Gray Codes for Set Partitions and Restricted Growth Tails

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Abstract

We survey results on generating set partitions, $S(n)$, and restricted growth tails, $T(n,k)$, with an emphasis on Gray code listings, in which the change between successive elements is minimal. Although there is a well-known bijection between $S(n)$ and $T(n,0)$, it preserves minimal changes in only one direction.

We show that a minimal Gray code listing for $T(n,k)$ is not always possible, although for a slight relaxation of the adjacency criterion, we can construct Gray codes for all $n > 0$, $k \geq 0$. One consequence is a Gray code for $S(n)$ in which only one element changes, to a cyclically adjacent block, between successive partitions on the list. This generalizes earlier work of Knuth for $S(n)$ and Ehrlich for $T(n,0)$. Our construction for $T(n,k)$ yields Gray codes which can be required to be cyclic, or to go from the lexicographically minimum to maximum elements, properties not possessed by the Gray codes of Ehrlich and Knuth.

1 Introduction

For a given integer $n > 0$, a *set partition* is a decomposition of $\{1, \ldots, n\}$ as a disjoint union of nonempty subsets called *blocks* ([8], p. 18). The set of all partitions of $\{1, \ldots, n\}$ is denoted $S(n)$; $S(4)$ is listed in Figure 1(a). The *restricted growth functions* (RG functions) of length n, denoted $R(n)$, are those strings $a_1 \ldots a_n$ of non-negative integers satisfying $a_1 = 0$ and $a_i \leq 1 + \max\{a_1, \ldots, a_{i-1}\}$ ([8], p. 18).

With each $\pi \in S(n)$, associate a string $a_1 \cdots a_n$ as follows. Order the blocks of π according to their smallest element, for example, the blocks of $\pi = \{ \{9\}, \{1, 2, 7\}, \{4, 10, 11\}, \{3, 5, 6, 8\} \}$ would be ordered $\{1, 2, 7\}, \{3, 5, 6, 8\}, \{4, 10, 11\}, \{9\}$. Label the blocks of π in order by $0, 1, 2, \ldots$ and for $1 \leq i \leq n$, let a_i be the

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(a) $S(4)$	(b) $L(4)$ in lex. order	(c) Knuth's Gray code	(d) modified Knuth	(e) Ehrlich's algorithm
$\{1, 2, 3, 4\}$	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\{1, 2, 3\}, \{4\}$	0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 1
$\{1, 2, 4\}, \{3\}$	0 0 1 0	0 0 1 2	0 0 1 2	0 0 1 1
$\{1, 2\}, \{3, 4\}$	0 0 1 1	0 0 1 1	0 0 1 1	0 0 1 2
$\{1, 2\}, \{3\}, \{4\}$	0 0 1 2	0 0 1 0	0 0 1 0	0 0 1 0
$\{1, 3, 4\}, \{2\}$	0 1 0 0	0 1 2 0	0 1 1 0	0 1 1 0
$\{1, 3\}, \{2, 4\}$	0 1 0 1	0 1 2 1	0 1 1 1	0 1 1 2
$\{1, 3\}, \{2\}, \{4\}$	0 1 0 2	0 1 2 2	0 1 1 2	0 1 1 1
$\{1, 4,\}, \{2, 3\}$	0 1 1 0	0 1 2 3	0 1 2 2	0 1 2 1
$\{1\}, \{2, 3, 4\}$	0 1 1 1	0 1 1 2	0 1 2 3	0 1 2 2
$\{1\}, \{2, 3\}, \{4\}$	0 1 1 2	0 1 1 1	0 1 2 1	0 1 2 3
$\{1, 4\}, \{2\}, \{3\}$	0 1 2 0	0 1 1 0	0 1 2 0	0 1 2 0
$\{1\}, \{2, 4\}, \{3\}$	0 1 2 1	0 1 0 0	0 1 0 0	0 1 0 0
$\{1\}, \{2\}, \{3, 4\}$	0 1 2 2	0 1 0 1	0 1 0 1	0 1 0 2
$\{1\}, \{2\}, \{3\}, \{4\}$	0 1 2 3	0 1 0 2	0 1 0 2	0 1 0 1

Figure 1: Listings of $S(4)$ and $R(4)$.

label of the block containing i. The associated string for π above is 0 0 1 2 1 1 0 1 3 2 2. Note that $a_1 \ldots a_n \in R(n)$ and, in fact, this mapping is a bijection between $S(n)$ and $R(n)$ ([8], p. 18-19). For $n = 4$, the bijection is illustrated in the first two columns of Figure 1.

A Gray code for a combinatorial family is a listing of the objects in the family so that successive objects differ in some pre-specified, usually small, way [3]. Although any listing algorithm for one of $S(n)$ or $R(n)$ can be used for the other, small changes between objects of one family may be magnified by the bijection. For example, the partitions $\pi_1 = \{1, 2, 5\}, \{3, 6\}, \{4\}$ and $\pi_2 = \{1, 3, 6\}, \{2, 5\}, \{4\}$ differ only in that element 1 changes sets. However, the RG functions associated with π_1 and π_2 are 0 0 1 2 0 1 and 0 1 0 2 1 0, which differ in several positions.

In [4], Kaye describes a Gray code $L(n)$ for $S(n)$, attributed to Knuth, where between successive partitions, only one element moves and that move is to an adjacent block. The listing is recursive with $L(1) = \{1\}$; the list $L(n)$ is obtained from $L(n-1)$ by replacing every $\pi = B_1, \ldots, B_{b(\pi)}$ on $L(n-1)$ by the sublist:

$$\begin{array}{l} B_1 \cup \{n\},\ B_2,\ \ldots,\ B_{b(\pi)}, \\ B_1,\ B_2 \cup \{n\},\ \ldots,\ B_{b(\pi)}, \\ \quad\vdots \\ B_1,\ B_2,\ \ldots,\ B_{b(\pi)} \cup \{n\}, \\ B_1,\ B_2,\ \ldots,\ B_{b(\pi)},\ \{n\}, \end{array}$$

in the given order if π has odd rank on $L(n-1)$ and in reverse order if π has even rank. In terms of RG functions, this corresponds to successively appending $0, 1, \ldots, b+1$, in the given or reversed order, to the RG function associated with π. (See Figure

1(c).) Although $L(n)$ is a Gray code listing of $S(n)$, the RG functions associated with successive elements of $L(n)$ may differ in a number of positions which can grow arbitrarily large with n.

On the other hand, if $R(n)$ can be listed so that successive elements differ in just one position, then in the corresponding listing of $S(n)$, only one element moves to a different block. In [6], Ruskey describes a modification of Knuth's algorithm in which the associated RG functions differ by at most 2, as follows: if π, of odd rank on $L(n-1)$, has b blocks, but its successor has only $b-1$ blocks, then to the RG function for π, append successively $0,\ 1,\ \ldots,\ b-1,\ b+1,\ b$. Similarly, if π has even rank and b blocks, but its predecessor has only $b-1$ blocks, then to the RG function for π, append successively the elements in the reverse order of the sublist above. (See Figure 1(d).)

Call two elements of $R(n)$ *strictly adjacent* in they differ in only one position and in that position by only 1. A listing of $R(n)$ is a *strict Gray code* if successive elements are strictly adjacent. Ehrlich observed that a strict Gray code for $R(n)$ is impossible for infinitely many values of n [1]. Nevertheless, he was able to find an efficient listing algorithm for $R(n)$ (loop-free) which has the following interesting property: successive elements differ in one position and the element in that position can change by 1, or, if it is the largest element in the string, it can change to 0, or conversely a 0 can change to the largest value v or to $v+1$. For example, 0 1 0 2 0 2 1 can change to 0 1 0 2 0 0 1, and conversely. In the associated list of set partitions, this change corresponds to moving one element to an adjacent block in the partition, where the first and last blocks are considered adjacent. The algorithm of Ehrlich is similar to that of Knuth in that to each $\mathbf{a} = a_1 \ldots a_{n-1}$ on the list for $R(n-1)$, we successively append the values $0,\ 1,\ \ldots,\ b+1$, but in a different order: $1,\ 2,\ \ldots,\ b+1,\ 0$ if $\mathbf{a}$ has odd rank, and the reverse of this order otherwise. (See Figure 1(e).)

In this paper we generalize the results of Ehrlich to the set of *restricted growth tails*, $T(n,k)$, which are strings of non-negative integers satisfying $a_1 \leq k$ and $a_i \leq 1 + \max\{a_1, \ldots, a_{i-1},\ k-1\}$. (These are a variation of the $T(n,m)$ used in [9] (p. 97) for ranking and unranking set partitions.) Note that $T(n,0) = R(n)$. In Section 2, we show that for all k there are infinitely many values of n for which $T(n,k)$ has no strict Gray code. However, if we relax this to allow that the largest element in a string can change to 0 (as long as it is at least k), we show that Gray codes are possible for all $n > 0,\ k \geq 0$.

Our construction gives several families of Gray codes for $T(n,k)$, and thus also for $R(n)$ and $S(n)$. In particular, they can be required to be *circular* (first and last elements are adjacent) or *min-max* (starts and ends at the lexicographically minimum and maximum element, respectively.) These are properties not possessed by the Gray codes of Knuth, Ruskey, and Ehrlich.

In Section 3, we consider the case where the number of blocks of the partition is fixed. It is shown that strict Gray codes are not possible in general, even under the modified criterion of Ehrlich.

Algorithm efficiency, historical notes, and open problems are discussed in Section 4.

2 Gray Codes for Restricted Growth Tails

For $n > 0$ and $k \geq 0$, the set of RG tails $T(n,k)$, defined in Section 1, can be decomposed according to the element in the first position as the disjoint union

$$T(n,k) = \bigcup_{i=0}^{k-1} i \cdot T(n-1,k) \quad \cup \quad k \cdot T(n-1,k+1). \tag{1}$$

Let $G(n,k)$ be the graph with vertex set $T(n,k)$ and with strictly adjacent RG tails adjacent in $G(n,k)$. Then a strict Gray code for $T(n,k)$ is a Hamilton path in $G(n,k)$. Note that $G(n,k)$ is bipartite: let

$$E(n,k) = \{a_1 \ldots a_n \in T(n,k) \mid \sum_{i=1}^{n} a_i \text{ is even}\}$$

and let $O(n,k) = T(n,k) \setminus E(n,k)$. Then two vertices in $E(n,k)$ or $O(n,k)$ cannot be adjacent in $G(n,k)$. Furthermore, the vertex $v^* = k\ k+1\ \cdots k+n-1$ of $G(n,k)$ has degree 1 since it is adjacent only to $k\ k+1\ \cdots\ k+n-2$.

Let $d(n,k) = |E(n,k)| - |O(n,k)|$. Then using the recurrence (1),

$$d(n,k) = \sum_{i=0}^{k-1} (-1)^i d(n-1,k) \quad + \quad (-1)^k d(n-1,k+1)$$

so that

$$d(n,k) = \begin{cases} d(n-1,k+1) & \text{if } k \text{ is even} \\ d(n-1,k) - d(n-1,k+1) & \text{otherwise.} \end{cases}$$

If $G(n,k)$ has a Hamilton path, then $|d(n,k)| \leq 1$. Furthermore, if $d(n,k) = 1$, then v^* must be in $E(n,k)$, and if $d(n,k) = -1$, then v^* must be in $O(n,k)$.

It can be shown by induction that

$$d(n,k) = \begin{cases} 0 & \text{if } k \text{ is even and } n \equiv 2,5 \pmod 6 \\ & \text{ or } k \text{ is odd and } n \equiv 1,4 \pmod 6\text{, or} \\ 1 & \text{if } k \text{ is even and } n \equiv 1,6 \pmod 6 \\ & \text{ or } k \text{ is odd and } n \equiv 5,6 \pmod 6\text{, or} \\ -1 & \text{if } k \text{ is even and } n \equiv 3,4 \pmod 6 \\ & \text{ or } k \text{ is odd and } n \equiv 2,3 \pmod 6. \end{cases}$$

Note that v^* is in $E(n,k)$ if and only if

$$k + (k+1) + \cdots + (k+n-1) \; = \; \binom{k+n}{2} - \binom{k}{2}$$

is even. Since $\binom{n}{2}$ is even exactly when $n \pmod 4 \in \{2,3\}$, the vertex v^* is in $E(n,k)$ if and only if

- k is even and $n \equiv 0,1 \pmod 4$, or

- k is odd and $n \equiv 0, 3 \pmod 4$.

Thus, if $(n, k) \in X$, where X is the set of ordered pairs for which

- k is even and $n \equiv 4, 6, 7, 9 \pmod{12}$, or
- k is odd and $n \equiv 3, 5, 6, 8 \pmod{12}$,

then there is no Hamilton path in $G(n, k)$.

We call two elements $a_1 \cdots a_n,\ b_1 \cdots b_n \in T(n, k)$ *weakly adjacent* if they differ in one position i and in that position, either

$$a_i = b_i - 1 \text{ or}$$

$$a_i = \max\{a_1, \ldots, a_n\}, \quad a_i \geq k \text{ and } b_i = 0.$$

Note that if $\mathbf{a},\ \mathbf{b} \in T(n, k)$ are weakly adjacent and if $j \leq k$, then $j \cdot \mathbf{a}$ and $j \cdot \mathbf{b}$ in $j \cdot T(n, k)$ are weakly adjacent.

Let $G'(n, k)$ be the graph with vertex set $T(n, k)$ and with weakly adjacent RG tails adjacent in $G'(n, k)$. A weak Gray code for $T(n, k)$ is a Hamilton path in $G'(n, k)$.

Theorem 1 *For $n \geq 1$ and $k \geq 0$, $G'(n, k)$ has a Hamilton path from α to β for each of the pairs α, β below.*
(a) $0\ 0\ \cdots\ 0$ to $k\ k+1\ \cdots\ k+n-1$.
(b) $k-1\ 0\ \cdots\ 0$ to $k\ k+1\ \cdots\ k+n-1$ (if $k \geq 1$).
(c) $i\ 0\ \cdots\ 0$ to $(i+1) \bmod (k+1)\ \ 0\ \cdots\ 0$ for $0 \leq i \leq k$ (if $k \geq 1$.)

Proof. We use induction on n. For the cases $T(1,0) = \{0\}$ and $T(1,1) = \{0,\ 1\}$, (a)-(c) are clearly satisfied. If $n = 1$ and $k \geq 2$, then $0,\ 1,\ \ldots,\ k$ is a cycle in $G'(n, k)$. This cycle contains paths satisfying (a)-(c).

For $n \geq 2$, decompose $T(n, k)$ as the disjoint union

$$T(n, k) = \bigcup_{i=0}^{k-1} i \cdot T(n-1, k) \quad \cup \quad k \cdot T(n-1,\ k+1).$$

If $k = 0$, then $T(n, 0) = 0 \cdot T(n-1, 1)$. By induction, $G'(n-1, 1)$ has a Hamilton path from $0\ 0 \cdots\ 0$ to $1\ 2\ \cdots\ n-1$. Adding a prefix of 0 to each vertex on this path gives a Hamilton path in $G'(n, 0)$ satisfying (a).

When $n \geq 2$ and $k \geq 1$, by induction, the following Hamilton paths exist:

1. p_i in $G'(n-1, k)$ from $i\ 0\ \cdots\ 0$ to $(i+1) \bmod (k+1)\ \ 0\ \cdots\ 0$ for $0 \leq i \leq k$.
2. q_i in $G'(n-1, k+1)$ from $i\ 0\ \cdots\ 0$ to $(i+1) \bmod (k+2)\ \ 0\ \cdots\ 0$ for $0 \leq i \leq k+1$.
3. r in $G'(n-1, k+1)$ from $k\ 0\ \cdots\ 0$ to $k+1\ \ k+2\ \cdots\ k+n-1$.

Then the required Hamilton paths in $G'(n, k)$ are (see Figures 2, 3, 4):

(a) $0 \cdot p_0,\ 1 \cdot p_1,\ \ldots,\ (k-1) \cdot p_{k-1},\ k \cdot r$.

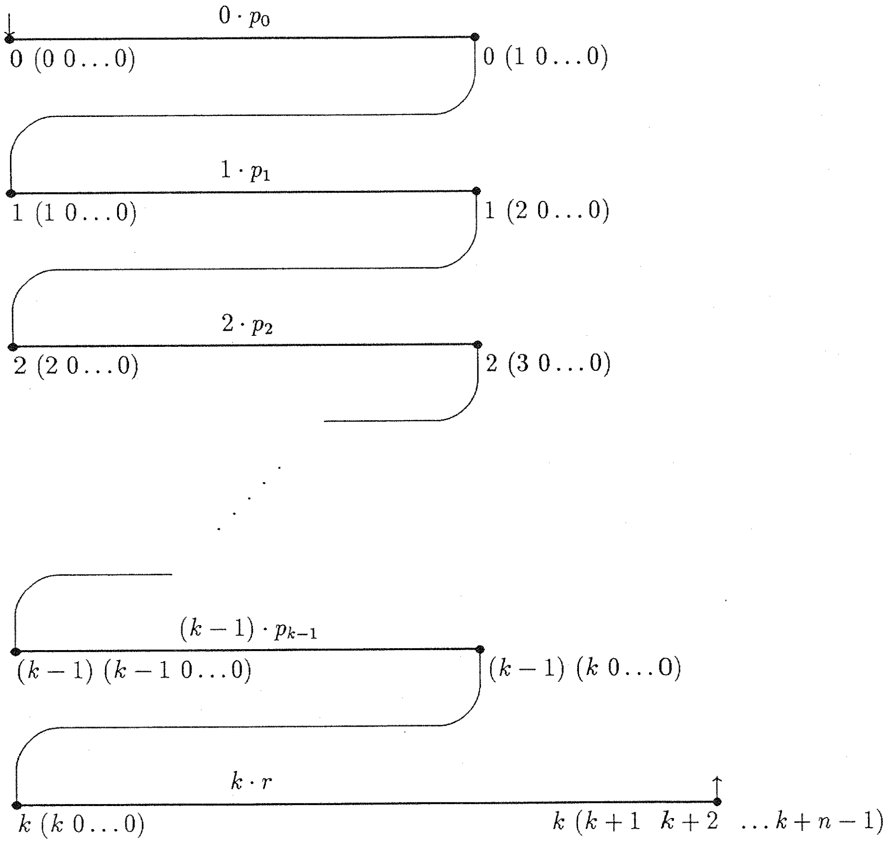

Figure 2: Construction of path satisfying (a).

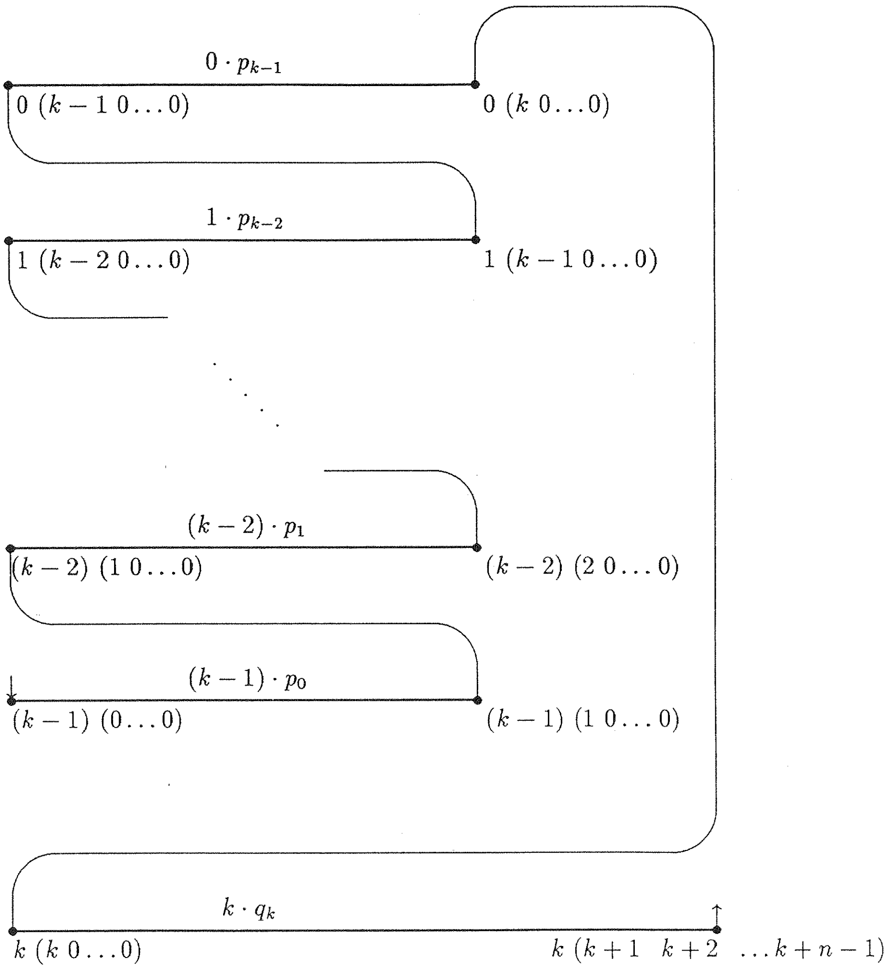

Figure 3: Construction of path satisfying (b).

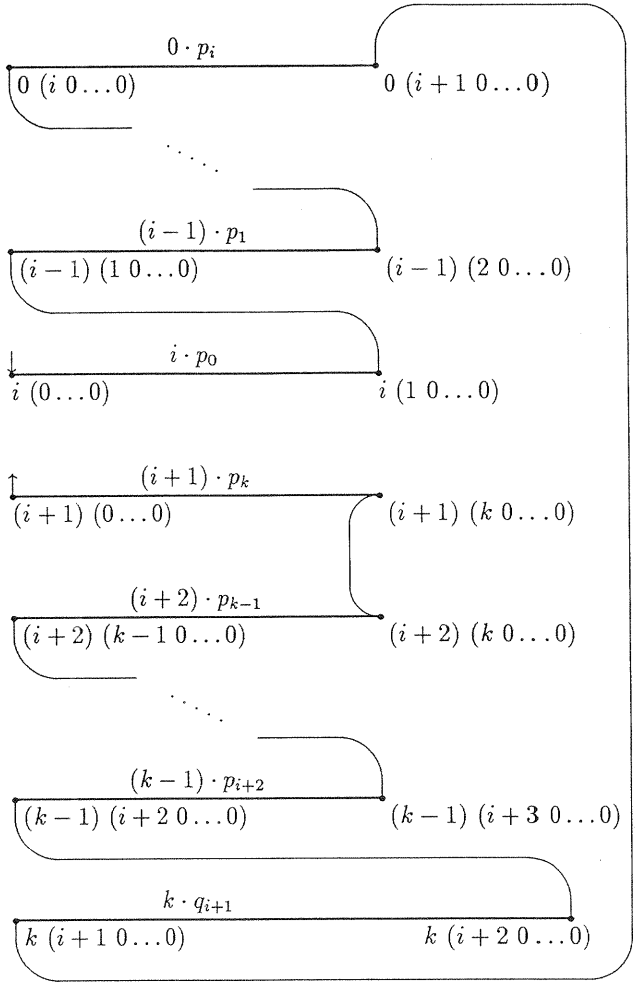

Figure 4: Construction of path satisfying (c) when $k \geq 2$ and $0 \leq i \leq k-2$.

(b) $(k-1)\cdot p_0,\ (k-2)\cdot p_1,\ \ldots, 1\cdot p_{k-2},\ 0\cdot p_{k-1},\ k\cdot q_k$.

(c) If $k=1$, then the path $0\cdot p_0,\ 1\cdot q_0^{-1}$, when $i=0$ and its reverse if $i=1$.

If $k\geq 2$ then

- $i\cdot p_0,\ (i-1)\cdot p_1,\ \ldots,\ 0\cdot p_i,\ k\cdot q_{i+1},\ (k-1)\cdot p_{i+2},\ \ldots,\ (i+2)\cdot p_{k-1},\ (i+1)\cdot p_k$, when $0\leq i\leq k-2$,
- $(k-1)\cdot p_k^{-1},\ (k-2)\cdot p_{k-1}^{-1},\ \ldots, 0\cdot p_1^{-1},\ k\cdot q_0^{-1}$ when $i=k-1$, and
- $k\cdot q_0,\ (k-1)\cdot p_1,\ (k-2)\cdot p_2,\ \ldots,\ 1\cdot p_{k-1},\ 0\cdot p_k$, when $i=k$.

□

Corollary 1 *$R(n)$ has cyclic (weak) Gray code and a min-max (weak) Gray code.*

Proof. By (1), $R(n)=T(n,0)=0\cdot T(n-1,1)$. By Theorem 1(a) and (c), $T(n-1,1)$ has weak Gray codes from $0\cdots 0$ to $1\,2\cdots n{-}1$ and from $0\cdots 0$ to $1\,0\cdots 0$. Prepending 0 to the elements of these lists gives weak Gray codes for $R(n)$ from from $0\,0\cdots 0$ to $0\ 1\ 2\cdots n-1$ (min-max) and from $0\ 0\cdots 0$ to $0\ 1\ 0\cdots 0$ (cyclic). Examples when $n=4$ are given in Figure 5. □

min-max	**circular**
0 0 0 0	0 0 0 0
0 0 0 1	0 0 0 1
0 0 1 1	0 0 1 1
0 0 1 2	0 0 1 2
0 0 1 0	0 0 1 0
0 1 1 0	0 1 1 0
0 1 1 2	0 1 1 1
0 1 1 1	0 1 1 2
0 1 0 1	0 1 2 2
0 1 0 0	0 1 2 3
0 1 0 2	0 1 2 0
0 1 2 2	0 1 2 1
0 1 2 1	0 1 0 1
0 1 2 0	0 1 0 2
0 1 2 3	0 1 0 0

Figure 5: New (weak) Gray codes for $R(4)=T(4,0)=0\cdot T(3,1)$.

3 Partitions Into a Fixed Number of Blocks

For $n \geq 1$ and $0 \leq b \leq n-1$, let $S_b(n)$ be the set of partitions of $\{1,\ldots,n\}$ into $b+1$ blocks. The bijection between $S(n)$ and $R(n)$ restricts to a bijection between $S_b(n)$ and

$$R_b(n) = \{a_1 \ldots a_n \in R(n) \mid \max\{a_1,\ldots,a_n\} = b\}.$$

Ehrlich presents an algorithm for generating $S_b(n)$ in which successive partitions differ only in that two elements have moved to different blocks [1]. Ruskey describes a Gray code for $R_b(n)$ in which successive elements differ in one position, but possibly by more than 1 in that position [7]. We show in this section that in general, $R_b(n)$ has neither a strict nor a weak Gray code.

For $n \geq 1$, $R_0(n) = \{0\,0 \cdots 0\}$, $R_{n-1}(n) = \{0\,1 \cdots n-1\}$, and for $0 < b < n-1$, $R_b(n)$ can be partitioned as the disjoint union

$$R_b(n) = \bigcup_{i=0}^{b} R_b(n-1) \cdot i \quad \cup \quad R_{b-1}(n-1) \cdot b. \tag{2}$$

Theorem 2 *For infinitely many positive values of n and b, $R_b(n)$ has neither a strict nor a weak Gray code.*

Proof. Let $G_b(n)$ be the subgraph of $G(n,0)$ induced by $R_b(n)$. Then $G_b(n)$ is bipartite with bipartition $E_b(n)$, $O_b(n)$, where $E_b(n) = E(n,0) \cap R_b(n)$, $O_b(n) = O(n,0) \cap R_b(n)$, and $d_b(n) = |E_b(n)| - |O_b(n)|$.

Clearly $d_0(n) = 1$ and $d_{n-1}(n) = (-1)^{n(n-1)/2}$. Using the recurrence (2) for $n \geq 1$ and $0 < b < n-1$,

$$d_b(n) = \begin{cases} -d_{b-1}(n-1) & \text{if } b \text{ is odd} \\ d_b(n-1) + d_{b-1}(n-1) & \text{otherwise.} \end{cases}$$

(See Table 1.) A strict Gray code for $R_b(n)$ is a Hamilton path in $G_b(n)$, which can exist only if $|d_b(n)| \leq 1$. However, it can be shown by induction that

$$d_b(n) \;=\; (-1)^{\lceil b/2 \rceil} \binom{n-1-\lceil b/2 \rceil}{\lfloor b/2 \rfloor}$$

and thus that for $1 < b < n-1$, $d_b(n) > 1$ if $\lceil b/2 \rceil$ is even and $d_b(n) < 1$ if $\lceil b/2 \rceil$ is odd. Thus, no strict Gray code can exist in these cases.

Let $G'_b(n)$ be the graph with vertex set $R_b(n)$ and with weakly adjacent elements of $R_b(n)$ adjacent in $G'_b(n)$. Note that $G'_b(n)$ differs from $G_b(n)$ only in that for $1 \leq i \leq n$, strings $a_1 \ldots a_{i-1} b\ a_{i+1} \ldots a_n$ and $a_1 \ldots a_{i-1} 0\ a_{i+1} \ldots a_n$ are now adjacent. If b is odd, $G'_b(n)$ is still bipartite and $E_b(n)$, $O_b(n)$, and $d_b(n)$ are the same as for $G_b(n)$. So, no weak Gray code can exist in these cases. $\square$

4 Concluding Remarks

It remains open whether there is a strict Gray code for $T(n,k)$ when the $(n,k) \notin X$. Similarly, when b is even, does $S_b(n)$ have a weak Gray code?

$n\backslash b$	0	1	2	3	4	5	6	7
1	1							
2	1	-1						
3	1	-1	-1					
4	1	-1	-2	1				
5	1	-1	-3	2	1			
6	1	-1	-4	3	3	-1		
7	1	-1	-5	4	6	-3	-1	
8	1	-1	-6	5	10	-6	-4	1
9	1	-1	-7	6	15	-10	-10	4
10	1	-1	-8	7	21	-15	-20	10
11	1	-1	-9	8	28	-21	-35	20

Table 1: Table of values for $d_b(n)$.

Assume that a listing algorithm must generate N objects of size $O(n)$, where usually $n \ll N$. The efficiency is measured by the time to generate the first object and the delay between successive elements generated. In a *loop-free* algorithm [1], the start-up time is $O(n)$ and the worst-case delay is constant, independent of n and N. Next best is to have a constant amortized time (CAT) algorithm in which the total time is $O(N)$, even though some delays may be large [7].

In his 1973 paper, Ehrlich [1] gave a loop-free implementation of his Gray code algorithm for $R(n)$ and a loop-free algorithm for generating $S_b(n)$, results which have been overlooked by some later papers. Kaye's 1976 paper contains a CAT implementation of Knuth's Gray code for $S(n)$ [4]. The solutions manual by Fill and Reingold [2] for the book by Reingold, Nievergelt, and Deo [5] also presents a Gray Code for $S_b(n)$, which they attibute to Brian Hansche. Ruskey gives a CAT implementation of his Gray code for $S_b(n)$ [7]. We conjecture that our new Gray codes for RG tails, $T(n, k)$, can be implemented by a CAT algorithm.

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