On the determination of the precision of observations
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1

In order to establish the principles of the method of least squares, we have assumed that the probability of an error of observation is given by the formula

$$\frac{h}{\sqrt{\pi}} \mathrm{e}^{-h^2\Delta^2}$$

where π represents the half circumference of the unit circle, e the base of hyperbolic logarithms, and h a constant which one may consider (*Theoria Motus Corporum Coelestium*, article 178) as measuring the precision of the observations. It is not necessary to know the value of h in order to determine, by means of the method of least squares, the most likely values of the quantities on which the observations depend, since the ratio of the precision of these results to the precision of the observations is also independent of h.

Nonetheless, since knowledge of the quantity h is interesting and instructive, I shall show how it may be determined form the observations.

2

Let us begin with some remarks which will clarify the question and denote by $\theta(t)$ the definite integral

$$\int_0^t \frac{2\mathrm{e}^{-t}\,dt}{\sqrt{\pi}}.$$

Some specific values of this function will give an idea of its behaviour

$$\begin{array}{lllllllll}
t & = & 0.476\,9363 & = & \rho & \theta(t) & = & 0.5 \\
t & = & 0.595\,1161 & = & \rho \times 1.247\,790 & \theta(t) & = & 0.6 \\
t & = & 0.732\,8691 & = & \rho \times 1.536\,618 & \theta(t) & = & 0.7 \\
t & = & 0.906\,1939 & = & \rho \times 1.900\,032 & \theta(t) & = & 0.8 \\
t & = & 1 & = & \rho \times 2.096\,716 & \theta(t) & = & 0.842\,7008 \\
t & = & 1.163\,0872 & = & \rho \times 2.348\,664 & \theta(t) & = & 0.9 \\
t & = & 1.821\,3864 & = & \rho \times 2.818\,930 & \theta(t) & = & 0.99 \\
t & = & 2.327\,6754 & = & \rho \times 4.880\,475 & \theta(t) & = & 0.999 \\
t & = & 2.751\,0654 & = & \rho \times 5.768\,204 & \theta(t) & = & 0.9999 \\
t & = & \infty & = & \infty & \theta(t) & = & 1
\end{array}$$

The probability that the error of one observation is included in the limits $+\Delta$ and $-\Delta$, or disregarding the sign, that it does not exceed Δ, will be equal to

$$\frac{h}{\sqrt{\pi}} \int_{-\Delta}^{\Delta} \mathrm{e}^{-h^2\Delta^2}$$

it will be equal to twice the integral in question if this is taken between the limits $x = 0$ and $x = \Delta$, and consequently it will be equal to $\theta(h\Delta)$.

Thus the probability that the error is not less than $\frac{\rho}{h}$ is equal to $\frac{1}{2}$ that is equal to the probability of its contrary. Thus I shall call this quantity $\frac{\rho}{h}$ the probable error, and shall denote it by r.

On the other hand, the probability that the error exceeds $2.348664r$ is only one tenth; the probability that the error exceeds $3.818390r$ is only one one-hundredth; and so on.

3

Let us suppose that the errors actually made in m observations are α, β, γ, etc., and let us see what consequences one can derive regarding the values of h and r.

By taking two hypotheses on the exact value of h, and supposing it equal to H or H', the probabilities that the observations will be affected by the errors α, β, γ etc. will be, for the two cases, in the ratio of

$$\begin{array}{llll} H\mathrm{e}^{-H^2\alpha^2} & \times\ H\mathrm{e}^{-H^2\beta^2} & \times\ H\mathrm{e}^{-H^2\gamma^2} & \times \dots \\ H'\mathrm{e}^{-H'^2\alpha^2} & \times\ H'\mathrm{e}^{-H'^2\beta^2} & \times\ H'\mathrm{e}^{-H'^2\gamma^2} & \times \dots \end{array}$$

that is to say as

$$H^m \mathrm{e}^{-H^2(\alpha^2+\beta^2+\gamma^2+\dots)}$$

is to

$$H'^m \mathrm{e}^{-H'^2(\alpha^2+\beta^2+\gamma^2+\dots)}$$

It is clear that the probabilities that H or H' be the true values of H are in the same ratio (*Theoria Motus Corporum Coelestium*, article 176); Consequently the probability of an arbitrary value of H is

$$H^m \mathrm{e}^{-h^2(\alpha^2+\beta^2+\gamma^2+\dots)},$$

and the most likely value of H is that for which this function becomes a maximum. But one finds by well known rules that h is then equal to

$$\sqrt{\frac{m}{2(\alpha^2+\beta^2+\gamma^2+\dots)}}$$

hence the most probable value of r will be

$$\rho\sqrt{\frac{2(\alpha^2+\beta^2+\gamma^2+\dots)}{m}}$$

or

$$0.674\,4897 \times \sqrt{\frac{1}{m}(\alpha^2+\beta^2+\gamma^2+\dots)}$$

This is a general result, whether m is large or small.

It is easy to understand that the values found for h and r are less certain the smaller the number m.

Let us now find the degree of precision which one should assign to the values of h and r when m is fairly large number.

Let us denote by H the most probable value of h, which we have found to be

$$\sqrt{\frac{m}{2(\alpha^2+\beta^2+\gamma^2+\dots)}}$$

and note that the probability that H is the true value of h is to the probability that $(H-\lambda)$ is the true value as

$$H^m \mathrm{e}^{-\frac{m}{2}} = \text{ is to } (H+\lambda)^m . \mathrm{e}^{-\frac{m(H+\lambda)^2}{2H^2}},$$

or as

$$1 \text{ is to } \mathrm{e}^{\frac{\lambda^2 m}{H^2}\left(1-\frac{1}{2}\cdot\frac{\lambda}{H}+\frac{1}{4}\cdot\frac{\lambda^2}{H^2}+\frac{1}{5}\cdot\frac{\lambda^3}{H^3}+\dots\right)}$$

The second term will bear a substantial ratio to the first only if $\frac{\lambda}{H}$ is a small fraction, and in this case we may replace the indicated ratio by

$$1 : H\mathrm{e}^{-\frac{\lambda^2 m}{H^2}}.$$

This means that the probability that the true value of h is included between $(H+\lambda)$ and $(H+\lambda+d)$ is approximately equal to

$$K\mathrm{e}^{-\frac{\lambda^2 m}{H^2}}\, d\lambda,$$

where K is a constant such that the integral

$$\int K\mathrm{e}^{-\frac{\lambda^2 m}{H^2}}\, d\lambda$$

taken between the admissible limits of λ, becomes equal to one.

In the present case because of the large value of m, $\mathrm{e}^{-\frac{\lambda^2 m}{H^2}}$ becomes extremely small when $\frac{\lambda}{H}$is different from a small fraction, and it will become permissible to take the integral from ∞ to $+\infty$; thus one obtains

$$K = \frac{1}{H}\sqrt{\frac{m}{\pi}}$$

Consequently, the probability that the true value of h lies between $(H-\lambda)$ and $(H+\lambda)$ will be equal to

$$\theta\left(\frac{\lambda}{H}\sqrt{m}\right),$$

and it will be equal to $\frac{1}{2}$ when $\frac{\lambda}{H}\sqrt{m} = \rho$.

Thus it is an even bet that the true value of h lies between

$$H\left(1-\frac{\rho}{\sqrt{m}}\right) \text{ and } H\left(1+\frac{\rho}{\sqrt{m}}\right),$$

or that the true value of r lies between

$$\frac{H}{1-\frac{\rho}{\sqrt{m}}} \text{ and } \frac{H}{1-\frac{\rho}{\sqrt{m}}},$$

when H denotes the most likely value of r found in the preceding section. These limits may be called the probable limits of the true values of h and r. It is clear that we may take here as probable limits of r

$$R\left(1-\frac{\rho}{\sqrt{m}}\right) \text{ and } R\left(1+\frac{\rho}{\sqrt{m}}\right).$$

5

In the preceding discussion, we considered α, β, γ, etc. as definite given quantities, in order to evaluate the probability that the true value of h and of r would lie between certain limits.

One may consider the question from another point of view, by assuming that the errors of the observations follow a given probability law; one can then evaluate the probability that the sum of the squares of m errors of observations will fall between certain limits. Laplace has already solved this problem in the case where m is a very large number, as well as the problem of determining the probability that the sum of m errors of observation lies between certain limits.

It is easy to generalize this investigation; I shall restrict myself here to indicating the result.

Let us denote by $\phi(x)$ the probability of an error x in the observation, so that

$$\int_{-\infty}^{\infty} \phi x.dx = 1$$

Then let us denote by K_n the value of the integral

$$\int_{-\infty}^{\infty} \phi x.x^n\, dx.$$

Finally let

$$S_n = \alpha^n + \beta^n + \gamma^n + \dots$$

where α, β, γ, etc. represent m arbitrary errors of observation; the terms of this sum will always be taken positively, even if n is odd.

Then mK_n will be the most likely value of S_n, and the probability that the true value of S_n falls between the limits $(mK_n - \lambda)$ and $(mK_n + \lambda)$ will be equal to

$$\theta \, \frac{\lambda}{\sqrt{2m(K_{2n} - K_n^2)}};$$

Consequently the probable limits of S_n will be

$$mK_n - \rho\sqrt{2m(K_{2n} - K_n^2)}$$

and

$$mK_n + \rho\sqrt{2m(K_{2n} - K_n^2)}$$

This results, in a general way, to any probability distribution for errors. Applying it to the particular case in which

$$\phi x = \frac{h}{\sqrt{\pi}}.\mathrm{e}^{-h^2x^2},$$

we find

$$K_n = \frac{\Pi\frac{1}{2}(n-1)}{h^n\sqrt{\pi}},$$

where the symbol Π is taken to have the same meaning as in *Disquisitiones generales circa seriem infinitem** (*Comm. no. Soc. Gotting.*, Volume III, note 5, article 28).

Hence

$$\begin{array}{llllllll} K & = 1, & K_1 & = \frac{1}{h\sqrt{\pi}}, & K_2 & = \frac{1}{2h^2}, & K_3 & = \frac{1}{h^3\sqrt{\pi}}, \\ K_4 & = \frac{1.3}{4n^4}, & K_5 & = \frac{1.2}{h^5\sqrt{\pi}}, & K_6 & = \frac{1.3.5}{8h^6}, & K_7 & = \frac{1.2.3}{h^7\sqrt{\pi}}, \end{array}$$

and consequently the most likely value of S_n will be

$$\frac{m\Pi\frac{1}{2}(n-1)}{h^n\sqrt{\pi}}$$

and the most likely limits for the true values of S_n will be

$$\frac{m\Pi\frac{n-1}{2}}{h^n\sqrt{\pi}}\left\{1 - \rho\sqrt{\frac{2}{m}\left[\frac{\Pi(n-\frac{1}{2}).\sqrt{\pi}}{[\Pi\frac{1}{2}(n-1)]^2}\right] - 1}\right\}$$

and

$$\frac{m\Pi\frac{n-1}{2}}{h^n\sqrt{\pi}}\left\{1 + \rho\sqrt{\frac{2}{m}\left[\frac{\Pi(n-\frac{1}{2}).\sqrt{\pi}}{[\Pi\frac{1}{2}(n-1)]^2}\right] - 1}\right\}$$

Thus if we put as above

$$\frac{\rho}{h} = r,$$

*$\Pi(n) = \Gamma(n+1)$ is Gauss's notation for the factorial function.

where r represents the probable error of observations, the most likely value of

$$\rho \sqrt[n]{\frac{S_n\sqrt{\pi}}{m\Pi\left(\frac{n-1}{2}\right)}}$$

will obviously be r; and the probable limits for the value of this quantity will be

$$r\left\{1-\frac{\rho}{n}\sqrt{\frac{2}{m}\left[\frac{\Pi(n-\frac{1}{2}).\sqrt{\pi}}{[\Pi\frac{1}{2}(n-1)]^2}\right]-1}\right\}$$

and

$$r\left\{1+\frac{\rho}{n}\sqrt{\frac{2}{m}\left[\frac{\Pi(n-\frac{1}{2}).\sqrt{\pi}}{[\Pi\frac{1}{2}(n-1)]^2}\right]-1}\right\}$$

Hence it is an even bet that r will lie between the limits

$$\rho \sqrt[n]{\frac{S_n\pi}{m|pi\left(\frac{n-1}{2}\right)}}\left\{1-\frac{\rho}{n}\sqrt{\frac{2}{m}.\frac{\Pi\left(n-\frac{1}{2}\right)\sqrt{\pi}}{\Pi\left(\frac{n-1}{2}\right)^2}-1}\right\}$$

and

$$\rho \sqrt[n]{\frac{S_n\pi}{m|pi\left(\frac{n-1}{2}\right)}}\left\{1+\frac{\rho}{n}\sqrt{\frac{2}{m}.\frac{\Pi\left(n-\frac{1}{2}\right)\sqrt{\pi}}{\Pi\left(\frac{n-1}{2}\right)^2}-1}\right\}$$

For $n=2$, the limits will be

$$\rho\sqrt{\frac{2S_n}{m}.\left(1-\frac{\rho}{m}\right)}$$

and

$$\rho\sqrt{\frac{2S_n}{m}.\left(1+\frac{\rho}{m}\right)}$$

which agrees perfectly with those which we found in section 4.

In general, if n is even, one will have the limits

$$\rho\sqrt{2}\sqrt[n]{\frac{S_n}{m.1.3.5.7.\dots.(n-1)}}\times\left\{1-\frac{\rho}{n}\sqrt{\frac{2}{m}\frac{(n+1)(n+3)\dots(2n-1)}{1.3.5.\dots.(n-1)}-1}\right\}$$

and

$$\rho\sqrt{2}\sqrt[n]{\frac{S_n}{m.1.3.5.7.\dots.(n-1)}}\times\left\{1+\frac{\rho}{n}\sqrt{\frac{2}{m}\frac{(n+1)(n+3)\dots(2n-1)}{1.3.5.\dots.(n-1)}-1}\right\}$$

and, if n is odd, the following limits

$$\rho\sqrt{2}\sqrt[n]{\frac{S_n\sqrt{\pi}}{m.1.3.5.7.\dots.(\frac{n-1}{2})}}\times\left\{1-\frac{\rho}{n}\sqrt{\frac{2}{m}\frac{1.3.5.7.\dots.(2n-1)\pi}{[2.4.6.\dots.(n-1)]^2}-2}\right\}$$

and

$$\rho\sqrt{2}\sqrt[n]{\frac{S_n\sqrt{\pi}}{m.1.3.5.7.\ldots.(\frac{n-1}{2})}} \times \left\{1+\frac{\rho}{n}\sqrt{\frac{2}{m}\frac{1.3.5.7.\ldots.(2n-1)\pi}{[2.4.6.\ldots.(n-1)]^2}-2}\right\}$$

6

I add here the numerical values for the most simple cases

Probable limits of r:

I.	$0.845\,3473 \times \frac{S_1}{m}\pi\left(1 \pm 0.\frac{509\,5841}{\sqrt{m}}\right)$
II.	$0.674\,4897 \times \sqrt{\frac{S_2}{m}}.\left(1 \pm \frac{0.476\,9363}{\sqrt{m}}\right)$
III.	$0.577\,1897 \times 3\sqrt{\frac{S_3}{m}}.\left(1 \pm \frac{0.497\,1987}{\sqrt{m}}\right)$
IV.	$0.512\,5017 \times 4\sqrt{\frac{S_4}{m}}.\left(1 \pm \frac{0.550\,7186}{\sqrt{m}}\right)$
V.	$0.465\,5532 \times 5\sqrt{\frac{S_5}{m}}.\left(1 \pm \frac{0.635\,5080}{\sqrt{m}}\right)$
VI.	$0.429\,4972 \times 6\sqrt{\frac{S_6}{m}}.\left(1 \pm \frac{0.755\,7764}{\sqrt{m}}\right)$

Thus one sees by comparison that the second method of determining r is the best; for 100 errors of observation, treated by this formula, will give a result as reliable as

114	observations	from	formula	I
109	"	"	"	III
133	"	"	"	IV
178	"	"	"	V
251	"	"	"	VI

However formula I presents the advantage of lending itself to numerical calculations; and since its degree of precision is only slightly inferior to that of formula II, one can always do it, unless one already knows the sum of the squares of the errors or one wishes to know it.

7

The following procedure is more convenient but much less exact.

Let us arrange the absolute values of the m errors of observations in order of magnitude, and denote by M the centre term if their number is odd or the arithmetic mean between the two centre terms if their number is even.

One can show (which we shall not do here) that, just for a large number of observations, the most likely value of M is r, and that the probable limits of M are

$$r\left(1-\mathrm{e}^{\rho^2}.\sqrt{\frac{\pi}{8m}}\right) \text{ and } r\left(1+\mathrm{e}^{\rho^2}.\sqrt{\frac{\pi}{8m}}\right)$$

or that the probable limits of r are

$$M\left(1-\mathrm{e}^{\rho^2}.\sqrt{\frac{\pi}{8m}}\right) \text{ and } M\left(1+\mathrm{e}^{\rho^2}.\sqrt{\frac{\pi}{8m}}\right)$$

Consequently this procedure is not much less exact than the application of formula VI, and it would be necessary to consider the errors in 249 observations in order to obtain a result as sure as in applying formula II to one hundred errors of observations.

8

The application of these methods to the errors committed in 48 observations of the right ascension of the Pole star by Bessel gave

$$S_1 = 60.46''$$
$$S_2 = 110.600''$$
$$S_3 = 250.341118''.$$

From this one deduces the most probable values of r:

According to formula I	$1.065''$	Probable error	$\pm\,0.068''$
II	$1.024''$		$\pm\,0.070''$
III	$1.001''$		$\pm\,0.072''$

and according to section 7:

	$1.045''$		$\pm\,0.113''$

an agreement of results which could hardly be hoped for. Bessel gives $1.067''$ and seems, consequently, to have calculated according to formula I.

Taken from *Work (1803–1826) on the Theory of Least Squares*, trans. H F Trotter, Technical Report No.5, Statistical Techniques Research Group, Princeton, NJ: Princeton University 1957.