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The spiral property of q-Eulerian numbers of type B

Zhe Wang   Zhi-Yong Zhu

School of Mathematics and Statistics
Northeastern University at Qinhuangdao
Hebei 066004, P.R. China
wangzhepapers@stumail.neu.edu.cn   zhiyongzhu@stumail.edu.cn

Abstract

We give a direct proof of the spiral property of the q-Eulerian numbers of type B, which arise from q-counting signed permutations in the hyperoctahedral group by the negative index. For a given nonnegative real number q, the spiral property implies that the q-polynomial of type B is unimodal and the maximum coefficient appears exactly in the middle.

1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$ and $\pm[n] = [n] \cup \{-1, -2, \ldots, -n\}$. Denote by B_n the hyperoctahedral group of rank n. Given $\pi \in B_n$. Elements of B_n are signed permutations of $\pm[n]$ with the property that $\pi(-i) = -\pi(i)$ for all $i \in [n]$. The number of *descents* of π is defined by

$$\mathrm{des}_B(\pi) = \#\{i \in \{0, 1, 2, \ldots, n-1\} : \ \pi(i) > \pi(i+1)\},$$

where $\pi(0) = 0$. The negative index of π is defined by $N(\pi) = \#\{i \in [n] : \pi(i) < 0\}$. The *$q$-Eulerian polynomials of type B* are given as follows:

$$B_n(x, q) = \sum_{\pi \in B_n} x^{\mathrm{des}_B(\pi)} q^{N(\pi)} = \sum_{k=0}^{n} B_{n,k}(q) x^k.$$

Following [1, Theorem 3.4], the polynomials $B_n(x, q)$ satisfy the recurrence relation

$$B_n(x, q) = [1 + (1+q)nx - x]B_{n-1}(x, q) + (1+q)\left(x - x^2\right)\frac{\partial}{\partial x}B_{n-1}(x, q), \qquad (1)$$

with the initial condition $B_0(x, q) = 1$. The exponential generating function of $B_n(x, q)$ is given as follows:

$$\sum_{n=0}^{\infty} B_n(x, q)\frac{t^n}{n!} = \frac{(1-x)\mathrm{e}^{t(1-x)}}{1 - x\mathrm{e}^{t(1-x)(1+q)}}.$$

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Various generalizations or variations of $B_n(x,q)$ have been extensively studied. For example, Fulman, Kim, Lee and Petersen [3] recently studied the joint distribution of descents and sign for elements of the hyperoctahedral group, where the sign of an element $\pi \in B_n$ is the product of $(-1)^{N(\pi)}$ and the sign of the underlying unsigned permutation. Below are the polynomials $B_n(x,q)$ for $n \leqslant 4$:

$$\begin{aligned} B_1(x,q) &= 1+qx,\ B_2(x,q) = 1+(1+4q+q^2)x+q^2x^2, \\ B_3(x,q) &= 1+(4+12q+6q^2+q^3)x+(1+6q+12q^2+4q^3)x^2+q^3x^3, \\ B_4(x,q) &= 1+(11+32q+24q^2+8q^3+q^4)x+(11+56q+96q^2+56q^3+11q^4)x^2 \\ &\quad +(1+8q+24q^2+32q^3+11q^4)x^3+q^4x^4. \end{aligned}$$

Let $f(x)=\sum_{i=0}^n f_i x^i$ be a polynomial with nonnegative coefficients. We say that $f(x)$ is *unimodal* if

$$f_0 \leqslant f_1 \leqslant \cdots \leqslant f_k \geqslant f_{k+1} \geqslant \cdots \geqslant f_n$$

for some k, where the index k is called the *mode* of $f(x)$. Following [2, 6], the polynomial $f(x)$ is said to be *spiral* if

$$f_n \leqslant f_0 \leqslant f_{n-1} \leqslant f_1 \leqslant \cdots \leqslant f_{\lfloor n/2 \rfloor}.$$

It is clear that the spiral property is stronger than unimodality. We say that $f(x)$ is *real-rooted* if it has real roots only. And we say that $f(x)$ is *symmetric* if $f_j = f_{n-j}$ for each $0 \leq j \leq n$. The real-rootedness of $B_n(x,q)$ implies the unimodality of it; see [1, Corollary 3.7] for details. In particular, when $q=1$, the polynomial $B_n(x,1)$ is symmetric. The spiral property of q-Eulerian numbers of type B was first proved in [4, Corollary 42] by using the bi-γ-positivity of certain colored Eulerian polynomials. In this note we give a direct proof of this property. The main result of this note is the following.

Theorem 1. *For any $n \geq 1$, we have the following results:*

(*A*) *when $0<q<1$, the polynomial $B_n(x,q)$ is spiral;*

(*B*) *when $q>1$, the polynomial $x^nB_n(1/x,q)$ is spiral.*

Example 2. The first few $2^nB_n(x,1/2)$ are given as follows:

$$2B_1(x,1/2)=2+x,\ 2^2B_2(x,1/2)=4+13x+x^2,\ 2^3B_3(x,1/2)=8+93x+60x^2+x^3.$$

The first few $B_n(x,2)$ are given as follows:

$$B_1(x,2)=1+2x,\ B_2(x,2)=1+13x+4x^2,\ B_3(x,2)=1+60x+93x^2+8x^3.$$

The first few $B_n(x,3)$ are given as follows:

$$B_1(x,2)=1+3x,\ B_2(x,2)=1+22x+9x^2,\ B_3(x,2)=1+121x+235x^2+27x^3.$$

In [5], the sequences $\{B_{n,k}(2)\}_{k=0}^n$ and $\{B_{n,k}(3)\}_{k=0}^n$ appear as A225117 and A225118, respectively.

2 The proof of Theorem 1

Proof. (A) We first consider the case $0 < q < 1$. In order to show that

$$B_{n,n}(q) < B_{n,0}(q) < B_{n,n-1}(q) < B_{n,1}(q) < \cdots < B_{n,\lfloor \frac{n}{2}-1 \rfloor}(q) < B_{n,\lceil \frac{n}{2} \rceil}(q)$$

when n is odd, one has $B_{n,\frac{n+1}{2}}(q) < B_{n,\frac{n-1}{2}}(q)$, and it suffices to prove the following inequalities:

$$B_{n,n-k}(q) < B_{n,k}(q) < B_{n,n-k-1}(q) \tag{2}$$

for any $0 \le k \le \lceil \frac{n-3}{2} \rceil$, and in addition

$$B_{n,\frac{n+1}{2}}(q) < B_{n,\frac{n-1}{2}}(q) \tag{3}$$

when n is odd. We proceed to prove the inequalities (2) and (3) by induction on n. It is clear that these inequalities hold for $1 \le n \le 3$. We now assume that they hold for all integers up to n. We aim to show that

$$B_{n+1,n+1-k}(q) < B_{n+1,k}(q) < B_{n+1,n-k}(q) \tag{4}$$

for any $0 \le k \le \lceil \frac{n-2}{2} \rceil$, and when $n+1$ is odd,

$$B_{n+1,\frac{n+2}{2}}(q) < B_{n+1,\frac{n}{2}}(q). \tag{5}$$

For $k = 0$, we have $B_{n+1,0}(q) - B_{n+1,n+1}(q) = 1 - q^{n+1} > 0$. It follows from (1) that

$$B_{n,k}(q) = (k + kq + 1)B_{n-1,k}(q) + [(n-k) + (n+1-k)q]B_{n-1,k-1}(q).$$

For $k = n$, we have $B_{n+1,n}(q) = (n+nq+1)B_{n,n}(q) + (1+2q)B_{n,n-1}(q) > B_{n,n-1}(q)$. Therefore $B_{n+1,n}(q) > B_{n+1,0}(q)$ with $B_{n,n-1}(q) > B_{n,0}(q) = B_{n+1,0}(q)$.

For $1 \le k \le \lceil \frac{n-2}{2} \rceil$, we can get

$$B_{n+1,n+1-k}(q) = [(n+2-k) + (n+1-k)q]B_{n,n+1-k}(q) + [k + (k+1)q]B_{n,n-k}(q); \tag{6}$$

$$B_{n+1,k}(q) = (k + kq + 1)B_{n,k}(q) + [(n+1-k) + (n+2-k)q]B_{n,k-1}(q); \tag{7}$$

$$B_{n+1,n-k}(q) = [n+1-k+(n-k)q]B_{n,n-k}(q) + [k+1+(k+2)q]B_{n,n-k-1}(q). \tag{8}$$

It follows from (6) and (7) that

$$\begin{aligned} B_{n+1,k}(q) - B_{n+1,n+1-k}(p) = {} & (k+kq)[B_{n,k}(q) - B_{n,n-k}(q)] \\ & + [n-k+1+(n-k+1)q][B_{n,k-1}(q) - B_{n,n-k+1}(q)] \\ & + [B_{n,k}(q) - B_{n,n-k+1}(q)] + q[B_{n,n-k}(q) - B_{n,k-1}(q)]. \end{aligned}$$

By induction, we see that the difference in every pair of parentheses in the above expression is positive. This implies that for $1 \le k \le \lceil \frac{n-2}{2} \rceil$,

$$B_{n+1,k}(q) - B_{n+1,n+1-k}(q) > 0. \tag{9}$$

Similarly, for $1 \leq k \leq \lceil \frac{n-2}{2} \rceil$, in view of (7) and (8) we find

$$\begin{aligned}B_{n+1,n-k}(q) - B_{n+1,k}(q) = &(k+1+kq)(B_{n,n-k-1}(q) - B_{n,k}(q))\\ &+ [n-k+1+(n-k)q](B_{n,n-k}(q) - B_{n,k-1}(q))\\ &+ 2q(B_{n,n-k-1}(q) - B_{n,k-1}(q)).\end{aligned}$$

Again, by the inductive hypothesis, we deduce that for $1 \leq k \leq \lceil \frac{n-2}{2} \rceil$,

$$B_{n+1,\,n-k}(q) - B_{n+1,\,k}(q) > 0. \tag{10}$$

Combining (9) and (10) gives (4) for $0 \leq k \leq \lceil \frac{n-2}{2} \rceil$. It remains to verify (5) when $n+1$ is odd. By the recurrence relation for $B_{n,\,k}(q)$, we have

$$\begin{aligned}B_{n+1,\,\frac{n+2}{2}}(q) &= \left(\frac{n+4}{2} + \frac{n+2}{2}q\right) B_{n,\,\frac{n+2}{2}}(q) + \left(\frac{n}{2} + \frac{n+2}{2}q\right) B_{n,\,\frac{n}{2}}(q),\\ B_{n+1,\,\frac{n}{2}}(q) &= \left(\frac{n+2}{2} + \frac{n}{2}q\right) B_{n,\,\frac{n}{2}}(q) + \left(\frac{n+2}{2} + \frac{n+4}{2}q\right) B_{n,\,\frac{n-2}{2}}(q).\end{aligned}$$

This yields

$$\begin{aligned}B_{n+1,\,\frac{n}{2}}(q) - B_{n+1,\,\frac{n+2}{2}}(q) = &\left(\frac{n+2}{2} + \frac{n+2}{2}q\right)[B_{n,\,\frac{n-2}{2}}(q) - B_{n,\,\frac{n+2}{2}}(q)]\\ &+ [B_{n,\,\frac{n}{2}}(q) - B_{n,\,\frac{n+2}{2}}(q)] + q[B_{n,\,\frac{n-2}{2}}(q) - B_{n,\,\frac{n}{2}}(q)].\end{aligned}$$

Again, by the inductive hypothesis, we obtain (5). The completes the proof of (2).

(B) Consider the case $q > 1$. We shall prove that

$$B_{n,\,0}(q) < B_{n,\,n}(q) < B_{n,\,1}(q) < B_{n,\,n-1}(q) < \cdots < B_{n,\,\lfloor \frac{n}{2}+1 \rfloor}(q) < B_{n,\,\lceil \frac{n}{2} \rceil}(q)$$

and when n is odd, one has $B_{n,\,\frac{n-1}{2}}(q) < B_{n,\,\frac{n+1}{2}}(q)$.

According to [1, Proposition 3.10], one has

$$B_{n,k}(q) = q^n B_{n,n-k}\left(\frac{1}{q}\right). \tag{11}$$

Let $p = 1/q$. Comparison with (4) and (11) yields

$$B_{n+1,\,k}(p) < B_{n+1,\,n+1-k}(p) < B_{n,\,k+1}(p).$$

Comparison with (5) and (11) yields

$$B_{n+1,\,\frac{n}{2}}(p) < B_{n+1,\,\frac{n+2}{2}}(p)$$

when n is odd. This completes the proof. □

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References

[1] F. Brenti, q-Eulerian polynomials arising from Coxeter groups, *European J. Combin.* 15 (1994), 417–441.

[2] W. Y. C. Chen, R. L. Tang and A. F. Y. Zhao, Derangement polynomials and excedances of type B, *Electron. J. Combin.* 16(2) (2009), #R15.

[3] J. Fulman, G. B. Kim, S. Lee and T. K. Petersen, On the joint distribution of descents and signs of permutations, *Electron. J. Combin.* 28 (2021), P3.37.

[4] S.-M. Ma, J. Ma, J. Yeh and Y.-N. Yeh, Excedance-type polynomials and gamma-positivity. arXiv:2102.00899.

[5] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, https://oeis.org.

[6] X.-D. Zhang, On the spiral property of the q-derangement numbers, *Discrete Math.* 159 (1996), 295–298.

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