Kunihiko Kodaira: Mathematician, Friend, and Teacher

F. Hirzebruch

Kunihiko Kodaira died July 26, 1997, and a memorial article appeared in the March 1998 *Notices*, pp. 388–389.

Kunihiko Kodaira

Kunihiko Kodaira was friend and teacher for me. My wife and I remember our last visit to the Kodairas' house in Tokyo. He was working at the kitchen table on textbooks for secondary schools. Seiko Kodaira had to push the papers away when preparing the meal. In 1995 I congratulated him on his eightieth birthday. He answered in his charming way. But when we came to Tokyo in 1996, he was already in the hospital. We could not talk to him anymore.

I have read the obituaries by D. C. Spencer in the *Notices of the AMS* (March 1998) and by M. F. Atiyah for the London Mathematical Society. Both say much about our mutual friend; I do not have to repeat it.

I want to report about the influence which Kodaira had on my mathematical work. I shall emphasize the period from 1952 to 1954 when I was a member of the Institute for Advanced Study in Princeton. On Monday, August 18, 1952, I arrived in Hoboken, New Jersey, on the *Ryndam* of the Holland America Line. D. C. Spencer and Newton Hawley picked me up. On Saturday, August 23, I wrote to my parents that I had worked every day in the Institute with Kodaira, Spencer, and Hawley. When I read this letter again after forty-six years, I was surprised to see that my mathematical training in Princeton under Kodaira and Spencer started immediately after my arrival in spite of the Princeton summer.

When I arrived, I knew nothing about sheaves and very little about algebraic geometry and characteristic classes. This improved fast. Our heavy work was made easier by a fine picnic given by Kunihiko and Seiko Kodaira.

In 1975 Kodaira's *Collected Works* appeared in three volumes (Iwanami Shoten, Publishers, and Princeton University Press) with a preface by his student Walter L. Baily Jr. giving a survey and appreciation of Kodaira's work until then.

At the end of this paper I shall reproduce twenty-six entries from the table of contents of the *Collected Works* using the numbering there. These are mostly the papers quoted in my book *Topological Methods in Algebraic Geometry* (Translation and Appendix One by R. L. E. Schwarzenberger, Appendix Two by A. Borel), which was published by Springer-Verlag in 1966 as the English version of *Neue topologische Methoden in der algebraischen Geometrie* (Springer-Verlag, 1956). I added reference [28] ("Work done at Princeton University, 1952"). These are the lecture notes of his course at Princeton University which I attended, at least partially, in the winter 1952–53. I do not know how much of [28] he covered in his course, but this

Friedrich Hirzebruch is professor of mathematics and retired founding director of the Max Planck Institute for Mathematics in Bonn.

rich material certainly occurred in the many conversations and private seminars of Kodaira, Spencer, and me. In September 1952 Spencer picked me up by car quite regularly at 9 a.m. and drove me to the Institute, where we worked until 5 p.m., mostly with Kodaira, whose course began at the end of September. I also added [37], which is the announcement of his great result that the Hodge manifolds are all projective algebraic, which is fully presented in [38]. I added [63, 66, 68], which together with [60] are the four papers of the famous series "On the structure of compact complex analytic surfaces", which I quoted in my paper *Hilbert modular surfaces* (Enseign. Math. **19** (1973), 183–281) and which I used so much in teaching and research. This paper on Hilbert modular surfaces grew out of my International Mathematical Union lectures, Tokyo, February–March 1972. I remember vividly that Kodaira and Kawada picked us up at the airport. This was the first journey to Japan by my wife and me. Kodaira, having returned to Japan in 1967, was in full action as dean at the University of Tokyo. He introduced me to many of his brilliant students, who later became research visitors in Bonn. For the first time we enjoyed Kunihiko's and Seiko's hospitality in Japan. Finally, I added [64] to the list because it gave rise to my paper *The signature of ramified coverings*, published in the volume *Global Analysis* (University of Tokyo Press, Princeton University Press, 1969), dedicated to Kodaira at the time when he left the United States for Japan.

I have explained how my selection of papers from Kodaira's *Collected Works* was motivated. Among them are eight joint papers by Kodaira and Spencer. Atiyah characterized the collaboration between the two, from which I profited so much: "The Kodaira-Spencer collaboration was more than just a working relationship. The two had very different personalities which were complementary. Kodaira's shyness and reticence were balanced by Spencer's dynamism. In the world of university politics Spencer was able to exercise his talents on Kodaira's behalf, providing a protective environment in which Kodaira's mathematical talents could flourish."

I now begin to go into more detail concerning some of the selected papers. In [28] results of earlier papers are incorporated. Sheaves do not occur yet. The theory of harmonic integrals is used to study the vector space of all meromorphic differentials W of degree n on an n-dimensional Kähler manifold V_n of dimension n which satisfy $(W)+S \geq 0$, where S is a given divisor of V_n. The dimension of this space equals $\dim \mid K+S \mid +1$ if K is a canonical divisor, where $\mid K+S \mid$ is the complete linear system of divisors linearly equivalent to $K+S$ and is called the *adjoint system* of S. Several Riemann-Roch type formulas for $\dim \mid K+S \mid$ are proved. Following Severi, Kodaira introduces the numerical characteristic $a(V_n)$ by the formula

$$(1)\quad a(V_n) = g_n(V_n) - g_{n-1}(V_n) + \ldots + (-1)^{n-1} g_1(V_n),$$

where $g_i(V_n)$ is the dimension of the space of holomorphic differentials of degree i. He formulates a Riemann-Roch theorem for adjoint systems (assuming now that V_n is projective algebraic and E is a smooth hyperplane section for some embedding). He proves (Theorem 2.3.1 in [28])

$$(2)\qquad \dim \mid K+E \mid = a(V_n) + a(E) - 1.$$

Not much later we would say that it is better to consider the holomorphic Euler number

$$(3)\qquad \chi(V_n) = \sum_{i=0}^{n} (-1)^i g_i(V_n)$$

($g_0 = 1$ if V_n is connected) and for a divisor D the number

$$(4)\qquad \chi(V_n, D) = \sum_{i=0}^{n} (-1)^i \dim H^i(V_n, \Omega(D)),$$

where $\Omega(D)$ is the sheaf of local meromorphic functions f with $(f)+D \geq 0$. Then $\dim H^0(V_n, \Omega(D)) = \dim \mid D \mid +1$. By the Kodaira vanishing theorem [35] the spaces $H^i(V_n, \Omega(K+E))$ are zero for $i > 0$. Hence $\dim \mid K+E \mid +1 = \chi(V_n, K+E)$, but by Serre duality $\chi(V_n, K+E) = (-1)^n \chi(V_n, -E)$ (true for any divisor E and for the individual terms in the alternating sum). Serre duality is mentioned in [34] and [35]. Therefore (2) becomes

$$(5)\qquad \chi(V_n, -E) = \chi(V_n) - \chi(E),$$

which follows from an easy exact sequence of sheaves.

But let us go back to [28]. Kodaira proves that $\dim \mid D + hE \mid$ is a polynomial $\nu(h, D)$ in h for large h (often called a Hilbert polynomial) and that $\nu(0, D)$ depends only on D. It is called the virtual dimension of $\mid D \mid$. Now we follow [31]. There are two distinct ways in which arithmetic genera may be defined. In the first place we may define the *arithmetic genus* $P_a(V_n)$ to be the virtual dimension $\nu(0, K)$ of $\mid K \mid$ increased by $1-(-1)^n$ and alternatively the *arithmetic genus* $p_a(V_n)$ by $(-1)^n \nu(0,0)$. In [31] the authors point out that $p_a(V_n) = P_a(V_n)$ has not been established before for $n \geq 5$. They prove it using sheaves. In [28] Kodaira showed $P_a(V_n) = a(V_n)$ in general and $P_a(V_n) = p_a(V_n)$ for a special class of varieties, including complete intersections in projective spaces. A little later we would say

$$\dim \mid D + hE \mid +1 = \chi(V_n, D + hE)$$

for h large by the Kodaira vanishing theorem. But $\chi(V_n, D+hE)$ is a polynomial for all h. Therefore $\nu(0, D) + 1 = \chi(V_n, D)$ and

$$\nu(0,0) + 1 = (-1)^n (\nu(0, K) + 1)$$

At the Institute for Advanced Study, August 1952. Left to right: F. Hirzebruch, Mrs. Kodaira, K. Kodaira.

by Serre duality. But this is the equality $p_a = P_a$. I pointed out "A little later we would say…." Here I must mention the papers [31]–[36] from which I could learn so much (of course, before the papers were written). These papers were all communicated to the *Proceedings of the National Academy of Sciences* by S. Lefschetz. Let us look briefly at [32] and [35].

In [32] Kodaira works with a compact complex analytic variety V of complex dimension n and a holomorphic line bundle F over V and studies the sheaf (faisceau, stack) $\Omega^p(F)$ over V of germs of holomorphic p forms with coefficients in F. I quote from [32]: "The faisceau $\Omega^p(F)$ introduced recently by D. C. Spencer and, independently, by J.-P. Serre turned out to be of importance to applications of faisceaux to the theory of compact analytic varieties. However, for these applications, we need a basic theorem to the effect that the cohomology groups $H^q(V; \Omega^p(F))$ of V with coefficients in $\Omega^p(F)$ have finite dimension. The purpose of the present short note is to give an outline of a proof of this basic theorem." Kodaira uses a Hermitian metric and has to generalize Hodge theory on Kähler manifolds to this more general case using the complex Laplace-Beltrami operator studied earlier by Garabedian and Spencer in the case where F is trivial. The Laplace-Beltrami operator is elliptic. Solution spaces are finite dimensional. With respect to this operator $H^q(V; \Omega^p(F))$ can be identified with the vector space $H^{p,q}(F)$ of all harmonic forms of type (p, q) on V with coefficients in F. In [35] Kodaira writes, "In the present note we shall prove by a differential-geometric method due to Bochner some sufficient conditions for the vanishing of $H^q(V; \Omega^p(F))$ in terms of the characteristic class of the bundle F." In particular he proves: If the characteristic class of F is positive in the sense of Kodaira (representable by a Kähler form), then $H^q(V, \Omega^n(F))$ and $H^{n-q}(V, \Omega^0(-F))$ (Serre duality) both vanish for $1 \le q \le n$. Bochner's papers (*Curvature and Betti numbers, I and II*) appeared in the *Annals of Mathematics* in 1948 and 1949. In [36] Kodaira and Spencer study the holomorphic Euler numbers

$$\chi_V^p(F) = \sum_q (-1)^q \dim H^q(V; \Omega^p(F)) \tag{6}$$

and prove the "form term formula"

$$\chi_V^p(F) = \chi_V^p(F - \{S\}) + \chi_S^p(F_S) + \chi_S^{p-1}(F_S - \{S\}_S) \tag{7}$$

(for a line bundle or divisor F and a smooth hypersurface S), which was very important for me when I studied the polynomial

$$\chi_y(V, F) = \sum_{p=0}^{n} \chi_V^p(F) y^p \tag{8}$$

(also for a holomorphic vector bundle F), where the χ_y-genus $\chi_y(V)$ is the polynomial obtained if F is the trivial line bundle. These polynomials occur in the proof of my Riemann-Roch theorem.

The χ_y-genus is a generalization of the holomorphic Euler number $\chi(V)$ (for $y = 0$) and $\chi(V)$ equals $(-1)^n a(V) + 1$ if $n = \dim V$ (see (3)).

At this point let me emphasize the Paris-Princeton relations of the early 1950s. I recommend reading the letter of Serre to Borel of April 16, 1953 (published in Serre's *Collected Papers*, Vol. 1, No. 20, Springer-Verlag, 1986), and Serre's comments (Vol. 1, p. 588).

In my recent lecture "Learning Complex Analysis in Münster-Paris, Zürich and Princeton from 1945 to 1953" (Journée en l'Honneur d'Henri Cartan, June 14, 1997; *Gazette des Mathématiciens* **74** (1997), 27–39) I talk about Paris-Princeton on pp. 35–36.

In the introduction of my book I speak of four definitions of the arithmetic genus

$$p_a(V), P_a(V), a(V) = g_n - g_{n-1} + \ldots + (-1)^{n-1} g_1$$

and the Todd genus. The basic reference is J. A. Todd, *The arithmetical invariants of algebraic loci*, Proc. London Math. Soc. **43** (1937), 190–225, where Todd uses the characteristic classes K_i of Eger and Todd, which are $(2n - 2i)$-dimensional cycles ($K_1 = K$), to express $(-1)^n P_a(V) + 1$ as a polynomial in the K_i. The proof relies on an unproved lemma of Severi from which Todd concludes that such polynomials must exist. He characterizes them by requesting that they give the correct values on the complete intersection of smooth hypersurfaces of degrees $n_1, n_2, \ldots, n_d$ in the projective space of dimension $2d$. Todd's formalism of his polynomials is very difficult to read. Kodaira ([28], (6.1.1)) gave a formula for the Todd polynomial which is close to my multiplicative sequences and stems from his careful analysis of Todd's paper. However, I do not remember whether I realized this in the old days. Clearly, the power series $(e^x - 1)/x$ is recognizable in his formula, as it is in Todd's formula (22),which can be interpreted as a formula for the arithmetic genus of the complete intersection V of smooth hypersurfaces of degrees $n_1, \ldots, n_d$ in $P_{2d}(\mathbb{C})$ involving the total Todd class of the normal bundle of V in $P_{2d}(\mathbb{C})$ and the total Todd class of $P_{2d}(\mathbb{C})$. Staying close to Todd's formalism, Kodaira proves that the Todd polynomial gives the arithmetic genus $P_a(V)$ for a class of varieties including complete intersections in projective spaces. At one point, relating the characteristic classes of the tangent bundle of a hypersurface to those of the ambient variety and of the normal bundle, he needs the help of S. S. Chern, who had just proved his "duality theorem" for Chern classes. Kodaira was aware of the fact that the K_i of Eger and Todd coincide up to the factor $(-1)^i$ with the Chern classes $c_i \in H^{2i}(V, \mathbb{Z})$. With the use of multiplicative sequences, the inductive proof for Kodaira's result that the Todd polynomials give the arithmetic genus on complete intersections became very simple. At a time when I had formulated the Riemann-Roch theorem but could not yet prove it, I also conjectured as a special case of the Riemann-Roch theorem that the polynomial $\chi_y(V)$ is the genus belonging to the power series

$$\text{(9)} \qquad \frac{x(y+1)}{1-e^{-x(y+1)}} - xy = \frac{x}{f_y(x)} ,$$

and I proved it for complete intersections. This leads me to the following story. (Compare my *Collected Papers*, Vol. 1, Commentaries, Springer-Verlag, 1987, p. 785.)

A. Weil wrote to Kodaira on October 22, 1953, asking in particular for the Hodge numbers of the complete intersection of two quadrics. Kodaira answered on November 4, 1953, explaining my result on the χ_y-genus of complete intersections by which all Hodge numbers of complete intersections are known. At the end of this letter Kodaira writes:

"Recently I could prove that every Hodge variety (i.e. a Kähler variety whose fundamental form $i\Sigma g_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}$ is homologous to an integral cocycle) is an algebraic variety imbedded in a projective space. I believe that my proof is correct; however, I am afraid that my result is too good. I would appreciate very much your comment on this result."

Here Kodaira announces one of his most famous and deep results ([37], communicated by S. Bochner on February 23, 1954, and [38]). I do not know Weil's answer. He must have reacted, because Kodaira wrote him on November 18, 1953, thanking him for a letter and explaining to him in all detail my formulas, which make the calculation of the $h^{p,q}$ of complete intersections more explicit and which are published in the *Proceedings of the International Congress of Mathematicians, Amsterdam, 1954* (my *Collected Papers*, Vol. 1, No. 12, formula (1)).

In the introduction to [38] Kodaira recalls that Hodge had introduced in 1951 the Kähler manifolds of special type and A. Weil had called them Hodge manifolds (A. Weil, *On Picard varieties*, Amer. J. Math. **74** (1952), 865–894). A. Weil proves theorems on Hodge manifolds and recalls Hodge's result that the Picard variety of a Hodge manifold is projective algebraic.

For the proof of his fundamental result Kodaira has to use results of earlier papers, for example, his vanishing theorem. He proves that for a Hodge variety V there exists a real $(1,1)$-form β such that, for any complex line bundle F whose characteristic class contains a closed real $(1,1)$-form $\gamma > \beta$, the holomorphic sections of F define a biregular mapping of V into a projective space (Theorem 3 in [38]).

Kodaira on Princeton campus, 1952.

Kodaira's fundamental theorem generalizes classical results characterizing those complex tori which are projective algebraic. He gives several applications. One was especially important for me. Section 18 of my book *Topological Methods in Algebraic Geometry* carries the title "Some fundamental theorems of Kodaira". I quote Theorem 18.3.1 (Kodaira):

"A complex analytic fiber bundle L over the projective algebraic manifold V with the projective space $P_r(\mathbb{C})$ as fiber and $\mathrm{PGL}(r+1, \mathbb{C})$ as structure group is itself a projective algebraic manifold."

This is used for the proof of my Riemann-Roch theorem, which was completed on December 10, 1953, and announced in the *Proceedings of the National Academy of Sciences* (communicated by S. Lefschetz on December 21, 1953). I had to reduce everything to complex split manifolds where the structural group is the triangular group contained in the general linear group. Then the arithmetic genus can be expressed by virtual signatures which (by the signature theorem as a consequence of Thom's cobordism theory) can be expressed by characteristic classes. But for certain inductive processes I had to stay in the category of projective algebraic manifolds. For a projective algebraic manifold the total space of the flag manifold bundle associated to the tangent bundle is a split manifold. It is projective algebraic by repeated applications of Kodaira's theorem 18.3.1. In my announcement I refer to Kodaira in footnote 9 ("Kodaira, K., not yet published"). I also needed results on the behavior of genera in fiber bundles. The best result is in Appendix Two (by A. Borel) of my book :

"Let $\xi = (E, B, F, \pi)$ be a complex analytic fiber bundle with connected structure group, where E, B, F are compact connected, and F is Kählerian. Let W be a complex analytic fiber bundle over B. Then $\chi_y(E, \pi^* W) = \chi_y(B, W)\chi_y(F)$."

The cooperation with A. Borel in Princeton was of great importance for Kodaira, Spencer, and me in learning characteristic classes and in many other ways, as can be seen, for example, by remarks of Kodaira in [38].

Of course, I am very proud to have one joint paper with Kodaira [41], which was published only in 1957, though I had announced the result already in my talk in Amsterdam in 1954 (loc. cit.). One of my main discoveries (standard joke) is the formula

$$\frac{x}{1-e^{-x}} = e^{x/2}\cdot\frac{x/2}{\sinh x/2},$$

which showed that the Todd genus is expressible by the first Chern class c_1 and the Pontryagin classes. The latter ones do not depend on the complex analytic structure. For a divisor D the number $\chi(V,D)$ depends on the cohomology class $d + c_1/2$ where d is the cohomology class of D and otherwise only on the oriented differentiable manifold V. This we used in [41]. This remark led to the introduction of the $\widehat{A}$-genus which is defined for oriented differentiable manifolds. It equals $\chi(V,D)$ if $2d + c_1 = 0$. From here a new development started whose beginning for me was Atiyah's lecture at the Bonn *Arbeitstagung* in 1962, where it was conjectured that for a spin-manifold the $\widehat{A}$-genus is the index of the Dirac operator. This was proved a little later by Atiyah and Singer as a special case of their general index theorem for linear elliptic operators. The index theorem also included my Riemann-Roch theorem as a special case even for complex manifolds (used by Kodaira in [60]). The paper [41] was for me a sign of the importance of the $\widehat{A}$-genus.

One more word about the χ_y-genus. If S and T are smooth hypersurfaces in the projective algebraic manifold V and if the divisor $S+T$ is also represented by a smooth hypersurface such that the intersections $S\cdot T$ and $S\cdot T\cdot(S+T)$ are transversal and hence smooth, then

$$\begin{aligned}(10)\quad \chi_y(S+T) &= \chi_y(S)+\chi_y(T)\\ &\quad +(y-1)\chi_y(S\cdot T) - y\chi_y(S\cdot T\cdot(S+T)),\end{aligned}$$

which I deduced from the four-term formula (7). The functional equation (10) is also true for the χ_y-genus in terms of characteristic classes using the power series (9). It follows from a corresponding elementary functional equation of $f_y(x)$. Kodaira often proved and used (10) for $y=0$. (See his concept of A-functional in [28], Sections 2.7 and 6.3). It is clear that (10) is useful for the study of complete intersections (inductive proofs).

Kodaira's and Spencer's joint work on deformations of complex analytic structures ([43], [48], and several other papers) is perhaps the greatest achievement of their cooperation. It is an enlightment to read in the introduction of [43], "… we *define* a differentiable family of compact complex structures (manifolds) as a fiber space $\mathcal{V}$ over a connected differentiable manifold M whose structure is a mixture of differentiable and complex structures." Kodaira and Spencer introduce the sheaf Θ on $\mathcal{V}$, the corresponding sheaf of cohomology $\mathcal{H}^1(\Theta)$ on M, and a homomorphism (the Kodaira-Spencer map) $\rho: T_M \to \mathcal{H}^1(\Theta)$ where T_M is the sheaf of germs of differentiable vector fields of M. The complex structure V_t, $t\in M$, is independent of t if and only if ρ vanishes. By restricting Θ to V_t (fixed fiber over $t\in M$) they obtain the homomorphism $\rho_t:(T_M)_t \to H^1(V_t,\Theta_t)$ of Frölicher and Nijenhuis, where $(T_M)_t$ is the tangent space of M at t and Θ_t is the sheaf of germs of holomorphic vector fields on V_t. The vanishing of ρ_t for all t does not imply the vanishing of ρ, as "jumps" show, for example, from the smooth quadric surface to the singular quadric with a node and the node blown up (Atiyah, Brieskorn). Now I quote again from the introduction of [43]: "Next we extend Riemann's concept of number of moduli to higher dimensional complex manifolds (Section 11). The main point here is to avoid the use of the concept of the space of moduli of complex manifolds which cannot be defined in general for higher dimensional manifolds (Section 14, (γ)). Moreover, a necessary condition for the existence of a number $m(V_0)$ of moduli of a complex manifold V_0 is that $H^1(V_0,\Theta_0)$ contain only one deformation space; hence $m(V_0)$ is not defined for all compact complex manifolds…." Kodaira and Spencer find it surprising that $m(V_0) = \dim H^1(V_0,\Theta_0)$ for so many examples and consider a better understanding of this fact as the main problem in deformation theory. I do not want to say more about their deformation theory. Surveys are in Baily's preface to Kodaira's *Collected Works* and in the introduction by K. Ueno and T. Shioda to the volume *Complex Analysis and Algebraic Geometry* (Iwanami Shoten, Publishers, and Cambridge University Press, 1977), dedicated to Kodaira on the occasion of his sixtieth birthday. Anyhow, this report is personal and concerns those aspects of Kodaira's work related to my own. Hence, for lovers of Riemann-Roch, I write what this theorem gives for Θ_0 in dimension $n=1$ (Riemann) and $n=2$ (Max Noether).

$n=1$:

$$\dim H^0(V_0,\Theta_0) - \dim H^1(V_0,\Theta_0) = 3-3g.$$

The number of moduli equals $3g-3+$ dimension of the group of automorphisms of V_0.

$n=2$:

$$\begin{aligned}&\dim H^0(V_0,\Theta_0) - \dim H^1(V_0,\Theta_0)\\ &\qquad + \dim H^2(V_0,\Theta_0) = -10\,\chi(V_0) + 2c_1^2,\end{aligned}$$

where χ is the holomorphic Euler number.

Let us remark that by Serre duality

$$H^i(V_0,\Theta_0) \simeq H^{n-i}(V_0,\Omega^1(K)),$$

which, for $n=1$ and $i=1$, is the isomorphism to the space of holomorphic quadratic differentials (see the obituary by Spencer). We have

$$\chi(V_0, \Theta_0) = (-1)^n \chi^1_{V_0}(K).$$

These numbers can be calculated by the Riemann-Roch theorem as linear combinations of Chern numbers. For a Kähler manifold with trivial canonical bundle, $\dim H^i(V_0, \Theta_0)$ equals the Hodge number $h^{1,n-i}$. For a $K3$-surface we have $h^{1,1} = 20$. Kodaira and Spencer discuss many more examples. For the complex projective space $P_n(\mathbb{C})$ we have $\dim H^1(P_n(\mathbb{C}), \Theta_0) = 0$ in agreement with the result in [41].

With the exception of three papers, the whole Volume III of Kodaira's *Collected Works* is concerned with complex analytic surfaces. His work in this area is overwhelming. He can use his earlier papers on complex manifolds and on deformations. I have used the papers in Volume III very often. Looking, for example, at my joint paper with A. Van de Ven, *Hilbert modular surfaces and the classification of algebraic surfaces* (Invent. Math. **23** (1974), 1–29), I find that we used the following:

1. *Rough classification of surfaces, Kodaira dimension* ([68], Theorem 55). Kodaira proves that the compact complex surfaces free from exceptional curves can be divided into seven classes. Class 5 comprises the minimal algebraic surfaces of general type. Class 7 surfaces are mysterious surfaces with first Betti number equal to 1. Van de Ven and I specialize Kodaira's classification to algebraic surfaces, where this classification in broad outline was known to the Italian school, but many of the proofs are due to Kodaira.
2. *Kodaira's proof of Castelnuovo's criterion for the rationality of algebraic surfaces.*
3. *Study of elliptic surfaces, their multiple fibers, and a formula for the canonical divisor.*
4. *Classification of the exceptional fibers in elliptic surfaces.*
5. *The fact that all $K3$-surfaces are homeomorphic and hence simply connected.* Kodaira proves more ([60], Theorem 13): Every $K3$-surface is a deformation of a nonsingular quartic surface in a projective 3-space.

The surfaces in Class 7 are also called VII_0-surfaces ([60], Theorem 21). Masahisa Inoue (*New surfaces with no meromorphic functions II*, in the volume dedicated to Kodaira's sixtieth birthday) has constructed such surfaces using my resolution of the cusp singularities of Hilbert modular surfaces. Such a surface has only finitely many curves on it. They are rational and arranged in two disjoint cycles.

Now a last case where a paper of Kodaira was especially close to my interest. In [64] he constructed algebraic surfaces with positive signature whose total spaces are differentiable fiber bundles with compact Riemann surfaces as base and fiber. In the early 1950s we did not know a single surface with signature greater than 1 and often talked about it at Princeton. How the situation developed over the years can be seen, for example, in the book *Geradenkonfigurationen und Algebraische Flächen* (by Gottfried Barthel, Thomas Höfer, and me, Vieweg, 1987). Also, Kodaira's surfaces give examples in which the signature of the total space of a fibration is not equal to the product of the signatures of base and fiber. The multiplicativity of the signature in fiber bundles (of oriented manifolds) was proved by S. S. Chern, J.-P. Serre, and me under the assumption that the fundamental group of the base operates trivially on the real cohomology of the fiber (*On the index of a fibered manifold*, Proc. Amer. Math. Soc. **8** (1957), 587–596).

Grauert, Hirzebruch, Remmert return to Germany after the Conference on Analytic Functions at the Institute for Advanced Study, September 1957. From left to right: F. Hirzebruch, Joan Frankel (Mrs. Theodore Frankel), D. C. Spencer, K. Kodaira, A. Borel, W.-L. Chow, H. Grauert. Foreground: R. Remmert.

Far from attempting to give a thorough appreciation of Kodaira's great mathematical work, I wanted to show how much I am indebted to him and where our mathematical lives crossed.

Selected Papers from Kodaira's Collected Works

[22] *The theorem of Riemann-Roch on compact analytic surfaces*, Amer. J. Math. **73** (1951), 813–875.

[24] *The theorem of Riemann-Roch for adjoint systems on 3-dimensional algebraic varieties*, Ann. of Math. **56** (1952), 298–342.

[28] *The theory of harmonic integrals and their applications to algebraic geometry,* Work done at Princeton University, 1952.

[31] *On arithmetic genera of algebraic varieties* (collaborated with D. C. Spencer), Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 641–649.

[32] *On cohomology groups of compact analytic varieties with coefficients in some analytic faisceaux*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 865–868.

[33] *Groups of complex line bundles over compact Kähler varieties* (collaborated with D. C. Spencer), Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 868–872.

[34] *Divisor class groups on algebraic varieties* (collaborated with D. C. Spencer), Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 872–877.

[35] *On a differential-geometric method in the theory of analytic stacks*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 1268–1273.

[36] *On a theorem of Lefschetz and the Lemma of Enriques-Severi-Zariski* (collaborated with D. C. Spencer), Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 1273–1278.

[37] *On Kähler varieties of restricted type*, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 313–316.

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