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Basic retracts and counting of lattices

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Abstract

In this paper, we treat nullity of a poset as the nullity of its cover graph. Using nullity, we introduce the concepts of a basic retract associated to a poset and a fundamental basic block associated to a dismantlable lattice. We prove that a lattice in which all the reducible elements are comparable is dismantlable but the converse is not true. We establish recurrence relations and count up to isomorphism all fundamental basic blocks and all basic retracts/blocks associated to the lattices (with respect to arbitrary large nullity and/or the number of reducible elements) in which all the reducible elements are comparable.

1 Introduction and preliminaries

In 1940, Birkhoff [4] posed the open problem of counting all non-isomorphic lattices and posets on n elements. Klarner [21], Stanley [31], Quackenbush [28] and Erne et al. [14] have answered Birkhoff's problems for certain classes of posets. In 1980, Kleitman and Winston [22] provided asymptotic bounds for the number of lattices on n elements. Pawar and Waphare [27] have counted lattices with equal numbers of elements and edges. Thakare et al. [33] have counted lattices with n elements and up to $n+1$ edges. Czédli et al. [11] have counted the number of slim, semimodular lattices (see also [9], [10] and [12]).

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Kyuno [25], Koda [23], Chaunier and Lygeros [6], and Lygeros and Zimmermann [26] have counted lattices and posets on n elements, for small values of n. In 2002, Brinkmann and Mckay [5] have counted posets on up to 16 points, and Heitzig and Reinhold [19] have counted lattices on up to 18 points. In 2015, Jipsen and Lawless [20] have counted lattices on up to 19 points. Recently, Gebhardt and Tawn [15] have counted lattices on up to 20 points, and Kohonen [24] presented an algorithm and counted modular lattices on up to 30 elements.

In this paper, we study the class of all lattices such that each member of the class has all the reducible elements comparable. We refer to this class as the class of RC-lattices. Thus, if L is an RC-lattice then all the reducible elements of L form a chain. A chain is regarded as an RC-lattice. We count, up to isomorphism, four subclasses of the class of RC-lattices. These four classes are, namely, a class of all non-isomorphic fundamental basic blocks containing r reducible elements, a class of all non-isomorphic fundamental basic blocks having nullity l, a class of all non-isomorphic basic blocks having nullity l, and a class of all non-isomorphic basic retracts having nullity l. In the last section we carry out, by establishing some recurrence relations, the actual counting of those four classes easily, for arbitrarily large parameter values r and l, and provide four corresponding integer sequences. Interestingly, the first three of the four sequences, respectively, resemble the known sequences $A006129, A121251$ and $A121316$ (see the On-line Encyclopedia of Integer Sequences, OEIS [30]).

In Section 4, we prove that every RC-lattice is a dismantlable lattice, but the converse is not true. Using the theory of partitions and with the help of counting all non-isomorphic basic retracts having nullity l, one can carry out the enumeration of all non-isomorphic RC-lattices having nullity l. Many people are trying to solve Birkhoff's open problems for small values of n. We are making an attempt to solve the problem for the class of RC-lattices for arbitrary large values of n. For this purpose, here we count the number of those four subclasses of RC-lattices for arbitrary large values of the parameters r and l, as mentioned above.

A partially ordered set (or poset) is a set P of elements together with a binary relation $\leq$ on P which is reflexive, antisymmetric and transitive. An element $x \in P$ is an *upper bound* for a subset $S \subseteq P$ if $s \leq x$ for all $s \in S$. An element $x \in P$ is the *least upper bound* for $S \subseteq P$ if x is an upper bound for S and for any other upper bound $z \in P$ of S, $x \leq z$. A *lower bound* and the *greatest lower bound* for $S \subseteq P$ are defined dually. If two elements a and b of P have the least upper bound (greatest lower bound) then we denote this by $a \vee b$ ($a \wedge b$), also called the join (meet) of a and b.

A *lattice* is a poset in which every pair of elements has a least upper bound and a greatest lower bound. A *sublattice* is a subset S of a lattice such that for any $a, b \in S$, $a \vee b \in S$ and $a \wedge b \in S$. Two elements $a, b \in P$ are *comparable* if either $a \leq b$ or $b \leq a$. Two elements of P are said to be *incomparable* if they are not comparable. A *chain* is a lattice in which any two elements are comparable. An *antichain* is a poset in which any two elements are incomparable. The *width* of a poset is the length of a maximal antichain in it. For $a, b \in P$, if $a < b$ and $a \leq c \leq b$ imply $c \in \{a, b\}$, then we say that b *covers* a, or b is an *upper cover of* a, or a is a *lower cover of* b, and this is denoted by $a \prec b$. If a is not covered by b then it is denoted by $a \not\prec b$. The graph on a poset P with edges as covering relations $\prec$ is called the *cover graph*, and is denoted by $C(P)$. We say that a poset P is *connected* if $C(P)$ is a connected graph.

An element x in a lattice L is *join-reducible (meet-reducible)* in L if there exist $y, z \in L$, both distinct from x, such that $y \vee z = x$ $(y \wedge z = x)$; x is *reducible* if it is either join reducible or meet reducible; x is *join-irreducible (meet-irreducible)* if it is not join-reducible (meet-reducible); x is *doubly irreducible* if it is both join-irreducible and meet-irreducible. We denote the set of doubly irreducible elements in L by $Irr(L)$. Two posets P_1 and P_2 are said to be isomorphic, denoted by $P_1 \cong P_2$, if there exists a bijective function $\phi : P_1 \to P_2$ satisfying $x \leq y$ in P_1 if and only if $\phi(x) \leq \phi(y)$ in P_2.

Throughout this paper, we assume that all the posets are finite and connected. The definitions, terminology and notation are taken from [16] and [34]. We now begin with the following.

Proposition 1.1. *[29] If L is a lattice and $A \subseteq Irr(L)$ then the complement of A in L, $L \setminus A$, is a sublattice of L.*

M_2 Crown

Figure 1

In 1973, Ajtai [1] introduced primitive lattices and studied the properties of primitive lattices. Then Rival [29] also studied primitive lattices with a different name, namely, dismantlable lattices. A finite lattice L of order n is called *dismantlable* if there exists a chain $L_1 \subset L_2 \subset \cdots \subset L_n (= L)$ of sublattices of L such that $|L_i| = i$, for all i. For example, M_2 (see Fig. 1) is a dismantlable lattice. In Section 4, we prove that every RC-lattice is a dismantlable lattice.

Thakare et al. [33] introduced an adjunct operation on lattices as follows.

Definition 1.2. If L_1 and L_2 are two disjoint finite lattices and (a, b) is a pair of elements in L_1 such that $a < b$ and $a \not\prec b$, we define the partial order "$\leq$" on $L = L_1 \cup L_2$ with respect to the pair (a, b) as followsr:.

$x \leq y$ in L if either $x, y \in L_1$ and $x \leq y$ in L_1; or $x, y \in L_2$ and $x \leq y$ in L_2;

or $x \in L_1$, $y \in L_2$ and $x \leq a$ in L_1; or $x \in L_2$, $y \in L_1$ and $b \leq y$ in L_1.

It is easy to see that L is a lattice containing L_1 and L_2 as sublattices. The procedure of obtaining L in this way is called an *adjunct operation of L_2 to L_1 or* *adjunct of* L_1 *with* L_2. It is denoted by $L = L_1]_a^b L_2$ or $L = L_1]_\alpha L_2$, where $\alpha = (a, b)$ is called an adjunct pair.

For example, M_2 (see Fig. 1) is the adjunct sum of a 3-chain with a 1-chain, where the adjunct pair is $(0, 1)$. The lattices shown in Fig. 2 are the adjunct of two chains.

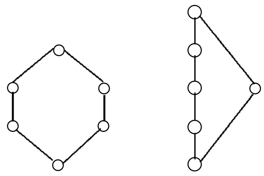

Figure 2

Using the adjunct operation, Thakare et al. [33] proved the following structure theorem for dismantlable lattices.

Theorem 1.3 ([33]). *A finite lattice is dismantlable if and only if it is an adjunct of chains.*

Corollary 1.4 ([33]). *A dismantlable lattice with* n *elements has* $n + r - 2$ *coverings if and only if it is an adjunct of* r *chains.*

Using Theorem 1.3, Thakare et al. [33] carried out enumeration of non-isomorphic lattices having nullity up to 2 and containing n elements. In the last section we count, up to isomorphism, all fundamental basic blocks and all basic retracts/blocks of given nullity, in which all the reducible elements are comparable. For this purpose, let us note the following.

The *nullity* of a graph G is given by $m - n + c$, where m is the number of edges in G, n is the number of vertices in G, and c is the number of connected components of G. This nullity equals the multiplicity of the eigenvalue 0 in the spectrum of the adjacency matrix of the graph G (see Cvetkovič and Gutman [8], Cheng and Liu [7], and Gutman and Borovićanin [17]).

2 Nullity of a poset

We define the *nullity of a poset* to be the nullity of its cover graph. Therefore the nullity of a poset P, denoted by $\eta(P)$, is given by $|E(P)| - |P| + c$, where $E(P)$ is the set of all edges in $C(P)$ and c is the number of components of $C(P)$. Note that, for a connected poset, $c = 1$. Clearly, the nullity of a chain is 0. The lattices shown in Fig. 2 are of nullity 1. Pawar and Waphare [27] have counted non-isomorphic lattices having nullity 1 and containing n elements. Using Corollary 1.4, we have the following.

Theorem 2.1. *A dismantlable lattice L containing n elements is of nullity l if and only if L is an adjunct of $l+1$ chains.*

Proof. Suppose a dismantlable lattice L containing n elements is of nullity l. If L contains m edges then the nullity $l = m - n + 1$ and hence $m = n + l - 1$. Therefore by Corollary 1.4, L is an adjunct of $l+1$ chains. Conversely, suppose L is an adjunct of $l+1$ chains. Again by Corollary 1.4, the number of edges in L is $m = n + l - 1$. Thus $l = m - n + 1$ and hence the nullity of L is l. □

Lemma 2.2 ([33]). *Every lattice with n elements and $n+r$ coverings with $-1 \leq r \leq 3$ is dismantlable.*

Theorem 2.3. *Any lattice of nullity at most 4 is dismantlable.*

Proof. By Lemma 2.2, every lattice with n elements and $n + r$ coverings (or edges) with $-1 \leq r \leq 3$ is dismantlable. If L is a lattice on n elements, containing m edges and having nullity l, then $l = m-n+1$. If $m = n+r$ and $-1 \leq r \leq 3$ then $l = r+1$ and $0 \leq l \leq 4$. Hence the proof follows. □

Let G be a loopless connected graph. An *ear of a graph G* is a subgraph of G which is a maximal path in which all internal vertices are of degree 2 in G. If G is a path itself then that path is the only ear of G. An ear which does not contain any internal vertex is called a *trivial ear*. Therefore a trivial ear is just an edge in G and, in this case, the length of the ear is one. An ear which is not an edge is called a *non-trivial ear* in G. An ear $E : a - x_1 - x_2 - \cdots - x_r - b$ is said to be an *ear associated to the pair* (a, b) of length $r + 1$. Also for each i, we say x_i is associated to the pair (a, b). Hereinafter by a path (or an ear) in a poset/lattice, we mean the path (or the ear) in the cover graph of that poset/lattice.

An element in a poset P is called *doubly irreducible* if it has at most one lower cover and at most one upper cover in P. Let $Irr(P)$ denote the set of all doubly irreducible elements in the poset P. Let $Red(P) = P \setminus Irr(P)$.

Definition 2.4. Let P be a poset. Let $x \in Irr(P)$. Then x is called a *retractible* element of P if it satisfies either of the following two conditions.

1. There are no $y, z \in Red(P)$ such that $y \prec x \prec z$.
2. There are $y, z \in Red(P)$ such that $y \prec x \prec z$ and there is no other directed path from y to z in P.

It can be easily observed that there are two retractible elements in each of the posets in Fig. 2, whereas no element of M_2 is retractible.

Recall that the posets under consideration are all connected. In the following, we obtain the condition under which removal of a doubly irreducible element preserves the nullity of the poset.

Theorem 2.5. *Let P be a poset with $|P| > 1$. Let $x \in Irr(P)$. Then $\eta(P \setminus \{x\}) = \eta(P)$ if and only if x is a retractible element of P.*

Proof. Let $x \in Irr(P)$. Suppose $P' = P \setminus \{x\}$ and $\eta(P') = \eta(P)$. If x satisfies the condition (1) of Definition 2.4 then we are done. If not, then there are $y, z \in Red(P)$ such that $y \prec x \prec z$. Suppose there is another path from y to z; then $\eta(P') = |E(P')| - |P'| + 1 = (|E(P)| - 2) - (|P| - 1) + 1 = |E(P)| - |P| = \eta(P) - 1$, a contradiction. Therefore x must satisfy condition (2) of Definition 2.4. Thus x is a retractible element of P.

Conversely, suppose x is a retractible element of P. Suppose the condition (1) of Definition 2.4 is true. If $y \prec x \prec z$ in P then either $y \in Irr(P)$ or $z \in Irr(P)$. In any case, $\eta(P') = |E(P')| - |P'| + 1 = (|E(P)| - 1) - (|P| - 1) + 1 = |E(P)| - |P| + 1 = \eta(P)$. Now suppose condition (2) of Definition 2.4 is true. But then we also get $\eta(P') = |E(P')| - |P'| + 1 = (|E(P)| - 1) - (|P| - 1) + 1 = |E(P)| - |P| + 1 = \eta(P)$, since $y \prec z$ in P'. □

Corollary 2.6. *Let L be a lattice with $|L| > 1$. Let $x \in L$. Then $L' = L \setminus \{x\}$ is a sublattice of L, maintaining the nullity if and only if x is a retractible element of L.*

Proof. Suppose $x \in L$ and $L' = L \setminus \{x\}$ is a sublattice of L with $\eta(L') = \eta(L)$. If x is meet reducible in L then there are $a, b \in L$ with $a \wedge b = x$. But then $L' = L \setminus \{x\}$ will not be a sublattice of L, since $a \wedge b$ will not be maintained in L', which is a contradiction. Hence x is not meet reducible in L. Similarly, it can be proved that x is not join reducible in L. Hence $x \in Irr(L)$. The remainder of the proof follows from Theorem 2.5.

Conversely, suppose x is a retractible element of L. Therefore $x \in Irr(L)$ and satisfies both the conditions of Definition 2.4. As $x \in Irr(L)$, by Proposition 1.1, $L' = L \setminus \{x\}$ is a sublattice of L. The rest of the proof follows from Theorem 2.5. □

Let P be a poset and let $x \in P$. We denote an element y by x^- if $y \prec x$ and by x^+ if $x \prec y$. The *indegree* of an element x in a poset P is given by $|\{y \in P : y \prec x\}|$, and the *outdegree* of an element x in a poset P is given by $|\{z \in P : x \prec z\}|$. The *degree* of an element x in a poset P is the sum of the indegree and the outdegree of x in P. Note that the degree of x in P is same as the degree of x in $C(P)$. An element x of a poset is called *pendant* if the degree of x is 1. Let $Irr^*(P) = \{x \in Irr(P) :\ x$ has exactly one upper cover and exactly one lower cover in $P\}$. Note that, for a connected poset P, if $x \in Irr(P)$ then either $x \in Irr^*(P)$ or x is a pendant vertex in P.

Theorem 2.7. *Let P be a poset. Let $x \in Irr^*(P)$. Then the respective indegrees as well as the respective outdegrees of any reducible element in both P and $P \setminus \{x\}$ are the same if and only if x is a retractible element of P.*

Proof. Suppose the respective indegrees as well as the respective outdegrees of any reducible element in both P and $P \setminus \{x\}$ are the same. Suppose $y \prec x \prec z$. Now either at least one of y and z belongs to $Irr(P)$ or both $y, z \in Red(P)$. If at least one of y and z belongs to $Irr(P)$ then x satisfies the condition (1) of Definition 2.4. If both $y, z \in Red(P)$ then either there is another path from y to z in P or there is no other directed path from y to z in P. If there is another path from y to z in P then the outdegree of y and the indegree of z decrease by one if x is removed from P, since $x \in Irr^*(P)$, which is not possible by assumption. Thus there are $y, z \in Red(P)$ with $y \prec x \prec z$ and there is no other directed path from y to z in P. Therefore x satisfies the condition (2) of Definition 2.4. Hence x is a retractible element of P.

Conversely, suppose x is a retractible element of P. Therefore either there are no $y, z \in Red(P)$ with $y \prec x \prec z$, or there are $y, z \in Red(P)$ with $y \prec x \prec z$ and there is no other directed path from y to z in P. Let $a \in Red(P)$. Let d_1 and d_2 be the indegree and the outdegree of a in P respectively. Then we have the following three cases. Case 1 : Suppose $a \prec x$. Let $x \prec b$. If $b \in Red(P)$ then x must satisfy the condition (2) of Definition 2.4. Therefore there is no other directed path from a to b in P. Hence $a \prec b$ in $P \setminus \{x\}$. Also, if $b \notin Red(P)$ then x satisfies the condition (1) of Definition 2.4. But then $b \in Irr(P)$. As $x \in Irr^*(P)$ and $a \prec x \prec b$, $a \prec b$ in $P \setminus \{x\}$. Thus removal of x from P does not change the values of d_1 and d_2. Case 2 : Suppose $x \prec a$. The proof in this case is similar to that of Case 1. Case 3 : Suppose neither $a \prec x$ nor $x \prec a$. Let $x^- \prec x \prec x^+$. Clearly $a \neq x^-$ as well as $a \neq x^+$. Therefore $x^- \prec x^+$ in $P \setminus \{x\}$. Thus, the removal of x from P does not change the values of d_1 and d_2. Hence the proof is complete. □

3 Basic retract associated to a poset

We now introduce the concept of a basic retract as follows.

Definition 3.1. A poset P is a *basic retract* if no element of $Irr^*(P)$ is retractible in P.

For example, a 1-chain, a 2-chain, an M_2 and a crown (see Fig. 1) are basic retracts.

It follows that, if $x \in Irr^*(P)$ then $P \setminus \{x\}$ is a subposet of P with $\eta(P \setminus \{x\}) \geq \eta(P) - 1$. By Theorem 2.5 and using Definition 5.2, we have the following.

Proposition 3.2. *A poset P, which is not a chain, is a basic retract if and only if removal of any element of $Irr^*(P)$ reduces $\eta(P)$ by one.*

We now introduce the concept of a basic retract associated to a poset as follows.

Definition 3.3. Let P be a poset. Consider a (Hasse) diagram of P. If $Irr^*(P) = \emptyset$ then we say that P is a basic retract associated to itself; otherwise, go on removing elements of $Irr^*(P)$ one by one as long as $\eta(P)$ does not alter. Ultimately we get a subposet P' of P such that no element of $Irr^*(P')$ is retractible in P'. The resultant subposet P' of P is called a *basic retract associated to P*.

For example, a crown is the basic retract associated to itself. An M_2 is the basic retract associated to the lattices shown in Fig. 2. In Fig. 3, P' is the basic retract associated to both the posets P_1 and P_2.

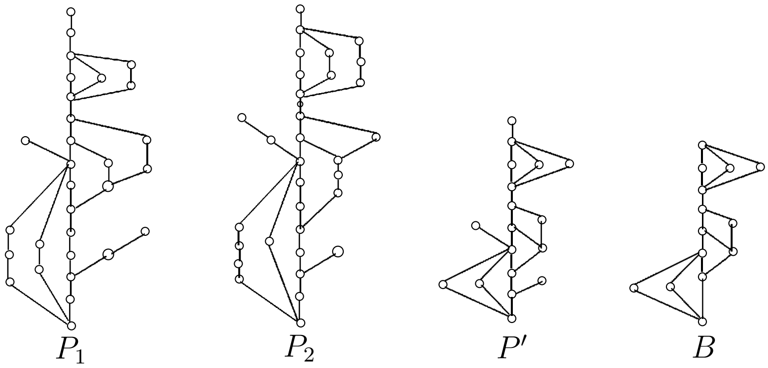

Figure 3

Duffus and Rival [13] introduced the following.

Definition 3.4. *[13]* An order-preserving map $g : P \to Q$ is a *retraction* of poset P onto subposet Q of P provided that $g(x) = x$ for all $x \in Q$. If there is a retraction of P onto Q, then Q is a *retract* of P.

By Definition 3.4, it follows that, a basic retract associated to a poset P is a retract of P. In the following, we prove some properties of basic retracts associated to the posets.

Theorem 3.5. *Let B be a basic retract associated to a poset P. Then*

1. *B is a sublattice of P whenever P is a lattice.*
2. *$\eta(B) = \eta(P)$.*
3. *$Red(B) = Red(P)$.*
4. *$Irr(B) \subseteq Irr(P)$.*
5. *If an ear is trivial in B associated to a pair (a, b) then there is no another path from a to b in P and hence there is a unique ear associated to (a, b) in P. Conversely, if there is no another path from a to b in P then there is no non-trivial ear associated to (a, b) in B.*
6. *If $x \in Irr^*(B)$ and x is associated to a pair (a, b) in B then x is associated to the pair (a, b) in P also. Moreover, every ear in B is either of length 1 or 2.*
7. *If $x \in Irr^*(B)$ then $x^-, x^+ \in Red(B)$. Moreover, $\eta(B \setminus \{x\}) = \eta(B) - 1$. Further, outdegree of x^- and indegree of x^+ both decreases by one.*
8. *Number of trivial ears in B is greater equal that in P.*
9. *A non-trivial ear in P associated to (a, b) if exists, becomes a trivial ear in B if and only if there is no another path from a to b in P.*

10. *If there is a non-trivial ear associated to (a, b) in B then the number of non-trivial ears (or the number of doubly irreducibles) associated to (a, b) in B is equal to the number of non-trivial ears associated to (a, b) in P.*

11. *The number of ears associated to (a, b) in B is equal to the number of ears associated to (a, b) in P.*

Proof. 1. Follows from the repeated use of Proposition 1.1.

2. Follows from the repeated use of Theorem 2.5.

3. Follows from the repeated use of Theorem 2.7.

4. The proof is obvious.

5. Let $E : a \prec b$ be a trivial ear in B associated to the pair (a, b). Let E' be the ear associated to (a, b) in P containing E. If $E' = E$ then clearly there is no other path from a to b in P. If $E' \neq E$ then E' is non-trivial ear. If there is another path from a to b in P then there is an element say x of E' such that $x \in B$ and x is associated to (a, b) in B. This is not possible, since $E : a \prec b$ is a trivial ear in B associated to the pair (a, b). Therefore, there is no another path from a to b in P. Hence E' is a unique ear associated to the pair (a, b) in P. The converse follows from the definition of a basic retract associated to a poset.

6. First part is obvious. Now suppose there is an ear E associated to (a, b) in B of length at least three. Let $x \prec y$ be the elements of E. Then $\eta(B \setminus \{x\}) = \eta(B)$, a contradiction. Therefore, every ear in B is of length at most two.

7. Suppose $x \in Irr^*(B)$. Let E be the ear containing x. If either x^- or x^+ or both are in $Irr^*(B)$ then as $x^- \prec x \prec x^+$ in B, the length of E is at least three, a contradiction by (6). Hence $x^-, x^+ \in Red(B)$. Now $\eta(B \setminus \{x\}) = \eta(B) - 1$ follows from Proposition 3.2. The remaining proof is obvious.

8. Note that if an ear associated to a pair (a, b) is trivial in P then it is also trivial in B. Therefore the proof is obvious.

9. Let $E : a \prec x_1 \prec x_2 \prec \cdots \prec x_r \prec b$ be a non-trivial ear associated to the pair (a, b) in P. Using the contrapositive, suppose there is another path from a to b in P. Then by the definition of a basic retract associated to a poset, $a \prec x_1 \prec b$ is a non-trivial ear associated to (a, b) in B. The converse follows from the definition of a basic retract, as there is no other path from a to b in P, so we can remove each x_i from E to obtain B, and hence E becomes a trivial ear associated to (a, b) in B.

10. Suppose an ear E associated to (a, b) is non-trivial in B. Let $m \geq 1$ be the number of non-trivial ears associated to (a, b) in B. Let n be the number of non-trivial ears associated to (a, b) in P. From the first part of (6), it is clear

that $m \leq n$. Now, as there is a non-trivial ear E associated to (a, b) in B, by the converse part of (5), there is another path from a to b in P. Therefore by (9), there are n non-trivial ears associated to (a, b) in B. Therefore $n \leq m$. Thus $m = n$.

11. Suppose there is a trivial ear in B associated to (a, b). Since there cannot be more than one trivial ear in B associated to (a, b), by (5), there is a unique ear associated to (a, b) in P. Now suppose there is a non-trivial ear in B associated to (a, b). But then the proof follows from (10).

□

Lemma 3.6. *Let P_1 and P_2 be connected posets. Let $\phi : P_1 \to P_2$ be an isomorphism. Then:*

1. *An element x is pendant in P_1 if and only if $\phi(x)$ is pendant in P_2.*
2. *$x \in Irr^*(P_1)$ if and only if $\phi(x) \in Irr^*(P_2)$.*
3. *$x \in Irr(P_1)$ if and only if $\phi(x) \in Irr(P_2)$.*
4. *If x is pendant or $x \in Irr^*(P_1)$, that is, if $x \in Irr(P_1)$, then $P_1 \setminus \{x\} \cong P_2 \setminus \{\phi(x)\}$.*

Proof. 1. Suppose an element x is a pendant vertex in P_1. Then either $x \prec z$ for exactly one $z \in P_1$, or $y \prec x$ for exactly one $y \in P_1$. In the former case we have $\phi(x) \prec \phi(z)$, since $\phi : P_1 \to P_2$ is an isomorphism. Now if there exists $v \in P_2$ with $\phi(x) \prec v$ and $v \neq \phi(z)$ then there exists $u \in P_1$ with $\phi(u) = v$. That is, $\phi(u) \neq \phi(z)$. Also $\phi(x) \prec v = \phi(u)$. Therefore we get $u \neq z$ and $x \prec u$, since $\phi^{-1} : P_2 \to P_1$ is also an isomorphism. This is a contradiction. Therefore there is no $v \in P_2$ with $\phi(x) \prec v$ and $v \neq \phi(z)$. The other case can be proved on the similar lines. Hence $\phi(x)$ is pendant vertex in P_2. The converse follows from the fact that $\phi^{-1} : P_2 \to P_1$ is also an isomorphism.

2. Suppose $x \in Irr^*(P_1)$. Therefore there exists $y, z \in P_1$ such that $y \prec x \prec z$. Therefore $\phi(y) \prec \phi(x) \prec \phi(z)$, since $\phi : P_1 \to P_2$ is an isomorphism. Therefore $\phi(x) \in Irr^*(P_2)$. Note that this proof also along the similar lines as that of (1) above. The converse follows from the fact that $\phi^{-1} : P_2 \to P_1$ is also an isomorphism.

3. We know that for a connected poset P, $x \in Irr(P)$ if and only if either $x \in Irr^*(P)$ or x is a pendant vertex in P. Therefore the proof in this case follows from (1) and (2) above.

4. Suppose x is pendant in P_1 or $x \in Irr^*(P_1)$. Then by (1) and (2) above, either $\phi(x)$ is pendant in P_2 or $\phi(x) \in Irr^*(P_2)$. In any case, we have $P_1 \setminus \{x\} \cong P_2 \setminus \{\phi(x)\}$, since $P_1 \cong P_2$.

□

Corollary 3.7. *Let P_1 and P_2 be connected posets. Let $\phi : P_1 \to P_2$ be an isomorphism. Then an element x is retractible in P_1 if and only if $\phi(x)$ is retractible in P_2. Moreover, if x is retractible in P_1 then $P_1 \setminus \{x\} \cong P_2 \setminus \{\phi(x)\}$.*

Proof. Suppose an element x is retractible in P_1. Therefore $x \in Irr(P_1)$. By Theorem 2.5, $\eta(P_1 \setminus \{x\}) = \eta(P_1)$. By (3) of Lemma 3.6, $\phi(x) \in Irr(P_2)$. As $\phi : P_1 \to P_2$ is an isomorphism, $|P_1| = |P_2|$. Also $|E(P_1)| = |E(P_2)|$, since $x \prec y$ in P_1 if and only if $\phi(x) \prec \phi(y)$ in P_2. Therefore ϕ preserves the nullity, that is, $\eta(P_1) = \eta(P_2)$. Now by (4) of Lemma 3.6, $P_1 \setminus \{x\} \cong P_2 \setminus \{\phi(x)\}$. Therefore $\eta(P_1 \setminus \{x\}) = \eta(P_2 \setminus \{\phi(x)\})$. Hence $\eta(P_2 \setminus \{\phi(x)\}) = \eta(P_2)$. Therefore by Theorem 2.5, $\phi(x)$ is retractible in P_2. The converse follows from the fact that $\phi^{-1} : P_2 \to P_1$ is also an isomorphism. Moreover, if x is retractible in P_1 then $x \in Irr(P_1)$. Therefore by (4) of Lemma 3.6, $P_1 \setminus \{x\} \cong P_2 \setminus \{\phi(x)\}$. ☐

Using Definition 3.3 and by repeated application of (4) of Lemma 3.6, we have the following.

Proposition 3.8. *If R_1, R_2 are basic retracts associated to the posets P_1, P_2 respectively and $P_1 \cong P_2$ then $R_1 \cong R_2$.*

However, the converse of Proposition 3.8 is not true, since the posets given in Fig. 2 are not isomorphic to each other but the basic retracts associated to them are isomorphic. In fact those basic retracts are each isomorphic to M_2.

Corollary 3.9. *If R_1 and R_2 are basic retracts associated to a poset P then $R_1 \cong R_2$.*

Proof. Consider an identity map $\psi : P \to P$. Then ψ is an isomorphism. Therefore using Proposition 3.8, $R_1 \cong R_2$. ☐

Thus, by Corollary 3.9, up to isomorphism, there is a unique basic retract associated to any poset.

Definition 3.10. A subposet B of a poset P is said to be a *basic block* associated to P if B is obtained from the basic retract associated to P by successive removal of all the pendant vertices and all the retractible elements formed due to removal of all the pendant vertices.

For example, a 1-chain is the basic block associated to a chain. M_2 is the only basic block associated to the lattices of nullity one. It is clear that, if $C(P)$ is a tree then a 1-chain is a basic block associated to the poset P. In Fig. 3, B is the basic block associated to the posets P_1, P_2 and P'. In Fig. 4, we depict the basic blocks associated to lattices of nullity two.

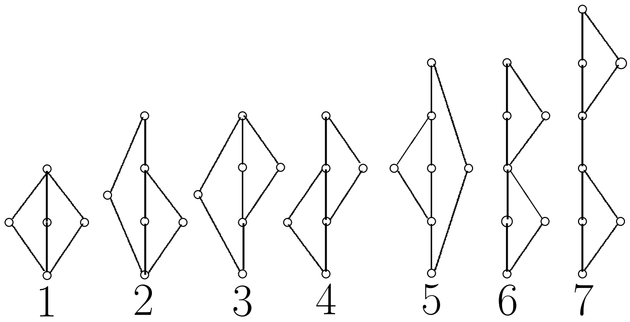

Figure 4

Now suppose R_1, R_2 are basic retracts associated to the posets P_1, P_2 respectively. Suppose B_1, B_2 are basic blocks associated to the posets P_1, P_2 respectively. Therefore B_1, B_2 are obtained from R_1, R_2 respectively by successive removal of all the pendant vertices and all the retractible elements formed due to removal of all the pendant vertices. If $P_1 \cong P_2$ then by Proposition 3.8, $R_1 \cong R_2$. Suppose B_1', B_2' are obtained from R_1, R_2 respectively by successive removal of all the pendant vertices. Then B_1, B_2 are obtained from B_1', B_2' respectively by removal of all the retractible elements formed due to removal of all the pendant vertices. Therefore by repeated application of (4) of Lemma 3.6, $R_1 \cong R_2$ implies that $B_1' \cong B_2'$. Hence by repeated application of second part of Corollary 3.7, $B_1 \cong B_2$. Thus, we have the following.

Proposition 3.11. *If B_1, B_2 are basic blocks associated to the posets P_1, P_2 respectively and $P_1 \cong P_2$ then $B_1 \cong B_2$.*

However, the converse of Proposition 3.11 is not true, since the posets given in Fig. 2 are not isomorphic to each other but the basic blocks associated to them are isomorphic to M_2. By Proposition 3.11, we have the following.

Corollary 3.12. *If B_1 and B_2 are basic blocks associated to a poset P then $B_1 \cong B_2$.*

Thakare et al. [33] enumerated all lattices of nullity two. Therefore by observation we have the following.

Proposition 3.13. *There are exactly seven non-isomorphic basic blocks (given in Fig. 4) associated to the lattices of nullity two.*

Let P and Q be disjoint posets. Let $P \cup Q$ be the union with the inherited order on P and Q such that $p < q$ for all $p \in P$ and $q \in Q$. Then it forms a poset called the *linear sum* of P and Q denoted by $P \oplus Q$.

By Definition 3.10, it follows that, if P is a lattice, then a basic block associated to P is the basic retract associated to P, without pendant vertices. Thakare et al. [33] introduced a *block* as a lattice in which 0 and 1 are reducible elements. If L is a lattice other than a chain then basic retract associated to L is either B or $B \oplus \{1\}$ or $\{0\} \oplus B$ or $\{0\} \oplus B \oplus \{1\}$, where B is a block. In fact B is a basic block associated to L. Consequently, as M_2 is the only basic block associated to the lattices of nullity one, there are four non-isomorphic basic retracts associated to the lattices of nullity one. In general, we have the following.

Theorem 3.14. *The number of non-isomorphic basic retracts associated to lattices having nullity at least one is four times the number of non-isomorphic basic blocks associated to them.*

Proof. Let $\mathscr{L}$ be a class of lattices having nullity at least one. Let $\mathscr{R}$ be a class of all non-isomorphic basic retracts associated to the lattices in $\mathscr{L}$. Let $\mathscr{B}$ be a class of all non-isomorphic basic blocks associated to the lattices in $\mathscr{L}$. Using Definition 3.3 and Definition 3.10, if $R \in \mathscr{R}$, and $B \in \mathscr{B}$ is obtained from R, then R takes one of the following forms. Either B or $B \oplus \{1\}$ or $\{0\} \oplus B$ or $\{0\} \oplus B \oplus \{1\}$. Now suppose $B_1, B_2 \in \mathscr{B}$ are basic blocks obtained from the basic retracts $R_1, R_2 \in \mathscr{R}$ respectively. Then we have $B_1 \cong B_2$ if and only if $R_1 \cong R_2$. Therefore $|\mathscr{R}| = 4 \times |\mathscr{B}|$. □

Note that a 1-chain and a 2-chain are the only basic retracts having nullity zero. Using Proposition 3.13 and Theorem 3.14, we have the following.

Corollary 3.15. *There are exactly twenty-eight non-isomorphic basic retracts associated to the lattices of nullity two.*

In order to enumerate up to isomorphism all the basic retracts associated to dismantlable lattices of a given nullity, we introduce a fundamental basic block in the next section.

4 Fundamental basic blocks

Definition 4.1. A dismantlable lattice B is said to be a *fundamental basic block* if it is a basic block and all the adjunct pairs in the adjunct representation of B into chains are distinct.

For example, M_2 (see Fig. 1) is fundamental basic block, whereas M_3 (see Fig. 4(1)) is not a fundamental basic block. We treat a 1-chain as a fundamental basic block. Using Proposition 3.13, we obtain, by observation, the following.

Corollary 4.2. *There are exactly six non-isomorphic fundamental basic blocks (see Fig. 4 (2 to 7)) of nullity two.*

Lemma 4.3 ([33])**.** *Let L be a dismantlable lattice with an adjunct representation $L = C_0]_{a_1}^{b_1} C_1]_{a_2}^{b_2} C_2 \ldots]_{a_l}^{b_l} C_l$. Then the set of meet irreducibles and join irreducibles are respectively given by $L \setminus \{a_1, a_2, \ldots, a_l\}$ and $L \setminus \{b_1, b_2, \ldots, b_l\}$.*

From Lemma 4.3, it follows that for a dismantlable lattice(which is not a chain) of nullity l containing r reducible elements, $2 \le r \le 2l$. Also $l \ge [\frac{r+1}{2}]$. Note that for a real number x, $[x]$ denotes the greatest integer less than or equal to x.

In the following, we prove that every RC-lattice is a dismantlable lattice.

Theorem 4.4. *A lattice in which all the reducible elements are comparable is a dismantlable lattice.*

Proof. Let L be a lattice containing r reducible elements which are all comparable. Then $r \geq 0$ but $r \neq 1$. If $r = 0$, then L is a chain and we are done. Suppose L is not a chain. Let $|L| = n \geq 4$. We use induction on n. If $n = 4$, then $L = M_2$ (see Fig. 1) and we are done. Now suppose $n > 4$, and the result is true for all lattices L' with $|L'| < n$. Let C be a maximal chain containing all the reducible elements of L and let $x \in L \setminus C$. Then there exists $y \in C$ such that x is incomparable with y in L. Let $a = x \wedge y$ and $b = x \vee y$. Let $C' : x_1 \prec x_2 \prec \cdots \prec x_t$ be the chain containing x with $a \prec x_1$ and $x_t \prec b$ in L. Clearly $x_i \in Irr^*(L)$, for all i, $1 \leq i \leq t$. If $L \setminus C' = C$, then $L = C]_a^b C'$ and we are done; Otherwise, let $L' = L \setminus C'$. But then L' contains at least two reducible elements with $|L'| < n$. Therefore by induction hypothesis, L' is a dismantlable lattice. Hence by Theorem 1.3, L' is an adjunct of chains. Suppose $L' = C]_{a_1}^{b_1} C_1]_{a_2}^{b_2} C_2 \cdots]_{a_k}^{b_k} C_k$ for some $k \geq 1$, where for each i, C_i is a chain. But then $L = L']_a^b C'$. Thus by Theorem 1.3, L is a dismantlable lattice. □

However, the converse of Theorem 4.4 is not true. For example, a lattice $L = (\{0\} \oplus M_2 \oplus \{1\})]_0^1 M_2$ is dismantlable lattice (having nullity three) but all the reducible elements in it are not comparable. This example also suggests that the lattices having nullity greater than two need not be RC-lattices.

Hereafter, we will restrict ourselves to the study of RC-lattices for achieving the desired countings. The following result follows from the fact that adjunct operation preserves the existing coverings of the lattices.

Lemma 4.5. *Let C_0, C_1 and C_2 be chains. Let α_1 and α_2 be adjunct pairs lying in C_0. Then $(C_0]_{\alpha_1} C_1)]_{\alpha_2} C_2 = (C_0]_{\alpha_2} C_2)]_{\alpha_1} C_1$.*

Proof. Let $\alpha_1 = (a_1, b_1)$ and $\alpha_2 = (a_2, b_2)$ be adjunct pairs lying in C_0. Without loss, assume that $a_1 \leq a_2$, and if $a_1 = a_2$ then $b_1 \leq b_2$. Then we have the following three cases. Either $a_1 < b_1 \leq a_2 < b_2$, or $a_1 < a_2 < b_2 < b_1$, or $a_1 \leq a_2 < b_1 \leq b_2$ in C_0.

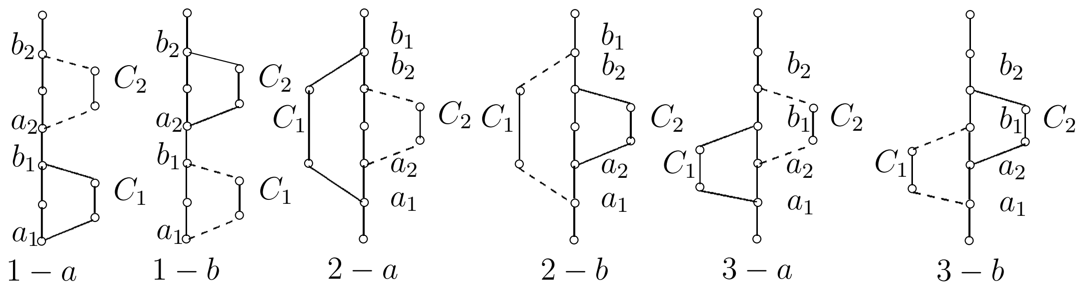

Figure 5

Consider the first case that $a_1 < b_1 \leq a_2 < b_2$ in C_0. Then using Definition 1.2 and from Fig. 5$(1-a, 1-b)$, we have $x \leq y$ in $(C_0]_{\alpha_1} C_1)]_{\alpha_2} C_2$ if and only if $x \leq y$ in $(C_0]_{\alpha_2} C_2)]_{\alpha_1} C_1$. Also from Fig. 5$(1-a, 1-b)$, it follows that, x and y are incomparable in $(C_0]_{\alpha_1} C_1)]_{\alpha_2} C_2$ if and only if x and y are incomparable in $(C_0]_{\alpha_2} C_2)]_{\alpha_1} C_1$. Thus $(C_0]_{\alpha_1} C_1)]_{\alpha_2} C_2 = (C_0]_{\alpha_2} C_2)]_{\alpha_1} C_1$. The proof for the remaining two cases follows from Fig. 5$(2-a, 2-b$, and $3-a, 3-b)$ on the similar lines. □

Using Lemma 4.5, we have the following.

Corollary 4.6. *Let* $L = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2\cdots]_{\alpha_i}C_i\cdots]_{\alpha_j}C_j\cdots]_{\alpha_k}C_k$*, where* C_0 *is a maximal chain containing all the reducible elements of* L*. Then for any* $i \neq j$*,* $L = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2\cdots]_{\alpha_j}C_j\cdots]_{\alpha_i}C_i\cdots]_{\alpha_k}C_k$*.*

Using Theorem 4.4 and Corollary 4.6, we have the following.

Proposition 4.7. *Let* L *be an RC-lattice. Let* C *be a maximal chain containing all the reducible elements of* L*. Then there exist chains* $C_1, C_2, \ldots, C_k$ *in* L *such that* $L = C]_{\alpha_1}C_1]_{\alpha_2}C_2\cdots]_{\alpha_k}C_k$*, where the adjunct pairs follow lexicographic(or dictionary) order defined on* $C \times C$*.*

Proposition 4.8. *Let* L *be an RC-lattice of nullity* l *containing* $r \geq 2$ *reducible elements. If* $l = [\frac{r+1}{2}]$ *then the multiplicity of each adjunct pair in an adjunct representation of* L *into chains is one.*

Proof. By Theorem 4.4, it is clear that L is a dismantlable lattice. As L is a lattice of nullity l, by Theorem 2.1, L is an adjunct of $l+1$ chains. Therefore by Proposition 4.7, $L = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2\cdots]_{\alpha_l}C_l$, where C_0 is a maximal chain containing all the r reducible elements, and for each i, C_i is a chain and α_i is an adjunct pair. Suppose for some i, the adjunct pair α_i has multiplicity more than one. Therefore there must be at least one $j \neq i$ such that $\alpha_i = \alpha_j$. Then by Corollary 4.6, $L = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2\cdots]_{\alpha_{i-1}}C_{i-1}]_{\alpha_l}C_l]_{\alpha_{i+1}}C_{i+1}\cdots]_{\alpha_{l-1}}C_{l-1}]_{\alpha_i}C_i$. But then by Proposition 1.1, $L \setminus C_i$ is a sublattice of L. Moreover, $L \setminus C_i$ contains r reducible elements. Also by Theorem 2.1, nullity of $L \setminus C_i$ is $l-1$. Thus, $L \setminus C_i$ is a dismantlable lattice of nullity $l-1$ containing r reducible elements. Therefore $l-1 \geq [\frac{r+1}{2}]$. This contradicts the fact that $l = [\frac{r+1}{2}]$. ☐

Using Definition 4.1 and by Proposition 4.8, we have the following.

Corollary 4.9. *If* $r \geq 2$ *and* $l = [\frac{r+1}{2}]$ *then every basic block associated to an RC-lattice of nullity* l *containing* r *reducible elements is a fundamental basic block.*

Note that, in Corollary 4.9, the condition $l = [\frac{r+1}{2}]$ is necessary, since if $r = 2$ and $l = 2 > [\frac{r+1}{2}] = 1$ then we get M_3, which is not a fundamental basic block.

Now in order to count the number of non-isomorphic basic retracts/blocks associated to the RC-lattices, we introduce the following.

Definition 4.10. Let L be an RC-lattice. Let B be a basic block associated to L. If B itself is a fundamental basic block, then we say that B is a fundamental basic block associated to L; otherwise, let (a, b) be an adjunct pair in an adjunct representation of B. If the interval $(a, b) \subseteq Irr(B)$ then remove all but two non-trivial ears associated to (a, b) in B; otherwise, remove all but one non-trivial ear associated to (a, b) in B. Perform the operation of removal of non-trivial ears associated to (a, b), for each adjunct pair (a, b) in an adjunct representation of B. The resultant *sublattice* of B is called a *fundamental basic block associated to* L.

For example, M_2 (see Fig. 1) is a fundamental basic block associated to M_3 (see Fig. 4 (1)). In Fig. 6, F is the fundamental basic block associated to the RC-lattice L. Note that F is also the fundamental basic block associated to R as well as B.

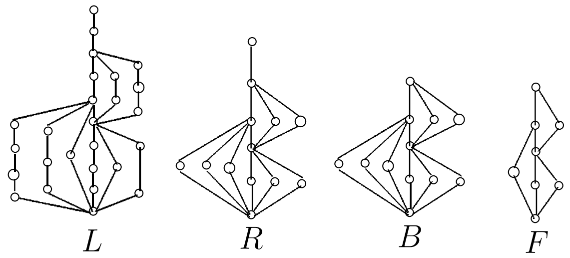

Figure 6

Note that, in Fig. 6, R and B are respectively the basic retract and the basic block associated to the RC-lattice L.

By Proposition 3.11, if B_1, B_2 are basic blocks associated to the RC-lattices L_1, L_2 respectively and $L_1 \cong L_2$ then $B_1 \cong B_2$. If F_1, F_2 are fundamental basic blocks associated to the RC-lattices L_1, L_2 respectively then F_1, F_2 are obtained from B_1, B_2 respectively using Definition 4.10. Hence, by repeated application of (4) of Lemma 3.6, $B_1 \cong B_2$ implies that $F_1 \cong F_2$. Thus, we have the following.

Proposition 4.11. *If F_1, F_2 are fundamental basic blocks associated to the RC-lattices L_1, L_2 respectively and $L_1 \cong L_2$ then $F_1 \cong F_2$.*

However, the converse of Proposition 4.11 is not true. This is because M_2 is the fundamental basic block associated to the RC-lattices shown in Fig. 2, but those lattices are not isomorphic to each other. Using Proposition 4.11, we have the following.

Corollary 4.12. *If F_1 and F_2 are fundamental basic blocks associated to an RC-lattice L then $F_1 \cong F_2$.*

From Proposition 4.11, it follows that, if two fundamental basic blocks are non-isomorphic, then the RC-lattices associated by these fundamental basic blocks are also non-isomorphic. Also, using Corollary 4.12, it follows that, up to isomorphism, there is a *unique* fundamental basic block associated to any RC-lattice. It can be observed that, the fundamental basic block associated to an RC-lattice is having the same or a smaller width as compared to the basic block associated to that lattice.

Let L be an RC-lattice of nullity l containing $r \geq 2$ reducible elements. By Theorem 4.4, L is a dismantlable lattice. Let $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ be the distinct adjunct pairs in the adjunct representation of L, containing C as a maximal chain containing all the r reducible elements of L. By Proposition 4.7, we can assume that $(a_1, b_1) < (a_2, b_2) < \ldots < (a_k, b_k)$ with respect to the dictionary order defined on $C \times C$. Let n_i be the multiplicity of an adjunct pair (a_i, b_i) for each i, $1 \leq i \leq k$. Let

$T_k = (n_1, n_2, \ldots, n_k)$. By Theorem 2.1, it is clear that $l = \sum_{i=1}^{k} n_i$. Now L is adjunct sum of $l+1$ chains and hence contain l adjunct pairs (repetition is allowed, if any). In the following, we prove that, if all the adjunct pairs in an adjunct representation of a lattice (in which all the $r \geq 2$ reducible elements are comparable) are distinct, then the nullity of that lattice cannot exceed $\binom{r}{2}$.

Proposition 4.13. *For an RC-lattice of nullity l containing $r \geq 2$ reducible elements with $T_k = 1^k = (1, 1, \ldots, 1)$, $[\frac{r+1}{2}] \leq k = l \leq \binom{r}{2}$.*

Proof. Let L be an RC-lattice of nullity l containing $r \geq 2$ reducible elements. By Theorem 4.4, L is a dismantlable lattice. We know that $[\frac{r+1}{2}] \leq l$. Also, if $T_k = (n_1, n_2, \ldots, n_k)$ then by Theorem 2.1, $l = \sum_{i=1}^{k} n_i$. Therefore for $T_k = 1^k = (1, 1, \ldots, 1)$, $l = k$. Now the multiplicity of each adjunct pair in an adjunct representation of L is one. Therefore the number of adjunct pairs is k. But L contains r reducible elements and one adjunct pair corresponds to two reducible elements. Therefore $k \leq \binom{r}{2}$. Thus $[\frac{r+1}{2}] \leq l = k \leq \binom{r}{2}$. □

Using Definition 4.1 and by Proposition 4.13, we have the following.

Corollary 4.14. *For any fundamental basic block of nullity l containing $r \geq 2$ reducible elements which are all comparable, $[\frac{r+1}{2}] \leq l \leq \binom{r}{2}$.*

5 Recurrence relations and Enumerations

In this section for the counting purpose, we are going to establish some recurrence relations in terms of the number of all non-isomorphic fundamental basic blocks (of given nullity) such that each one of them has r reducible elements which are all comparable. In this connection, let us see the following.

Definition 5.1. For $r \geq 0$, let $\mathscr{F}_r$ be the class of all non-isomorphic fundamental basic blocks such that each one of them has r reducible elements which are all comparable.

Let $F \in \mathscr{F}_r$, where $r \geq 0$. Then $\mathscr{F}_1 = \emptyset$. Also F is an RC-lattice. Moreover, F is a basic retract as no element of $Irr^*(F)$ is retractible in F.

Proposition 5.2. *Let $F \in \mathscr{F}_{r+1}$, where $r \geq 1$. Let $R = F \setminus \{1\}$ be a poset obtained from F by deleting 1 of F. Let B be a basic block associated to R. Then R is a basic retract and $B \in \mathscr{F}_j$ for some j, $0 \leq j \leq r$ with $j \neq 1$.*

Proof. Let x and y be incomparable elements in F with $x \vee y = 1$. As F is an RC-lattice, both $x, y \notin Red(F)$. Therefore either $x \in Irr^*(F)$ or $y \in Irr^*(F)$. Suppose $x \in Irr^*(F)$. But then x must be pendant in $R = F \setminus \{1\}$. Moreover, if $z \prec x$ then

$z \in Red(F)$. Thus $x \in Irr^*(F)$ if and only if $x \in Irr^*(R)$ or x is pendant in R. Therefore no element of $Irr^*(R)$ is retractible in R, since F being a basic retract, no element of $Irr^*(F)$ is retractible in F. Hence R is a basic retract. Now B is a basic block associated to R. Therefore B is obtained from R by successive removal of all the pendant vertices and all the retractible elements formed due to removal of all the pendant vertices. As F is a fundamental basic block, B is also a fundamental basic block. The remaining part of the proof is obvious. $\square$

Let $a_r = |\mathscr{F}_r|$, for all $r \geq 0$. Then $a_0 = 1$, since $\mathscr{F}_0$ consists of a 1-chain only. $a_1 = 0$, since $\mathscr{F}_1$ is an empty class. $a_2 = 1$, since $\mathscr{F}_2$ consists of an M_2 (see Fig. 1) only. We now obtain a recursive formula which produces a_r in the following.

Theorem 5.3. *For $r \geq 2$, $a_{r+1} = \left(\sum_{j=0}^{r} \binom{r}{j} 2^j a_j\right)$ - a_r with $a_0 = a_2 = 1$ and $a_1 = 0$.*

Proof. Let $F \in \mathscr{F}_{r+1}$, where $r \geq 2$. Then F is an RC-lattice. Consider the poset $F \setminus \{1\}$ obtained from F by deleting 1 of F. Let B be the basic block associated to $F \setminus \{1\}$. Then by Proposition 5.2, $F \setminus \{1\}$ is a basic retract, and B is a fundamental basic block containing j $(0 \leq j \leq r,\ j \neq 1)$ reducible elements of F except 1. Note that, for $F_1, F_2 \in \mathscr{F}_{r+1}$, if B_1, B_2 are basic blocks associated to the posets $F_1 \setminus \{1\}$, $F_2 \setminus \{1\}$ respectively and $F_1 \cong F_2$ then $F_1 \setminus \{1\} \cong F_2 \setminus \{1\}$, and hence by Proposition 3.11, $B_1 \cong B_2$. Let C be a maximal chain in F containing all the reducible elements of F. Let $C_r : x_1 \prec x_2 \prec \cdots \prec x_r$ be a r-chain. Then we have the following three cases.

Case 1 : $B \in \mathscr{F}_0$. That is, B is a 1-chain, say x_1 (in this case, $C(F \setminus \{1\})$ is a tree). Then F can be constructed using B in unique way as $F = (C_r \oplus \{x\} \oplus \{1\})]_{x_1}^1\{y_1\}]_{x_2}^1\{y_2\} \cdots]_{x_r}^1\{y_r\}$. Note that in this case we assume that C_r is precisely $C \cap Red(F) \setminus \{1\}$, and $C = C_r \oplus \{x\} \oplus \{1\}$.

Case 2 : $B \in \mathscr{F}_j$ for some $j = 2, 3, \cdots, r-1$. In this case we assume that $C \cap Red(B) \subset C_r$ with $|C \cap Red(B)| = j$. Let $x_{i_1}, x_{i_2}, \ldots, x_{i_j} \in Red(B)$. These j elements clearly have $\binom{r}{j}$ choices. Now for fixed j, every member F of $\mathscr{F}_{r+1}$ can be obtained in a unique way using a member B of $\mathscr{F}_j$ as follows.

For fixed $B \in \mathscr{F}_j$, let $F' = ((B \cup C_r) \oplus \{1\})]_{z_1}^1\{y_1\}]_{z_2}^1\{y_2\} \cdots]_{z_{r-j}}^1\{y_{r-j}\}$ or $F' = ((B \cup C_r) \oplus \{x\} \oplus \{1\})]_{z_1}^1\{y_1\}]_{z_2}^1\{y_2\} \cdots]_{z_{r-j}}^1\{y_{r-j}\}$, where $z_1, z_2, \ldots, z_{r-j} \in C_r \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_j}\}$ with $z_1 < z_2 < \cdots < z_{r-j}$. The latter construction is required whenever $i_j = r$ or $z_{r-j} = x_r$. Then either $F = F'$ or $F = F']_{w_1}^1\{y_{r-j+1}\}]_{w_2}^1\{y_{r-j+2}\} \cdots]_{w_i}^1\{y_{r-j+i}\}$ for $1 \leq i \leq j$, where for each k, $1 \leq k \leq i$, $w_k \in \{x_{i_1}, x_{i_2}, \ldots, x_{i_j}\}$ with $w_1 < w_2 < \cdots < w_i$. For fixed i, $1 \leq i \leq j$, there are $\binom{j}{i}$ independent choices for the latter construction. Thus, if we vary i, then by this kind of construction using B, we get in all $1 + \left(\binom{j}{1} + \cdots + \binom{j}{j}\right) = 2^j$ non-isomorphic members of $\mathscr{F}_{r+1}$. Now for fixed j and for variable choices of the j elements of $Red(B)$, we get in all $\binom{r}{j} \times 2^j$ non-isomorphic members of $\mathscr{F}_{r+1}$. Therefore, for fixed j and for variable $B \in \mathscr{F}_j$, we get in all $\binom{r}{j} \times 2^j \times |\mathscr{F}_j|$ non-isomorphic members of $\mathscr{F}_{r+1}$. Hence in this case, if we vary j, we get in all $\sum_{j=2}^{r-1}(\binom{r}{j} \times 2^j \times |\mathscr{F}_j|)$ non-isomorphic members of $\mathscr{F}_{r+1}$.

Case 3 : $B \in \mathscr{F}_r$. In this case we assume that $C_r = Red(B)$. Now every member F of $\mathscr{F}_{r+1}$ can be obtained in a unique way using a member B of $\mathscr{F}_r$ as follows. For fixed $B \in \mathscr{F}_r$, $F = (B \oplus \{1\})]_{w_1}^1\{y_1\}]_{w_2}^1\{y_2\}\cdots]_{w_i}^1\{y_i\}$ or $F = (B \oplus \{x\} \oplus \{1\})]_{w_1}^1\{y_1\}]_{w_2}^1\{y_2\}\cdots]_{w_i}^1\{y_i\}$ for $1 \leq i \leq r$, where for each k, $1 \leq k \leq i$, $w_k \in C_r$ with $w_1 < w_2 < \cdots < w_i$. The latter construction is required whenever $w_k = x_r$ for some k. For fixed i, $1 \leq i \leq r$, there are $\binom{r}{i}$ independent choices for these constructions. Thus, if we vary i, we get in all $\binom{r}{1} + \binom{r}{2} + \cdots + \binom{r}{r} = 2^r - 1$ non-isomorphic members of $\mathscr{F}_{r+1}$. Hence in this case, for variable $B \in \mathscr{F}_r$, we get in all $(2^r - 1) \times |\mathscr{F}_r|$ non-isomorphic members of $\mathscr{F}_{r+1}$.

Therefore by Case 1, Case 2 and Case 3, for $r \geq 2$, the total number of non-isomorphic fundamental basic blocks in $\mathscr{F}_{r+1}$, which can be obtained using all non-isomorphic members of $\mathscr{F}_j$, $0 \leq j \leq r, j \neq 1$ is given by $1 + \left(\sum_{j=2}^{r-1}\binom{r}{j}2^j a_j\right) + (2^r - 1)a_r$. But $a_1 = 0$ and $a_0 = a_2 = 1$. Hence for $r \geq 2$, $a_{r+1} = |\mathscr{F}_{r+1}| = \left(\sum_{j=0}^{r}\binom{r}{j}2^j a_j\right) - a_r$. $\square$

Note that the formula given by Theorem 5.3 also works for $r \geq 0$. In the following, we obtain another form of a recursive formula for a_r.

Corollary 5.4. *For $r \geq 1$, $a_{r+1} = \sum_{k=1}^{r}\sum_{j=0}^{k}\binom{r}{j}\binom{r-j}{k-j}a_{r-j}$ with $a_0 = 1$ and $a_1 = 0$.*

Proof. By Theorem 5.3,

$$
\begin{aligned}
a_{r+1} &= \left(\sum_{j=0}^{r}\binom{r}{j}2^j a_j\right) - a_r = \binom{r}{r}2^r a_r + \binom{r}{r-1}2^{r-1}a_{r-1} \\
&\quad + \binom{r}{r-2}2^{r-2}a_{r-2} + \cdots + \binom{r}{r-(r-1)}2^{r-(r-1)}a_{r-(r-1)} + \binom{r}{r-r}2^{r-r}a_{r-r} - a_r \\
&= \binom{r}{r}(2^r - 1)a_r + \binom{r}{r-1}2^{r-1}a_{r-1} + \binom{r}{r-2}2^{r-2}a_{r-2} + \cdots \\
&\quad + \binom{r}{r-(r-1)}2^{r-(r-1)}a_{r-(r-1)} + \binom{r}{r-r}2^{r-r}a_{r-r} \\
&= \binom{r}{0}(2^r - 1)a_r + \binom{r}{1}(2^{r-1})a_{r-1} + \binom{r}{2}(2^{r-2})a_{r-2} + \cdots + \binom{r}{r-1}(2^{r-(r-1)})a_{r-(r-1)} \\
&\quad + \binom{r}{r}(2^{r-r})a_{r-r} \\
&= \binom{r}{0}\left(\binom{r}{1} + \binom{r}{2} + \cdots + \binom{r}{r}\right)a_r + \binom{r}{1}\left(\binom{r-1}{0} + \binom{r-1}{1} + \cdots + \binom{r-1}{r-1}\right)a_{r-1} \\
&\quad + \binom{r}{2}\left(\binom{r-2}{0} + \binom{r-2}{1} + \cdots + \binom{r-2}{r-2}\right)a_{r-2} + \cdots \\
&\quad + \binom{r}{r-1}\left(\binom{r-(r-1)}{0} + \binom{r-(r-1)}{1}\right)a_{r-(r-1)} + \binom{r}{r}\left(\binom{r-r}{0}\right)a_{r-r} \\
&= \left(\binom{r}{0}\binom{r}{1}a_r + \binom{r}{1}\binom{r-1}{0}a_{r-1}\right) + \left(\binom{r}{0}\binom{r}{2}a_r + \binom{r}{1}\binom{r-1}{1}a_{r-1} + \binom{r}{2}\binom{r-2}{0}a_{r-2}\right) + \cdots \\
&\quad + \left(\binom{r}{0}\binom{r}{r}a_r + \binom{r}{1}\binom{r-1}{r-1}a_{r-1} + \cdots + \binom{r}{r}\binom{r-r}{0}a_{r-r}\right) \\
&= \left(\sum_{j=0}^{1}\binom{r}{j}\binom{r-j}{1-j}a_{r-j}\right) + \left(\sum_{j=0}^{2}\binom{r}{j}\binom{r-j}{2-j}a_{r-j}\right) + \cdots + \left(\sum_{j=0}^{r}\binom{r}{j}\binom{r-j}{r-j}a_{r-j}\right) \\
&= \sum_{k=1}^{r}\sum_{j=0}^{k}\binom{r}{j}\binom{r-j}{k-j}a_{r-j}.
\end{aligned}
$$

Thus, $a_{r+1} = \sum_{k=1}^{r}\sum_{j=0}^{k}\binom{r}{j}\binom{r-j}{k-j}a_{r-j}$. $\square$

Definition 5.5. Let $\mathscr{F}_r(l)$ be the subclass of $\mathscr{F}_r$ such that each fundamental basic block in it is of nullity l. For $1 \leq k \leq r$ and for $[\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}$, let $\mathscr{C}_k^l = \{F \in \mathscr{F}_{r+1}(l) :$ Indegree of 1 in F is $k+1\}$.

Note that $\mathscr{F}_0(0)$ consists of a 1-chain only. $\mathscr{F}_1(l) = \emptyset$ for all $l \geq 0$. $\mathscr{F}_2(1)$ as well as $\mathscr{C}_1^1$ consists of M_2 (see Fig. 1) only. Also, if $l < [\frac{r+1}{2}]$ or $\binom{r}{2} < l$ then $\mathscr{F}_r(l) = \emptyset$.

Proposition 5.6. *Let $F \in \mathscr{F}_{r+1}(l)$, where $r \geq 1$. Let $R = F \backslash \{1\}$ be a poset obtained from F by deleting 1 of F. Let B be a basic block associated to R. If indegree of 1 in F is $k+1$, where $1 \leq k \leq r$ then $B \in \mathscr{F}_{r-j}(l-k)$ for some j, $0 \leq j \leq k$ with $j \neq r-1$.*

Proof. As $F \in \mathscr{F}_{r+1}(l)$, F is an RC-lattice having nullity l. Let C be a maximal chain in F containing all the reducible elements of F. If indegree of 1 in F is $k+1$, then there exist $x_1, x_2, \ldots, x_k \in Irr^*(F)$ with $x_i \notin C$ and $x_i \prec 1$ for all i, $1 \leq i \leq k$. Also for each i, $1 \leq i \leq k$, x_i is pendant in $R = F \setminus \{1\}$. Now B is a basic block associated to R. Therefore B is obtained from R by successive removal of all the pendant vertices, and all the retractible elements formed due to removal of all the pendant vertices. Hence $x_i \notin B$ for all i, $1 \leq i \leq k$. As F is a fundamental basic block, B is also a fundamental basic block. Now F is a dismantlable lattice of nullity l. Hence by Theorem 2.1, F is adjunct of $l+1$ chains. But then B would be adjunct of $(l+1)-k$ chains. Hence again by Theorem 2.1, B is of nullity $l-k$. The remaining part of the proof is obvious. □

Note that, in the above formula, if $l < k$ then the class $\mathscr{F}_{r-j}(l-k)$ would be considered as empty. For $r = 2$, $|\mathscr{C}_1^2| = 2$, $|\mathscr{C}_1^3| = 0$, $|\mathscr{C}_2^2| = 1$ and $|\mathscr{C}_2^3| = 1$. In the following, we obtain the formula for $|\mathscr{C}_k^l|$ in terms of the number of non-isomorphic fundamental basic blocks of nullity $l-k$, in which the reducible elements are all comparable.

Theorem 5.7. *For fixed $r \geq 1$, for $1 \leq k \leq r$, and for $[\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}$,*

$$|\mathscr{C}_k^l| = \sum_{j=0}^{k} \binom{r}{j}\binom{r-j}{k-j} |\mathscr{F}_{r-j}(l-k)|.$$

Proof. For fixed $r \geq 1$, $1 \leq k \leq r$ and $[\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}$, let $F \in \mathscr{C}_k^l$. Therefore $F \in \mathscr{F}_{r+1}(l)$ with indegree of 1 in F is $k+1$. Consider the poset $F \setminus \{1\}$ obtained from F by deleting 1 of F. Let B be a basic block associated to $F \setminus \{1\}$. Then by Proposition 5.2, $F \backslash \{1\}$ is a basic retract. Also by Proposition 5.6, B is a fundamental basic block in $\mathscr{F}_{r-j}(l-k)$ for some $j = 0, 1, 2, \ldots, k$ with $j \neq r-1$. Note that, for $F_1, F_2 \in \mathscr{F}_{r+1}$, if B_1, B_2 are basic blocks associated to the posets $F_1 \setminus \{1\}$, $F_2 \setminus \{1\}$ respectively and $F_1 \cong F_2$ then $F_1 \setminus \{1\} \cong F_2 \setminus \{1\}$, and hence by Proposition 3.11, $B_1 \cong B_2$. Therefore, for any $F_1, F_2 \in \mathscr{C}_k^l$, if the corresponding basic blocks B_1 and B_2 are not isomorphic then $F_1 \ncong F_2$.

Note that the removal of 1 from F means the removal of k 1-chains corresponding to k adjunct pairs of the type $(a, 1)$, where $a \neq 1$ and $a \in Red(F)$. As F is a fundamental basic block, at most k out of r (excluding 1) reducible elements of F may become irreducible(in fact retractible) in $F\setminus\{1\}$ after removal of all the pendant vertices. Therefore $r-k \leq |Red(B)| \leq r$ and $Red(B) \subseteq Red(F)$. Let C be a maximal chain in F containing all the reducible elements of F. Let $C_r : x_1 \prec x_2 \prec \cdots \prec x_r$ be a r-chain. Then we have the following three cases.

Case 1: $B \in \mathscr{F}_r(l-k)$. That is $j = 0$. In this case $C_r = Red(B)$. Therefore every member F of $\mathscr{C}_k^l$ can be obtained in a unique way using a member B of $\mathscr{F}_r(l-k)$ as follows. For fixed $B \in \mathscr{F}_r(l-k)$, $F = (B \oplus \{1\})]_{z_1}^1\{y_1\}]_{z_2}^1\{y_2\}\cdots]_{z_k}^1\{y_k\}$ or $F = (B \oplus \{x\} \oplus \{1\})]_{z_1}^1\{y_1\}]_{z_2}^1\{y_2\}\cdots]_{z_k}^1\{y_k\}$, where $z_1, z_2, \ldots, z_k \in C_r$ with $z_1 < z_2 < \cdots < z_k$. The latter construction is required whenever $z_k = x_r$. Note that these k reducible elements can be chosen out of r reducible elements of C_r in $\binom{r}{k}$ ways, and nullity of F becomes l. Thus, using fixed $B \in \mathscr{F}_r(l-k)$, we get in all $\binom{r}{k}$ non-isomorphic members of $\mathscr{C}_k^l$. Hence in this case, for variable $B \in \mathscr{F}_r(l-k)$, we get in all $\binom{r}{k} \times |\mathscr{F}_r(l-k)|$ non-isomorphic members of $\mathscr{C}_k^l$.

Case 2: $B \in \mathscr{F}_{r-j}(l-k)$ for some $j = 1, 2, \ldots, k \leq r-2$. In this case we assume that $C \cap Red(B) \subset C_r$ with $|C \cap Red(B)| = r-j$. Let $x_{i_1}, x_{i_2}, \ldots, x_{i_{r-j}} \in Red(B)$. For fixed j, these $r-j$ reducible elements clearly have $\binom{r}{j}$ choices. Now every member F of $\mathscr{C}_k^l$ can be obtained in a unique way using a member B of $\mathscr{F}_{r-j}(l-k)$ as follows. For fixed $B \in \mathscr{F}_{r-j}(l-k)$, let $F' = ((B \cup C_r) \oplus \{1\})]_{z_1}^1\{y_1\}]_{z_2}^1\{y_2\}\cdots]_{z_j}^1\{y_j\}$ or $F' = ((B \cup C_r) \oplus \{x\} \oplus \{1\})]_{z_1}^1\{y_1\}]_{z_2}^1\{y_2\}\cdots]_{z_j}^1\{y_j\}$, where $z_1, z_2, \ldots, z_j \in C_r \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_{r-j}}\}$ with $z_1 < z_2 < \cdots < z_j$. The latter construction is required whenever $i_{r-j} = r$ or $z_j = x_r$. Note that nullity of $F' \in \mathscr{F}_{r+1}$ becomes $(l-k)+j$. If $j = k$ then $F = F'$; otherwise, $F = F']_{w_1}^1\{y_{j+1}\}]_{w_2}^1\{y_{j+2}\}\cdots]_{w_{k-j}}^1\{y_{j+(k-j)}\}$, where for each p, $1 \leq p \leq k-j$, $w_p \in \{x_{i_1}, x_{i_2}, \ldots, x_{i_{r-j}}\}$ with $w_1 < w_2 < \cdots < w_{k-j}$. For fixed j, $1 \leq j \leq k \leq r-2$, there are $\binom{r-j}{k-j}$ independent choices for the latter construction. Now for fixed j and for variable choices of the $r-j$ elements of $Red(B)$, we get in all $\binom{r}{j} \times \binom{r-j}{k-j}$ non-isomorphic members of $\mathscr{C}_k^l$. Therefore, for fixed j and for variable $B \in \mathscr{F}_{r-j}(l-k)$, we get in all $\binom{r}{j} \times \binom{r-j}{k-j} \times |\mathscr{F}_{r-j}(l-k)|$ non-isomorphic members of $\mathscr{C}_k^l$. Hence in this case, if we vary j, we get in all $\sum_{j=1}^{k}\left(\binom{r}{j} \times \binom{r-j}{k-j} \times |\mathscr{F}_{r-j}(l-k)|\right)$ non-isomorphic members of $\mathscr{C}_k^l$, where $k \leq r-2$.

Case 3: $B \in \mathscr{F}_0(l-k)$. That is, B is a 1-chain, say $B = x_1$, and $j = r$. Hence $k = r$. In this case $F \in \mathscr{C}_k^l$ can be constructed using B in a unique way as per the following. $F = (C_r \oplus \{x\} \oplus \{1\})]_{x_1}^1\{y_1\}]_{x_2}^1\{y_2\}\cdots]_{x_r}^1\{y_r\}$.

Therefore by Cases 1, 2 and 3, for $r \geq 1$,

$$|\mathscr{C}_k^l| = \binom{r}{k}|\mathscr{F}_r(l-k)| + \sum_{j=1}^{k\leq r-2}\binom{r}{j}\binom{r-j}{k-j}|\mathscr{F}_{r-j}(l-k)| + 1.$$

Note that, for $j = r-1$, $\mathscr{F}_{r-j}(l-k) = \emptyset$. Also $|\mathscr{F}_0(0)| = 1$. Hence for $r \geq 1$, $|\mathscr{C}_k^l| = \sum_{j=0}^{k}\binom{r}{j}\binom{r-j}{k-j}|\mathscr{F}_{r-j}(l-k)|$. □

For fixed $r \geq 1$ and fixed l with $[\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}$, the collection $\{\mathscr{C}_k^l : 1 \leq k \leq r\}$

forms a partition of $\mathscr{F}_{r+1}(l)$. Therefore by Theorem 5.7, we have the following.

Corollary 5.8. *For fixed* $r \geq 1$ *and for* $[\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}$,

$$|\mathscr{F}_{r+1}(l)| = \sum_{k=1}^{r} \sum_{j=0}^{k} \binom{r}{j} \binom{r-j}{k-j} |\mathscr{F}_{r-j}(l-k)|.$$

Remark 5.9. Coincidentally, the sequence obtained using Corollary 5.8 matches with the sequence $A054548$ (see the On-line Encyclopedia of Integer Sequences, OEIS [30]). This sequence gives the number of labeled graphs on $r \geq 1$ unisolated vertices and l edges, where $0 \leq l \leq \binom{r}{2}$. This problem was posed by Harary and Palmer [18] in 1973.

Note that if $l < [\frac{r+1}{2}]$ or $\binom{r}{2} < l$, then $\mathscr{F}_r(l) = \emptyset$. Now the collection $\{\mathscr{F}_{r+1}(l) : [\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}\}$ forms a partition of $\mathscr{F}_{r+1}$. Therefore by Corollary 5.8, we also have the following.

Corollary 5.10. *For* $r \geq 1$, $a_{r+1} = \sum_{l=[\frac{r+2}{2}]}^{\binom{r+1}{2}} \sum_{k=1}^{r} \sum_{j=0}^{k} \binom{r}{j} \binom{r-j}{k-j} |\mathscr{F}_{r-j}(l-k)|$.

For $1 \leq k \leq r$, let $\mathscr{F}^k_{r+1} = \{B \in \mathscr{F}_{r+1}$: Indegree, say d, of 1 in B is $k+1\}$. It is clear that $2 \leq d \leq r+1$, for any $B \in \mathscr{F}_{r+1}$. Also, for fixed $1 \leq k \leq r$, the collection $\{\mathscr{C}^l_k : [\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}\}$ forms a partition of $\mathscr{F}^k_{r+1}$. Therefore by Theorem 5.7, we have the following.

Corollary 5.11. *For fixed* $r \geq 1$ *and for* $1 \leq k \leq r$,

$$|\mathscr{F}^k_{r+1}| = \sum_{l=[\frac{r+2}{2}]}^{\binom{r+1}{2}} \sum_{j=0}^{k} \binom{r}{j} \binom{r-j}{k-j} |\mathscr{F}_{r-j}(l-k)|.$$

The collection $\{\mathscr{F}^k_{r+1} : 1 \leq k \leq r\}$ also forms a partition of $\mathscr{F}_{r+1}$. Therefore Corollary 5.10 can also be obtained using Corollary 5.11. It can be observed that the fundamental basic block associated to an RC-lattice is having the same or a smaller width as compared to the basic block associated to that lattice. In order to obtain the formula which gives the number of non-isomorphic basic blocks of nullity l, containing the reducible elements which are all comparable, using the number of non-isomorphic fundamental basic blocks of nullity $m \leq l$, we see the following.

Definition 5.12. Let $\mathscr{B}_r$ be the class of all non-isomorphic basic blocks such that each basic block in it has r reducible elements which are all comparable. Let $\mathscr{B}_r(l)$ be the subclass of $\mathscr{B}_r$ such that each basic block in it is of nullity l.

By Corollary 4.9, if $l = m = [\frac{r+1}{2}]$ then $\mathscr{B}_r(l) = \mathscr{F}_r(m)$. In general, if $l \geq m$ then $|\mathscr{B}_r(l)| \geq |\mathscr{F}_r(m)|$. Let p^r_n denote the number of (weak) compositions of n into

r (non-negative) parts. Then p_n^r is the number of non-negative integer solutions to the equation $n = x_1 + x_2 + \cdots + x_r$. The number of solutions is actually the number of distinct r-tuples, $(x_1, x_2, \ldots, x_r)$ satisfying the equation $n = x_1 + x_2 + \cdots + x_r$, where for each i, $x_i \geq 0$. It is known that $p_n^r = \binom{n+r-1}{r-1}$. We now obtain the formula for the number of non-isomorphic basic blocks of nullity l containing the reducible elements which are all comparable.

Theorem 5.13. *For fixed $r \geq 2$ and for $[\frac{r+1}{2}] \leq m \leq l \leq \binom{r}{2}$,*

$$|\mathscr{B}_r(l)| = \sum_{m=[\frac{r+1}{2}]}^{l} \binom{l-1}{m-1} |\mathscr{F}_r(m)|.$$

Proof. For fixed $r \geq 2$ and for fixed $[\frac{r+1}{2}] \leq l \leq \binom{r}{2}$, let $B \in \mathscr{B}_r(l)$. Suppose F is the fundamental basic block associated to B. Clearly $Red(F) = Red(B)$. If $m = \eta(F)$ then it is clear that $m \leq l$. Let $s = l - m$. Therefore any $B \in \mathscr{B}_r(l)$ can be obtained in a unique way using a member F of $\mathscr{F}_r(m)$ with the help of exactly s adjunct of the 1-chains, where all the adjunct pairs are the adjunct pairs of F. Further, by Proposition 4.11, for any $B_1, B_2 \in \mathscr{B}_r(l)$, if the corresponding fundamental basic blocks F_1 and F_2 are not isomorphic then $B_1 \not\cong B_2$.

By Theorem 2.1, as nullity of F is m, it is an adjunct of $m+1$ chains. Suppose C is a maximal chain containing all the r reducible elements of F. Then using Definition 4.1 and by Proposition 4.7, $F = C]_{\alpha_1}\{c_1\}]_{\alpha_2}\{c_2\}\cdots]_{\alpha_m}\{c_m\}$ with all the adjunct pairs $\alpha_i = (a_i, b_i)$ are distinct with $(a_1, b_1) < (a_2, b_2) < \ldots < (a_m, b_m)$ with respect to the dictionary order defined on $C \times C$. Here note that, for any isomorphism ϕ of $F \in \mathscr{F}_r(m)$ to itself, $\phi(a_i) = a_i$, $\phi(b_i) = b_i$ and hence $\phi(a_i, b_i) = (a_i, b_i)$, for each i, $1 \leq i \leq m$.

For each $F \in \mathscr{F}_r(m)$, let $\mathscr{A}_F = \{B \in \mathscr{B}_r(l) : F$ is the fundamental basic block associated to $B\}$. Also, for each i, $1 \leq i \leq m$, let n_i be the multiplicity of an adjunct pair (a_i, b_i) in $B \in \mathscr{B}_r(l)$. Then there is a one-to-one correspondence between the set $\mathscr{A}_F$ and the set $S = \{(n_1, n_2, \ldots, n_m) : n_1 + n_2 + \cdots + n_m = l,\ n_i \geq 1\}$. Now S is equivalent to the set $S' = \{(n_1, n_2, \ldots, n_m) : n_1 + n_2 + \cdots + n_m = s,\ n_i \geq 0\}$. Therefore $|\mathscr{A}_F| = |S| = |S'| = p_s^m$. But $p_s^m = \binom{s+m-1}{m-1} = \binom{l-1}{m-1}$. Hence $|\mathscr{A}_F| = \binom{l-1}{m-1}$. Thus for fixed m, the number of non-isomorphic basic blocks in $\mathscr{B}_r(l)$ which can be obtained from all the non-isomorphic members of $\mathscr{F}_r(m)$ is

$$\sum_{F \in \mathscr{F}_r(m)} |\mathscr{A}_F| = \sum_{F \in \mathscr{F}_r(m)} \binom{l-1}{m-1} = \binom{l-1}{m-1} |\mathscr{F}_r(m)|.$$

Hence $|\mathscr{B}_r(l)| = \displaystyle\sum_{m=[\frac{r+1}{2}]}^{l} \binom{l-1}{m-1} |\mathscr{F}_r(m)|.$ □

Definition 5.14. Let $\mathscr{F}(l)$ be the class of all non-isomorphic fundamental basic blocks of nullity l such that the reducible elements in each fundamental basic block in it are all comparable. Let $\mathscr{B}(l)$ be the class of all non-isomorphic basic blocks of nullity l such that the reducible elements in each basic block in it are all comparable.

For $r \geq 2$, it follows that, $a_r = \sum_{l=[\frac{r+1}{2}]}^{\binom{r}{2}} |\mathscr{F}_r(l)|$. Now let $f_l = |\mathscr{F}(l)|$ and $b_l = |\mathscr{B}(l)|$, for all $l \geq 0$. Then for $l \geq 1$, it follows that, $f_l = \sum_{r=2}^{2l} |\mathscr{F}_r(l)|$ and $b_l = \sum_{r=2}^{2l} |\mathscr{B}_r(l)|$. Note that 1-chain is the only basic block or fundamental basic block of nullity 0. Let r_l denote the number of all non-isomorphic basic retracts having nullity l. Then we have the following.

Remark 5.15. Using Theorem 5.3, Corollary 5.8, Theorem 5.13, and Theorem 3.14, we obtain four important integer sequences viz., $\langle a_r \rangle$, $\langle f_l \rangle$, $\langle b_l \rangle$ and $\langle r_l \rangle$ respectively in the following.

1. For $r \geq 0$, $\langle a_r \rangle : 1, 0, 1, 4, 41, 768, 27449, 1887284, 252522481, 66376424160, 34509011894545, \ldots$.
   Coincidentally, this is the sequence $A006129$ (see OEIS [30]), representing the number of labeled graphs on r unisolated vertices. This sequence also represents the counting of edge covers of labeled complete graph on r vertices (see [32]).

2. For $l \geq 0$, $\langle f_l \rangle : 1, 1, 6, 62, 900, 16824, 384668, 10398480, 324420840, 11472953760, 453518054216, \ldots$.
   Coincidentally again, this is the sequence $A121251$ (see OEIS [30]), representing the number of labeled graphs with unisolated vertices and containing $l \geq 0$ edges (see also [3]).

3. For $l \geq 0$, $\langle b_l \rangle : 1, 1, 7, 75, 1105, 20821, 478439, 12977815, 405909913, 14382249193, 569377926495, \ldots$.
   Note that this is also a known sequence $A121316$ (see OEIS [30]). This sequence represents the counting of labeled multigraphs without isolated vertices and containing $l \geq 1$ edges. This sequence also represents the number of generic l-rook placements with $l \geq 1$ rooks below the main diagonal (of the chess board). These are precisely generalised Eulerian numbers (see Banaian et al. [2]). According to Banaian et al. [2], there is a close connection between the mathematics of juggling and the mathematics of rook placements (see also the sequence $A269744$ in OEIS [30]).

4. For $l \geq 0$, $\langle r_l \rangle : 2, 4, 28, 300, 4420, 83284, 1913756, \ldots$.
   Note that for $l \geq 1$, $r_l = 4 \times b_l$.

6 Conclusion

In Theorem 3.5, we proved that basic retract associated to a poset preserves the nullity of the poset. In Theorem 4.4, we proved that every RC-lattice is a dismantlable lattice. But the converse is not true. It follows that, up to isomorphism, there is a unique basic block/retract associated to any poset. Also, up to isomorphism, there is a unique fundamental basic block associated to any RC-lattice.

In the end, we have obtained the four desired integer sequences (see Remark 5.15). However, it still remains to establish the equivalences of the sequences $\langle|\mathscr{F}_r(l)|\rangle$, $\langle a_r\rangle$, $\langle f_l\rangle$, and $\langle b_l\rangle$ with the sequences $A054548$, $A006129$, $A121251$, and $A121316$ (see OEIS [30]) respectively.

The main purpose behind introducing the concepts of a basic retract/block associated to posets, and a fundamental basic block associated to RC-lattices, is the counting of lattices and posets. We feel that the problem of counting lattices (in particular, RC-lattices) and posets can be dealt with using the theory of partitions and with the help of basic retracts.

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