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Cycle-up-down permutations

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To the memory of Philippe Flajolet.

Abstract

A permutation is defined to be *cycle-up-down* if it is a product of cycles that, when written starting with their smallest element, have an up-down pattern. We prove bijectively and analytically that these permutations are enumerated by the Euler numbers, and we study the distribution of some statistics on them, as well as on up-down permutations and on all permutations. The statistics include the number of cycles of even and odd length, the number of left-to-right minima, and the number of extreme elements.

1 Introduction

Let $[n] = \{1, 2, \dots, n\}$, and let $\mathcal{S}_n$ be the set of permutations of $[n]$. A permutation π of $[n]$ can be written in one-line notation as $\pi = \pi_1\pi_2 \dots \pi_n$ or as a product of cycles as

$$\pi = (a_{11}, a_{12}, \dots)(a_{21}, a_{22}, \dots)(a_{31}, a_{32}, \dots) \dots .$$

We will write commas in the cycle notation but not in the one-line notation, in order to distinguish them. A cycle is said to be in *standard form* if its smallest element is in first position. Every permutation π has a unique expression as a product of cycles in standard form where the first entries of the cycles are in increasing order. For example, $\pi = 2517364 = (1, 2, 5, 3)(4, 7)(6)$. A *cyclic permutation* is a permutation that consists of only one cycle.

An *up-down* permutation of $[n]$ is a permutation π satisfying

$$\pi_1 < \pi_2 > \pi_3 < \pi_4 > \dots .$$

Let Λ_n denote the set of up-down permutations of length n. It is well known [8] that the size of Λ_n is the Euler number E_n. The exponential generating function (EGF for short) for the Euler numbers is

$$E(z) = \sec z + \tan z = \sum_{n\geq 0} E_n \frac{z^n}{n!},$$

and the first values of E_n are $1, 1, 1, 2, 5, 16, 61, 272, \dots$. A permutation satisfying $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \dots$ is called a *down-up* permutation.

In this paper we are concerned with a variation of the above definition. Instead of requiring the one-line notation to be up-down, we will study permutations whose cycles, when written in standard form, have an up-down pattern. A precise definition of these permutations, which we call *cycle-up-down permutations*, is given in Section 2. It is also shown, both analytically and bijectively, that they are enumerated by the Euler numbers. In Section 3 we obtain refined generating functions for these families of permutations with respect to several statistics. We also find statistics that are preserved by our bijections between up-down and cycle-up-down permutations. In Section 4 we study the distribution of some of these permutation statistics on $\mathcal{S}_n$, obtaining some formulas that involve the Stirling numbers of the first kind. Finally, Section 5 gives another interpretation of cycle-up-down permutations in terms of perfect matchings.

Let us now introduce some notation and definitions that will be needed later on. Let $A = \{a_1, a_2, \dots, a_n\}$ be a finite subset of the natural numbers, and assume that $a_1 < \dots < a_n$. Given a permutation σ of A, let $\overline{\sigma}$ be the permutation obtained by replacing each entry a_i of the one-line notation of σ with a_{n+1-i}. For example, $\overline{2634} = 6243$. Following [5], we will call this operation a *switch*.

We say that π_i is a *left-to-right (LR) minimum* (respectively *maximum*) of a permutation π if $\pi_i < \pi_j$ (respectively $\pi_i > \pi_j$) for all $1 \leq j < i$. Note that π_1 is both an LR minimum and an LR maximum. If $i \geq 2$, we say that π_i is an *extreme element* if it is either an LR minimum or an LR maximum (see [6, p. 98]). The *min-max sequence* of π is defined to be the sequence $\pi_{i_1}\pi_{i_2}\dots\pi_{i_k}$ where $\pi_{i_1} = 1$ is the smallest entry of π, π_{i_2} is the largest entry to the right of π_{i_1}, π_{i_3} is the smallest entry to the right of π_{i_2}, and so on. Note that we always have $i_k = n$. For example, the min-max sequence of $\pi = 48127635$ is 1735. An *excedance* (respectively *deficiency*) of π is a value i such that $\pi_i > i$ (respectively $\pi_i < i$). We use the term *odd cycle* (respectively *even cycle*) to mean a cycle of odd (respectively even) length.

2 Cycle-up-down permutations

A cycle is said to be *up-down* if, when written in standard form, say $(b_1, b_2, \dots)$, one has $b_1 < b_2 > b_3 < \cdots$. Note that up-down cycles of even length can alternatively be described as fully alternating cycles, meaning that in any representation of the cycle, the entries alternate going up and down. On the other hand, up-down cycles of odd length have this property only when written in standard form.

We define a permutation π to be *cycle-up-down* (*CUD* for short) if it is a product of up-down cycles. For example, $(1,5,2,7)(3)(4,8,6)(9)$ is CUD, but $(1,3,5)(2,4)(6)$ is not. Let Δ_n be the set of CUD permutations of $[n]$. The main result of this section is the enumeration of CUD permutations.

Proposition 2.1. *The number of CUD permutations of* $[n]$ *is* E_{n+1}.

In subsection 2.2 we give a bijective proof of this fact. In order to construct a bijection, it will be convenient to separate odd cycles from even cycles. Then, CUD permutations having only cycles of even length are precisely those permutations π with no fixed points where the image of each excedance is a deficiency, and viceversa. The following proposition enumerates these permutations.

Proposition 2.2. *The number of CUD permutations of* $[2m]$ *all of whose cycles are even is* E_{2m}.

If we only allow odd cycles, we obtain a similar result.

Proposition 2.3. *The number of CUD permutations of* $[m]$ *all of whose cycles are odd is* E_m.

In subsections 2.1 and 2.2 we provide analytic and combinatorial proofs, respectively, of the three propositions given above. The proofs via generating functions are shorter and more straightforward, while the bijective proofs give a better insight into CUD permutations.

2.1 Proofs via generating functions

Analytic proof of Proposition 2.1. The number of CUD cyclic permutations of length n is E_{n-1}, because each such permutation can be written as $(1,b_2,b_3,\dots,b_n)$, with $(n+1-b_2)(n+1-b_3)\dots(n+1-b_n)$ being an up-down permutation of $[n-1]$. Thus, the corresponding EGF is

$$\sum_{n\geq 1} E_{n-1}\frac{z^n}{n!} = \int_0^z E(u)\,du. \tag{1}$$

Since any CUD permutation is an unordered product of up-down cycles, the EGF for CUD permutations is

$$\exp\left(\int_0^z E(u)\,du\right),$$

via the set construction (see [3]). Thus, using that $\sum_{n\geq 0} E_{n+1}\frac{z^n}{n!} = E'(z)$, the statement of the proposition, in terms of generating functions, is equivalent to

$$\exp\left(\int_0^z E(u)\,du\right) = E'(z).$$

It can be easily checked that this equation holds, since both sides are equal to $1/(1-\sin z)$. □

Analytic proof of Proposition 2.2. The EGF for CUD cyclic permutations of even length is

$$\sum_{m\geq 1} E_{2m-1}\frac{z^{2m}}{(2m)!} = \int_0^z \tan u\, du.$$

Thus, the statement of the proposition is equivalent to

$$\exp\left(\int_0^z \tan u\, du\right) = \sec z,$$

which can be easily checked. □

Remark. It is now clear that if we want to count permutations π where the image of excedances are deficiencies and viceversa, we only need to modify the proof of Proposition 2.2 by allowing fixed points. The EGF for these permutations is then

$$\exp\left(z + \int_0^z \tan u\, du\right) = e^z \sec z,$$

which appears in [7, A003701].

Analytic proof of Proposition 2.3. The EGF for CUD cyclic permutations of odd length is

$$\sum_{m\geq 0} E_{2m}\frac{z^{2m+1}}{(2m+1)!} = \int_0^z \sec u\, du.$$

Thus, the statement of the proposition is equivalent to

$$\exp\left(\int_0^z \sec u\, du\right) = E(z),$$

which can be easily checked. □

2.2 Bijective proofs

Bijective proof of Proposition 2.2. Given $\pi = \pi_1\pi_2\dots\pi_n \in \Lambda_{2m}$, let $\pi_{i_1} > \pi_{i_2} > \cdots > \pi_{i_k}$ be its left to right minima. Note that all the i_j must be odd. Then

$$g(\pi) = (\pi_{i_1},\dots,\pi_{i_2-1})(\pi_{i_2},\dots,\pi_{i_3-1})\dots(\pi_{i_k},\dots,\pi_{2m}) \tag{2}$$

is a CUD permutation with only even cycles, and g is a bijection.

To find the preimage of a CUD permutation with only even cycles, write it as a product of cycles in standard form ordered by decreasing first entry. Removing the parentheses gives the one-line notation of an up-down permutation which is its preimage by g.

For example, if $\pi = 47261538$, we have $g(\pi) = (4,7)(2,6)(1,5,3,8)$. □

Bijective proof of Proposition 2.3. It will be convenient to consider permutations of a finite set $A = \{a_1, a_2, \ldots, a_n\}$ of natural numbers with $a_1 < a_2 < \cdots < a_n$. We describe a bijection f between up-down permutations of A and CUD permutations of A with only odd cycles. Given an up-down permutation $\pi = \pi_1\pi_2\ldots\pi_n$, let k be such that $\pi_k = a_1$. Define recursively

$$f(\pi) = (\pi_k, \pi_{k-1}, \ldots, \pi_1)\, f(\overline{\pi_{k+1}\pi_{k+2}\ldots\pi_n}),$$

where $\sigma \mapsto \overline{\sigma}$ is the switch operation defined in Section 1.

For example, if $\pi = 471938562$, then

$$\begin{aligned} f(471938562) = (1,7,4)f(\overline{938562}) = (1,7,4)f(283659) = (1,7,4)(2)f(\overline{83659}) \\ = (1,7,4)(2)f(59683) = (1,7,4)(2)(3,8,6,9,5), \end{aligned}$$

which is a CUD permutation. Conversely, given a CUD permutation of A with odd cycles, write it as a product of cycles in standard form ordered by increasing first entry, and reverse the previous steps. For example, if $\sigma = (1,8,5,7,2)(3,6,4)$, then

$$\sigma = (1,8,5,7,2)f(463) = (1,8,5,7,2)f(\overline{436}) = f(27581436).$$

□

The bijection f is similar to Chebikin's decomposition of up-down permutations into a set of up-down permutations of odd length [1, p. 26].

Bijective proof of Proposition 2.1. We can combine the bijections in the above two proofs to give a bijection $\phi : \Lambda_{n+1} \to \Delta_n$. In the above proof of Proposition 2.2 we described a bijection g from up-down permutations of even length to CUD permutations with only even cycles. Consider the straightforward generalization of this bijection to permutations of a finite set A of natural numbers.

Given $\pi \in \Lambda_{n+1}$, let k be such that $\pi_k = 1$, and for $1 \le i \le n+1$, let $\pi'_i = \pi_i - 1$. Define $\phi(\pi)$ to be the CUD permutation which has cycle decomposition given by

$$g(\pi'_1 \ldots \pi'_{k-1})f(\overline{\pi'_{k+1} \ldots \pi'_{n+1}}).$$

For example, if $\pi = 6\,9\,3\,8\,5\,12\,1\,10\,2\,11\,4\,7$, then

$$\phi(\pi) = g(5\,8\,2\,7\,4\,11)f(\overline{9\,1\,10\,3\,6}).$$

Now, noticing that the left-to-right minima of $5\,8\,2\,7\,4\,11$ are 5 and 2, we have

$$\phi(\pi) = (5,8)(2,7,4,11)f(3\,10\,1\,9\,6) = (5,8)(2,7,4,11)(1,10,3)(6)(9).$$

To see that ϕ is a bijection, observe that separating even and odd cycles of $\phi(\pi)$ determines the index k such that $\pi_k = 1$. Then we use that g and f are bijections. □

An alternative bijective proof of Proposition 2.1 can be obtained by modifying a bijection of Johnson [5] involving the so-called *zigzig sequences*. We next describe the resulting map $\varphi : \Lambda_{n+1} \to \Delta_n$. Given $\pi = \pi_1\pi_2 \ldots \pi_{n+1} \in \Lambda_{n+1}$, let π_k be its rightmost extreme element. If it is an LR minimum (i.e. 1 in this case), let $\tau = \pi$; if it is an LR maximum (i.e. $n+1$ in this case), let $\tau = \overline{\pi}$. We define $(\tau_k, \tau_{k+1}, \ldots, \tau_{n+1})$ to be the first cycle of $\varphi(\pi)$, and we delete the entries $\tau_k\tau_{k+1} \ldots \tau_{n+1}$ from τ. Now we look at the rightmost extreme element τ_j of what remains of τ. Again, if it is an LR maximum, we switch τ to make τ_j an LR minimum. The second cycle of $\varphi(\pi)$ is $(\tau_j, \tau_{k+1}, \ldots, \tau_{k-1})$, we delete the entries $\tau_j\tau_{j+1} \ldots \tau_{k-1}$ and repeat this process until τ has only one entry, which at this point is necessarily $n+1$ (this entry does not get added as a cycle).

For example, if $\pi = 351827496$, then its rightmost extreme element is 9, so we take $\tau = \overline{\pi} = 759283614$, add the cycle $(1, 4)$ to $\varphi(\pi)$, and remove 1 and 4 from τ. Now the rightmost extreme element is 2, so $\tau = 7592836$, we add the cycle $(2, 8, 3, 6)$ to $\varphi(\pi)$, and delete 2836 from τ. The rightmost extreme element of 759 is 9, so $\tau = \overline{759} = 795$, and the cycle (5) gets added to $\varphi(\pi)$. The rightmost extreme element of 79 is 9, so $\tau = \overline{79} = 97$, the cycle (7) gets added to $\varphi(\pi)$, and we end here, obtaining $\varphi(\pi) = (1, 4)(2, 8, 3, 6)(5)(7)$.

The inverse map can be described as follows. Given $\sigma \in \Delta_n$, write its cycles in standard form ordered by decreasing first element. Removing the parentheses (this correspondence is due to Foata [4]) gives the one-line notation of a permutation. Let τ be the permutation obtained by adding $n + 1$ in front of it. Let k be the largest index such that $\tau_1\tau_2 \ldots \tau_k$ is alternating, that is, either up-down or down-up. Replace τ with $\overline{\tau_1\tau_2 \ldots \tau_k}\tau_{k+1} \ldots \tau_n$. Consider again the largest alternating prefix of τ and switch it. Repeat until τ is alternating. If it is up-down, then $\varphi^{-1}(\sigma) = \tau$; if it is down-up, then $\varphi^{-1}(\sigma) = \overline{\tau}$.

Going back to the above example $\sigma = (1, 4)(2, 8, 3, 6)(5)(7)$, after applying Foata's correspondence and inserting 9 at the beginning we get $\tau = 975283614$. It is alternating up to $\tau_2 = 7$, so we let $\tau = \overline{97}5283614 = 795283614$. Now it is alternating up to $\tau_3 = 5$, so we let $\tau = \overline{795}283614 = 759283614$. This is already an alternating permutation. Since it is down-up, we recover $\varphi^{-1}(\sigma) = \overline{\tau} = 351827496$.

3 Statistics on up-down and CUD permutations

For $\pi \in \mathcal{S}_n$, we define the following statistics:

- $c(\pi)$ = number of cycles of π,
- $c_o(\pi)$ = number of odd cycles of π,
- $c_e(\pi)$ = number of even cycles of π,
- $\text{fp}(\pi)$ = number of fixed points of π,
- $\text{lrm}(\pi)$ = number of left-to-right minima of π,
- $\text{mm}(\pi)$ = length of the min-max sequence of π,

- $\mathrm{extr}(\pi)$ = number of extreme elements of π,
- $\mathrm{exc}(\pi)$ = number of excedances of π.

3.1 Refined generating functions

It is possible to refine the above enumerations of CUD permutations by adding a variable t that marks the number of cycles and a variable x that marks the number of fixed points.

For CUD permutations we get the multivariate generating function

$$\begin{aligned}\sum_{n\geq 0}\sum_{\pi\in\Delta_n} x^{\mathrm{fp}(\pi)}t^{c(\pi)}\frac{z^n}{n!} &= \exp\left[t\left((x-1)z+\int_0^z E(u)\,du\right)\right]\\ &= e^{(x-1)tz}E'(z)^t\\ &= \frac{e^{(x-1)tz}}{(1-\sin z)^t}.\end{aligned}$$

In particular, the EGF for CUD derangements is

$$\frac{e^{-z}}{1-\sin z},$$

and the first few terms are $0, 1, 1, 5, 15, 71, 341, 1945, 12135, \ldots$. The EGF for CUD permutations with respect to the number of cycles is

$$(1-\sin z)^{-t}. \tag{3}$$

The coefficients of $z^n/n!$ in the expansion of (3) are the polynomials c_n in [5, Eq. (7.1)].

It is also easy to keep track of odd and even cycles separately:

$$\sum_{n\geq 0}\sum_{\pi\in\Delta_n} t_e^{c_e(\pi)}t_o^{c_o(\pi)}\frac{z^n}{n!} = \exp\left(\int_0^z (t_o\sec u + t_e\tan u)\,du\right) = (\sec z+\tan z)^{t_o}(\sec z)^{t_e}. \tag{4}$$

The number of odd cycles and the number of excedances in a permutation $\pi \in \Delta_n$ are related by

$$c_o(\pi) + 2\,\mathrm{exc}(\pi) = n,$$

since in each cycle of length $2k$ and each cycle of length $2k+1$, k of the entries are excedances. Consequently, the bivariate generating function for CUD permutations with respect to the number of excedances is

$$\sum_{n\geq 0}\sum_{\pi\in\Delta_n} t^{\mathrm{exc}(\pi)}\frac{z^n}{n!} = \frac{[\sec(z\sqrt{t})+\tan(z\sqrt{t})]^{1/\sqrt{t}}}{\cos(z\sqrt{t})}.$$

3.2 Statistics preserved by the bijections

Here we show that the bijections f, g, and ϕ defined in Section 2.2 preserve some permutation statistics involving cycles, left-to-right minima, min-max sequences, and extreme elements.

Proposition 3.1. *Let $\pi \in \Lambda_n$ and let $f(\pi) \in \Delta_n$ be the corresponding CUD permutation with only odd cycles. Then*

$$c(f(\pi)) = \mathrm{mm}(\pi).$$

Proof. We proceed by induction on n. The result is obviously true for $n = 1$. Recall the recursive definition of f: if $\pi \in \Lambda_n$ and $\pi_k = 1$, then

$$f(\pi) = (\pi_k, \pi_{k-1}, \ldots, \pi_1)\, f(\overline{\pi_{k+1}\pi_{k+2}\ldots\pi_n}).$$

Thus, we have that

$$c(f(\pi)) = 1 + c(f(\overline{\pi_{k+1}\pi_{k+2}\ldots\pi_n})) = 1 + \mathrm{mm}(\overline{\pi_{k+1}\pi_{k+2}\ldots\pi_n}) = \mathrm{mm}(\pi),$$

where the second equality holds by induction, and the last equality follows from the definition of the statistic mm. □

As a consequence of this result we obtain the EGF for up-down permutations with respect to the statistic mm:

$$\sum_{n\geq 0}\sum_{\pi\in\Lambda_n} t^{\mathrm{mm}(\pi)}\frac{z^n}{n!} = \sum_{n\geq 0}\sum_{\sigma\in\Delta_n} t^{c(\sigma)}\frac{z^n}{n!} = \exp\left(t\int_0^z \sec u\, du\right) = (\sec z + \tan z)^t, \quad (5)$$

where the second step follows from the analytic proof of Proposition 2.3. The coefficients of $z^n/n!$ in the expansion of (5) are the polynomials a_n in [5, Eq. (2.1)].

Proposition 3.2. *Let $\pi \in \Lambda_{2m}$ and let $g(\pi) \in \Delta_{2m}$ be the corresponding CUD permutation with only even cycles. Then*

$$c(g(\pi)) = \mathrm{lrm}(\pi).$$

Proof. It follows immediately from the definition of g, since the opening parentheses for the cycles of $g(\pi)$ are inserted at each left-to-right minimum of π. □

Proposition 3.3. *Let $\pi \in \Lambda_{n+1}$ and let $\sigma = \phi(\pi) \in \Delta_n$. Then*

$$\begin{aligned} c_e(\sigma) &= \mathrm{lrm}(\pi) - 1, \\ c_o(\sigma) &= \mathrm{mm}(\pi) - 1, \\ c(\sigma) &= \mathrm{lrm}(\pi) + \mathrm{mm}(\pi) - 2. \end{aligned}$$

Proof. Recall the definition of ϕ: if $\pi \in \Lambda_{n+1}$ and $\pi_k = 1$, then

$$\sigma = \phi(\pi) = g(\pi'_1 \dots \pi'_{k-1}) f(\overline{\pi'_{k+1} \dots \pi'_{n+1}}),$$

where $\pi'_i = \pi_i - 1$ for all i.

Since the range of f (respectively, g) are CUD permutations with only odd (respectively, even) cycles, we have that

$$c_e(\sigma) = c(g(\pi'_1 \dots \pi'_{k-1})) = \mathrm{lrm}(\pi'_1 \dots \pi'_{k-1}) = \mathrm{lrm}(\pi_1 \dots \pi_{k-1} 1) - 1 = \mathrm{lrm}(\pi) - 1$$

and

$$\begin{aligned} c_o(\sigma) &= c(f(\overline{\pi'_{k+1} \dots \pi'_{n+1}})) \\ &= \mathrm{mm}(\overline{\pi'_{k+1} \dots \pi'_{n+1}}) \\ &= \mathrm{mm}(1\pi_{k+1} \dots \pi_{n+1}) - 1 = \mathrm{mm}(\pi) - 1. \end{aligned}$$

The third equation is obtained by adding the other two. □

We know from equation (4) that the bivariate EGF for CUD permutations where t marks the number of even cycles is

$$G(t, z) = (\sec z + \tan z)(\sec z)^t.$$

It follows from the first part of Proposition 3.3 that the generating function for up-down permutations with respect to the number of LR minima is

$$\begin{aligned} \sum_{n \geq 1} \sum_{\pi \in \Lambda_n} t^{\mathrm{lrm}(\pi)} \frac{z^n}{n!} = t \int_0^z G(t, u)\, du = t \int_0^z (\sec u + \tan u)(\sec u)^t\, du \\ = t \int_0^z (\sec u)^{t+1}\, du + (\sec z)^t - 1. \end{aligned}$$

The coefficients of $z^{2n}/(2n)!$ in the expansion of $(\sec z)^t$ are the polynomials b_n in [5, Eq. (5.1)].

Under the bijection φ, described at the end of Section 2.2 and adapted from [5], the number of cycles corresponds to a different statistic:

Proposition 3.4. *Let* $\pi \in \Lambda_{n+1}$ *and let* $\varphi(\pi) \in \Delta_n$*. Then*

$$c(\varphi(\pi)) = \mathrm{extr}(\pi).$$

Proof. The positions of the extreme elements of $\pi \in \Lambda_{n+1}$ are not affected by the switches that take place in the construction of $\varphi(\pi)$. Since a new cycle of $\varphi(\pi)$ is created for each extreme element of π, the result follows. □

From the above proposition and equation (3), it follows that the EGF for up-down permutations with respect to the number of extreme elements is

$$\sum_{n \geq 1} \sum_{\pi \in \Lambda_n} t^{\mathrm{extr}(\pi)} \frac{z^n}{n!} = \int_0^z (1 - \sin u)^{-t}\, du.$$

It also follows from Propositions 3.3 and 3.4 that the statistics extr and $\mathrm{lrm} + \mathrm{mm} - 2$ are equidistributed on up-down permutations.

4 Statistics on all permutations

Some of the above statistics have an interesting distribution not only in Λ_n but also in $\mathcal{S}_n$. Let $\left[{n \atop k}\right]$ denote the signless Stirling numbers of the first kind. It is well known (see for example [8, Sec. 1.3]) that

$$|\{\pi \in \mathcal{S}_n : \mathrm{lrm}(\pi) = k\}| = |\{\pi \in \mathcal{S}_n : c(\pi) = k\}| = \begin{bmatrix} n \\ k \end{bmatrix}.$$

Proposition 4.1. *For $n \geq k \geq 1$,*

$$|\{\pi \in \mathcal{S}_n : \mathrm{mm}(\pi) = k\}| = \begin{bmatrix} n \\ k \end{bmatrix}.$$

Proof. We define a bijection $h : \mathcal{S}_n \to \mathcal{S}_n$ that transforms the min-max sequence of a permutation π into the sequence of left-to-right minima of $h(\pi)$. Given $\pi \in \mathcal{S}_n$, let $\pi_{i_1}\pi_{i_2}\dots\pi_{i_k}$ be the min-max sequence of π. Let $\tau = \pi$ initially. Then, for each $j = 1, 2, \dots, k-1$, apply the switch operation (defined in Section 1) to the entries $\tau_{i_j+1}\tau_{i_j+2}\dots\tau_n$ of τ. Finally, if $\tau_1\tau_2\dots\tau_n$ is the resulting permutation after the $k-1$ iterations, let $h(\pi) = \tau_n \dots \tau_2\tau_1$.

For example, if $\pi = 48\mathbf{1}\mathbf{2}\mathbf{7}6\mathbf{3}\mathbf{5}$ (the min-max sequence is in boldface), we first switch the entries 27635, getting $\tau = 48\mathbf{1}\mathbf{7}\mathbf{2}\mathbf{3}6\mathbf{5}$, then switch the entries 365, getting $\tau = 48\mathbf{1}\mathbf{7}\mathbf{2}6\mathbf{3}\mathbf{5}$. The switch of the 5 at the end does not change the permutation, so we obtain $h(\pi) = \mathbf{5}\mathbf{3}6\mathbf{2}7\mathbf{1}84$, where now the elements in boldface are the LR minima. ☐

In fact, the following generalization of Proposition 4.1 holds as well. For any infinite sequence $s = s_1s_2s_3\dots$, where each $s_i \in \{\min, \max\}$, define the statistic m_s on permutations $\pi \in \mathcal{S}_n$ as the length of the sequence $\pi_{i_1}\pi_{i_2}\dots\pi_{i_k}$ where $\pi_{i_1} = s_1(\pi_1\pi_2\dots\pi_n)$, $\pi_{i_2} = s_2(\pi_{i_1+1}\pi_{i_1+2}\dots\pi_n)$, and in general $\pi_{i_j} = s_j(\pi_{i_{j-1}+1}\pi_{i_{j-1}+2}\dots\pi_n)$ for each j until $\pi_{i_k} = \pi_n$ for some k. For example, if $s = \min\min\dots$, then $m_s(\pi) = \mathrm{lrm}(\pi)$, and if $s = \min\max\min\max\dots$, then $m_s(\pi) = \mathrm{mm}(\pi)$. Then, for any sequence s as above, we have

$$|\{\pi \in \mathcal{S}_n : m_s(\pi) = k\}| = \begin{bmatrix} n \\ k \end{bmatrix}.$$

The bijection from Proposition 4.1 can be easily generalized to prove this fact if we start with $\tau = \pi$ or $\tau = \overline{\pi}$ depending on whether s_1 equals min or max, respectively, and then for each $j = 1, 2, \dots, k-1$ we switch the entries $\tau_{i_j+1}\tau_{i_j+2}\dots\tau_n$ only if $s_j \neq s_{j+1}$.

Proposition 4.2. *For $n, k \geq 1$ with $n \geq k+1$,*

$$|\{\pi \in \mathcal{S}_n : \mathrm{extr}(\pi) = k\}| = 2^k \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

Proof. We give a bijection ℓ between the sets

$$\{(\pi, s) : \pi \in \mathcal{S}_{n-1}, \mathrm{lrm}(\pi) = k, s \in \{0,1\}^k\}$$

and

$$\{\sigma \in \mathcal{S}_n : \mathrm{extr}(\sigma) = k\}.$$

Given a pair (π, s) in the first set, the digit s_j being 0 or 1 will indicate whether the jth LR minimum of π will become an LR minimum or an LR maximum of $\ell(\pi, s)$, respectively. Let $\pi_{i_1} > \pi_{i_2} > \cdots > \pi_{i_k}$ be the LR minima of π. Define $s_{k+1} = 0$ for convenience. Let $\tau = \tau_0\tau_1 \dots \tau_{n-1} = n\pi_1 \dots \pi_{n-1}$. For $j = k, k-1, \dots, 1$, switch the prefix $\tau_0\tau_1 \dots \tau_{i_j}$ of τ if $s_j \neq s_{j+1}$. Let $\sigma = \ell(\pi, s)$ be the permutation τ obtained at the end. It is clear from the construction that the positions of the extreme elements of σ are the same as the LR minima of π.

For example, if $(\pi, s) = (\mathbf{86742}5\mathbf{1}3, 10011)$ (the LR minima of π are in boldface), we start with $\tau = 9\mathbf{86742}5\mathbf{1}3$. Since $s_5 = 1 \neq 0 = s_6$, we first switch the prefix 9867451, getting $\tau = 1\mathbf{25478}69\mathbf{3}$. Now $s_4 = 1 = s_5$, so no switches are done for $j = 4$. Since $s_3 = 0 \neq 1 = s_4$, we now switch the prefix 12547, getting $\tau = \mathbf{752418}69\mathbf{3}$. The last switch happens for $j = 1$ because $s_1 \neq s_2$, ending with $\sigma = \ell(\pi, s) = \tau = 5\mathbf{72418}69\mathbf{3}$.

The inverse map has a similar description. Given $\sigma \in \mathcal{S}_n$ with k extreme elements, we have $\ell^{-1}(\sigma) = (\pi, s)$ where, for $1 \leq j \leq k$, s_j is 0 or 1 depending on whether the jth extreme element of σ is an LR minimum or an LR maximum, respectively. To obtain π, let $\sigma_{i_1}, \sigma_{i_2}, \dots, \sigma_{i_k}$ be the extreme elements of σ from left to right. For $j = k, k-1, \dots, 1$, switch the first i_j entries of σ if $s_j \neq s_{j+1}$, where again we define $s_{k+1} = 0$ for convenience. Finally, we recover π by deleting the first entry, which is necessarily n. □

Let us finish this section by considering two permutation statistics based on the concept of up-down cycles. For $\pi \in \mathcal{S}_n$, let $\mathrm{ud}(\pi)$ be the number of up-down cycles in π, and let $\mathrm{nud}(\pi) = c(\pi) - \mathrm{ud}(\pi)$ be the number of cycles of π that are not up-down. Using equation (1) and the fact that the EGF for all cycles is $-\ln(1-z)$, we can derive the EGF for all permutations with respect to the number of up-down and non-up-down cycles:

$$\sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n} v^{\mathrm{ud}(\pi)} w^{\mathrm{nud}(\pi)} \frac{z^n}{n!} = \exp\left(-w\ln(1-z) + (v-w)\int_0^z E(u)\,du\right)$$
$$= \frac{E'(z)^{v-w}}{(1-z)^w} = \frac{1}{(1-z)^w(1-\sin z)^{v-w}}.$$

5 Another interpretation of CUD permutations

A permutation of $[n]$ can be represented as a directed graph on n vertices labeled $1, 2, \dots, n$, with an edge from i to j if and only if $\pi_i = j$. Consider the following way of drawing the graph. Put the n vertices on a horizontal line, ordered from left to right by increasing label. If $\pi_i = j > i$ (respectively, $\pi_i = j < i$), then draw a solid (respectively, dotted) arc between i to j above (respectively, below) the horizontal

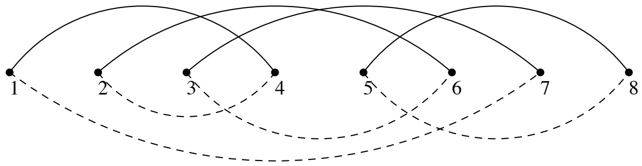

Figure 1: The CUD permutation $(1,4,2,6,3,7)(5,8)$.

line. Note that the orientation of the arcs is implicit, so we can draw undirected arcs (see Figure 1).

Using this drawing, CUD permutations having only even cycles correspond precisely to those graphs with the property that each vertex i is of one of two types, depending on whether i is an *up* or a *down* element of its cycle: it is incident to a solid and a dotted edge that connect it to vertices to its left, or it is incident to a solid and a dotted edge that connect it to vertices to its right. Equivalently, the edges of each type form a perfect matching of the vertices, and the two matchings agree on what vertices are *opening vertices* (matched with a vertex to their right) or *closing vertices* (matched with a vertex to their left).

Since the number of CUD permutations of $[2m]$ having only even cycles is E_{2m}, as shown in Proposition 2.2, this representation as a pair of "matching" perfect matchings allows us to recover the continued fraction for the secant numbers (see [2, p. 145]):

$$\sum_{m\geq 0} E_{2m} z^m = \cfrac{1}{1-\cfrac{z}{1-\cfrac{2^2 z}{1-\cfrac{3^2 z}{1-\cfrac{4^2 z}{\cdots}}}}}.$$

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