# Principles of 2-Sliding Mode Design [○]

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**Abstract**

Second order sliding modes are used to keep exactly a constraint of the second relative degree or just to avoid chattering, i.e. in the cases when the standard (first order) sliding mode implementation might be involved or impossible. Design of a number of new 2-sliding controllers is demonstrated by means of the proposed homogeneity-based approach. A recently developed robust exact differentiator being applied, robust output-feedback controllers with finite-time convergence are produced, capable to control any general uncertain single-input-single-output process with relative degree 2. An effective simple procedure is developed to attenuate the 1-sliding mode chattering. Simulation of new controllers is presented.

*Key words*: Second-order sliding mode; Robustness; Homogeneity; Chattering; Output feedback.

## 1. Introduction

Sliding mode control (Utkin, 1992; Zinober, 1994; Edwards and Spurgeon, 1998) is considered to be one of the main methods effective under uncertainty conditions. The approach is based on keeping exactly a properly chosen constraint by means of high-frequency control switching, and is known as robust and very accurate. Unfortunately, its standard application is restricted: if an output $\sigma$ is to be zeroed, the standard sliding mode may keep $\sigma = 0$ only if the output's relative degree is 1 (i.e. if the control appears explicitly already in $\dot{\sigma}$). High frequency control switching may also cause the dangerous chattering effect (Fridman, 2001, 2003; Lee & Utkin, 2006; Boiko, Fridman & Castellanos, 2004; Boiko, 2005).

Consider a smooth dynamic system with a smooth output function $\sigma$, and let the system be closed by some possibly-dynamical discontinuous feedback. Then, provided the successive total time derivatives $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$ are continuous functions of the closed-system state-space variables, and the set $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$ is non-empty and consists locally of Filippov trajectories (Filippov, 1988), the motion on the set $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$ is said to exist in *r-sliding mode* (*rth order sliding mode*) (Levant, 1993, 2003). The *r*th derivative $\sigma^{(r)}$ is mostly supposed to be discontinuous or non-existent.

The standard sliding mode is of the first order ($\dot{\sigma}$ is discontinuous). Higher-order sliding modes (HOSM) remove the above-mentioned relative-degree restriction and, being properly used, can practically eliminate the chattering. Asymptotically stable HOSMs appear in many systems with traditional sliding-mode control (Fridman, 2001, 2003) and are deliberately introduced in systems with dynamical sliding modes (Sira-Ramírez, 1993). Finite-time convergent HOSMs preserve the features of the standard (first order) sliding modes and improve their accuracy in the presence of switching delays and discrete measurements (Levant, 1993).

While finite-time-convergent arbitrary-order sliding-mode controllers are mostly still theoretically studied (Levant, 2001, 2003; Floquet, Barbot and Perruquetti, 2003), 2-sliding controllers with finite-time convergence have already been successfully implemented for solution of real problems (Bartolini, Ferrara & Punta, 2000; Bartolini, Pisano, Punta & Usai, 2003; Levant, Pridor, Gitizadeh, Yaesh & Ben-Asher, 2000; Sira-Ramírez, 2002; Orlov, Aguilar & Cadiou, 2003; Khan, Goh & Spurgeon, 2003; Shkolnikov & Shtessel, 2002; Shkolnikov, Shtessel, Lianos, & Thies, 2000; Shtessel & Shkolnikov, 2003; Shtessel, Shkolnikov & Brown, 2004). There are only few widely used 2-sliding controllers: the sub-optimal controller (Bartolini et al., 1998), the terminal sliding mode controllers (Man, Paplinski & Wu, 1994), and the twisting controller (Levant 1993). 2-3 more controllers are presented in (Levant, 1993, Khan et al., 2003), in particular, the super-twisting controller is used in differentiators (Levant, 1998, 2003).

Generally speaking, 2-sliding controllers are used to keep at zero outputs of the relative degree 2 or to avoid chattering while zeroing outputs of the relative degree 1. The main difficulty of their implementation is the necessity to use the first time derivative of the output, which causes possible sensitivity of the approach to sampling noises. These three subjects are studied in the present paper, and some standard solutions are proposed.

1. The homogeneity-based approach (Levant, 2005a) is proposed in this paper to regularize the construction of new finite-time convergent 2-sliding controllers featuring the highest possible accuracy of 2-sliding control (Levant, 1993). The simplicity of such controller design is demon-

strated, and a few new controllers are developed, some of which feature new advantageous properties. In particular, the so-called quasi-continuous controllers have better performance, being continuous everywhere except the 2-sliding set $\sigma = \dot{\sigma} = 0$. The main features of known 2-sliding controllers (Levant, 1993; Bartolini et al., 98, 2003) are proved to be valid for general finite-time-convergent homogeneous 2-sliding controllers.

2. A number of approaches were developed to avoid the dangerous chattering often featuring standard (1-sliding) modes. In particular, high-gain control with saturation is used to approximate the sign-function in a boundary layer around the switching manifold (Slotine & Li, 1991), the sliding-sector method (Furuta & Pan, 2000) is suitable to control disturbed linear time-invariant systems.

Another standard way is to avoid chattering by means of 2-sliding mode control, treating the input $u$ of the system as a new state variable, while using its time derivative $\dot{u}$ as the actual control ((Emelyanov, Korovin & Levant, 1993; Levant, 1993; Bartolini et al., 1998). In that case $\dot{u}$ has to dominate in the equation for $\ddot{\sigma}$. Regretfully, in general, the expression for $\ddot{\sigma}$ contains terms with $u$. Thus, $\dot{u}$ is to dominate over $u$ itself, which looks problematic. Fortunately, in the vicinity of the 2-sliding mode $u$ is close to the so-called equivalent control $u_{\text{eq}}(t, x)$ (Utkin, 1992), where $t$, $x$ denote the time and the state variables. The function $u_{\text{eq}}$ is defined from the equation $\dot{\sigma} = 0$ and is independent of $\dot{u}$. Thus, the approach is always valid in some vicinity of the 2-sliding set $\sigma = \dot{\sigma} = 0$, or, in other words, in a vicinity of the set $\sigma = 0$, $u = u_{\text{eq}}(t, x)$. Since $u_{\text{eq}}$ is typically unknown, the system stability is nevertheless difficult to guarantee. A simple effective procedure previously established only for the twisting controller is proved in this paper to resolve the problem by means of all known and newly-developed 2-sliding controllers.

3. In practice the sampled values of the output $\sigma$ are corrupted by noises. The controller robustness with respect to sampling noises is especially important, since most of 2-sliding controllers explicitly use possibly unavailable $\dot{\sigma}$ or sign $\dot{\sigma}$. Historically the first way to get the lacking information was to use finite differences instead of the derivative. That approach is simple and valid for all 2-sliding homogeneous controllers, but, unfortunately, it is sensitive to large noises or small sampling intervals. Following Levant (2005a), this paper suggests to use a recently developed robust exact differentiator (Levant, 1993, 2003) as a standard part of all 2-sliding homogeneous controllers, producing robust output-feedback control. The resulting output-feedback controllers preserve the ultimate accuracy and finite-time convergence of the original controllers and do not require any information on the noises. The corresponding asymptotic accuracies are estimated.

All contemporary applications are computer-based and use discrete-time sampling. The corresponding general-case asymptotic accuracies are also calculated.

The results of this paper were partially presented at a conference (Levant, 2002), and can be also considered as a demonstration of the general homogeneity-based HOSM controller design (Levant, 2005a) in the simplest case of the sliding order 2. Simulation demonstrates the feasibility of new controllers.

## 2. The problem statement

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t, x), \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}$ is control, $\sigma$ is the only measured output, smooth functions $a$, $b$, $\sigma$ and the dimension $n$ are unknown. The relative degree of the system (Isidori, 1989) is assumed to be 2. The task is to make the output $\sigma$ vanish in finite time and to keep $\sigma \equiv 0$ by means of a discontinuous globally-bounded feedback control. System trajectories are supposed to be infinitely extendible in time for any bounded input. The system is understood in the Filippov sense.

Calculating the second total time derivative $\ddot{\sigma}$ along the trajectories of (1) achieve that under these conditions

$$\ddot{\sigma} = h(t,x) + g(t,x)u, \quad h = \ddot{\sigma}\big|_{u=0}, \quad g = \tfrac{\partial}{\partial u}\ddot{\sigma} \neq 0 \tag{2}$$

where the functions $g$, $h$ are some unknown smooth functions. Suppose that the input-output termed conditions

$$0 < K_{\text{m}} \le \tfrac{\partial}{\partial u}\ddot{\sigma} \le K_{\text{M}}, \qquad |\ddot{\sigma}\big|_{u=0}| \le C. \tag{3}$$

hold globally for some $K_{\text{m}}$, $K_{\text{M}}$, $C > 0$. Note that at least locally (3) is satisfied for any smooth system (1) with the well-defined relative degree 2.

Obviously, no continuous feedback controller can solve the stated problem. Indeed, any continuous control $u = \varphi(\sigma, \dot{\sigma})$ providing for $\sigma \equiv 0$, has to satisfy the equality $\varphi(0, 0) = - h(t, x)/g(t, x)$, whenever $\sigma = \dot{\sigma} = 0$ holds. The problem uncertainty prevents it, for the controller will not be effective for the simple autonomous linear system $\ddot{\sigma} = c + ku$, $K_{\text{m}} \le k \le K_{\text{M}}$, $|c| \le C$, with $\varphi(0,0) \neq - c/k$. In other words, the 2-sliding mode $\sigma = 0$ is to be established.

Assume now that (3) holds globally. Then (2), (3) imply the differential inclusion

$$\ddot{\sigma} \in [-C, C] + [K_{\text{m}}, K_{\text{M}}]u. \tag{4}$$

Most 2-sliding controllers may be considered as controllers for (4) steering $\sigma$, $\dot{\sigma}$ to 0 in (preferably) finite time. Since inclusion (4) does not "remember" the original system (1), such controllers are obviously robust with respect to any perturbations preserving (3).

Hence, the problem is to find such a feedback

$$u = \varphi(\sigma, \dot{\sigma}), \tag{5}$$

that all the trajectories of (4), (5) converge in finite time to the origin $\sigma = \dot{\sigma} = 0$ of the phase plane $\sigma$, $\dot{\sigma}$.

Differential inclusion (4), (5) is understood here in the Filippov sense (Filippov, 1988), which means that the right-hand vector set is enlarged in a special way in order to satisfy certain convexity and semi-continuity conditions (Levant, 2005a). In particular, in the case when $\varphi$ is continuous almost everywhere, a set $\Phi(\sigma, \dot{\sigma})$ is substituted for $u$ in (4), $\Phi$ being the convex closure of all possible limit values of $\varphi(\sigma_1, \dot{\sigma}_1)$ obtained when the continuity point

$(\sigma_1, \dot\sigma_1)$ approaches $(\sigma, \dot\sigma)$. The function $\varphi$ is assumed to be a locally bounded Borel-measurable function (actually all functions used in sliding-mode control satisfy that restriction). Solution is any absolutely continuous vector function $(\sigma(t), \dot\sigma(t))$ satisfying (4), (5) for almost all $t$.

## 3. 2-sliding homogeneity and finite-time stability

The general notion of homogeneous differential equation, review of the corresponding theoretical results and numerous references can be found in (Bacciotti & Rosier, 2001). Introduce a few auxiliary notions and theorems adopted from (Levant, 2005a), where they are formulated for general homogeneous Filippov differential inclusions with negative homogeneity degrees. The combined time-coordinate transformation

$$G_\kappa: \quad (t, \sigma, \dot\sigma) \mapsto (\kappa t, \kappa^2\sigma, \kappa\dot\sigma) \qquad (6)$$

transfers solutions of (4), (5) into the solutions of the transformed inclusion

$$\ddot\sigma \in [-C, C] + [K_m, K_M]\, \varphi(\kappa^2\sigma, \kappa\dot\sigma).$$

Inclusion (4), (5) and controller (5) itself are called 2-*sliding homogeneous* (Levant, 2005a) if these two differential inclusions are equivalent for any $\sigma$, $\dot\sigma$ and $\kappa > 0$ (i.e. have the same solutions). The mapping $d_\kappa(\sigma, \dot\sigma) = (\kappa^2\sigma, \kappa\dot\sigma)$, $\kappa > 0$ is called the homogeneity dilation (Bacciotti et al., 2001). Obviously, (5) is 2-sliding homogeneous if

$$\varphi(\kappa^2\sigma, \kappa\dot\sigma) \equiv \varphi(\sigma, \dot\sigma) \qquad (7)$$

holds for any $\kappa > 0$. Such a function $\varphi$ is called 2-sliding homogeneous with the homogeneity degree 0. For example, the following controllers are 2-sliding homogeneous:

$$u = -\operatorname{sign}\sigma = -\operatorname{sign}\kappa^2\sigma,$$
$$u = (2\sigma - \dot\sigma^2)/(|\sigma| + \dot\sigma^2) = (2\kappa^2\sigma - (\kappa\dot\sigma)^2)/(|\kappa^2\sigma| + (\kappa\dot\sigma)^2).$$

Surely these controllers do not solve the stated problem.

Since the Filippov solutions do not depend on the values of $\varphi$ on any set of the zero measure, any changes of $\varphi$ on such a set do not change the homogeneity properties of the controller. It is assumed in this paper that (7) holds for any $\sigma$, $\dot\sigma$ and $\kappa > 0$. Note that (7) requires global boundedness of $\varphi$ (excluding possibly a zero-measure set), otherwise it is unbounded in any vicinity of 0 and Filippov's definition is not applicable to (1), (5).

1°. Differential inclusion (4), (5) is called further *globally uniformly finite-time stable at* 0 if it is Lyapunov stable and for any $R > 0$ exists $T > 0$ such that any trajectory starting within the disk $\|(\sigma, \dot\sigma)\| < R$ stabilizes at zero in the time $T$.

2°. Differential inclusion (4), (5) is called further *globally uniformly asymptotically stable* at 0, if it is Lyapunov stable and for any $R > 0$ and $\varepsilon > 0$ exists $T > 0$, such that any trajectory starting within the disk $\|(\sigma, \dot\sigma)\| < R$ enters the disk $\|(\sigma, \dot\sigma)\| < \varepsilon$ in the time $T$ to stay there forever.

A set $D$ is called *dilation-retractable* if $0 \in D$, and $d_\kappa D \subset D$ for any $\kappa < 1$. For example any disk centered at the origin is dilation-retractable.

3°. The homogeneous differential inclusion (4), (5) is called further *contractive* if there are 2 compact sets $D_1$, $D_2$ and $T > 0$ such that $D_2$ lies in the interior of $D_1$ and contains the origin; $D_1$ is dilation-retractable; and all trajectories starting at the time 0 within $D_1$ are localized in $D_2$ at the time moment $T$.

Most of known 2-sliding controllers (Levant, 1993, Bartolini et al., 2003) satisfy these properties. The following Theorem is actually true for any homogeneous differential inclusion of a negative homogeneity degree (Levant, 2005a).

**Theorem 1**. *Let controller* (5) *be* 2-*sliding homogeneous, then properties* 1°, 2° *and* 3° *are equivalent*.

Explain the Theorem in few words. Since 1° implies 2° and 3°, it is sufficient to show that 3° implies 1°. The homogeneity of the system (4), (5) means that it is invariant with respect to the transformation (6). Thus, the behavior of the system trajectories is geometrically the same in all points, which can be transferred one into another by means of a simple linear transformation $d_\kappa(\sigma, \dot\sigma) = (\kappa^2\sigma, \kappa\dot\sigma)$, $\kappa > 0$ (the dilation). The only difference is that the corresponding motion near the origin requires proportionally less time, according to (6). Hence, using the contractivity property 3°, a chain of embedded domains can be constructed, retracting to the origin. The corresponding system motion is actually shown to be a finite-time collapse towards the origin.

It is natural to call the controller $u = \phi(\sigma, \dot\sigma)$ a small homogeneous perturbation of (5) if the difference $\phi - \varphi$ is a 2-sliding homogeneous function with the homogeneity degree 0, small in some fixed vicinity of the origin.

**Corollary 1.** *Global uniform finite-time stability of the* 2-*sliding homogeneous controller* (5) *is robust with respect to small homogeneous perturbations of the controller*.

Indeed, it follows from the robustness of property 3°. The following theorems consider the robustness of homogeneous controllers with respect to sampling errors, which are supposed to be some bounded Lebesgue-measurable functions of time *of any nature*. No features of the noises are assumed to be known.

**Theorem 2** (Levant, 2005a). *Let the noise magnitudes of measurements of* $\sigma$, $\dot\sigma$ *be less than* $\beta_0\delta$ *and* $\beta_1\delta^{1/2}$ *respectively with some positive constants* $\beta_0$ *and* $\beta_1$. *Then any finite-time-stable 2-sliding-homogeneous controller* (5) *provides in finite time for keeping the inequalities* $|\sigma| < \gamma_0\delta$, $|\dot\sigma| < \gamma_1\delta^{1/2}$ *with the same positive constants* $\gamma_0$, $\gamma_1$ *for any* $\delta>0$.

Controller (5) requires availability of $\dot\sigma$. That information can be obtained by means of the real-time robust finite-time-convergent exact differentiator (Levant, 1998, 2003) as follows:

$$u = -\varphi(z_0, z_1), \qquad (8)$$
$$\dot z_0 = -\lambda_2 L^{1/2} |z_0 - \sigma|^{1/2} \operatorname{sign}(z_0 - \sigma) + z_1, \qquad (9)$$
$$\dot z_1 = -\lambda_1 L \operatorname{sign}(z_0 - \sigma). \qquad (10)$$

Here $z_0$, $z_1$ are the estimations of $\sigma$ and $\dot\sigma$ respectively. The parameters of the differentiator $\lambda_i$ are to be chosen in advance, in particular $\lambda_1 = 1.1$, $\lambda_2 = 1.5$ is a good choice (Levant, 1998). $L$ is the only differentiator parameter to be tuned, and it has to satisfy the only condition $|\ddot\sigma| \le L$, i.e. $L \ge C + K_M \sup|\varphi|$ (recall that $\varphi$ is bounded).

**Theorem 3** (Levant, 2005a). *Suppose that controller* (5) *is 2-sliding homogeneous and finite-time stable, then the output-feedback controller* (8) - (10) *provides in finite time for keeping* $\sigma = \dot\sigma = 0$. *If* $\sigma$ *is sampled with a constant sampling interval* $\tau > 0$, *the inequalities* $|\sigma| < \gamma_0\tau^2$, $|\dot\sigma| < \gamma_1\tau$ *are established with some positive constants* $\gamma_0$, $\gamma_1$. *If* $\sigma$ *is sampled (continuously) with a noise being a Lebesgue-measurable function of time of the magnitude* $\varepsilon > 0$, *the inequalities* $|\sigma| < \mu_0\varepsilon$, $|\dot\sigma| < \mu_1\varepsilon^{1/2}$ *are established with some positive constants* $\mu_0$, $\mu_1$.

There is another way to estimate $\dot\sigma$ with discrete sampling. Indeed, let

$$u = \varphi(\sigma, \Delta\sigma_i/\tau) = \varphi(\tau^2\sigma, \Delta\sigma_i), \tag{11}$$

where $\sigma_i = \sigma(t_i, x(t_i))$, $\Delta\sigma_i = \sigma_i - \sigma_{i-1}$, $t \in [t_i, t_{i+1})$.

**Theorem 4** (Levant, 2005a). *Let controller* (5) *be 2-sliding homogeneous and finite-time stable, then in the absence of measurement noises controller* (11) *provides in finite time for keeping the inequalities* $|\sigma| < \gamma_0\tau^2$, $|\dot\sigma| < \gamma_1\tau$ *with some positive constants* $\gamma_0$, $\gamma_1$.

Note that 1-sliding mode provides only for the accuracy proportional to $\tau$. The accuracy described in Theorems 3, 4 is the best possible with discontinuous control and the relative degree 2 (Levant, 1993).

Due to the finite-time convergence of the controllers, Theorems 2 - 4 have obvious local analogues in the case when (3) holds only locally. Recall that 2-sliding point is a point where $\sigma = \dot\sigma = 0$. Then in the absence of noises all trajectories of (1), (5) (respectively (1), (8) – (10) or (1), (11)) starting from some vicinity of a 2-sliding point with well-defined relative degree 2 converge in finite time to the 2-sliding mode $\sigma \equiv 0$, or the corresponding inequalities are established in the case of noisy measurements or discrete sampling. The long-term motion is determined by the system properties, especially by its zero dynamics (Isidori, 1989). Note that in the case, when $\frac{\partial}{\partial u}\ddot\sigma$ is negative the same controller (5) is to be used, but with the opposite sign.

The popular sub-optimal controller (Bartolini et al., 1998, 2003) is defined by the formula

$$u = -r_1 \operatorname{sign}(\sigma - \sigma^*/2) + r_2 \operatorname{sign}\sigma^*, \quad r_1 > r_2 > 0,$$

where $\sigma^*$ is the value of $\sigma$ detected at the closest time in the past when $\dot\sigma$ was 0. The initial value of $\sigma^*$ is 0. The corresponding convergence conditions are

$$2[(r_1 + r_2)K_m - C] > (r_1 - r_2)K_M + C, \quad (r_1 - r_2)K_m > C.$$

Usually the moments when $\dot\sigma$ changes its sign are detected using finite differences. The control $u$ depends actually on the whole history of measurements of $\dot\sigma$ and $\sigma$, and does not have the feedback form (5). Nevertheless, the homogeneity transformation (6) preserves its trajectories, and it is natural to call it 2-sliding homogeneous in the broad sense. Also the statements of Theorems 2 - 4 remain valid for this controller.

Results similar to Theorems 1 - 4 are formulated and proved in (Levant, 2005a) for any relative degree $r$ and $r$-sliding homogeneous controllers. At the same time design of new $r$-sliding controllers is rather difficult with $r > 2$ due to the complicated geometry of $\mathbf{R}^r$ (Levant, 2003, 2005b, Floquet et al., 2003).

## 4. Design of 2-sliding controllers

As follows from the previous Section it is sufficient to build a 2-sliding-homogeneous contractive controller. Design of such 2-sliding controllers is greatly facilitated by the simple geometry of the 2-dimensional phase plane with coordinates $\sigma$, $\dot\sigma$: any smooth curve locally divides the plane in two parts.

A number of known 2-sliding controllers may be considered as particular cases of a generalized 2-sliding homogeneous controller

$$\begin{aligned} u = &-r_1 \operatorname{sign}(\mu_1\dot\sigma + \lambda_1|\sigma|^{1/2}\operatorname{sign}\sigma) \\ &- r_2 \operatorname{sign}(\mu_2\dot\sigma + \lambda_2|\sigma|^{1/2}\operatorname{sign}\sigma), \quad r_1, r_2 > 0. \end{aligned} \tag{12}$$

Drawing the two lines $\mu_i\dot\sigma + \lambda_i|\sigma|^{1/2}\operatorname{sign}\sigma = 0$, $\mu_i^2 + \lambda_i^2 > 0$, $\mu_i, \lambda_i \ge 0$, $i = 1, 2$, $\mu_1^2 + \mu_2^2 > 0$, $\lambda_1^2 + \lambda_2^2 > 0$, in the phase plane, and considering various possible cases, one can readily check that $r_1$, $r_2$ can always be chosen so that controller (12) be finite-time stable. Indeed, if, for example, $\mu_1, \lambda_1 > 0$, then a 1-sliding mode can easily be organized on the line $\mu_1\dot\sigma + \lambda_1|\sigma|^{1/2}\operatorname{sign}\sigma = 0$. Sliding mode conditions being fulfilled at one point of the line, they automatically hold along the whole line due to the homogeneity properties. If for each $i$ one of the coefficients is zero, the twisting controller (Levant, 1993)

$$u = -(r_1 \operatorname{sign}\sigma + r_2 \operatorname{sign}\dot\sigma),$$

is built. Its convergence condition is

$$(r_1 + r_2)K_m - C > (r_1 - r_2)K_M + C, \quad (r_1 - r_2)K_m > C.$$

A typical trajectory in the plane $\sigma$, $\dot\sigma$ is shown in Fig. 1a. Controller (12) may be considered as a generalization of the twisting controller, when the switching takes place on parabolas $\mu_1\dot\sigma + \lambda_1|\sigma|^{1/2}\operatorname{sign}\sigma = 0$ instead of the coordinate axes.

Thus, the resulting controller satisfies Theorem 1, and its discrete-sampling version

$$\begin{aligned} u = &-r_1 \operatorname{sign}(\mu_1\Delta\sigma_i + \lambda_1\tau|\sigma_i|^{1/2}\operatorname{sign}\sigma_i) \\ &- r_2 \operatorname{sign}(\mu_2\Delta\sigma_i + \lambda_2\tau|\sigma_i|^{1/2}\operatorname{sign}\sigma_i). \end{aligned} \tag{13}$$

provides for the accuracy described in Theorem 4, i.e. $\sigma \sim \tau^2$, $\dot\sigma \sim \tau$. Similarly, the noisy measurements lead to the accuracy provided by Theorems 2, 3.

Consider special interesting cases of controller (12), (13). With $\mu_1 = \lambda_2 = 0$, $r_2 > r_1 > 0$, achieve the drift controller (Levant, 1993) from (13) (it does not converge with continuous measurements). A homogeneous form of the controller with prescribed convergence law (Levant 1993) arises when $\mu_1 = \mu_2$, $\lambda_1 = \lambda_2$:

$$u = -\alpha\,\mathrm{sign}(\dot{\sigma} + \beta|\sigma|^{1/2}\mathrm{sign}\,\sigma), \qquad (14)$$

where $\alpha = r_1 + r_2$, $\beta = \lambda_1/\mu_1$. This controller is a 2-sliding homogeneous analogue of the terminal sliding mode controller originally featuring a singularity at $\sigma = 0$ (Man et al., 1994). The complete elementary proof of the following simple proposition was actually never published.

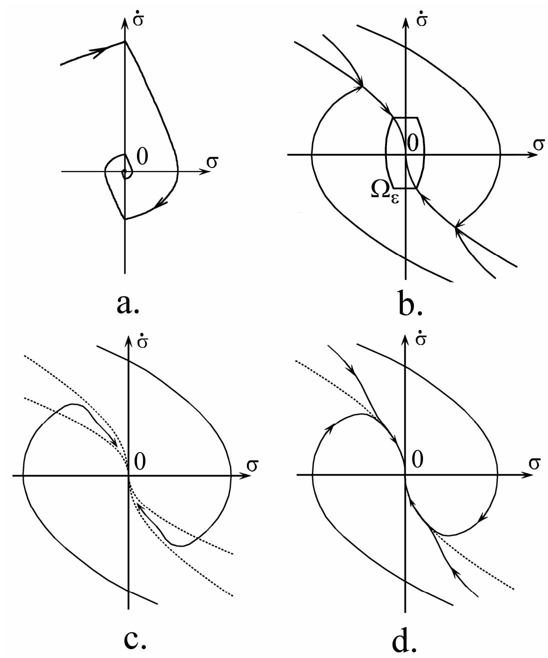


**Fig. 1. Convergence of various 2-sliding homogeneous controllers**

**Proposition 1.** *With* $\alpha K_m - C > \beta^2/2$ *controller* (14) *provides for the establishment of the finite-time stable* 2-*sliding mode* $\sigma \equiv 0$.

**Proof.** Differentiating the function $\Sigma = \dot{\sigma} + \beta|\sigma|^{1/2}\mathrm{sign}\,\sigma$ along the trajectory, obtain

$$\dot{\Sigma} \in [-C, C] - \alpha\,[K_m, K_M]\,\mathrm{sign}\,\Sigma + \frac{1}{2}\beta\,\dot{\sigma}\,|\sigma|^{-1/2}.$$

Checking the condition $\dot{\Sigma}\,\mathrm{sign}\,\Sigma < \mathrm{const} < 0$ in a vicinity of each point on the curve $\Sigma = 0$, obtain using $\dot{\sigma} = -\beta|\sigma|^{1/2}\mathrm{sign}\,\sigma$ that the 1-sliding-mode existence condition holds at each point except of the origin, if $\alpha K_m - C > \beta^2/2$.

The trajectories of the inclusion inevitably hit the curve $\Sigma = 0$ due to geometrical reasons. Indeed, each trajectory, starting with $\Sigma > 0$, terminates sooner or later at the semi-axis $\sigma = 0$, $\dot{\sigma} < 0$, if $u = -\alpha\,\mathrm{sign}\,\Sigma$ keeps its constant value $-\alpha$ (Fig. 1b). Thus, on the way it inevitably hits the curve $\Sigma = 0$. The same is true for the trajectory starting with $\Sigma < 0$. Since that moment the trajectory slides along the curve $\Sigma = 0$ towards the origin and reaches it in finite time. Obviously, each trajectory starting from a disk centered at the origin comes to the origin in a finite time, the convergence time being uniformly bounded in the disk.

Consider the region $\Omega_\varepsilon$ confined by the lines $\dot{\sigma} = \pm\varepsilon$ and the trajectories of the differential equations $\ddot{\sigma} = -C + K_m\alpha$ with initial conditions $\sigma = \varepsilon^2/\beta^2$, $\dot{\sigma} = -\varepsilon$, and $\ddot{\sigma} = C - K_m\alpha$ with initial conditions $\sigma = -\varepsilon^2/\beta^2$, $\dot{\sigma} = \varepsilon$ (Fig. 1b). No trajectory starting from the origin can leave $\Omega_\varepsilon$. Since $\varepsilon$ can be taken arbitrarily small, the trajectory cannot leave the origin. The same reasoning proves the Lyapunov stability of the origin. ■

The 2-sliding stability analysis is greatly simplified by the fact that all the trajectories in the plane $\sigma$, $\dot{\sigma}$ which pass through a given continuity point of $u = \varphi(\sigma, \dot{\sigma})$ are confined between the properly chosen trajectories of the homogeneous differential equations $\ddot{\sigma} = \pm C + K_M\varphi(\sigma, \dot{\sigma})$ and $\ddot{\sigma} = \pm C + K_m\varphi(\sigma, \dot{\sigma})$. These border trajectories cannot be crossed by other paths, if $\varphi$ is locally Lipschitzian, and may be often chosen as boundaries of appropriate dilation-retractable regions. The rule is to take trajectories satisfying $\ddot{\sigma} = C + K_M\varphi(\sigma, \dot{\sigma})$ and $\ddot{\sigma} = -C + K_m\varphi(\sigma, \dot{\sigma})$ with $\varphi(\sigma, \dot{\sigma}) > 0$, and $\ddot{\sigma} = -C + K_M\varphi(\sigma, \dot{\sigma})$ and $\ddot{\sigma} = C + K_m\varphi(\sigma, \dot{\sigma})$ when $\varphi(\sigma, \dot{\sigma}) < 0$. Recall that a region is dilation-retractable iff, with each its point $(\sigma, \dot{\sigma})$, it contains all the points of the parabolic segment $(\kappa^2\sigma, \kappa\dot{\sigma})$, $0 \le \kappa \le 1$.

As follows from Corollary 1 a small violation of the conditions of Proposition 1 preserves the finite-time stability of controller (14), if $\alpha K_m - C < \beta^2/2$, but still $\alpha K_m > C$.

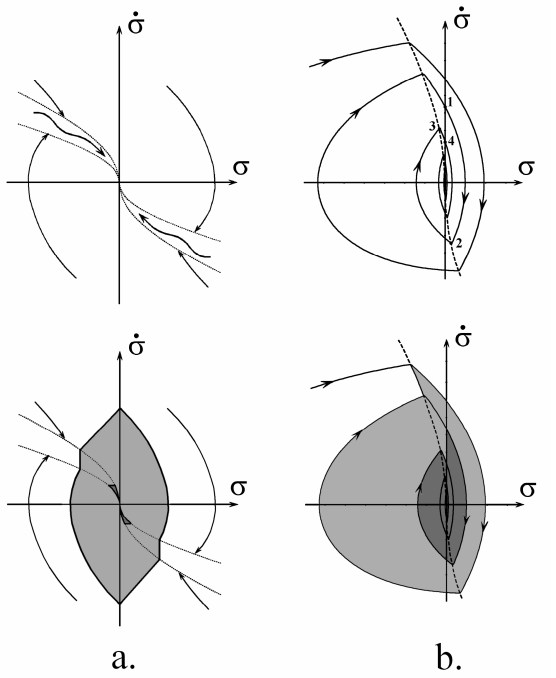


**Fig. 2. Contraction property of some 2-sliding homogeneous controllers**

**Proposition 2.** *Let* $C < \alpha K_m \le C + \beta^2/2$. *Then with sufficiently small* $C + \beta^2/2 - \alpha K_m$ *controller* (14) *provides for the twisting-like convergence to the finite-time stable* 2-*sliding mode* $\sigma \equiv 0$.

**Proof.** The corresponding trajectories and dilation-retractable sets are shown in Fig. 2b. Define once more $\Sigma = \dot{\sigma} + \beta|\sigma|^{1/2}\mathrm{sign}\,\sigma$. The boundaries of the regions are the trajectories of the differential equation $\ddot{\sigma} = -C + K_m\alpha$ with $\Sigma > 0$ and $\ddot{\sigma} = C - K_m\alpha$ with $\Sigma < 0$. Consider the combined curve 1-2-3-4 (Fig. 2b). With sufficiently small $C + \beta^2/2 - \alpha K_m$ point 4 is closer to zero than point 1. This requirement forms the convergence condition. Due to the homogeneity of the system, this geometric condition does not depend on the placement of point 1 on the axis $\dot\sigma$. Direct calculation will replace it with an algebraic condition. The absolute value $|\ddot\sigma|$ is separated from zero by $K_m\alpha - C$. Therefore, the convergence time to the smaller set is estimated by $3\,\dot\sigma_M/(K_m\alpha - C)$, where $\dot\sigma_M$ is the maximal value of $|\dot\sigma|$ in the corresponding dilation-retractable set. Hence, the finite time stability is obtained due to Theorem 1. ■

A new controller is obtained when $(\lambda_1, \mu_1) \neq (\lambda_2, \mu_2)$ and all components are not zero, especially interesting are the cases when

$$r_1 = r_2\,,\ 4r_1K_m - 2C > \max(\lambda_1^2/\mu_1^2, \lambda_2^2/\mu_2^2).$$

In that case the trajectory is confined between two parabolic segments (Fig. 1c, 2a). The control vanishes in that region. The corresponding dilation-retractable sets are shown in Fig. 2a. The convergence time to the smaller set is estimated by $2\,\dot\sigma_M/(2K_mr_1 - C)$, where $\dot\sigma_M$ is the maximal value of $|\dot\sigma|$ in the corresponding dilation-retractable set. The idea of that controller is close to that of the sliding sector control (Furuta et al., 2000). The corresponding proposition as well as its proof are obvious and are omitted.

An important class of HOSM controllers comprises recently proposed so-called *quasi-continuous* controllers, featuring control continuous everywhere except of the HOSM $\sigma = \dot\sigma = ... = \sigma^{(r-1)} = 0$ itself. Since the *r*-sliding condition has the codimension *r*, with the sliding order $r > 1$ the general-case trajectory does never hit the *r*-sliding manifold. Hence, the condition $\sigma = \dot\sigma = ... = \sigma^{(r-1)} = 0$ is never fulfilled in practice with $r \geq 2$, and the control remains continuous function of time all the time. As a result, the chattering is significantly reduced. Following is the 2-sliding controller from such a family of arbitrary-order sliding controllers (Levant, 2005b):

$$u = -\,\alpha\,\frac{\dot\sigma + \beta|\sigma|^{1/2}\operatorname{sign}\sigma}{|\dot\sigma| + \beta|\sigma|^{1/2}}\,. \qquad (15)$$

This control is continuous everywhere except of the origin. It vanishes on the parabola $\dot\sigma + \beta|\sigma|^{1/2}\operatorname{sign}\sigma = 0$. With sufficiently large $\alpha$ there are such numbers $\rho_1, \rho_2$, $0 < \rho_1 < \beta < \rho_2$ that all the trajectories enter the region between the curves $\dot\sigma + \rho_i|\sigma|^{1/2}\operatorname{sign}\sigma = 0$ and cannot leave it (Fig. 1c). Note that no explicit conditions on the parameter choice are obtained in (Levant, 2005b).

**Proposition 3.** *Let*

$$\alpha, \beta > 0,\ \alpha K_m - C > 0 \qquad (16)$$

*and the inequality*

$$\alpha K_m - C - 2\,\alpha K_m\,\frac{\beta}{\rho+\beta} - \frac{1}{2}\rho^2 > 0 \qquad (17)$$

*hold for some positive* $\rho > \beta$ (*which is always true with sufficiently large* $\alpha$)*, then controller* (15) *provides for the establishment of the finite-time stable* 2*-sliding mode* $\sigma \equiv 0$.

Conditions of the proposition can be solved for $\alpha$, but the resulting expressions are redundantly cumbersome.

**Proof.** Denote $\rho = -\dot\sigma/|\sigma|^{1/2}$. Calculations show that $u = \alpha\,(\rho - \beta)/(|\rho| + \beta)$ and

$$\dot\rho \in ([-C, C] - [K_m, K_M]\alpha\,\frac{\rho-\beta}{|\rho|+\beta} + \frac{1}{2}\rho^2\operatorname{sign}\sigma)\,|\sigma|^{-1/2}.$$

Due to the symmetry of the problem, it is enough to consider the case $\sigma > 0$, $-\infty < \rho < \infty$. With negative or small positive $\rho$ the rotation velocity $\dot\rho$ is always positive due to (16), thus there is such positive $\rho_1 < \beta$ that the trajectories enter the region $\rho > \rho_1$. It is needed to show now that there is $\rho_2 > \beta$ such that in some vicinity of $\rho = \rho_2$ the inequality $\dot\rho < 0$ holds. That is exactly condition (17). Thus, conditions (16), (17) provide for the establishment and keeping of the inequality $\rho_1 < \rho < \rho_2$.■

Following is another example of a quasi-continuous 2-sliding controller:

$$u = \begin{cases} \min\{\alpha, \max[-\alpha, -\gamma(\dot\sigma/|\sigma|^{1/2} + \beta\operatorname{sign}\sigma)\}, & \sigma \neq 0 \\ -\alpha\operatorname{sign}\dot\sigma, & \sigma = 0 \end{cases}, \qquad (18)$$

where $\alpha, \gamma, \beta > 0$, $\alpha K_m - C > \beta^2/2$, $\gamma\beta > \alpha$. The definition of the control *u* with $\sigma = 0$ is made by continuity and does not influence the system trajectories, since it only influences values on a zero-measure set. Enlarging $\gamma$ one compels the trajectories to get closer to the parabola $\dot\sigma + \beta|\sigma|^{1/2}\operatorname{sign}\sigma = 0$ without increasing the control magnitude (Fig. 1d). Also here the discontinuity is concentrated at $\sigma = \dot\sigma = 0$. Thus, in the presence of measurement errors the motion takes place in some vicinity of the mode $\sigma = \dot\sigma = 0$ without entering it, and the control signal turns out to be continuous.

Moreover, let $\zeta(y)$ be any monotonously growing positive continuous function of a non-negative argument, then the following controller generalizes (18)

$$u_\zeta = \min\{\alpha, \max[-\alpha, -\gamma\zeta(|\dot\sigma|/|\sigma|^{1/2})\operatorname{sign}\dot\sigma + \zeta(\beta)\operatorname{sign}\sigma)]\},$$

$$u = \begin{cases} u_\varsigma(\sigma,\dot\sigma), & \sigma \neq 0 \\ -\alpha\operatorname{sign}\dot\sigma, & \sigma = 0 \end{cases}. \qquad (19)$$

**Proposition 4.** *Let* $\alpha, \beta, \gamma > 0$, $\alpha K_m - C > \zeta(\beta)^2/2$, $\gamma\zeta(\beta) > \alpha$ *and* $\gamma$ *be sufficiently large, then controller* (19) *provides for the establishment of the finite-time stable* 2*-sliding mode* $\sigma \equiv 0$.

The proof is very similar to that of Proposition 3. Enlarging $\gamma$ one compels the trajectories to get closer to the parabola $\dot\sigma + \zeta(\beta)\,|\sigma|^{1/2}\operatorname{sign}\sigma = 0$ without increasing the control magnitude (Fig. 1d). Like other listed controllers, this one also provides for the finite-time convergence to the 2-sliding mode and the accuracies corresponding to Theorems 2 - 4.

**Chattering attenuation.** The standard problem of classical (first order) sliding-mode control is attenuation of the chattering effect (Slotine et al, 1991; Furuta et al., 2000; Fridman, 2003). 2-sliding mode control provides effective tools for the reduction or even practical elimination of the chattering without compromising the benefits of the standard sliding mode (Boiko & Fridman, 2005; Boiko, Fridman, Iriarte, Pisano & Usai, 2006). Let the relative degree of the system (1) be 1, i.e. (2) is replaced by

$$\dot\sigma = h(t,x) + g(t,x)u, \quad 0 < K_m \le g \le K_M, \qquad |h| \le C, \quad (20)$$

where the functions $g$, $h$ are some unknown smooth functions. Let also the control $u = -k \operatorname{sign} \sigma$ solve the problem of establishing and keeping $\sigma \equiv 0$. In particular,

$$kK_m - C > 0 \quad (21)$$

is assumed. Consider $\dot u$ as a new control, in order to overcome the chattering. Differentiating (20) achieve

$$\ddot\sigma = h_1(t,x,u) + g(t,x)\,\dot u,$$
$$h_1 = h'_t + h'_x \cdot (a + bu) + (g'_t + g'_x \cdot (a + bu))\,u.$$

Assume that with $|u| \le k_1$, $k_1 > k$, the function $h_1(t,x,u)$ is bounded:

$$\sup_{|u|\le k_1} |h_1(t,x,u)| \le C_1 . \quad (22)$$

Any listed controller $u = \phi(\alpha, \sigma, \dot\sigma)$ can be used here in order to overcome the chattering and improve the sliding accuracy of the standard sliding mode. Indeed, define

$$\dot u = \begin{cases} -u, |u| > k \\ \phi(\alpha,\sigma,\dot\sigma), |u| \le k. \end{cases} \quad (23)$$

**Theorem 5.** *Let* $\phi$ *be anyone of controllers* (14), (15), (18), (19) *and the controller parameters be chosen in accordance with the corresponding Propositions* 1, 2, 3 *or* 4. *Then with sufficiently large* $\alpha$ *controller* (23) *provides for the establishment of the finite-time-stable* 2*-sliding mode* $\sigma \equiv 0$. *Also the statements of Theorems* 2 - 4 *are valid with sufficiently small noises or sampling intervals.*

Controller (23) keeps $|u| \le k$, and on certain time intervals $u \equiv k$ or $u \equiv -k$ is kept in 1-sliding mode. Note that though Theorem is not formulated for *arbitrary* 2-sliding homogeneous controllers, it is valid for all standard controllers (Levant, 1993, Bartolini et al., 1998).

**Proof.** It follows from (20), (21) that the inequality $|\dot\sigma| < kK_m - C$ implies $|u| < k$. Thus, within the set $|\dot\sigma| < kK_m - C$ the system is driven by the controller $\dot u = \phi(\alpha, \sigma, \dot\sigma)$.

**Lemma 1.** *Any trajectory of the system* (1), (23) *hits in finite time the manifold* $\sigma = 0$ *or enters the set* $\sigma\dot\sigma < 0$, $|u| \le k$.

Indeed, suppose that $\sigma$ does not change its sign. Obviously, the inequality $|u| \le k$ is established in finite time. If the condition $\sigma\dot\sigma < 0$ is attained, the statement of the Lemma is true. Suppose that $\sigma\dot\sigma \ge 0$ holds, then, according to (23), $u$ moves towards $u = -k \operatorname{sign} \sigma$ with $|\dot u| \ge \min(\alpha, k)$, both if $|u| > k$ or $|u| \le k$. The remark that $u = -k \operatorname{sign} \sigma$ can be established only with $\sigma\dot\sigma < 0$ proves the Lemma. ■

**Lemma 2.** *With sufficiently large* $\alpha$ *any trajectory of the system* (1), (23) *hits in finite time the manifold* $\sigma = 0$.

**Proof.** Denote $S$ the set defined by the inequalities $|\dot\sigma| < kK_m - C$, $\sigma\dot\sigma < 0$. There is a specific set $\Theta$ for each controller, adjacent to the axis $\sigma = 0$, and lying in the strip $S$, such that any trajectory entering it, either converges in finite time to $\sigma = \dot\sigma = 0$, or hits the axis $\sigma = 0$; also no trajectory can enter $S$ outside of $\Theta$. For example, $\Theta$ is defined by the inequalities $(\dot\sigma + \beta|\sigma|^{1/2} \operatorname{sign} \sigma)\sigma \le 0$, $|\dot\sigma| < kK_m - C$ for controller (14). Any trajectory starting in $S$ either leaves it in finite time, or enters $\Theta$. Thus, there are 2 options: from some moment on a trajectory stays out of $S$, which means that $|\dot\sigma| \ge kK_m - C$, $\sigma\dot\sigma < 0$, or it enters $\Theta$. In both cases the trajectory hits $\sigma = 0$. ■

The following Lemma is obviously true for any convergent 2-sliding controller.

**Lemma 3.** *There is a vicinity* $\Omega$ *of the origin within the strip* $|\dot\sigma| < kK_m - C$, *which is invariant with respect to the controller* $\dot u = \phi(\alpha, \sigma, \dot\sigma)$.

**Proof.** Consider the auxiliary problem, when (22) holds independently of the control value, and the corresponding differential inclusion. Since all trajectories starting in a closed disk centered at the origin, converge to the origin in finite time, the set, which comprises these transient trajectory segments, is an invariant compact for the controller $\dot u = \phi(\alpha, \sigma, \dot\sigma)$ (Filippov, 1988). Applying now the homogeneity transformation, the set can be retracted into the strip $|\dot\sigma| < kK_m - C$, where (22) is really kept. ■

**Lemma 4.** *With sufficiently large* $\alpha$ *any trajectory starting on the manifold* $\sigma = 0$ *with* $|u| \le k$ *enters the invariant set* $\Omega$.

**Proof.** Any trajectory starting with $\sigma = 0$ and $\dot\sigma \ne 0$ inevitably enters the region $\sigma\dot\sigma > 0$, $|u| < k$. Within this region $\dot u = -\alpha \operatorname{sign} \sigma$. Hence, the control $u$ moves towards the value $-k \operatorname{sign} \sigma$, and on the way the trajectory hits the set $\dot\sigma = 0$, which still features $|u| < k$. As follows from (20), $|u| \le k$ implies the global bound $|\dot\sigma| \le kK_M + C$. That restriction is true also at the initial point on the axis $\sigma = 0$. Simple calculation shows that the inequality $|\sigma| \le \frac{1}{2}(kK_M + C)^2/(\alpha K_m - C_1)$ takes place at the moment when $\dot\sigma$ vanishes. With sufficiently large $\alpha$ that point inevitably belongs to $\Omega$. ■

Once the trajectory enters $\Omega$, it continues to converge to the 2-sliding mode according to the corresponding 2-sliding homogeneous dynamics. This proves the convergence to the 2-sliding mode. In the presence of small noises and sampling intervals the resulting motion will take place in a small vicinity of the 2-sliding mode $\sigma = \dot\sigma = 0$. Thus, if this motion does not leave $\Omega$, the homogeneous dynamics is still in charge, and the statements of Theorems 1 - 4 are true. ■

## 5. Simulation results

A number of new controllers from the previous Section are demonstrated here. Consider an academic example of a variable-length pendulum with motions restricted to some vertical plane. A load of a known mass $m$ moves without friction along the pendulum rod (Fig. 3a). Its distance from $O$ equals $R(t)$ and is not measured. An engine transmits a torque $u$, which is considered as control. The task is to track some function $x_c$ given in real time by the angular coordinate $x$ of the rod.

The system is described by the equation

$$\ddot x = -2\,\frac{\dot R}{R}\,\dot x - g\frac{1}{R}\sin x + \frac{1}{mR^2}\,u, \quad (24)$$

where $m = 1$ and $g = 9.81$ is the gravitational constant. Let $0 < R_m \le R \le R_M$, $\dot{R}$, $\ddot{R}$, $\dot{x}_c$ and $\ddot{x}_c$ be bounded, $\sigma = x\text{-}x_c$ be available. Following are the "unknown" functions $R$ and $x_c$ considered in the simulation:

$$R = 0.8 + 0.1 \sin 8t + 0.3 \cos 4t,$$
$$x_c = 0.5 \sin 0.5t + 0.5 \cos t .$$

Let $\sigma = x\text{-}x_c$. The relative degree of the system equals 2. The assumptions (3) are fulfilled here only locally, and the controllers to be applied are effective only for some bounded set of initial conditions. Choosing the controller parameter $\alpha$ the convergence region can be made arbitrarily large.

Both main applications of 2-sliding modes are demonstrated: the straight-forward implementation of 2-sliding-mode controllers leading to discontinuous control and possibly dangerous chattering, and the standard practical removal of chattering by means of 2-sliding mode. In the latter case a redefinition of the output and input are needed. Note that the 3-sliding-mode controllers (Levant, 2003, 2005b) are probably more effective in that case, but are out of the scope of this paper.

The Euler integration method was used, being the only method valid for sliding-mode simulation. The parameters of the controllers were found by simulation, since the direct calculation of the constants $C$, $K_m$, $K_M$ is difficult and, inevitably, not precise.

*Discontinuous control*

The controllers include a real-time differentiator and have the form

$$u = \varphi(z_0, z_1), \qquad \sigma = x\text{-}x_c ,$$
$$\dot{z}_0 = -\ 10.61\, |\, z_0 - \sigma|^{1/2} \operatorname{sign}(z_0 - \sigma) + z_1, \qquad (25)$$
$$\dot{z}_1 = -\ 55\, L \operatorname{sign}(z_0 - \sigma), \qquad (26)$$

where $z_0$, $z_1$ are real-time estimations of $\sigma$, $\dot{\sigma}$ respectively. Differentiator (25), (26) is exact for input signals $\sigma$ with second derivative not exceeding 50 in absolute value.

The initial conditions $x(0) = \dot{x}(0) = 0$ were taken, $z_0(0) = x(0) - x_c(0) = -\ 0.5$, $z_1(0) = 0$, the sampling step $\tau$ and the integration steps being the same, $\tau = 0.0001$.

Consider a controller of the form (14)

$$u = -\ 10 \operatorname{sign}(\, z_1 + 11 |\, z_0|^{1/2} \operatorname{sign} z_0\, ). \qquad (27)$$

The magnitude of the control is not sufficiently large here to establish a 1-sliding mode on the curve $\dot{\sigma} + 11|\sigma| \operatorname{sign} \sigma = 0$, nevertheless the 2-sliding mode $\sigma = \dot{\sigma} = 0$ is established here in finite time according to Proposition 2. Note that *it is the first time when such a twisting-type convergence is demonstrated for this controller*. The phase trajectories in the plane $\sigma$, $\dot{\sigma}$ and the first 0.1 seconds of the differentiator convergence are shown in Fig. 3b,c respectively, the corresponding accuracies being $|\sigma| = |\, x - x_c\, | \le 4.2 \cdot 10^{-5}$, $|\, \dot{\sigma}\, | = |\, \dot{x} - \dot{x}_c\, | \le 2.7 \cdot 10^{-2}$. After the sampling step $\tau$ was reduced from $10^{-4}$ to $10^{-5}$ the resulting accuracies changed to $|\, x - x_c\, | \le 4.8 \cdot 10^{-7}$, $|\, \dot{\sigma}\, | = |\, \dot{x} - \dot{x}_c\, | \le 3.1 \cdot 10^{-3}$ which corresponds to Theorem 4. The differentiator convergence is demonstrated in Fig. 3f.

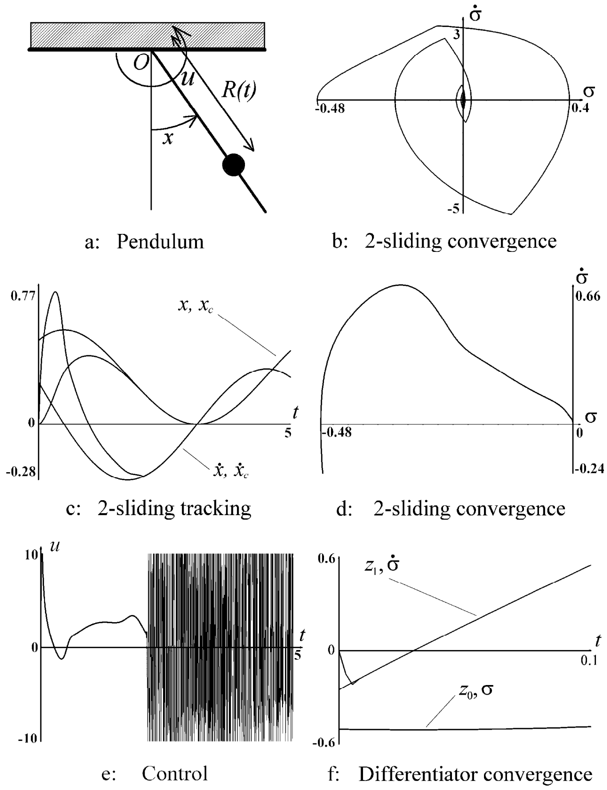


**Fig. 3: Pendulum and performance of controllers (25), (26), (27) (b, f) and (25), (26), (28) (c, d, e)**

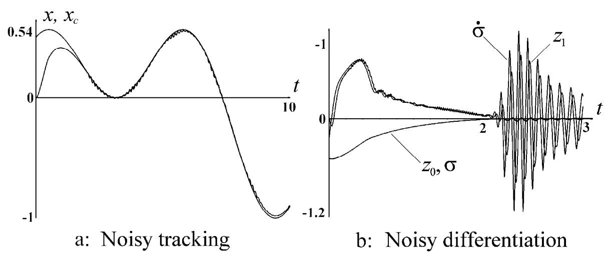


**Fig. 4: Performance of the controller (28), (25), (26) with noisy measurements**

Consider the quasi-continuous controller

$$u = -\ 10\ \frac{\dot{\sigma} + |\, \sigma\, |^{1/2} \operatorname{sign} \sigma}{|\, \dot{\sigma}\, | + |\, \sigma\, |^{1/2}} . \qquad (28)$$

The trajectory and 2-sliding tracking performance in the absence of noises are shown in Figs. 3d, and 3c respectively, the corresponding accuracies being $|\sigma| = |x - x_c| \le 5.4 \cdot 10^{-6}$, $|\, \dot{x} - \dot{x}_c\, | \le 1.0 \cdot 10^{-2}$ with $\tau = 0.0001$. Control (28) is demonstrated in Fig. 3e. It is seen from the graph that the control remains continuous until the entrance into the 2-sliding mode $\sigma = \dot{\sigma} = 0$.

The tracking results of (25), (26), (28) and the differentiator performance in the presence of a noise with the magnitude 0.01 are demonstrated in Figs. 4c,d respectively, the tracking accuracy being $|\sigma| = |\, x - x_c\, | \le 0.036$. The noise was a periodic nonsmooth function with nonzero average. The performance does not significantly change, when the frequency of the noise varies from 10 to 100000. These results correspond to Theorem 3.

*Chattering attenuation*

The controller includes a real-time second-order differentiator (Levant, 2003)

$$\dot z_0 = v_0 = -\,13.4\,|\,z_0 - \sigma|^{2/3}\operatorname{sign}(z_0 - \sigma) + z_1, \qquad (29)$$

$$\dot z_1 = v_1 = -\,26.0\,|\,z_0 - v_0|^{1/2}\operatorname{sign}(z_0 - v_0) + z_2, \qquad (30)$$

$$\dot z_2 = -\,330\operatorname{sign}(z_2 - v_1), \qquad \sigma = x - x_c\,, \qquad (31)$$

where $z_0$, $z_1$, $z_2$ are real-time estimations of $\sigma$, $\dot\sigma$, $\ddot\sigma$ respectively. Differentiator (29) - (31) is exact for input signals $\sigma$ with the third derivative not exceeding 300 in absolute value. The initial values $z_0(0) = \sigma(0)$, $z_1(0) = z_2(0) = 0$ were taken.

Consider a controller of the form (19), (23) which establishes in finite time the 2-sliding mode $\Sigma = \dot\Sigma = 0$, where $\Sigma = \sigma + \dot\sigma$. Substituting the estimations $z_0$, $z_1$, $z_2$ of the derivatives $\sigma$, $\dot\sigma$, $\ddot\sigma$ obtain the controller

$$\dot u = \min\{30, \max[-30, -\,40\,(\ln(|s_1|/|s_0|^{1/2} + 1)\cdot\operatorname{sign} s_1 + \ln 2\cdot\operatorname{sign} s_0)]\} \ \text{ with } |u| < 10,\ |s_1| < 100|s_0|^{1/2}, \qquad (32)$$

$$\dot u = -30\operatorname{sign} s_1 \ \text{ with } |u| < 10,\ |s_1| \ge 100|s_0|^{1/2}, \qquad (33)$$

$$\dot u = -\,u \ \text{ with } |u| \ge 10, \qquad s_0 = z_0 + z_1,\ \ s_1 = z_1 + z_2, \qquad (34)$$

where the function $\zeta(*) = \ln(* + 1)$ is chosen, $u(0) = 0$. Without changing the control values, (33) is constructed so that the overflow be avoided during the computer simulation (or practical implementation).

In the absence of noises the controller provides in finite time for keeping $\sigma + \dot\sigma \equiv 0$. As a result the asymptotically stable 3-sliding mode $\sigma = \dot\sigma = \ddot\sigma = 0$ is established. Note that since (19) is quasi-continuous, the applied controller (29) – (34) produces the control $u$, whose derivative $\dot u$ remains continuous until the entrance into the 2-sliding mode $\Sigma = \dot\Sigma = 0$. The graph of $\dot u$ is very similar to Fig. 3e and is omitted. The graphs of 3-sliding deviations $\sigma$, $\dot\sigma$, $\ddot\sigma$ and the control $u$ are demonstrated in Figs 5a,c respectively. The 2-sliding convergence in the plane $\Sigma$, $\dot\Sigma$ is demonstrated in Fig. 5b. It is seen from Fig. 5d that $\sigma + \dot\sigma \equiv 0$ is kept in 2-sliding mode. Due to this equality the accuracies achieved at $t = 10$ are of the same order: $|\sigma| = |\,x - x_c| \le 4.0\cdot10^{-4}$, $|\dot\sigma| = |\,\dot x - \dot x_c| \le 4.0\cdot10^{-4}$, $|\ddot\sigma| = |\,\ddot x - \ddot x_c| \le 4.0\cdot10^{-4}$.

## 6. Conclusions

2-sliding homogeneity and contractivity are shown to provide for all needed features of 2-sliding mode controllers (Theorems 1-4). Construction of controllers is not difficult due to the simplicity of the plane geometry. New finite-time stable 2-sliding controllers were obtained in such a way, significantly increasing the choice of known 2-sliding controllers. In particular, the quasi-continuous controllers (15), (18), (19) have probably better performance than the standard 2-sliding controllers popular today.

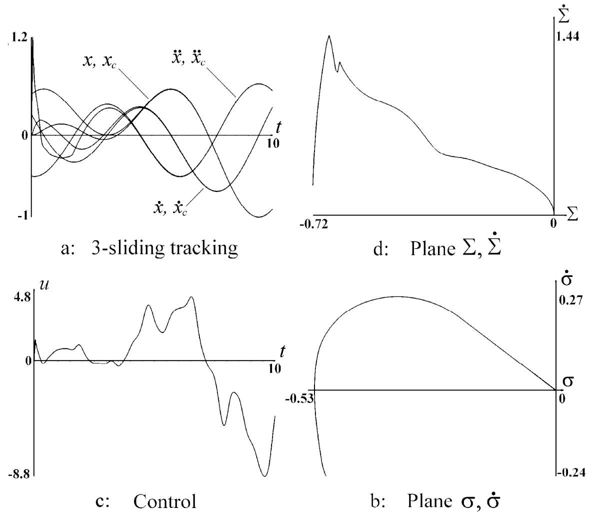


**Fig. 5: Chattering attenuation**

The number of such 2-sliding homogeneous controllers is obviously infinite, and one can adjust a controller to his needs. Unfortunately, design of higher-order sliding controllers is much more difficult due to the higher dimension of the problem (Levant, 2003, 2005b).

A simple efficient procedure was developed for the chattering attenuation in the systems with standard (first order) sliding modes based on their replacement by 2-sliding modes (Theorem 5). The resulting continuous lipschitzian control can be used to keep auxiliary constraints, and, when solving practical control problems, is readily combined with different control technique (Levant et al., 2000).

The main results known for the known 2-sliding controllers are extended to any finite-time-stable 2-sliding-homogeneous controllers.

A real-time robust exact differentiator having been used as a standard part of the 2-sliding controllers, the full single-input-single-output control is achieved based on the input measurements only. The resulting controllers are locally applicable to any uncertain smooth process of relative degree 2, and they are globally applicable, if the boundedness conditions (3) hold globally.

The resulting robust output-feedback controllers preserve the ultimate accuracy of the original 2-sliding controllers with direct measurements of the input derivative (Theorem 3). In the absence of noises the tracking accuracy proportional to $\tau^2$ is provided, $\tau$ being a sampling period. That is the best possible accuracy with discontinuous second output derivative (Levant, 1993). In the presence of a measurement noise the tracking accuracy is proportional to the unknown noise magnitude. That result does not depend on the noise features.

The differentiator is to be used whenever the sampling step can be taken small. At the same time in the practically important case, when the sampling step is sufficiently large compared with the noises, the differentiator is successfully replaced by the first finite difference (Theorem 4).

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