Ramanujan's Constant (e^{pv163}) And Its Cousins

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"Mathematics, rightly viewed, possesses not only truth, but supreme beauty." – Bertrand Russell

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I. Introduction

It is quite easy to introduce this topic such that the educated layman with an interest in mathematics can appreciate it. Given *e = 2.718281…*, the base of natural logarithms, one can easily show that,

$$e^{pv163} = 262{,}537{,}412{,}640{,}768{,}743.99999999999925\ldots$$

The mathematical constants *e* and *p* are *transcendental numbers*, that is, they can never be the roots of finite equations with coefficients in the rational field. Yet, here we have a combination of *e* and *p* that is almost an integer. One perhaps can assume it to be mere coincidence; of the infinity of possible numbers of the form e^{pvd} for some positive integer *d*, it may be expected there will be some that will be close to an integer.

However, when the above number, as well as others, shows a certain "internal structure", namely,

$$e^{pv67} = 5280^3 + 743.9999986\ldots$$
$$e^{pv163} = 640320^3 + 743.99999999999925\ldots$$

including some relations that involve square roots,

$$e^{pv22} \sim 2^6(1+v2)^{12} + 23.99988\ldots$$
$$e^{pv58} \sim 2^6((5+v29)/2)^{12} + 23.999999988\ldots$$

and,

$e^{pv42} \sim 4^4(21+8v6)^4 - 104.0000062\ldots$
$e^{pv130} \sim 12^4(323+40v65)^4 - 104.0000000000012\ldots$

one cannot dismiss it as just coincidence. Something interesting is going on.

It turns out the answer has to do with *modular functions* and what are called *class polynomials*, namely the Hilbert, Weber, and Ramanujan class polynomials, respectively, for the three pairs of examples above.

II. Hilbert Class Polynomials

The modular function involved in the first pair of examples is known as the *j-function*. This function, *j(q)*, has the series expansion,

$$j(q) = 1/q + 744 + 196884q + 21493760q^2 + 864299970q^3 + \ldots$$

where $q = e^{2pit}$ and t is the *half-period ratio*. The expansion somehow "explains" why in the first pair of examples we find the approximations for 744.

The *j(q)* is dependent on the t. If t involves a quadratic irrational $\sqrt{d}$, then an important result is that *j(q)* is an algebraic number of degree *n*, where *n* is the *class number* of *d*. We can say two things: First, this algebraic number is in fact an algebraic integer, or it is defined by an equation with the leading coefficient as 1 (a monic equation). Second, this equation, the *Hilbert class polynomial*, is a *solvable equation*, or solvable using only a finite number of arithmetic operations and root extractions. Thus, for any transcendental number e^{pvd}, if we know the class number *n* of *d*, then we can find an approximation to it involving an algebraic number of degree *n* solvable in radicals.

Class numbers are involved in the study of number fields, though we need not go into the details here. The list of numbers *d*, or *discriminants*, belonging to particular class numbers *n* has been made for smaller *n*. One is referred to Mathworld, http://mathworld.wolfram.com/ClassNumber.html for a listing of up to class number 25.

For class number 1, there are 9 discriminants, namely, 3, 4,7, 8, 11, 19, 43, 67, 163, also called the *Heegner numbers*. The last five give the more impressive approximations to e^{pvd} involving integers, since *j(q)* would be an algebraic integer of degree 1, which is simply a plain integer. However, there are also discriminants of class number 2, so *j(q)* would be an algebraic integer of degree 2, or a root of a quadratic. And so on for the higher class numbers, examples of which we will give here.

I have tabulated the *j(q)* of the lower class numbers, with the entries for class numbers 1 and 2 computed by myself. They have been grouped in a manner that will be justified eventually. For the highest *d* of class number 1 and 2, I also included the error difference of the approximation. While e^{pv163} is visually impressive, instantly recognizable as a near integer and thus almost the root of a linear equation with a difference of only 7.5×10^{-13}, the transcendental number e^{pv427} is more numerically impressive, though how many of us can recognize at a glance that it misses being the root of a quadratic by a mere 1.3×10^{-23}?

<u>Class Number 1</u> (9 discriminants)

$e^{pv11} \sim 32^3 + 738$

$e^{pv19} \sim 96^3 + 744$

$e^{pv43} \sim 960^3 + 744$

$e^{pv67} \sim 5280^3 + 744$

$e^{pv163} \sim 640320^3 + 744$ $(7.5 \text{ x } 10^{-13})$

I have included only the five highest Heegner numbers. Technically, there are 13 discriminants with class number 1, though only 9 are maximal. Some of the others are,

$e^{pv16} \sim 66^3 - 744$

$e^{pv28} \sim 255^3 - 744$

<u>Class Number 2</u> (18 discriminants)

$e^{pv20} \sim 2^3(25+13v5)^3 - 744$

$e^{pv52} \sim 30^3(31+9v13)^3 - 744$

$e^{pv148} \sim 60^3(2837+468v37)^3 - 744$

$e^{pv24} \sim 12^3(1+\sqrt{2})^2(5+2\sqrt{2})^3 - 744$

$e^{pv40} \sim 6^3(65+27v5)^3 - 744$

$e^{pv88} \sim 60^3(155+108v2)^3 - 744$

$e^{pv232} \sim 30^3(140989+26163v29)^3 - 744$

$e^{pv35} \sim 16^3(15+7v5)^3 + 744$

$e^{pv91} \sim 48^3(227+63v13)^3 + 744$

$e^{pv115} \sim 48^3(785+351v5)^3 + 744$

$e^{pv187} \sim 240^3(3451+837v17)^3 + 744$

$e^{pv235} \sim 528^3(8875+3969v5)^3 + 744$

$e^{pv403} \sim 240^3(2809615+779247v13)^3 + 744$

$e^{pv427} \sim 5280^3(236674+30303v61)^3 + 744$ $(1.3 \text{ x } 10^{-23})$

The above numbers follow the pattern of Ramanujan's constant, approximately cubes of algebraic numbers plus 744 if the discriminant is odd, minus 744 if the discriminant is even. (Other than e^{pv24} but that's because it involves *d* that is a multiple of 3.) For the last four discriminants of class number 2, namely d = 15, 51, 123, 267, it took some time to find *j(q)* and it turned out that for these discriminants, *j(q)* is not a perfect cube. Mathworld would give e^{pv51} as,

$$e^{pv51} \sim 4*48^3(6263+1519\sqrt{17}) + 744$$

However, these discriminants also happened to be multiples of 3. I was aware of a technique (see paper by Yui and Zagier) to factor certain algebraic roots and express it in terms of *fundamental units*, and another factor which turns out to be a perfect cube. So we can have expressions for the *j(q)* of these discriminants using smaller numbers, namely the *square* of a fundamental unit and a perfect *cube*, given below:

$$e^{pv15} \sim 3^3((1+\sqrt{5})/2)^2(5+4\sqrt{5})^3 + 743$$

$$e^{pv51} \sim 48^3(4+\sqrt{17})^2(5+\sqrt{17})^3 + 744$$

$$e^{pv123} \sim 480^3(32+5\sqrt{41})^2(8+\sqrt{41})^3 + 744$$

$$e^{pv267} \sim 240^3(500+53\sqrt{89})^2(625+53\sqrt{89})^3 + 744 \quad (1.0 \text{ x } 10^{-17})$$

Note that:

$$(1/2)^2 - 5*(1/2)^2 = -1$$

$$4^2 - 17*1^2 = -1$$

$$32^2 - 41*5^2 = -1$$

$$500^2 - 89*53^2 = -1$$

<u>Class Number 3</u> (16 discriminants)

For the next class numbers 3, 4, 5, we will not give the complete list but just a few examples. Only e^{pv59} and e^{pv83} have been derived by myself, the others are from other authors. The drawback of Hilbert class polynomials is the size of their coefficients and a signature of these polynomials is that their constant term is a perfect cube, which may be an indication that the root is a perfect cube of an algebraic number. I have observed that for some discriminants, we indeed can have a simplification of these polynomials. For example, while for d = 23 the Hilbert class polynomial is given by (see paper by Morain),

$$y^3+3491750y^2-5151296875y+23375^3 = 0$$

by setting $y = x^3$, it will factor such that, after scaling, it simplifies to the equation below. The other class polynomials have been reduced in the same manner. I believe that for odd class numbers, the Hilbert class polynomial in the variable *y*, by setting $y = x^3$, can be factored so that it will have smaller coefficients. For even class numbers, especially if the discriminant is a multiple of 3, it may not simplify so easily. We have then,

$$e^{pv23} \sim 5^3x^3 + 744; \; (x^3-31x^2+26x-187=0)$$

e^{pv31} ~ $3^3x^3 + 744$; ($x^3-114x^2+93x-4301=0$)

e^{pv59} ~ $32^3x^3 + 744$; ($x^3-98x^2+67x-22=0$)

e^{pv83} ~ $160^3x^3 + 744$; ($x^3-87x^2+5x-2=0$)

Class Number 4 (54 discriminants)

e^{pv55} ~ $x^3 + 744$; ($x^4-2355x^3-8370x^2-5553900x-26484975=0$)

e^{pv56} ~ $4^3x^3 – 744$; ($x^4-646x^3+8347x^2-11286x+84337=0$)

Class Number 5 (25 discriminants)

e^{pv47} ~ $5^3x^3 + 744$; ($x^5-264x^4+484x^3-15419x^2+21714x-80707=0$)

and so on…

III. Weber Class Polynomials

The modular function involved in the second pair of examples is known as the *Weber modular function*. For brevity, perhaps we can call it as the *w-function* or *w(q)*. This function has the series expansion,

$$w(q) = 1/q + 24 + 276q + 2048q^2 + 11202q^3 + \dots$$

Again, we can see in the expansion why in the second pair of examples we have approximations to a certain constant, this time the integer 24. Just like with the j-function *j(q)*, the *w(q)* is also an algebraic number determined by an equation of degree *k* dependent on the class number *n* of the discriminant *d*. This equation, the *Weber class polynomial*, is also solvable in radicals.

However, there are some complications with regards to the degree since it is not necessarily $k = n$. The degree *k* is also dependent on the nature of the discriminant *d*, especially for odd *d* if it is of the form $8m+3$ or $8m+7$.

A. Odd Class Numbers

Let *x* be the real root of its class polynomial. If a discriminant *d* of form $8m+3$ has class number *n*, then the general form is,

e^{pvd} ~ $x^{24} - 24$

where *x* is a root of an equation of degree $3n$ with a constant term -2^n. (The exception to this rule seems to be d = 3, which has a Weber class polynomial that is not a cubic.) If a discriminant *d* of form $8m+7$ has class number *n*, then the general form is,

e^{pvd} ~ $2^{12}x^{24} – 24$

where *x* is a root of an equation of degree *n* with a constant term *-1*.

The Weber class polynomials for odd class numbers up to nine I found by myself using the Integer Relations applet found at http://www.cecm.sfu.ca/projects/IntegerRelations/ . I admit there was a certain satisfaction in finding it independently, as you use the approximations $(e^{pvd} + 24)^{1/24}$ and $(e^{pvd} + 24)^{1/24}/\sqrt{2}$, gradually increase the sensitivity of the applet, and it would churn out candidate polynomials with increasingly large coefficients, then suddenly there would be this polynomial with small coefficients, sometimes just single digits, with much higher accuracy than the one before and you know this was the one you were looking for.

This was a few years back. Since then, Annegret Weng has made available the Weber class polynomials of up to d = 422500. See http://www.exp-math.uni-essen.de/zahlentheorie/classpol/class.html . For the list below, we will give for *d* of form *8m+3* only for class number one, which would then have cubic class polynomials (with the exception of d = 3). For other odd class numbers *n*, it is understood that such *d* would have class polynomials of degree *3n*, which are quite tedious to write down. However, we can do so for *d* of form *8m+7*.

Class Number 1 (9 discriminants, 6 of the form *8m+3*)

$e^{pv11} \sim x^{24} - 24$; $(x^3-2x^2+2x-2=0)$

$e^{pv19} \sim x^{24} - 24$; $(x^3-2x-2=0)$

$e^{pv43} \sim x^{24} - 24$; $(x^3-2x^2-2=0)$

$e^{pv67} \sim x^{24} - 24$; $(x^3-2x^2-2x-2=0)$

$e^{pv163} \sim x^{24} - 24$; $(x^3-6x^2+4x-2=0)$

Class Number 3 (16 discriminants, 2 of the form *8m+7*)

$e^{pv23} \sim 2^{12}x^{24} - 24$; $(x^3-x-1=0)$

$e^{pv31} \sim 2^{12}x^{24} - 24$; $(x^3-x^2-1=0)$

Class Number 5 (25 discriminants, 4 of the form *8m+7*)

$e^{pv47} \sim 2^{12}x^{24} - 24$; $(x^5-x^3-2x^2-2x-1=0)$

$e^{pv79} \sim 2^{12}x^{24} - 24$; $(x^5-3x^4+2x^3-x^2+x-1=0)$

$e^{pv103} \sim 2^{12}x^{24} - 24$; $(x^5-x^4-3x^3-3x^2-2x-1=0)$

$e^{pv127} \sim 2^{12}x^{24} - 24$; $(x^5-3x^4-x^3+2x^2+x-1=0)$

Class Number 7 (31 discriminants, 5 of the form *8m+7*)

$e^{pv71} \sim 2^{12}x^{24} - 24$; $(x^7-2x^6-x^5+x^4+x^3+x^2-x-1=0)$

$e^{pv151} \sim 2^{12}x^{24} - 24$; $(x^7-3x^6-x^5-3x^4-x^2-x-1=0)$

$e^{pv223} \sim 2^{12}x^{24} - 24$; $(x^7-5x^6-x^4-4x^3-x^2-1=0)$

$e^{pv463} \sim 2^{12}x^{24} - 24$; $(x^7-11x^6-9x^5-8x^4-7x^3-7x^2-3x-1=0)$

$e^{pv487} \sim 2^{12}x^{24} - 24$; $(x^7-13x^6+4x^5-4x^4+7x^3-4x^2+x-1=0)$

and so on…

B. Even Class Numbers

While the discriminants *d* of odd class numbers seem to be always odd (other than class number 1 which has *d = 4, 8*), discriminants of even class numbers are a mix of odd and even. I have observed that given even discriminants *4p* or *8q* of even class numbers with *p* or *q* prime, the appropriate root of the Weber class polynomial seems to approximate e^{pvp} and $e^{pv(2q)}$. We can call the first as *group 1* and the second as *group 2*, a grouping I also used earlier. I will be limiting this section only to these and not the odd discriminants of even class numbers.

For the *w(q)* of class number 4 labeled *Others*, these were taken from Weber's book (*Lehrbuch der Algebra*), an old book I found in the library. These radicals are too beautiful to be locked up in the musty pages of an old book.

<u>Class Number 2</u>

Group 1

$e^{pv5} \sim 2^6(f)^6 - 24$

$e^{pv13} \sim 2^6((3+v13)/2)^6 - 24$

$e^{pv37} \sim 2^6(6+v37)^6 - 24$

Group 2

$e^{pv6} \sim 2^6(1+v2)^4 + 24$

$e^{pv10} \sim 2^6(f)^{12} + 24$

$e^{pv22} \sim 2^6(1+v2)^{12} + 24$

$e^{pv58} \sim 2^6((5+v29)/2)^{12} + 24$

where *f* is the *golden ratio* (1+v5)/2 = 1.61803… It's interesting how this number crops up in the expressions for *w(q)* whenever *d* is a multiple of 5, though perhaps it is to be expected since it is also a fundamental unit. (The other 11 discriminants, other than e^{pv15}, are almost-roots of sextics.)

<u>Class Number 4</u>

Group 1

$e^{pv17} \sim 2^6(Pv17)^{12} - 24$

$e^{pv73} \sim 2^6(Pv73)^{12} - 24$

$e^{pv97} \sim 2^6(Pv97)^{12} - 24$

$e^{pv193} \sim 2^6(Pv193)^{12} - 24$

Group 2

$e^{pv14} \sim 2^6(Pv14)^{12} + 24$

$e^{pv34} \sim 2^6(Pv34)^{12} + 24$

$e^{pv46} \sim 2^6(Pv46)^{12} + 24$

$e^{pv82} \sim 2^6(Pv82)^{12} + 24$

$e^{pv142} \sim 2^6(Pv142)^{12} + 24$

where,

$Pv17 = (1+v17+vr_1)/4$; $r_1 = 2(1+v17)$

$Pv73 = (5+v73+vr_2)/4$; $r_2 = 2(41+5v73)$

$Pv97 = (9+v97+vr_3)/4$; $r_3 = 2(81+9v97)$

$Pv193 = (13+v193+vr_4)/2$; $r_4 = 2(179+13v193)$

$Pv14 = (1+v2+vr_5)/2$; $r_5 = (-1+2v2)$

$Pv34 = (3+v17+vr_6)/4$; $r_6 = 2(5+3v17)$

$Pv46 = (3+v2+vr_7)/2$; $r_7 = (7+6v2)$

$Pv82 = (9+v41+vr_8)/4$; $r_8 = 2(53+9v41)$

$Pv142 = (9+5v2+vr_9)/2$; $r_9 = (127+90v2)$

Others:

$e^{pv70} \sim 2^6(Pv70)^{12} + 24$

$e^{pv130} \sim 2^6(Pv130)^{12} + 24$

$e^{pv190} \sim 2^6(Pv190)^{12} + 24$

$e^{pv85} \sim 2^6(Pv85)^6 - 24$

$e^{pv133} \sim 2^6(Pv133)^6 - 24$

$e^{pv253} \sim 2^6(Pv253)^6 - 24$

$e^{pv30} \sim 2^6(Pv30)^4 + 24$

$e^{pv42} \sim 2^6(Pv42)^4 + 24$

$e^{pv78} \sim 2^6(Pv78)^4 + 24$

$e^{pv102} \sim 2^6(Pv102)^4 + 24$

$e^{pv33} \sim 2^6(Pv33)^4 - 24$

$e^{pv57} \sim 2^6(Pv57)^4 - 24$

$e^{pv177} \sim 2^6(Pv177)^4 - 24$

$e^{pv21} \sim 2^6(Pv21)^2 - 24$

$e^{pv93} \sim 2^6(Pv93)^2 - 24$

where,

$Pv70 = (f)^2 (1+v2)$

$Pv130 = (f)^3 (3+v13)/2$

$Pv190 = (f)^3 (3+v10)$

$Pv30 = (f)^3 (3+v10)$ (*Curious, same as above.*)

$Pv42 = (7+2v14)(14+3v21)/7$

$Pv78 = (3+v13)^3 (5+v26)/8$

$Pv102 = (2+v2)^3 (3v2+v17)^2/v8$

$Pv85 = (f)^4 (9+v85)/2$

$Pv133 = (3+v7)^2 (5v7+3v19)/4$

$Pv253 = (5+v23)^2 (13v11+9v23)/4$

$Pv33 = (1+v3)^3 (3+v11)/4$

$Pv57 = (1+v3)^3 (13+3v19)/4$

$Pv177 = (1+v3)^9 (23+3v59)/32$

$Pv21 = (v3+v7)^3 (3+v7)^2/16$

$Pv93 = (3v3+v31)^3 (39+7v31)^2/16$

For the next class numbers 6, 8, 10, 12, while I have the complete list for groups 1 and 2, we will give only one example per group to illustrate a certain pattern.

Class Number 6

Let, $y = x - 1/x$

Group 1: $e^{pv29} \sim 2^6x^6 - 24$; $(y^3-9y^2+8y-16=0)$

Group 2: $e^{pv26} \sim 2^6x^{12} + 24$; $(y^3-2y^2+y-4=0)$

Class Number 8

Let, $y = x + 1/x$

Group 1: $e^{pv41} \sim 2^6x^{12} - 24$; $(y^4-5y^3+3y^2+3y+2=0)$

Group 2: $e^{pv62} \sim 2^6x^{12} + 24$; $(y^4-2y^3-17y^2-24y-8=0)$

Class Number 10

Let, $y = x - 1/x$

Group 1: $e^{pv181} \sim 2^6x^6 - 24$; $(y^5-573y^4-81y^3-3483y^2-3240y-3888=0)$

Group 2: $e^{pv74} \sim 2^6x^{12} + 24$; $(y^5-8y^4+14y^3-36y^2+41y-28=0)$

Class Number 12

Let, y = x + 1/x

Group 1: $e^{pv89} \sim 2^6x^{12} - 24$; ($y^6-5y^5-27y^4-25y^3+28y^2+44y+16=0$)

Group 2: $e^{pv274} \sim 2^6x^{12} + 24$; ($y^6-57y^5+168y^4-78y^3+45y^2-345y+202=0$)

and so on…

We can summarize our results. Given even discriminants *4p* or *8q* of even class numbers with *p* or *q* prime. For class numbers 2, 6, 10,… *(4m+2)*, let, y = x – 1/x, where *y* is the appropriate root of an equation of degree *(4m+2)/2 = 2m+1*:

Group 1: $e^{pvp} \sim 2^6x^6 - 24$; Group 2: $e^{pv(2q)} \sim 2^6x^{12} + 24$

For class numbers 4, 8, 12,… *(4m+4)*, let, y = x + 1/x, where *y* is the appropriate root of an equation of degree *(4m+4)/2 = 2m+2*:

Group 1: $e^{pvp} \sim 2^6x^{12} - 24$; Group 2: $e^{pv(2q)} \sim 2^6x^{12} + 24$

In other words, for what we defined as groups 1 and 2, we can observe two things: (a) Let *x* be the appropriate root of the Weber class polynomial. For group 1, e^{pvp} is closely approximated by 2^6x^6 for class number *4m+2* but 2^6x^{12} for class number *4m+4*. For group 2, there is no difference and $e^{pv(2q)}$ is closely approximated by 2^6x^{12}. (b) For class number *4m+2*, the Weber class polynomial in *x* is a *semi-palindromic* polynomial, the same whether read forwards or backwards but only if we disregard sign. However, for class number *4m+4*, it is a true *palindromic* polynomial.

IV. Ramanujan Class Polynomials

The modular function involved in the last pair of examples has a formal designation in another context, the *Monster group*, which we will be going into later. However, for purposes of consistency, perhaps it is permissible to call it as the *r-function* (for Ramanujan) since he did work on this function. This function, *r(q)*, has the series expansion,

$$r(q) = 1/q + 104 + 4372q + 96256q^2 + 1240002q^3 + \dots$$

and we see why the last pair of examples involved approximations to 104. Just like *j(q)* and *w(q)*, *r(q)* again is an algebraic number determined by an equation of degree *k* dependent on the class number *n* of some discriminant *d*. This equation, which perhaps we can call the *Ramanujan class polynomial*, is solvable in radicals.

The *r(q)* given below for class numbers 2 and 4 were known to Ramanujan, though for d = 14, 82, 42, 190, it doesn't seem to be found in his Notebooks.

<u>Class Number 2</u>

$e^{pv5} \sim (4\sqrt{2})^4 + 100$

$e^{pv13} \sim (12\sqrt{2})^4 + 104$

$e^{pv37} \sim (84\sqrt{2})^4 + 104$

$e^{pv6} \sim (4\sqrt{3})^4 - 106$

$e^{pv10} \sim 12^4 - 104$

$e^{pv22} \sim (12\sqrt{11})^4 - 104$

$e^{pv58} \sim 396^4 - 104$

<u>Class Number 4</u>

(Unknown for e^{pv17}, e^{pv73}, e^{pv97}, e^{pv193}.)

$e^{pv14} \sim 4^4(11+8v2)^2 - 104$

$e^{pv34} \sim 12^4(4+v17)^4 - 104$

$e^{pv46} \sim 12^4(147+104v2)^2 - 104$

$e^{pv82} \sim 12^4(51+8v41)^4 - 104$

$e^{pv142} \sim 12^4(467539+330600v2)^2 - 104$

$e^{pv30} \sim (4v3)^4(5+4v2)^4 - 104$

$e^{pv42} \sim 4^4(21+8v6)^4 - 104$

$e^{pv78} \sim (4v3)^4(75+52v2)^4 - 104$

$e^{pv102} \sim (4v3)^4(200+49v17)^4 - 104$

$e^{pv70} \sim (12v7)^4(5v5+8v2)^4 - 104$

$e^{pv130} \sim 12^4(323+40v65)^4 - 104$

$e^{pv190} \sim (12v19)^4(481+340v2)^4 - 104$

<u>Class Number 6</u>

The *r(q)* for class number 6 was found by myself, using an assumption and again the Integer Relations applet. I observed that, in the *r(q)* for class number 2 for what we defined as group 2 (namely *d = 6, 10, 22, 58*) for *d = 6 & 22*, *r(q)* was a quadratic irrational, while for *d = 10 & 58*, *r(q)* was an integer. The difference was that *d* of the former was of the form *2(4m-1)* while for the latter was *2(4m+1)*. Since we already know that the degree k of the $r(q)$ can be dependent on the nature of d, might it be the case that for $d = 2(4m+1)$ of class number n, then $e^{pvd} \sim x^4 - 104$, where x is a root of an equation of degree $n/2$?

It seems it was the case. A check to the validity of the four cubics below can be made considering the polynomial discriminants are given by $3d$. It is hoped that an interested reader can provide the missing polynomials for class number 8 and above.

$e^{pv26} \sim (4x)^4 - 104$; $(x^3-13x^2-9x-11=0)$

$e^{pv106} \sim (12x)^4 - 104$; $(x^3-271x^2+63x-49=0)$

$e^{pv202} \sim (12x)^4 - 104$; $(x^3-5871x^2+2815x-913=0)$

$e^{pv298} \sim (12x)^4 - 104$; $(x^3-64419x^2-16061x-1441=0)$

Class Number 8

$e^{pv178} \sim (12x)^4 - 104$; x = ?

$e^{pv226} \sim (12x)^4 - 104$; x = ?

$e^{pv466} \sim (12x)^4 - 104$; x = ?

$e^{pv562} \sim (12x)^4 - 104$; x = ?

I am aware of $r(q)$ only for class numbers 2, 4, 6 so far, or only for even n. Ramanujan nor others does not seem to have worked on class polynomials defining $r(q)$ for odd n. It should be interesting to know if indeed there are such polynomials.

V. Pi Formulas

We can use our modular functions $j(q)$, $r(q)$, and perhaps also $w(q)$ to come up with formulas for pi, or more accurately $1/\pi$. We have the following infinite series due to the Chudnovsky brothers (where the summation Σ is understood to go from n = 0 to infinity),

Let, $c = (-1)^n(6n)!/((n!)^3(3n)!)$

$1/(4\pi) = \Sigma\ c\ (154n+15)/(32^3)^{n+1/2}$

$1/(12\pi) = \Sigma\ c\ (342n+25)/(96^3)^{n+1/2}$

$1/(12\pi) = \Sigma\ c\ (16254n+789)/(960^3)^{n+1/2}$

$1/(12\pi) = \Sigma\ c\ (261702n+10177)/(5280^3)^{n+1/2}$

$1/(12\pi) = \Sigma\ c\ (545140134n+13591409)/(640320^3)^{n+1/2}$

which uses the *j(q)* of *d* = 11, 19, 43, 67, 163 of class number 1. The "signature" of the *d* can be found in the formula, other than the *j(q)* in the denominator. Consider the factorizations,

$154 = 2*7*11$

$342 = 2*3^2 *19$

$16254 = 2*3^3 *7*43$

$261702 = 2*3^2 *7*31*67$

$545140134 = 2*3^2 *7*11*19*127*163$

The general form of the formula seems to be:

$$1/(12\pi) = \Sigma\ c\ (An+B)/(C)^{n+1/2}$$

where A, B, C are algebraic numbers of degree *k*. Thus, one can also use the *j(q)* of the *d* of class number 2 and so on.

The inspiration for the formulas derived by the Chudnovskys was a set of beautiful formulas for $1/\pi$ (17 in all) found by Ramanujan and listed down in his notebooks with little explanation on how he came up with them. Most of them involve *d* of class number 2. What I'm interested are the two formulas:

$$1/(\pi\sqrt{8}) = 1/3^2\ \Sigma\ r\ (10n+1)/12^{4n}$$

$$1/(\pi\sqrt{8}) = 1/99^2\ \Sigma\ r\ (26390n+1103)/396^{4n}$$

where, $r = (4n)!/(n!^4)$. To recall,

$$e^{pv10\sim}\ 12^4 - 104$$

$$e^{pv58\sim}\ 396^4 - 104$$

The two formulas use the *r(q)* of the above. (Note that 10 = 2*5 and 26390 = 2*5*7*13*29.) I believe the general form is,

$$1/(\pi\sqrt{8}) = 1/D\ \Sigma\ r\ (An+B)/C^{4n}$$

where A, B, C, D are algebraic integers of degree *k* and C^4 is the *r(q)* of a *d* of class number *2k*. Thus, it will be restricted to *even* class numbers. It may then be a slightly different general form to the one found by the Borweins, though I'm not sure if they will turn out to be essentially the same. The next candidates will be *d* = 34, 82 with class number 4 and *r(q)* of algebraic degree 2,

$$e^{pv34\sim}\ 12^4(4+v17)^4 - 104$$

$e^{pv82} \sim 12^4(51+8v41)^4 - 104$

and which should have A, B, D also as algebraic numbers of degree 2 if my assumption is correct. I am not aware of pi formulas that use the Weber function *w(q)* though I believe one can perhaps find general forms in analogy with what was done for *j(q)* and *r(q)*.

VI. Monster Group and Conclusion

Before we go to a fascinating connection to group theory and conclude our paper, we can make a small clarification regarding the constant $e^{\pi\sqrt{163}}$. Ramanujan worked mostly on *d* with class number a power of two, and while $e^{\pi\sqrt{58}}$ is found in his notebooks, $e^{\pi\sqrt{163}}$ is not. Hermite was aware of it as being an almost-integer c. 1859.

The name is taken from an April Fool's joke by Martin Gardner where he claimed Ramanujan had conjectured that it was exactly an integer. In fact, numbers of the form $e^{\pi\sqrt{d}}$ for positive integer *d* are transcendental, as proven by Aleksandr Gelfond. However, this interesting property of $e^{\pi\sqrt{163}}$ seems to be in line with the body of Ramanujan's work which itself is most interesting, especially keeping in mind the conditions in which it was made. So it turns out the name for this constant is fitting indeed.

We pointed out earlier that there is a connection between the modular functions we have mentioned and what is called the *Monster group*. To recall, the series expansion of the j-function was,

$$j(q) = 1/q + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots$$

Now, the Monster group *M*, the largest of the 26 sporadic groups, is the group of rotations in 196883-dimensional space. Its irreducible representations are given by 1, 196883, 21296876…etc. It was noticed by John McKay in the late 70's that 196883 was awfully close to the coefficient 196884 of the j-function above. When John Conway was told by J.G. Thompson about this observation, he thought that it was "*moonshine*", or fanciful. However, when you realize that,

$$196884 = 1 + 196883$$
$$21493760 = 1 + 196883 + 21296876$$

and so on, or the coefficients of the j-function seemed to be simple linear combinations of the representations of the Monster, then something *really* interesting must be going on. The assumed relationship between the j-function and the Monster was known as the *Monstrous Moonshine Conjecture*, after a paper written by Conway and S. Norton in 1979 and was finally proven to be true by Richard Borcherds in 1992. And as if that amazing relationship was not enough, the proof used a theorem from *string theory*! Borcherds eventually won the Fields medal for proving this conjecture.

Thus, the coefficients of the j-function are also known as the McKay-Thompson series of class 1A for Monster. And what about our other modular function *w(q)*?

$$w(q) = 1/q + 24 + 276q + 2048q^2 + 11202q^3 + \dots$$

The coefficients of the w-function happen to be also connected to the Monster and is known as the McKay-Thompson series of class 2B for Monster. For *r(q)*?

$$r(q) = 1/q + 104 + 4372q + 96256q^2 + 1240002q^3 + \ldots$$

The list of coefficients is also known as the McKay-Thompson series of class 2A for Monster.

And so we have this profound connection between two seemingly different mathematical topics. Ramanujan would have loved this.

--End--

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