta \eta_1}, \frac{\delta V_1}{\delta \eta_2}, \dots \frac{\delta V_1}{\delta \eta_{3n}} \right), \\ T_2 &= F \left( \frac{\delta V_2}{\delta \eta_1}, \frac{\delta V_2}{\delta \eta_2}, \dots \frac{\delta V_2}{\delta \eta_{3n}} \right), \end{aligned} \right\} \tag{D$^5$.}$$

the function $F$ being always rational and integer, and homogeneous of the second dimension; and being therefore such that (besides other properties)

$$T = T_1 + T_2 + \frac{\delta T_1}{\delta \frac{\delta V_1}{\delta \eta_1}} \frac{\delta V_2}{\delta \eta_1} + \frac{\delta T_1}{\delta \frac{\delta V_1}{\delta \eta_2}} \frac{\delta V_2}{\delta \eta_2} + \dots + \frac{\delta T_1}{\delta \frac{\delta V_1}{\delta \eta_{3n}}} \frac{\delta V_2}{\delta \eta_{3n}}, \tag{E$^5$.}$$

$$\frac{\delta T}{\delta \frac{\delta V}{\delta \eta_1}} = \frac{\delta T_1}{\delta \frac{\delta V_1}{\delta \eta_1}} + \frac{\delta T_2}{\delta \frac{\delta V_2}{\delta \eta_1}}, \ldots \quad \frac{\delta T}{\delta \frac{\delta V}{\delta \eta_{3n}}} = \frac{\delta T_1}{\delta \frac{\delta V_1}{\delta \eta_{3n}}} + \frac{\delta T_2}{\delta \frac{\delta V_2}{\delta \eta_{3n}}}, \qquad (\text{F}^5.)$$

and

$$\frac{\delta T_2}{\delta \frac{\delta V_2}{\delta \eta_1}} \frac{\delta V_2}{\delta \eta_1} + \frac{\delta T_2}{\delta \frac{\delta V_2}{\delta \eta_2}} \frac{\delta V_2}{\delta \eta_2} + \cdots + \frac{\delta T_2}{\delta \frac{\delta V_2}{\delta \eta_{3n}}} \frac{\delta V_2}{\delta \eta_{3n}} = 2T_2. \qquad (\text{G}^5.)$$

By the principles of the eighth number, we have also,

$$\frac{\delta T}{\delta \frac{\delta V}{\delta \eta_1}} = \eta'_1, \quad \frac{\delta T}{\delta \frac{\delta V}{\delta \eta_2}} = \eta'_2, \quad \ldots \quad \frac{\delta T}{\delta \frac{\delta V}{\delta \eta_{3n}}} = \eta'_{3n}; \qquad (\text{H}^5.)$$

and since the meanings of $\eta'_1, \ldots \eta'_{3n}$ give evidently the symbolical equation,

$$\eta'_1 \frac{\delta}{\delta \eta_1} + \eta'_2 \frac{\delta}{\delta \eta_2} + \cdots + \eta'_{3n} \frac{\delta}{\delta \eta_{3n}} = \frac{d}{dt}, \qquad (\text{I}^5.)$$

we see that the equation ($\text{A}^5$.) still holds with the present more general marks of position of a moving system, and gives still the expression ($\text{B}^5$.), supposing only, as before, that the two parts of the whole characteristic function are chosen so as to vanish with the time.

It may not at first sight appear, that this rigorous transformation ($\text{B}^5$.), of the partial differential equation (F.), or of the analogous equation (T.) with coordinates not rectangular, is likely to assist much in discovering the form of the part $V_2$ of the characteristic function $V$, (the other part $V_1$ being supposed to have been previously assumed;) because it involves under the sign of integration, in the term $T_2$, the partial differential coefficients of the sought part $V_2$. But if we observe that these unknown coefficients enter only by their squares and products, we shall perceive that it offers a general method of improving an approximation in any problem of dynamics. For if the first part $V_1$ be an approximate value of the whole sought function $V$, the second part $V_2$ will be small, and the term $T_2$ will not only be also small, but will be in general of a higher order of smallness; we shall therefore in general improve an approximate value $V_1$ of the characteristic function $V$, by adding to it the definite integral,

$$V_2 = \int_0^t (T - T_1)\, dt; \qquad (\text{K}^5.)$$

though this is not, like ($\text{B}^5$.), a perfectly rigorous expression for the remaining part of the function. And in calculating this integral ($\text{K}^5$.), for the improvement of an approximation $V_1$, we may employ the following analogous approximations to the rigorous formulæ (D.) and (E.),

$$\left.\begin{array}{llll} \frac{\delta V_1}{\delta a_1} = -m_1 a'_1; & \frac{\delta V_1}{\delta a_2} = -m_2 a'_2; & \ldots & \frac{\delta V_1}{\delta a_n} = -m_n a'_n; \\ \frac{\delta V_1}{\delta b_1} = -m_1 b'_1; & \frac{\delta V_1}{\delta b_2} = -m_2 b'_2; & \ldots & \frac{\delta V_1}{\delta b_n} = -m_n b'_n; \\ \frac{\delta V_1}{\delta c_1} = -m_1 c'_1; & \frac{\delta V_1}{\delta c_2} = -m_2 c'_2; & \ldots & \frac{\delta V_1}{\delta c_n} = -m_n c'_n; \end{array}\right\} \qquad (\text{L}^5.)$$

and

$$\frac{\delta V_1}{\delta H} = t; \tag{M$^5$.}$$

or with any other marks of final and initial position, (instead of rectangular coordinates,) the following approximate forms of the rigorous equations (S.),

$$\frac{\delta V_1}{\delta e_1} = -\frac{\delta T_0}{\delta e'_1}, \quad \frac{\delta V_1}{\delta e_2} = -\frac{\delta T_0}{\delta e'_2}, \quad \dots \quad \frac{\delta V_1}{\delta e_{3n}} = -\frac{\delta T_0}{\delta e'_{3n}}, \tag{N$^5$.}$$

together with the formula (M$^5$.); by which new formulæ the manner of motion of the system is approximately though not rigorously expressed.

It is easy to extend these remarks to problems of relative motion, and to show that in such problems we have the rigorous transformation

$$V_{\prime 2} = \int_0^t (T_\prime - T_{\prime 1} + T_{\prime 2})\, dt, \tag{O$^5$.}$$

and the approximate expression

$$V_{\prime 2} = \int_0^t (T_\prime - T_{\prime 1})\, dt, \tag{P$^5$.}$$

$V_{\prime 1}$ being any approximate value of the function $V_\prime$ of relative motion, and $V_{\prime 2}$ being the correction of this value; and $T_{\prime 1}$, $T_{\prime 2}$, being homogeneous functions of the second dimension, composed of the partial differential coefficients of these two parts $V_{\prime 1}$, $V_{\prime 2}$, in the same way as $T_\prime$ is composed of the coefficients of the whole function $V_\prime$. These general remarks may usefully be illustrated by a particular but extensive application.

*Application of the foregoing method to the case of a Ternary or Multiple System, with any laws of attraction or repulsion, and with one predominant mass.*

20. The value (68.), for the relative living force $2T_\prime$ of a system, reduces itself successively to the following parts, $2T_\prime^{(1)}, 2T_\prime^{(2)}, \dots 2T_\prime^{(n-1)}$, when we suppose that all the $n-1$ first masses vanish, with the exception of each successively; namely, to the part

$$2T_\prime^{(1)} = \frac{m_1 m_n}{m_1 + m_n}(\xi_1'^2 + \eta_1'^2 + \zeta_1'^2), \tag{132.}$$

when only $m_1$, $m_n$, do not vanish; the part

$$2T_\prime^{(2)} = \frac{m_2 m_n}{m_2 + m_n}(\xi_2'^2 + \eta_2'^2 + \zeta_2'^2), \tag{133.}$$

when all but $m_2$, $m_n$, vanish; and so on, as far as the part

$$2T_\prime^{(n-1)} = \frac{m_{n-1} m_n}{m_{n-1} + m_n}(\xi_{n-1}'^2 + \eta_{n-1}'^2 + \zeta_{n-1}'^2), \tag{134.}$$

which remains, when only the two last masses are retained. The sum of these $n-1$ parts is not, in general, equal to the whole relative living force $2T_{\prime}$ of the system, with all the $n$ masses retained; but it differs little from that whole when the first $n-1$ masses are small in comparison with the last mass $m_n$; for the rigorous value of this difference is, by (68.), and by (132.) (133.) (134.),

$$\left.\begin{aligned} &2T_{\prime} - 2T_{\prime}^{(1)} - 2T_{\prime}^{(2)} - \cdots - 2T_{\prime}^{(n-1)} \\ &\qquad = \frac{2m_1}{m_n}(T_{\prime}^{(1)} - T_{\prime}) + \frac{2m_2}{m_n}(T_{\prime}^{(2)} - T_{\prime}) + \cdots + \frac{2m_{n-1}}{m_n}(T_{\prime}^{(n-1)} - T_{\prime}) \\ &\qquad + \frac{1}{m_n}\,\Sigma_{\prime}\,.m_i m_k\{(\xi_i' - \xi_k')^2 + (\eta_i' - \eta_k')^2 + (\zeta_i' - \zeta_k')^2\} : \end{aligned}\right\} \qquad (135.)$$

an expression which is small of the second order when the $n-1$ first masses are small of the first order. If, then, we denote by $V_{\prime}^{(1)}, V_{\prime}^{(2)}, \ldots V_{\prime}^{(n-1)}$, the relative actions, or accumulated relative living forces, such as they would be in the $n-1$ binary systems, $(m_1\, m_n)$, $(m_2\, m_n), \ldots (m_{n-1}\, m_n)$, without the perturbations of the other small masses of the entire multiple system of $n$ points; so that

$$V_{\prime}^{(1)} = \int_0^t 2T_{\prime}^{(1)}\,dt, \quad V_{\prime}^{(2)} = \int_0^t 2T_{\prime}^{(2)}\,dt, \quad \cdots \quad V_{\prime}^{(n-1)} = \int_0^t 2T_{\prime}^{(n-1)}\,dt, \qquad (\mathrm{Q}^5.)$$

the perturbations being neglected in calculating these $n-1$ definite integrals; we shall have, as an approximate value for the whole relative action $V_{\prime}$ of the system, the sum $V_{\prime 1}$ of its values for these separate binary systems,

$$V_{\prime 1} = V_{\prime}^{(1)} + V_{\prime}^{(2)} + \cdots + V_{\prime}^{(n-1)}. \qquad (\mathrm{R}^5.)$$

This sum, by our theory of binary systems, may be otherwise expressed as follows:

$$V_{\prime 1} = \frac{m_1 m_n w^{(1)}}{m_1 + m_n} + \frac{m_2 m_n w^{(2)}}{m_2 + m_n} + \cdots + \frac{m_{n-1} m_n w^{(n-1)}}{m_{n-1} + m_n}, \qquad (\mathrm{S}^5.)$$

if we put for abridgement

$$\left.\begin{aligned} w^{(1)} &= h^{(1)}\vartheta^{(1)} + \int_{r_0^{(1)}}^{r^{(1)}} r'^{(1)}\,dr^{(1)}, \\ w^{(2)} &= h^{(2)}\vartheta^{(2)} + \int_{r_0^{(2)}}^{r^{(2)}} r'^{(2)}\,dr^{(2)}, \\ &\cdots \\ w^{(n-1)} &= h^{(n-1)}\vartheta^{(n-1)} + \int_{r_0^{(n-1)}}^{r^{(n-1)}} r'^{(n-1)}\,dr^{(n-1)}. \end{aligned}\right\} \qquad (\mathrm{T}^5.)$$

In this expression,

$$\left.\begin{aligned} r'^{(1)} &= \pm\sqrt{2(m_1 + m_n)f^{(1)} + 2g^{(1)} - \frac{h^{(1)2}}{r^{(1)2}}}, \\ &\cdots \\ r'^{(n-1)} &= \pm\sqrt{2(m_{n-1} + m_n)f^{(n-1)} + 2g^{(n-1)} - \frac{h^{(n-1)2}}{r^{(n-1)2}}}, \end{aligned}\right\} \qquad (\mathrm{U}^5.)$$

$r^{(1)}, \dots r^{(n-1)}$ being abridged expressions for the distances $r^{(1,n)}, \dots r^{(n-1,n)}$, and $f^{(1)}, \dots f^{(n-1)}$ being abridgements for the functions $f^{(1,n)}, \dots f^{(n-1,n)}$, of these distances, of which the derivatives, according as they are negative or positive, express the laws of attraction or repulsion: we have also introduced $2n-2$ auxiliary quantities $h^{(1)}\ g^{(1)} \dots h^{(n-1)}\ g^{(n-1)}$, to be eliminated or determined by the following equations of condition:

$$\left.\begin{aligned} 0 &= \vartheta^{(1)} + \int_{r_0^{(1)}}^{r^{(1)}} \frac{\delta r'^{(1)}}{\delta h^{(1)}} dr^{(1)}, \\ 0 &= \vartheta^{(2)} + \int_{r_0^{(2)}}^{r^{(2)}} \frac{\delta r'^{(2)}}{\delta h^{(2)}} dr^{(2)}, \\ &\dots \\ 0 &= \vartheta^{(n-1)} + \int_{r_0^{(n-1)}}^{r^{(n-1)}} \frac{\delta r'^{(n-1)}}{\delta h^{(n-1)}} dr^{(n-1)}, \end{aligned}\right\} \qquad (\text{V}^5.)$$

and

$$\int_{r_0^{(1)}}^{r^{(1)}} \frac{dr^{(1)}}{r'^{(1)}} = \int_{r_0^{(2)}}^{r^{(2)}} \frac{dr^{(2)}}{r'^{(2)}} = \dots = \int_{r_0^{(n-1)}}^{r^{(n-1)}} \frac{dr^{(n-1)}}{r'^{(n-1)}}, \qquad (\text{W}^5.)$$

or

$$\frac{\delta w^{(1)}}{\delta g^{(1)}} = \frac{\delta w^{(2)}}{\delta g^{(2)}} = \dots = \frac{\delta w^{(n-1)}}{\delta g^{(n-1)}}, \qquad (\text{X}^5.)$$

along with this last condition,

$$\frac{m_1 g^{(1)}}{m_1 + m_n} + \frac{m_2 g^{(2)}}{m_2 + m_n} + \frac{m_3 g^{(3)}}{m_3 + m_n} + \dots + \frac{m_{n-1} g^{(n-1)}}{m_{n-1} + m_n} = \frac{H_{\prime}}{m_n}; \qquad (\text{Y}^5.)$$

and we have denoted by $\vartheta^{(1)}, \dots \vartheta^{(n-1)}$, the angles which the final distances $r^{(1)}, \dots r^{(n-1)}$, of the first $n-1$ points from the last or $n$th point of the system, make respectively with the initial distances corresponding, namely, $r_0^{(1)}, \dots r_0^{(n-1)}$. The variation of the sum $V_{\prime 1}$ is, by (S$^5$.),

$$\delta V_{\prime 1} = \frac{m_1 m_n \delta w^{(1)}}{m_1 + m_n} + \frac{m_2 m_n \delta w^{(2)}}{m_2 + m_n} + \dots + \frac{m_{n-1} m_n \delta w^{(n-1)}}{m_{n-1} + m_n}; \qquad (\text{Z}^5.)$$

in which, by the equations of condition, we may treat all the auxiliary quantities $h^{(1)}\ g^{(1)} \dots h^{(n-1)}\ g^{(n-1)}$ as constant, if $H_{\prime}$ be considered as given: so that the part of this variation $\delta V_{\prime 1}$, which depends on the variations of the final relative coordinates, may be put under the form,

$$\left.\begin{aligned} \delta_{\xi,\eta,\zeta} V_{\prime 1} = {} & \frac{m_1 m_n}{m_1 + m_n} \left( \frac{\delta w^{(1)}}{\delta \xi_1} \delta \xi_1 + \frac{\delta w^{(1)}}{\delta \eta_1} \delta \eta_1 + \frac{\delta w^{(1)}}{\delta \zeta_1} \delta \zeta_1 \right) \\ & + \frac{m_2 m_n}{m_2 + m_n} \left( \frac{\delta w^{(2)}}{\delta \xi_2} \delta \xi_2 + \frac{\delta w^{(2)}}{\delta \eta_2} \delta \eta_2 + \frac{\delta w^{(2)}}{\delta \zeta_2} \delta \zeta_2 \right) \\ & + \dots \\ & + \frac{m_{n-1} m_n}{m_{n-1} + m_n} \left( \frac{\delta w^{(n-1)}}{\delta \xi_{n-1}} \delta \xi_{n-1} + \frac{\delta w^{(n-1)}}{\delta \eta_{n-1}} \delta \eta_{n-1} + \frac{\delta w^{(n-1)}}{\delta \zeta_{n-1}} \delta \zeta_{n-1} \right). \end{aligned}\right\} \qquad (\text{A}^6.)$$

By the equations (T⁵.) (U⁵.), or by the theory of binary systems, we have, rigorously,

$$\left.\begin{array}{l} \left(\dfrac{\delta w^{(1)}}{\delta\xi_1}\right)^2 + \left(\dfrac{\delta w^{(1)}}{\delta\eta_1}\right)^2 + \left(\dfrac{\delta w^{(1)}}{\delta\zeta_1}\right)^2 = 2(m_1 + m_n)f^{(1)} + 2g^{(1)}; \\ \left(\dfrac{\delta w^{(2)}}{\delta\xi_2}\right)^2 + \left(\dfrac{\delta w^{(2)}}{\delta\eta_2}\right)^2 + \left(\dfrac{\delta w^{(2)}}{\delta\zeta_2}\right)^2 = 2(m_2 + m_n)f^{(2)} + 2g^{(2)}; \\ \cdots \\ \left(\dfrac{\delta w^{(n-1)}}{\delta\xi_{n-1}}\right)^2 + \left(\dfrac{\delta w^{(n-1)}}{\delta\eta_{n-1}}\right)^2 + \left(\dfrac{\delta w^{(n-1)}}{\delta\zeta_{n-1}}\right)^2 = 2(m_{n-1} + m_n)f^{(n-1)} + 2g^{(n-1)}; \end{array}\right\} \quad (\mathrm{B}^6.)$$

and the rigorous law of relative living force for the whole multiple system, is

$$T_\prime = U + H_\prime, \quad (50.)$$

in which

$$U = m_n(m_1 f^{(1)} + m_2 f^{(2)} + \cdots + m_{n-1} f^{(n-1)}) + \Sigma_\prime\,.m_i m_k f^{(i,k)}, \quad (\mathrm{C}^6.)$$

and

$$\left.\begin{array}{l} T_\prime = \dfrac{1}{2}\left(\dfrac{1}{m_1} + \dfrac{1}{m_n}\right)\left\{\left(\dfrac{\delta V_\prime}{\delta\xi_1}\right)^2 + \left(\dfrac{\delta V_\prime}{\delta\eta_1}\right)^2 + \left(\dfrac{\delta V_\prime}{\delta\zeta_1}\right)^2\right\} \\ \quad + \dfrac{1}{2}\left(\dfrac{1}{m_2} + \dfrac{1}{m_n}\right)\left\{\left(\dfrac{\delta V_\prime}{\delta\xi_2}\right)^2 + \left(\dfrac{\delta V_\prime}{\delta\eta_2}\right)^2 + \left(\dfrac{\delta V_\prime}{\delta\zeta_2}\right)^2\right\} \\ \quad + \cdots \\ \quad + \dfrac{1}{2}\left(\dfrac{1}{m_{n-1}} + \dfrac{1}{m_n}\right)\left\{\left(\dfrac{\delta V_\prime}{\delta\xi_{n-1}}\right)^2 + \left(\dfrac{\delta V_\prime}{\delta\eta_{n-1}}\right)^2 + \left(\dfrac{\delta V_\prime}{\delta\zeta_{n-1}}\right)^2\right\} \\ \quad + \dfrac{1}{m_n}\Sigma_\prime\left(\dfrac{\delta V_\prime}{\delta\xi_i}\dfrac{\delta V_\prime}{\delta\xi_k} + \dfrac{\delta V_\prime}{\delta\eta_i}\dfrac{\delta V_\prime}{\delta\eta_k} + \dfrac{\delta V_\prime}{\delta\zeta_i}\dfrac{\delta V_\prime}{\delta\zeta_k}\right). \end{array}\right\} \quad (\mathrm{D}^6.)$$

We have therefore, by changing in this last expression the coefficients of the characteristic function $V_\prime$ to those of its first part $V_{\prime 1}$, and by attending to the foregoing equations,

$$\left.\begin{array}{l} T_{\prime 1} = m_n\,\Sigma_\prime\,.m_i f^{(i)} + H_\prime \\ \quad + m_n\,\Sigma_\prime\,.\dfrac{m_i}{m_n + m_i}\dfrac{m_k}{m_n + m_k}\left(\dfrac{\delta w^{(i)}}{\delta\xi_i}\dfrac{\delta w^{(k)}}{\delta\xi_k} + \dfrac{\delta w^{(i)}}{\delta\eta_i}\dfrac{\delta w^{(k)}}{\delta\eta_k} + \dfrac{\delta w^{(i)}}{\delta\zeta_i}\dfrac{\delta w^{(k)}}{\delta\zeta_k}\right); \end{array}\right\} \quad (\mathrm{E}^6.)$$

and consequently

$$\left.\begin{array}{l} T_\prime - T_{\prime 1} = \Sigma_\prime\, m_i m_k \Big\{ f^{(i,k)} \\ \quad - \dfrac{m_n}{(m_n + m_i)(m_n + m_k)}\left(\dfrac{\delta w^{(i)}}{\delta\xi_i}\dfrac{\delta w^{(k)}}{\delta\xi_k} + \dfrac{\delta w^{(i)}}{\delta\eta_i}\dfrac{\delta w^{(k)}}{\delta\eta_k} + \dfrac{\delta w^{(i)}}{\delta\zeta_i}\dfrac{\delta w^{(k)}}{\delta\zeta_k}\right)\Big\}. \end{array}\right\} \quad (\mathrm{F}^6.)$$

The general transformation of the foregoing number gives therefore, rigorously, for the remaining part $V_{\prime 2}$ of the characteristic function $V_\prime$ of relative motion of the multiple system, the equation

$$\left.\begin{array}{l} V_{\prime 2} = \displaystyle\int_0^t T_{\prime 2}\, dt \\ \quad + \Sigma_\prime\, .m_i m_k \displaystyle\int_0^t \left\{ f^{(i,k)} - \frac{\dfrac{\delta w^{(i)}}{\delta \xi_i}\dfrac{\delta w^{(k)}}{\delta \xi_k} + \dfrac{\delta w^{(i)}}{\delta \eta_i}\dfrac{\delta w^{(k)}}{\delta \eta_k} + \dfrac{\delta w^{(i)}}{\delta \zeta_i}\dfrac{\delta w^{(k)}}{\delta \zeta_k}}{\dfrac{1}{m_n}(m_n + m_i)(m_n + m_k)} \right\} dt; \end{array}\right\} \tag{G$^6$.}$$

and, approximately, the expression

$$V_{\prime 2} = \Sigma_\prime\, .m_i m_k \int_0^t \left\{ f^{(i,k)} - \frac{1}{m_n}(\xi'_i \xi'_k + \eta'_i \eta'_k + \zeta'_i \zeta'_k) \right\} dt : \tag{H$^6$.}$$

with which last expression we may combine the following approximate formulæ belonging in rigour to binary systems only,

$$\xi'_i = \frac{\delta w^{(i)}}{\delta \xi_i}, \quad \eta'_i = \frac{\delta w^{(i)}}{\delta \eta_i}, \quad \zeta'_i = \frac{\delta w^{(i)}}{\delta \zeta_i}, \tag{I$^6$.}$$

$$\alpha'_i = -\frac{\delta w^{(i)}}{\delta \alpha_i}, \quad \beta'_i = -\frac{\delta w^{(i)}}{\delta \beta_i}, \quad \gamma'_i = -\frac{\delta w^{(i)}}{\delta \gamma_i}, \tag{K$^6$.}$$

and

$$t = \frac{\delta w^{(i)}}{\delta g^{(i)}}. \tag{L$^6$.}$$

We have also, rigorously, for binary systems, the following differential equations of motion of the second order,

$$\xi''_i = (m_n + m_i)\frac{\delta f^{(i)}}{\delta \xi_i}; \quad \eta''_i = (m_n + m_i)\frac{\delta f^{(i)}}{\delta \eta_i}; \quad \zeta''_i = (m_n + m_i)\frac{\delta f^{(i)}}{\delta \zeta_i}; \tag{M$^6$.}$$

which enable us to transform in various ways the approximate expression (H$^6$.). Thus, in the case of a ternary system, with any laws of attraction or repulsion, but with one predominant mass $m_3$, the *disturbing part* $V_{\prime 2}$ of the characteristic function $V_\prime$ of relative motion, may be put under the form

$$V_{\prime 2} = m_1 m_2 W, \tag{N$^6$.}$$

in which the coefficient $W$ may be approximately be expressed as follows:

$$W = \int_0^t \left\{ f^{(1,2)} - \frac{1}{m_3}(\xi'_1 \xi'_2 + \eta'_1 \eta'_2 + \zeta'_1 \zeta'_2) \right\} dt, \tag{O$^6$.}$$

or thus:

$$\left. \begin{aligned} W = \int_0^t \left( f^{(1,2)} + \xi_2 \frac{\delta f^{(1)}}{\delta \xi_1} + \eta_2 \frac{\delta f^{(1)}}{\delta \eta_1} + \zeta_2 \frac{\delta f^{(1)}}{\delta \zeta_1} \right) dt \\ - \frac{1}{m_3} \left( \xi_2 \frac{\delta w^{(1)}}{\delta \xi_1} + \eta_2 \frac{\delta w^{(1)}}{\delta \eta_1} + \zeta_2 \frac{\delta w^{(1)}}{\delta \zeta_1} + \alpha_2 \frac{\delta w^{(1)}}{\delta \alpha_1} + \beta_2 \frac{\delta w^{(1)}}{\delta \beta_1} + \gamma_2 \frac{\delta w^{(1)}}{\delta \gamma_1} \right), \end{aligned} \right\} \quad (\mathrm{P}^6.)$$

or finally,

$$\left. \begin{aligned} W = \int_0^t \left( f^{(1,2)} + \xi_1 \frac{\delta f^{(2)}}{\delta \xi_2} + \eta_1 \frac{\delta f^{(2)}}{\delta \eta_2} + \zeta_1 \frac{\delta f^{(2)}}{\delta \zeta_2} \right) dt \\ - \frac{1}{m_3} \left( \xi_1 \frac{\delta w^{(2)}}{\delta \xi_2} + \eta_1 \frac{\delta w^{(2)}}{\delta \eta_2} + \zeta_1 \frac{\delta w^{(2)}}{\delta \zeta_2} + \alpha_1 \frac{\delta w^{(2)}}{\delta \alpha_2} + \beta_1 \frac{\delta w^{(2)}}{\delta \beta_2} + \gamma_1 \frac{\delta w^{(2)}}{\delta \gamma_2} \right). \end{aligned} \right\} \quad (\mathrm{Q}^6.)$$

In general, for a multiple system, we may put

$$V_{\prime 2} = \Sigma_\prime . m_i m_k W^{(i,k)}; \quad (\mathrm{R}^6.)$$

and approximately,

$$\left. \begin{aligned} W^{(i,k)} = \int_0^t \left( f^{(i,k)} + \xi_k \frac{\delta f^{(i)}}{\delta \xi_i} + \eta_k \frac{\delta f^{(i)}}{\delta \eta_i} + \zeta_k \frac{\delta f^{(i)}}{\delta \zeta_i} \right) dt \\ - \frac{1}{m_n} \left( \xi_k \frac{\delta w^{(i)}}{\delta \xi_i} + \eta_k \frac{\delta w^{(i)}}{\delta \eta_i} + \zeta_k \frac{\delta w^{(i)}}{\delta \zeta_i} + \alpha_k \frac{\delta w^{(i)}}{\delta \alpha_i} + \beta_k \frac{\delta w^{(i)}}{\delta \beta_i} + \gamma_k \frac{\delta w^{(i)}}{\delta \gamma_i} \right), \end{aligned} \right\} \quad (\mathrm{S}^6.)$$

or

$$\left. \begin{aligned} W^{(i,k)} = \int_0^t \left( f^{(i,k)} + \xi_i \frac{\delta f^{(k)}}{\delta \xi_k} + \eta_i \frac{\delta f^{(k)}}{\delta \eta_k} + \zeta_i \frac{\delta f^{(k)}}{\delta \zeta_k} \right) dt \\ - \frac{1}{m_n} \left( \xi_i \frac{\delta w^{(k)}}{\delta \xi_k} + \eta_i \frac{\delta w^{(k)}}{\delta \eta_k} + \zeta_i \frac{\delta w^{(k)}}{\delta \zeta_k} + \alpha_i \frac{\delta w^{(k)}}{\delta \alpha_k} + \beta_i \frac{\delta w^{(k)}}{\delta \beta_k} + \gamma_i \frac{\delta w^{(k)}}{\delta \gamma_k} \right). \end{aligned} \right\} \quad (\mathrm{T}^6.)$$

*Rigorous transition from the theory of Binary to that of Multiple Systems, by means of the disturbing part of the whole Characteristic Function; and approximate expressions for the perturbations.*

21. The three equations (K$^6$.) when the auxiliary constant $g^{(i)}$ is eliminated by the formula (L$^6$.) are rigorously (by our theory) the three final integrals of the three known equations of the second order (M$^6$.), for the relative motion of the binary system $(m_i m_n)$; and give, for such a system, the three varying relative coordinates $\xi_i\ \eta_i\ \zeta_i$, as functions of their initial values and initial rates of increase $\alpha_i\ \beta_i\ \gamma_i\ \alpha'_i\ \beta'_i\ \gamma'_i$, and of the time $t$. In like manner the three equations (I$^6$.), when $g^{(i)}$ is eliminated by (L$^6$.), are rigorously the three intermediate integrals of the same known differential equations of motion of the same binary system. These integrals, however, cease to be rigorous when we introduce the perturbations of the relative motion of this partial or binary system $(m_i\, m_n)$, arising from the attractions or repulsions of

the other points $m_k$, of the whole proposed multiple system; but they may be corrected and rendered rigorous by employing the remaining part $V_{\prime 2}$ of the whole characteristic function of relative motion $V_\prime$, along with the principal part or approximate value $V_{\prime 1}$.

The equations (X¹.), (Y¹.) of the twelfth number, give rigorously

$$\xi'_i = \frac{1}{m_i}\frac{\delta V_\prime}{\delta \xi_i} + \frac{1}{m_n}\Sigma_\prime \frac{\delta V_\prime}{\delta \xi_i}, \quad \eta'_i = \frac{1}{m_i}\frac{\delta V_\prime}{\delta \eta_i} + \frac{1}{m_n}\Sigma_\prime \frac{\delta V_\prime}{\delta \eta_i}, \quad \zeta'_i = \frac{1}{m_i}\frac{\delta V_\prime}{\delta \zeta_i} + \frac{1}{m_n}\Sigma_\prime \frac{\delta V_\prime}{\delta \zeta_i}, \qquad (\mathrm{U}^6.)$$

and

$$-\alpha'_i = \frac{1}{m_i}\frac{\delta V_\prime}{\delta \alpha_i} + \frac{1}{m_n}\Sigma_\prime \frac{\delta V_\prime}{\delta \alpha_i}, \quad -\beta'_i = \frac{1}{m_i}\frac{\delta V_\prime}{\delta \beta_i} + \frac{1}{m_n}\Sigma_\prime \frac{\delta V_\prime}{\delta \beta_i}, \quad -\gamma'_i = \frac{1}{m_i}\frac{\delta V_\prime}{\delta \gamma_i} + \frac{1}{m_n}\Sigma_\prime \frac{\delta V_\prime}{\delta \gamma_i}, \qquad (\mathrm{V}^6.)$$

and therefore, by (A⁶.),

$$\left.\begin{aligned}
\frac{\delta w^{(i)}}{\delta \xi_i} &= \xi'_i - \Sigma_{\prime\prime}\,.\frac{m_k}{m_k+m_n}\frac{\delta w^{(k)}}{\delta \xi_k} - \frac{1}{m_i}\frac{\delta V_{\prime 2}}{\delta \xi_i} - \frac{1}{m_n}\Sigma_\prime \frac{\delta V_{\prime 2}}{\delta \xi_i},\\
\frac{\delta w^{(i)}}{\delta \eta_i} &= \eta'_i - \Sigma_{\prime\prime}\,.\frac{m_k}{m_k+m_n}\frac{\delta w^{(k)}}{\delta \eta_k} - \frac{1}{m_i}\frac{\delta V_{\prime 2}}{\delta \eta_i} - \frac{1}{m_n}\Sigma_\prime \frac{\delta V_{\prime 2}}{\delta \eta_i},\\
\frac{\delta w^{(i)}}{\delta \zeta_i} &= \zeta'_i - \Sigma_{\prime\prime}\,.\frac{m_k}{m_k+m_n}\frac{\delta w^{(k)}}{\delta \zeta_k} - \frac{1}{m_i}\frac{\delta V_{\prime 2}}{\delta \zeta_i} - \frac{1}{m_n}\Sigma_\prime \frac{\delta V_{\prime 2}}{\delta \zeta_i},
\end{aligned}\right\} \qquad (\mathrm{W}^6.)$$

and similarly

$$\left.\begin{aligned}
-\frac{\delta w^{(i)}}{\delta \alpha_i} &= \alpha'_i + \Sigma_{\prime\prime}\,.\frac{m_k}{m_k+m_n}\frac{\delta w^{(k)}}{\delta \alpha_k} + \frac{1}{m_i}\frac{\delta V_{\prime 2}}{\delta \alpha_i} + \frac{1}{m_n}\Sigma_\prime \frac{\delta V_{\prime 2}}{\delta \alpha_i},\\
-\frac{\delta w^{(i)}}{\delta \beta_i} &= \beta'_i + \Sigma_{\prime\prime}\,.\frac{m_k}{m_k+m_n}\frac{\delta w^{(k)}}{\delta \beta_k} + \frac{1}{m_i}\frac{\delta V_{\prime 2}}{\delta \beta_i} + \frac{1}{m_n}\Sigma_\prime \frac{\delta V_{\prime 2}}{\delta \beta_i},\\
-\frac{\delta w^{(i)}}{\delta \gamma_i} &= \gamma'_i + \Sigma_{\prime\prime}\,.\frac{m_k}{m_k+m_n}\frac{\delta w^{(k)}}{\delta \gamma_k} + \frac{1}{m_i}\frac{\delta V_{\prime 2}}{\delta \gamma_i} + \frac{1}{m_n}\Sigma_\prime \frac{\delta V_{\prime 2}}{\delta \gamma_i},
\end{aligned}\right\} \qquad (\mathrm{X}^6.)$$

the sign of summation $\Sigma_{\prime\prime}$ referring only to the disturbing masses $m_k$, to the exclusion of $m_i$ and $m_n$; and these equations (W⁶.), (X⁶.) are the rigorous formulæ, corresponding to the approximate relations (I⁶.), (K⁶.). In like manner, the formula (L⁶.) for the time of motion in a binary system, which is only an approximation when the system is considered as multiple, may be rigorously corrected for perturbation by adding to it an analogous term deduced from the disturbing part $V_{\prime 2}$ of the whole characteristic function; that is, by changing it to the following:

$$t = \frac{\delta w^{(i)}}{\delta g^{(i)}} + \frac{\delta V_{\prime 2}}{\delta H_\prime}, \qquad (\mathrm{Y}^6.)$$

which gives, for this other coefficient of $w^{(i)}$, the corrected and rigorous expression

$$\frac{\delta w^{(i)}}{\delta g^{(i)}} = t - \frac{\delta V_{\prime 2}}{\delta H_\prime}: \qquad (\mathrm{Z}^6.)$$

$V_{\prime 2}$ being here supposed so chosen as to be rigorously the correction to $V_{\prime 1}$. If therefore, by the theory of binary systems, or by eliminating $g^{(i)}$ between the four equations (K$^6$.) (L$^6$.), we have deduced expressions for the three varying relative coordinates $\xi_i$ $\eta_i$ $\zeta_i$ as functions of the time $t$, and of the six initial quantities $\alpha_i$ $\beta_i$ $\gamma_i$ $\alpha'_i$ $\beta'_i$ $\gamma'_i$, which may be thus denoted,

$$\left.\begin{aligned} \xi_i &= \phi_1(\alpha_i, \beta_i, \gamma_i, \alpha'_i, \beta'_i, \gamma'_i, t), \\ \eta_i &= \phi_2(\alpha_i, \beta_i, \gamma_i, \alpha'_i, \beta'_i, \gamma'_i, t), \\ \zeta_i &= \phi_3(\alpha_i, \beta_i, \gamma_i, \alpha'_i, \beta'_i, \gamma'_i, t); \end{aligned}\right\} \tag{A$^7$.}$$

we shall know that the following relations are rigorously and *identically* true,

$$\left.\begin{aligned} \xi_i &= \phi_1\left(\alpha_i, \beta_i, \gamma_i, -\frac{\delta w^{(i)}}{\delta \alpha_i}, -\frac{\delta w^{(i)}}{\delta \beta_i}, -\frac{\delta w^{(i)}}{\delta \gamma_i}, \frac{\delta w^{(i)}}{\delta g^{(i)}}\right), \\ \eta_i &= \phi_2\left(\alpha_i, \beta_i, \gamma_i, -\frac{\delta w^{(i)}}{\delta \alpha_i}, -\frac{\delta w^{(i)}}{\delta \beta_i}, -\frac{\delta w^{(i)}}{\delta \gamma_i}, \frac{\delta w^{(i)}}{\delta g^{(i)}}\right), \\ \zeta_i &= \phi_3\left(\alpha_i, \beta_i, \gamma_i, -\frac{\delta w^{(i)}}{\delta \alpha_i}, -\frac{\delta w^{(i)}}{\delta \beta_i}, -\frac{\delta w^{(i)}}{\delta \gamma_i}, \frac{\delta w^{(i)}}{\delta g^{(i)}}\right), \end{aligned}\right\} \tag{B$^7$.}$$

and consequently that these relations will still be rigorously true when we substitute for the four coefficients of $w^{(i)}$ their rigorous values (X$^6$.) and (Z$^6$.) for the case of a multiple system. We may thus retain in rigour for any multiple system the final integrals (A$^7$.) of the motion of a binary system, if only we add to the initial components $\alpha'_i$ $\beta'_i$ $\gamma'_i$ of relative velocity, and to the time $t$, the following perturbational terms:

$$\left.\begin{aligned} \Delta\alpha'_i &= \Sigma_{\prime\prime} \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \alpha_k} + \frac{1}{m_i} \frac{\delta V_{\prime 2}}{\delta \alpha_i} + \frac{1}{m_n} \Sigma_\prime \frac{\delta V_{\prime 2}}{\delta \alpha_i}, \\ \Delta\beta'_i &= \Sigma_{\prime\prime} \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \beta_k} + \frac{1}{m_i} \frac{\delta V_{\prime 2}}{\delta \beta_i} + \frac{1}{m_n} \Sigma_\prime \frac{\delta V_{\prime 2}}{\delta \beta_i}, \\ \Delta\gamma'_i &= \Sigma_{\prime\prime} \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \gamma_k} + \frac{1}{m_i} \frac{\delta V_{\prime 2}}{\delta \gamma_i} + \frac{1}{m_n} \Sigma_\prime \frac{\delta V_{\prime 2}}{\delta \gamma_i}, \end{aligned}\right\} \tag{C$^7$.}$$

and

$$\Delta t = -\frac{\delta V_{\prime 2}}{\delta H_\prime}. \tag{D$^7$.}$$

In the same way, if the theory of binary systems, or the elimination of $g^{(i)}$ between the four equations (I$^6$.) (L$^6$.), has given three intermediate integrals, of the form

$$\left.\begin{aligned} \xi'_i &= \psi_1(\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i, \gamma_i, t), \\ \eta'_i &= \psi_2(\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i, \gamma_i, t), \\ \zeta'_i &= \psi_3(\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i, \gamma_i, t), \end{aligned}\right\} \tag{E$^7$.}$$

we can conclude that the following equations are rigorous and identical,

$$\left.\begin{aligned}\frac{\delta w^{(i)}}{\delta \xi_i} &= \psi_1\left(\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i, \gamma_i, \frac{\delta w^{(i)}}{\delta g^{(i)}}\right),\\ \frac{\delta w^{(i)}}{\delta \eta_i} &= \psi_2\left(\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i, \gamma_i, \frac{\delta w^{(i)}}{\delta g^{(i)}}\right),\\ \frac{\delta w^{(i)}}{\delta \zeta_i} &= \psi_3\left(\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i, \gamma_i, \frac{\delta w^{(i)}}{\delta g^{(i)}}\right),\end{aligned}\right\} \qquad (\mathrm{F}^7.)$$

and must therefore be still true, when, in passing to a multiple system, we change the coefficients of $w^{(i)}$ to their rigorous values ($\mathrm{W}^6$.) ($\mathrm{Z}^6$.). The three intermediate integrals ($\mathrm{E}^7$.) of the motion of a binary system may therefore be adapted rigorously to the case of a multiple system, by first adding to the time $t$ the perturbational term ($\mathrm{D}^7$.), and afterwards adding to the resulting values of the final components of relative velocity the terms

$$\left.\begin{aligned}\Delta\xi_i' &= \Sigma_{\prime\prime}\cdot\frac{m_k}{m_k+m_n}\frac{\delta w^{(k)}}{\delta \xi_k} + \frac{1}{m_i}\frac{\delta V_{\prime 2}}{\delta \xi_i} + \frac{1}{m_n}\Sigma_\prime\frac{\delta V_{\prime 2}}{\delta \xi_i},\\ \Delta\eta_i' &= \Sigma_{\prime\prime}\cdot\frac{m_k}{m_k+m_n}\frac{\delta w^{(k)}}{\delta \eta_k} + \frac{1}{m_i}\frac{\delta V_{\prime 2}}{\delta \eta_i} + \frac{1}{m_n}\Sigma_\prime\frac{\delta V_{\prime 2}}{\delta \eta_i},\\ \Delta\zeta_i' &= \Sigma_{\prime\prime}\cdot\frac{m_k}{m_k+m_n}\frac{\delta w^{(k)}}{\delta \zeta_k} + \frac{1}{m_i}\frac{\delta V_{\prime 2}}{\delta \zeta_i} + \frac{1}{m_n}\Sigma_\prime\frac{\delta V_{\prime 2}}{\delta \zeta_i}.\end{aligned}\right\} \qquad (\mathrm{G}^7.)$$

22. To derive now, from these rigorous results, some useful approximate expressions, we shall neglect, in the perturbations, the terms which are of the second order, with respect to the small masses of the system, and with respect to the constant $2H_\prime$ of relative living force, which is easily seen to be small of the same order as the masses: and then the perturbations of these coordinates, deduced by the method that has been explained, become

$$\left.\begin{aligned}\Delta\xi_i &= \frac{\delta\xi_i}{\delta\alpha_i'}\Delta\alpha_i' + \frac{\delta\xi_i}{\delta\beta_i'}\Delta\beta_i' + \frac{\delta\xi_i}{\delta\gamma_i'}\Delta\gamma_i' + \frac{\delta\xi_i}{\delta t}\Delta t,\\ \Delta\eta_i &= \frac{\delta\eta_i}{\delta\alpha_i'}\Delta\alpha_i' + \frac{\delta\eta_i}{\delta\beta_i'}\Delta\beta_i' + \frac{\delta\eta_i}{\delta\gamma_i'}\Delta\gamma_i' + \frac{\delta\eta_i}{\delta t}\Delta t,\\ \Delta\zeta_i &= \frac{\delta\zeta_i}{\delta\alpha_i'}\Delta\alpha_i' + \frac{\delta\zeta_i}{\delta\beta_i'}\Delta\beta_i' + \frac{\delta\zeta_i}{\delta\gamma_i'}\Delta\gamma_i' + \frac{\delta\zeta_i}{\delta t}\Delta t,\end{aligned}\right\} \qquad (\mathrm{H}^7.)$$

in which we may employ, instead of the rigorous values ($\mathrm{C}^7$.) for $\Delta\alpha_i'$, $\Delta\beta_i'$, $\Delta\gamma_i'$, the following approximate values:

$$\left.\begin{aligned}\Delta\alpha_i' &= \Sigma_{\prime\prime}\frac{m_k}{m_n}\frac{\delta w^{(k)}}{\delta\alpha_k} + \frac{1}{m_i}\frac{\delta V_{\prime 2}}{\delta\alpha_i},\\ \Delta\beta_i' &= \Sigma_{\prime\prime}\frac{m_k}{m_n}\frac{\delta w^{(k)}}{\delta\beta_k} + \frac{1}{m_i}\frac{\delta V_{\prime 2}}{\delta\beta_i},\\ \Delta\gamma_i' &= \Sigma_{\prime\prime}\frac{m_k}{m_n}\frac{\delta w^{(k)}}{\delta\gamma_k} + \frac{1}{m_i}\frac{\delta V_{\prime 2}}{\delta\gamma_i}.\end{aligned}\right\} \qquad (\mathrm{I}^7.)$$

To calculate the four coefficients

$$\frac{\delta V_{\prime 2}}{\delta \alpha_i}, \quad \frac{\delta V_{\prime 2}}{\delta \beta_i}, \quad \frac{\delta V_{\prime 2}}{\delta \gamma_i}, \quad \frac{\delta V_{\prime 2}}{\delta H_\prime},$$

which enter into the values (I⁷.) (D⁷.), we may consider $V_{\prime 2}$, by (R⁶.) (T⁶.), and by the theory of binary systems, as a function of the initial and final relative coordinates, and initial components of relative velocities, involving also expressly the time $t$ and the $n-2$ auxiliary quantities of the form $g^{(k)}$; and then we are to consider those initial components and auxiliary quantities and the time, as depending themselves on the initial and final coordinates, and on $H_\prime$. But it is not difficult to prove, by the foregoing principles, that when $t$ and $g^{(k)}$ are thus considered, their variations are, in the present order of approximation,

$$\delta t = \frac{\Sigma_\prime \,.m \left(\dfrac{\delta^2 w}{\delta g^2}\right)^{-1} \delta_\prime \dfrac{\delta w}{\delta g} + \delta H_\prime}{\Sigma_\prime \,.m \left(\dfrac{\delta^2 w}{\delta g^2}\right)^{-1}} \tag{K$^7$.}$$

and

$$\delta g^{(k)} = \left(\frac{\delta^2 w^{(k)}}{\delta g^{(k)2}}\right)^{-1} \left(\delta t - \delta_\prime \frac{\delta w^{(k)}}{\delta g^{(k)}}\right), \tag{L$^7$.}$$

the sign of variation $\delta_\prime$ referring only to the initial and final coordinates; and also that

$$\frac{\delta^2 w^{(i)}}{\delta g^{(i)2}} \frac{\delta \xi_i}{\delta t} = \frac{\delta^2 w^{(i)}}{\delta \alpha_i \, \delta g^{(i)}} \frac{\delta \xi_i}{\delta \alpha'_i} + \frac{\delta^2 w^{(i)}}{\delta \beta_i \, \delta g^{(i)}} \frac{\delta \xi_i}{\delta \beta'_i} + \frac{\delta^2 w^{(i)}}{\delta \gamma_i \, \delta g^{(i)}} \frac{\delta \xi_i}{\delta \gamma'_i}, \tag{M$^7$.}$$

along with two other analogous relations between the coefficients of the two other coordinates $\eta_i$, $\zeta_i$; from which it follows that $t$ and $g^{(k)}$, and therefore $\alpha'_k$ $\beta'_k$ $\gamma'_k$, may be treated as constant, in taking the variation of the disturbing part $V_{\prime 2}$, for the purpose of calculating the perturbations (H⁷.): and that the terms involving $\Delta t$ are destroyed by other terms. We may therefore put simply

$$\left.\begin{aligned}
\Delta \xi_i &= \frac{\delta \xi_i}{\delta \alpha'_i} \Delta \alpha'_i + \frac{\delta \xi_i}{\delta \beta'_i} \Delta \beta'_i + \frac{\delta \xi_i}{\delta \gamma'_i} \Delta \gamma'_i, \\
\Delta \eta_i &= \frac{\delta \eta_i}{\delta \alpha'_i} \Delta \alpha'_i + \frac{\delta \eta_i}{\delta \beta'_i} \Delta \beta'_i + \frac{\delta \eta_i}{\delta \gamma'_i} \Delta \gamma'_i, \\
\Delta \zeta_i &= \frac{\delta \zeta_i}{\delta \alpha'_i} \Delta \alpha'_i + \frac{\delta \zeta_i}{\delta \beta'_i} \Delta \beta'_i + \frac{\delta \zeta_i}{\delta \gamma'_i} \Delta \gamma'_i,
\end{aligned}\right\} \tag{N$^7$.}$$

employing for $\Delta \alpha'_i$ the following new expression,

$$\left.\begin{aligned}
\Delta \alpha'_i = \Sigma_{\prime\prime} \,.m_k \Bigg\{ &\int_0^t \frac{\delta R^{(i,k)}}{\delta \alpha_i} dt + \frac{\delta \alpha'_i}{\delta \alpha_i} \int_0^t \frac{\delta R^{(i,k)}}{\delta \alpha'_i} dt \\
&+ \frac{\delta \beta'_i}{\delta \alpha_i} \int_0^t \frac{\delta R^{(i,k)}}{\delta \beta'_i} dt + \frac{\delta \gamma'_i}{\delta \alpha_i} \int_0^t \frac{\delta R^{(i,k)}}{\delta \gamma'_i} dt \Bigg\}
\end{aligned}\right\} \tag{O$^7$.}$$

together with analogous expressions for $\Delta\beta'_i$, $\Delta\gamma'_i$, in which the sign of summation $\Sigma_{\prime\prime}$ refers to the disturbing masses, and in which the quantity

$$R^{(i,k)} = f^{(i,k)} + \xi_i \frac{\delta f^{(k)}}{\delta \xi_k} + \eta_i \frac{\delta f^{(k)}}{\delta \eta_k} + \zeta_i \frac{\delta f^{(k)}}{\delta \zeta_k} \tag{P$^7$.}$$

is considered as depending on $\alpha_i\ \beta_i\ \gamma_i\ \alpha'_i\ \beta'_i\ \gamma'_i\ \alpha_k\ \beta_k\ \gamma_k\ \alpha'_k\ \beta'_k\ \gamma'_k\ t$ by the theory of binary systems, while $\alpha'_i\ \beta'_i\ \gamma'_i$, are considered as depending, by the same rules, on $\alpha_i\ \beta_i\ \gamma_i\ \xi_i\ \eta_i\ \zeta_i$ and $t$.

It may also be easily shown, that

$$\frac{\delta \xi_i}{\delta \alpha'_i} \frac{\delta \alpha'_i}{\delta \alpha_i} + \frac{\delta \xi_i}{\delta \beta'_i} \frac{\delta \alpha'_i}{\delta \beta_i} + \frac{\delta \xi_i}{\delta \gamma'_i} \frac{\delta \alpha'_i}{\delta \gamma_i} = -\frac{\delta \xi_i}{\delta \alpha_i}; \tag{Q$^7$.}$$

with other analogous equations: the perturbation of the coordinates $\xi_i$ may therefore be thus expressed,

$$\left.\begin{aligned} \Delta\xi_i = \Sigma_{\prime\prime}\,.m_k \Big\{ & \frac{\delta \xi_i}{\delta \alpha'_i} \int_0^t \frac{\delta R^{(i,k)}}{\delta \alpha_i} dt - \frac{\delta \xi_i}{\delta \alpha_i} \int_0^t \frac{\delta R^{(i,k)}}{\delta \alpha'_i} dt \\ & + \frac{\delta \xi_i}{\delta \beta'_i} \int_0^t \frac{\delta R^{(i,k)}}{\delta \beta_i} dt - \frac{\delta \xi_i}{\delta \beta_i} \int_0^t \frac{\delta R^{(i,k)}}{\delta \beta'_i} dt \\ & + \frac{\delta \xi_i}{\delta \gamma'_i} \int_0^t \frac{\delta R^{(i,k)}}{\delta \gamma_i} dt - \frac{\delta \xi_i}{\delta \gamma_i} \int_0^t \frac{\delta R^{(i,k)}}{\delta \gamma'_i} dt \Big\}, \end{aligned}\right\} \tag{R$^7$.}$$

and the perturbations of the two other coordinates may be expressed in an analogous manner.

It results from the same principles, that in taking the first differentials of these perturbations (R$^7$.), the integrals may be treated as constant; and therefore that we may either represent the change of place of the disturbed point $m_i$, in its relative orbit about $m_n$, by altering a little the initial components of velocity without altering the initial position, and then employing the rules of binary systems; or calculate at once the perturbations of place and of velocity, by employing the same rules, and altering at once the initial position and initial velocity. If we adopt the former of these two methods, we are to employ the expressions (O$^7$.), which may be thus summed up,

$$\left.\begin{aligned} \Delta\alpha'_i &= \Sigma_{\prime\prime}\,.m_k \frac{\delta}{\delta \alpha_i} \int_0^t R^{(i,k)} dt, \\ \Delta\beta'_i &= \Sigma_{\prime\prime}\,.m_k \frac{\delta}{\delta \beta_i} \int_0^t R^{(i,k)} dt, \\ \Delta\gamma'_i &= \Sigma_{\prime\prime}\,.m_k \frac{\delta}{\delta \gamma_i} \int_0^t R^{(i,k)} dt; \end{aligned}\right\} \tag{S$^7$.}$$

and if we adopt the latter method, we are to make,

$$\left.\begin{aligned} \Delta\alpha'_i &= \Sigma_{\prime\prime}\,.m_k \int_0^t \frac{\delta R^{(i,k)}}{\delta \alpha_i} dt, & \Delta\alpha_i &= -\,\Sigma_{\prime\prime}\,.m_k \int_0^t \frac{\delta R^{(i,k)}}{\delta \alpha'_i} dt, \\ \Delta\beta'_i &= \Sigma_{\prime\prime}\,.m_k \int_0^t \frac{\delta R^{(i,k)}}{\delta \beta_i} dt, & \Delta\beta_i &= -\,\Sigma_{\prime\prime}\,.m_k \int_0^t \frac{\delta R^{(i,k)}}{\delta \beta'_i} dt, \\ \Delta\gamma'_i &= \Sigma_{\prime\prime}\,.m_k \int_0^t \frac{\delta R^{(i,k)}}{\delta \gamma_i} dt, & \Delta\gamma_i &= -\,\Sigma_{\prime\prime}\,.m_k \int_0^t \frac{\delta R^{(i,k)}}{\delta \gamma'_i} dt. \end{aligned}\right\} \tag{T$^7$.}$$

The latter was the method of LAGRANGE: the former is suggested more immediately by the principles of the present essay.

*General introduction of the Time into the expression of the Characteristic Function in any dynamical problem.*

23. Before we conclude this sketch of our general method in dynamics, it will be proper to notice briefly a transformation of the characteristic function, which may be used in all applications. This transformation consists in putting, generally,

$$V = tH + S, \tag{U$^7$.}$$

and considering the part $S$, namely, the definite integral

$$S = \int_0^t (T + U)\, dt, \tag{V$^7$.}$$

as a function of the initial and final coordinates and of the time, of which the variation is, by our law of varying action,

$$\delta S = -H\, \delta t + \Sigma\, .m(x'\, \delta x - a'\, \delta a + y'\, \delta y - b'\, \delta b + z'\, \delta z - c'\, \delta c). \tag{W$^7$.}$$

The partial differential coefficients of the first order of this auxiliary function $S$, are hence,

$$\frac{\delta S}{\delta t} = -H; \tag{X$^7$.}$$

$$\frac{\delta S}{\delta x_i} = m_i x'_i, \quad \frac{\delta S}{\delta y_i} = m_i y'_i, \quad \frac{\delta S}{\delta z_i} = m_i z'_i; \tag{Y$^7$.}$$

and

$$\frac{\delta S}{\delta a_i} = -m_i a'_i, \quad \frac{\delta S}{\delta b_i} = -m_i b'_i, \quad \frac{\delta S}{\delta c_i} = -m_i c'_i. \tag{Z$^7$.}$$

These last expressions (Z$^7$.) are forms for the final integrals of motion of any system, corresponding to the result of elimination of $H$ between the equations (D.) and (E.); and the expressions (Y$^7$.) are forms for the intermediate integrals, more convenient in many respects than the forms already employed.

24. The limits of the present essay do not permit us here to develope the consequences of these new expressions. We can only observe, that the auxiliary function $S$ must satisfy the two following equations, in partial differentials of the first order, analogous to, and deduced from, the equations (F.) and (G.):

$$\frac{\delta S}{\delta t} + \Sigma\,.\frac{1}{2m}\left\{\left(\frac{\delta S}{\delta x}\right)^2 + \left(\frac{\delta S}{\delta y}\right)^2 + \left(\frac{\delta S}{\delta z}\right)^2\right\} = U, \tag{A$^8$.}$$

and

$$\frac{\delta S}{\delta t} + \Sigma \,. \frac{1}{2m} \left\{ \left( \frac{\delta S}{\delta a} \right)^2 + \left( \frac{\delta S}{\delta b} \right)^2 + \left( \frac{\delta S}{\delta c} \right)^2 \right\} = U_0; \tag{B$^8$.}$$

and that to correct an approximate value $S_1$ of $S$, in the integration of these equations, or to find the remaining part $S_2$, if

$$S = S_1 + S_2, \tag{C$^8$.}$$

we may employ the symbolic equation

$$\frac{d}{dt} = \frac{\delta}{\delta t} + \Sigma \,. \frac{1}{m} \left( \frac{\delta S}{\delta x} \frac{\delta}{\delta x} + \frac{\delta S}{\delta y} \frac{\delta}{\delta y} + \frac{\delta S}{\delta z} \frac{\delta}{\delta z} \right); \tag{D$^8$.}$$

which gives, rigorously,

$$\frac{dS_2}{dt} = U - U_1 + \Sigma \,. \frac{1}{2m} \left\{ \left( \frac{\delta S_2}{\delta x} \right)^2 + \left( \frac{\delta S_2}{\delta y} \right)^2 + \left( \frac{\delta S_2}{\delta z} \right)^2 \right\} \tag{E$^8$.}$$

if we establish by analogy the definition

$$U_1 = \frac{\delta S_1}{\delta t} + \Sigma \,. \frac{1}{2m} \left\{ \left( \frac{\delta S_1}{\delta x} \right)^2 + \left( \frac{\delta S_1}{\delta y} \right)^2 + \left( \frac{\delta S_1}{\delta z} \right)^2 \right\}; \tag{F$^8$.}$$

and therefore approximately

$$S_2 = \int_0^t (U - U_1)\, dt, \tag{G$^8$.}$$

the parts $S_1$ $S_2$ being chosen so as to vanish with the time. These remarks may all be extended easily, so as to embrace relative and polar coordinates, and other marks of position, and offer a new and better way of investigating the orbits and perturbations of a system, by a new and better form of the function and method of this Essay.

*March* 29, 1834.