Rogers-Ramanujan and the Baker-Gammel-Wills (Padé) conjecture

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Dedicated to the memory of Israel, Zivia and Ranan Lubinsky

Abstract

In 1961, Baker, Gammel and Wills conjectured that for functions f meromorphic in the unit ball, a subsequence of its diagonal Padé approximants converges uniformly in compact subsets of the ball omitting poles of f. There is also apparently a cruder version of the conjecture due to Padé himself, going back to the early twentieth century. We show here that for carefully chosen q on the unit circle, the Rogers-Ramanujan continued fraction

$$1 + \frac{qz|}{|1} + \frac{q^2z|}{|1} + \frac{q^3z|}{|1} + \cdots$$

provides a counterexample to the conjecture. We also highlight some other interesting phenomena displayed by this fraction.

1. Introduction

Let

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

be a formal power series, with complex coefficients. Given integers $m, n \geq 0$, the (m, n) Padé approximant to f is a rational function

$$[m/n] = P/Q$$

where P, Q are polynomials of degree at most m, n respectively, such that Q is not identically 0, and such that

$$(1.1) \qquad (fQ - P)(z) = O\left(z^{m+n+1}\right).$$

By this last relation, we mean that the coefficients of $1, z, z^2, \ldots, z^{m+n}$ in the formal power series on the left-hand side vanish. The basic idea is that $[m/n]$ is

a rational function with given upper bounds on its numerator and denominator degrees, chosen in such a way that its Maclaurin series reproduces as many terms as possible in the power series f.

It is easy to see that $[m/n]$ exists: we can reformulate (1.1) as a system of $m + n + 1$ homogeneous linear equations in the $(m + 1) + (n + 1)$ coefficients of the polynomials P and Q. As there are more unknowns than equations, there is a nontrivial solution, and it is easily seen from (1.1) that Q cannot be identically 0 in any nontrivial solution. While P and Q are not separately unique, the ratio $[m/n]$ is, and this is again an easy consequence of (1.1).

It was C. Hermite, who gave his student Henri Eugene Padé the approximant to study in the 1890's. Although the approximant was known earlier, by amongst others, Jacobi and Frobenius, it was perhaps Padé's thorough investigation of the structure of the Padé table, namely the array

$$
\begin{array}{ccccc}
[0/0] & [0/1] & [0/2] & [0/3] & \ldots \\
[1/0] & [1/1] & [1/2] & [1/3] & \ldots \\
[2/0] & [2/1] & [2/2] & [2/3] & \ldots \\
[3/0] & [3/1] & [3/2] & [3/3] & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

that has ensured the approximant being named after him.

Padé approximants have been applied in proofs of irrationality and transcendence in number theory, in practical computation of special functions, and in analysis of difference schemes for numerical solution of partial differential equations. However, the application which really brought them to prominence in the 1960's and 1970's, was in location of singularities of functions: in various physical problems, for example inverse scattering theory, one would have a means for computing the coefficients of a power series f. One could use these coefficients to compute, for example, the $[3/3]$ Padé approximant to f, and use the poles of the approximant as predictors of the location of poles or other singularities of f. Moreover, under certain conditions on f, which were often satisfied in physical examples, this process could be theoretically justified.

In addition to their wide variety of applications, they are also closely associated with continued fraction expansions, orthogonal polynomials, moment problems, the theory of quadrature, amongst others. See [3] and [5] for a detailed development of the theory, and [6] for their history.

One of the fascinating features of Padé approximants is the complexity of their convergence theory. There are power series f with zero radius of convergence, for which $[n/n](z)$ converges as $n \to \infty$ to a function single valued and analytic in the cut-plane $\mathbb{C}\backslash[0, \infty)$. On the other hand, there are

entire functions f for which

$$\limsup_{n\to\infty} \left| [n/n](z) \right| = \infty$$

for all $z \in \mathbb{C} \setminus \{0\}$.

Probably the most important general theorem that applies to functions meromorphic in the plane is that of Nuttall-Pommerenke. It asserts that if f is meromorphic throughout $\mathbb{C}$, and analytic at 0, then $\{[n/n]\}_{n=1}^{\infty}$ converges in planar measure. More generally, this holds if f has singularities of (logarithmic) capacity 0, and planar measure may be replaced by capacity. There are much deeper analogues of this theorem for functions with branchpoints, due to H. Stahl. Uniform convergence of sequences of Padé approximants has been established for Pólya frequency series, series of Stieltjes/Markov/Hamburger, and other special classes. For surveys and various perspectives on the convergence theory, see [3], [5], [18], [31], [34], [44], [45], [46], [49].

Long before the Nuttall-Pommerenke theorem was established, George Baker and his collaborators observed the phenomenon of spurious poles: several of the approximants could have poles which in no way were related to those of the underlying function. However, those poles affected convergence only in a small neighbourhood, and there were usually very few of these "bad" approximants. Thus, one might compute $[n/n]$, $n = 1, 2, 3, \ldots 50$, and find a definite convergence trend in 45 of the approximants, with five of the 50 approximants displaying pathological behaviour. The curious thing (contrary to expectation) is that the five bad approximants could be distributed anywhere in the 50, and need not be the first few. Nevertheless, after omitting the "bad" approximants, one obtained a clear convergence trend. This seemed to be a characteristic of the Padé method, and Baker et al. formulated a now famous conjecture [4]. There are now many forms of the conjecture; we shall concentrate on the following form:

BAKER-GAMMEL-WILLS CONJECTURE (1961). *Let f be meromorphic in the unit ball, and analytic at 0. There is an infinite subsequence $\{[n/n]\}_{n\in\mathcal{S}}$ of the diagonal sequence $\{[n/n]\}_{n=1}^{\infty}$ that converges uniformly in all compact subsets of the unit ball omitting poles of f.*

Thus, there is an infinite sequence of "good" approximants. In the first form of the conjecture, f was required to have a nonpolar singularity on the unit circle, but this was subsequently relaxed (cf. [3, p. 188 ff.]). There is also apparently a cruder form of the conjecture due to Padé himself, dating back to the 1900's; the author must thank J. Gilewicz for this historical information.

The main result of this paper is that the above form of the conjecture is false, and that a counterexample is provided by a continued fraction of Rogers-

Ramanujan. For q not a root of unity, let

$$(1.2) \qquad G_q(z) := \sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\cdots(1-q^j)} z^j$$

denote the Rogers-Ramanujan function. Of course, it is at this stage merely a formal power series. Moreover, let

$$(1.3) \qquad H_q(z) := G_q(z)/G_q(qz).$$

When H_q has an analytic (or meromorphic) continuation to a region beyond the domain of definition of G_q, we denote that continuation by H_q also. There is the well-known functional relation, which we shall establish in Section 3:

$$(1.4) \qquad H_q(z) = 1 + \frac{qz}{H_q(qz)}.$$

Iterating this leads to

$$(1.5) \qquad H_q(z) = 1 + \cfrac{qz}{1 + \cfrac{q^2 z}{1 + \ddots \cfrac{}{\frac{q^n z}{H_q(q^n z)}}}}$$

and hence to the formal infinite continued fraction

$$(1.6) \qquad H_q(z) = 1 + \frac{qz|}{|1} + \frac{q^2 z|}{|1} + \frac{q^3 z|}{|1} + \cdots .$$

(The continued fraction notation used should be self explanatory.) For $|q| < 1$, the continued fraction was considered independently by L. J. Rogers and S. Ramanujan in the early part of the twentieth century.

The truncations of a continued fraction are called its *convergents*. We shall use the notation

$$(1.7) \qquad \frac{\mu_n}{\nu_n}(z) = 1 + \frac{qz|}{|1} + \frac{q^2 z|}{|1} + \cdots + \frac{q^n z|}{|1}, \qquad n \geq 1$$

for the nth convergent, to emphasize that it is a rational function with numerator polynomial μ_n and denominator polynomial ν_n. We also set

$$\mu_0/\nu_0 := 1.$$

The continued fraction is said to *converge* if

$$\lim_{n \to \infty} \mu_n(z)/\nu_n(z)$$

exists.

At least when G_q has a positive radius of convergence, it does not really matter whether we define H_q by (1.3) or (1.6), for both have the same Maclaurin series, so both analytically continue that Maclaurin series inside their domain of convergence. When G_q has zero radius of convergence, we shall define H_q by (1.6).

We shall make substantial use of the fact that the sequence $\{\mu_n/\nu_n\}_{n=1}^{\infty}$ of convergents includes both the diagonal sequence $\{[n/n]\}_{n=1}^{\infty}$ and the sub-diagonal sequence $\{[n+1/n]\}_{n=1}^{\infty}$ to H_q. So as n increases, the convergents trace a stair step in the Padé table. For a proof of this, see [5] or [27].

Our counterexample is contained in:

THEOREM 1.1. *Let*

$$(1.8) \qquad\qquad\qquad q := \exp\left(2\pi i \tau\right)$$

where

$$(1.9) \qquad\qquad\qquad \tau := \frac{2}{99 + \sqrt{5}}.$$

Then H_q is meromorphic in the unit ball and analytic at 0. There does not exist any subsequence of $\{\mu_n/\nu_n\}_{n=1}^{\infty}$ that converges uniformly in all compact subsets of

$$\mathcal{A} := \{z : |z| < 0.46\}$$

omitting poles of H_q. In particular no subsequence of

$$\{[n/n]\}_{n=1}^{\infty} \qquad or \qquad \{[n+1/n]\}_{n=1}^{\infty}$$

can converge uniformly in all compact subsets of $\mathcal{A}$ omitting poles of H_q.

The crux of the counterexample is that, given any subsequence $\{\mu_n/\nu_n\}_{n\in\mathcal{S}}$ of the convergents, there is a compact subset of $\mathcal{A}$ not containing any poles of H_q, such that infinitely many of the convergents have a pole in the interior of the compact set. Moreover, there is a limit point of poles in the interior of that compact set, and uniform convergence is not possible.

There are several limits to our example. We are certain that with sufficient effort, one may replace 0.46 above by $\frac{1}{4} + \varepsilon$, for an arbitrarily small $\varepsilon > 0$ and a corresponding q on the unit circle. However, we cannot go below $\frac{1}{4}$. Indeed, an old theorem of Worpitzky guarantees that the full sequence of convergents $\{\mu_n/\nu_n\}_{n=1}^{\infty}$ converges uniformly in compact subsets of $\left\{z : |z| < \frac{1}{4}\right\}$. Thus one can still look for an example in which no subsequence of the convergents converges uniformly, or even pointwise, in any neighbourhood of 0.

Moreover, given any point in the unit ball at which H_q is analytic, there is a neighbourhood of it and a subsequence of the convergents that converges uniformly in that neighbourhood. So one can also look for an example without this property. We shall discuss this further in Section 8.

We shall see that for a.e. q on the unit circle (and in particular for the q above), H_q is meromorphic in the unit ball, with a natural boundary on the unit circle. Moreover, for a.e. q, G_q is analytic in the unit ball, with a natural boundary on the unit circle.

However, given $0 < s < \frac{1}{4}$, then for some exceptional q, there is the very striking feature, that G_q is analytic in $|z| < s$, with a natural boundary on the circle $\{z : |z| = s\}$, yet H_q defined by (1.3) admits an analytic continuation to at least the ball centre 0, radius $\frac{1}{4}$. So somehow, in the division in (1.3), the natural boundary of G_q is cancelled out, as if it were a removable singularity.

There are other striking features for a.e. q: if on a circle centre 0, H_q has poles of total multiplicity ℓ, then in any neighbourhood of that circle, all convergents μ_n/ν_n with n large enough, have at least 2ℓ poles, namely double as many as H_q.

This paper is organised as follows: in Section 2, we shall state in greater detail, our results on G_q, H_q and the convergence or divergence properties of the continued fraction. In Section 3, we shall present some identities involving the approximants and their proofs. In Section 4, we shall prove our results on the continued fraction when q is a root of unity. In Sections 5 and 6, we shall prove the results of Section 2. In Section 7, we shall prove Theorem 1.1. Finally in Section 8, we shall discuss some of the implications of this paper.

2. The continued fraction for H_q

We emphasise that the Rogers-Ramanujan c.f. (continued fraction) is not the first candidate we have examined as a possible counterexample to the Baker-Gammel-Wills conjecture. In the search for a counterexample, basic hypergeometric, or q series, have been most useful, just as they have had applications in so many branches of mathematics. What is somewhat exotic, however, is the range of the parameter q. In most studies of q-series, $|q| < 1$, and sometimes $|q| > 1$. However, many of the identities persist for $|q| = 1$, and it is in this range of q, that several interesting phenomena and counterexamples in the convergence theory of Padé approximation have been discovered. In other contexts, the case $|q| = 1$ has also proved to be interesting [43].

In [35], E. B. Saff and the author investigated the Padé table and continued fraction for the partial theta function

$$\sum_{j=0}^{\infty} q^{j(j-1)/2} z^j = 1 + \frac{z|}{|1} - \frac{qz|}{|1} + \frac{q(1-q)z|}{|1} - \frac{q^3 z|}{|1} + \frac{q^2(1-q)z|}{|1} \cdots$$

when $|q| = 1$. Subsequently K. A. Driver and the author [9], [11], [10], [12] undertook a detailed study of the Padé table and continued fraction for the more general Wynn's series [50]

$$\sum_{j=0}^{\infty} \left[\prod_{l=0}^{j-1} (A - q^{l+\alpha}) \right] z^j;$$

$$\sum_{j=0}^{\infty} \frac{z^j}{\prod_{l=0}^{j-1}(C - q^{l+\alpha})};$$

$$\sum_{j=0}^{\infty} \left[\prod_{l=0}^{j-1} \frac{A - q^{l+\alpha}}{C - q^{l+\gamma}} \right] z^j.$$

Here A, C, α and γ are suitably restricted parameters.

Amongst the interesting features is that some subsequence of the convergents converges uniformly inside the region of analyticity, so that Baker-Gammel-Wills is true for these series, while "most" subsequences have poles that cycle around the region of analyticity.

There are at least three aspects of the Rogers-Ramanujan c.f. that distinguish it from Wynn's series in the case where $|q| = 1$. Firstly the functional relation for H_q, namely

$$H_q(z) = 1 + \frac{qz}{H_q(qz)}$$

generates its c.f. by repeated application. For Wynn's series, there is not such a simple relationship between the c.f. and the functional equation. Secondly all the coefficients in the Rogers-Ramanujan c.f. have modulus 1, whereas a subsequence of the coefficients in the c.f. for Wynn's series converges to 0. Moreover the latter subsequence is associated with a subsequence of the convergents to the c.f. that converges throughout the region of analyticity. This already suggests that there may not be a uniformly convergent subsequence of the convergents for the Rogers-Ramanujan c.f. Thirdly, in the case where q is a root of unity, all of the Wynn's series reduce to rational functions, while the Rogers-Ramanujan c.f. corresponds to a function with branchpoints.

It is an immediate consequence of Worpitzky's theorem that the c.f. (1.6) converges for $|z| < \frac{1}{4}$, for each $|q| = 1$. In fact, we shall show using standard methods that (1.6) converges for $|z| < (2 + |1 + q|)^{-1}$. However beyond that circle, standard methods give very little, because of the oscillatory nature of the continued fraction coefficients $\{q^n\}_{n=1}^{\infty}$.

One must obviously distinguish the case where q is a root of unity, as the power series coefficients of G_q are not even defined in this case. Then, rather than defining H_q by (1.3), we shall define it as the function corresponding to the continued fraction (1.6). Using standard results for periodic c.f.'s, we shall prove in Section 4, the following:

THEOREM 2.1. *Let $\ell \geq 1$ and q be a primitive ℓ^{th} root of unity. Let*

(2.1) $$\mathcal{L} := \left\{ z \in \mathbb{C} : z^\ell \in \left(-\infty, -\frac{1}{4} \right] \right\}.$$

There exists a set $\mathcal{P}$ of at most $(\ell - 1)(2\ell - 1)/2$ points such that for $z \in \mathbb{C}\backslash(\mathcal{L} \cup \mathcal{P})$,

$$(2.2) \qquad \lim_{n \to \infty} \frac{\mu_n(z)}{\nu_n(z)} = \frac{\mu_{\ell-1}(z) - \frac{1}{2} + \sqrt{z^\ell + \frac{1}{4}}}{\nu_{\ell-1}(z)} =: H_q(z).$$

Here the branch of $\sqrt{}$ is the principal one, analytic in $\mathbb{C}\backslash(-\infty, 0]$ and positive in $(0, \infty)$.

Of course, $\mathcal{L}$ consists of ℓ distinct rays with an angle of $2\pi/\ell$ between successive rays, extending from the ℓ values of $(-\frac{1}{4})^{1/\ell}$ out to ∞. So the c.f. chooses the most natural choice for the branchcuts; see [44], [45] for the ways that continued fractions and Padé approximants choose branchcuts in far more general situations.

The set $\mathcal{P}$ contains the poles of H_q, that is, the at most $(\ell - 1)/2$ zeros of $\nu_{\ell-1}$, which need not lie on the branchcuts contained in the set $\mathcal{L}$. For example, if $\ell = 5$, $\nu_{\ell-1}(z) = (qz - 1)(z - 1)$ has zeros at 1 and $\overline{q}$, which are not in $\mathcal{L}$. Also, $\mathcal{P}$ contains additional points that arise in applying the standard theorems on periodic continued fractions. We have not been able to determine if these additional points are really points of divergence, or to determine where they lie. In all probability, our bound of $(\ell - 1)(2\ell - 1)/2$ on the number of points in $\mathcal{P}$ is too large.

Next, we turn to the more difficult case where q is not a root of unity. Clearly the series G_q of (1.2) at least has well-defined coefficients if q is not a root of unity, and its radius of convergence is

$$(2.3) \qquad R(q) := \liminf_{j \to \infty} \left| \prod_{k=0}^{j-1} (1 - q^k) \right|^{1/j}.$$

It was essentially proved in [19] (and we shall reproduce the proof in Lemma 6.2) that

$$(2.4) \qquad R(q) = \liminf_{j \to \infty} \left| 1 - q^j \right|^{1/j}.$$

If we write $q = e^{2\pi i \tau}$, this is readily reformulated in terms of the diophantine approximation properties of τ. Since $|1 - q^j| = 2|\sin[\pi(j\tau - k)]|$ for any integer k, we see that

$$(2.5) \qquad R(q) = \liminf_{j \to \infty} \|j\tau\|^{1/j},$$

where $\|x\|$ denotes the distance from x to the nearest integer.

It is known that $R(q) = 1$ for "most" q. Indeed the set

$$(2.6) \qquad \mathcal{G} := \{q : R(q) < 1\}$$

has linear measure 0, Hausdorff dimension 0, and even logarithmic dimension 2 [30]. G. Petruska has shown [38] that the related quantity

$$\limsup_{j \to \infty} \left| \prod_{k=0}^{j-1} (A - q^k) \right|^{1/j}$$

may assume any value in $[0, 1]$ as A and q range over the unit circle. Using his results, we can easily show that $R(q)$ may assume any value in $[0, 1]$. Curiously enough, the radius of convergence $R(q)$ of G_q need not coincide with the radius of meromorphy of H_q, that is, the largest circle centre 0 inside which H_q may be meromorphically continued. On the boundary of that circle, we show that H_q has a natural boundary:

THEOREM 2.2.   *Let* $|q| = 1$, *and assume that* q *is not a root of unity. Let* $\rho(q)$ *denote the radius of meromorphy of* H_q. *Then*

(a) H_q *has a natural boundary on the circle* $\{z : |z| = \rho(q)\}$ *and*

(2.7) $$1 \geq \rho(q) \geq \max \left\{ R(q), \frac{1}{2 + |1 + q|} \right\} \geq \frac{1}{4}.$$

(b) G_q *has a natural boundary on the circle* $\{z : |z| = R(q)\}$. *Moreover, as* q *ranges over the unit circle,* $R(q)$ *may assume any value in* $[0, 1]$.

(c) *For* $q \notin \mathcal{G}, R(q) = \rho(q) = 1$. *In particular, this is true for* a.e. q.

We are not sure if $\rho(q)$ may assume values < 1, but are inclined to believe that always $\rho(q) = 1$. At least for "most" q, the above result asserts that H_q is given by (1.3) inside its radius of meromorphy.

We are also interested in how H_q varies as q does, especially near roots of unity, as the branchcuts of H_q should then attract poles and zeros of the "nearby" meromorphic H_q. The following result partly justifies the latter:

THEOREM 2.3.   *Let* $|q_k| = 1, k \geq 1$, *and assume that*

(2.8) $$\lim_{k \to \infty} q_k = q.$$

(a) *Then uniformly in compact subsets of* $\{z : |z| < \frac{1}{2+|1+q|}\}$,

(2.9) $$\lim_{k \to \infty} H_{q_k}(z) = H_q(z).$$

(b) *Let* $\ell \geq 1$ *and let* q *be a primitive* ℓ^{th} *root of unity, and*

(2.10) $$\rho(q_k) > 2^{-2/\ell}, \qquad k \geq 1.$$

Let Ω_1 *and* Ω_2 *be open connected sets with* $\Omega_1 \subseteq \Omega_2$ *and* Ω_1 *containing a branchpoint of* H_q, *that is, containing one of the* ℓ *values of* $(-\frac{1}{4})^{1/\ell}$. *Assume moreover that*

(2.11) $$z \in \Omega_1 \Rightarrow zq^{\pm 1} \in \Omega_2.$$

Then for large enough k, H_{q_k} has a pole in Ω_2. If ℓ is odd, and $1 > r > 2^{-2/\ell}$ and $\delta > 0$, then for large k, H_{q_k} has a pole in $\{z : r < |z| < r + \delta\}$.

Thus (b) shows that every branchpoint of H_q attracts a growing number of poles of H_{q_k} as $k \to \infty$. Next, we turn to convergence of the c.f. Let us recall that we denoted the nth convergent for (1.6) by

$$\frac{\mu_n(z)}{\nu_n(z)} = 1 + \frac{qz|}{|1} + \frac{q^2z|}{|1} + \cdots + \frac{q^n z|}{|1}.$$

It is known [21] that

$$(2.12) \qquad \mu_n(z) = \sum_{k=0}^{[\frac{n+1}{2}]} z^k q^{k^2} \begin{bmatrix} n+1-k \\ k \end{bmatrix}$$

and

$$(2.13) \qquad \nu_n(z) = \mu_{n-1}(qz) = \sum_{k=0}^{[\frac{n}{2}]} z^k q^{k(k+1)} \begin{bmatrix} n-k \\ k \end{bmatrix}$$

where $[x]$ is the greatest integer $\leq x$, and

$$\begin{bmatrix} \alpha \\ l \end{bmatrix} = \frac{(1-q^\alpha)(1-q^{\alpha-1})\cdots(1-q^{\alpha-l+1})}{(1-q)(1-q^2)\cdots(1-q^l)}, \qquad l \geq 0, \alpha \in \mathbb{C},$$

is the Gaussian binomial coefficient. We shall reproduce the elegant proof due to Adiga et al. [1] in Section 2.

When G_q has positive radius of convergence, subsequences of the numerators $\{\mu_n\}_{n=1}^\infty$ and denominators $\{\nu_n\}_{n=1}^\infty$ of the continued fraction converge separately, depending on the behaviour of q^n. Of course, if q is not a root of unity, then $\{q^n\}_{n=1}^\infty$ is dense on the unit circle, and one may extract a subsequence converging to an arbitrary β on the unit circle.

THEOREM 2.4. *Let $q = e^{2\pi i \tau}$, τ irrational. Let $|\beta| = 1$ and $\mathcal{S}$ be any infinite sequence of positive integers with*

$$(2.14) \qquad \lim_{n \to \infty, n \in \mathcal{S}} q^n = \beta.$$

Then uniformly in compact subsets of $\{z : |z| < R(q)\}$,

$$(2.15) \qquad \lim_{n \to \infty, n \in \mathcal{S}} \mu_n(z) = \overline{G_q(\overline{\beta q z})} G_q(z);$$

$$(2.16) \qquad \lim_{n \to \infty, n \in \mathcal{S}} \nu_n(z) = \overline{G_q(\overline{\beta q z})} G_q(qz).$$

Moreover, uniformly in compact subsets of $\{z : |z| < R(q)\}$ omitting zeros of $G_q(\overline{\beta q z})$ and $G_q(qz)$,

$$(2.17) \qquad \lim_{n \to \infty, n \in \mathcal{S}} \frac{H_q(z) - \frac{\mu_n(z)}{\nu_n(z)}}{(-1)^n z^{n+1} q^{(n+1)(n+2)/2}} = \frac{G_q(\beta q^2 z)}{G_q(qz)^2 \overline{G_q(\overline{\beta q z})}}$$

and so in such sets omitting these zeros,

$$(2.18) \qquad \lim_{n \to \infty, n \in \mathcal{S}} \frac{\mu_n(z)}{\nu_n(z)} = H_q(z).$$

The crucial point in the last line is that the convergence takes place away from the zeros of both $G_q(z)$ and $G_q(\overline{\beta q z})$. The zeros of $G_q(\overline{\beta q z})$ need not be poles of H_q, and yet (2.16) shows that they attract poles of the convergents. Moreover as the zeros of H_q are simply rotations and reflections of the zeros of $G_q(z)$ it follows that if H_q has poles of total multiplicity ℓ on a given circle, then for all large enough n, $\frac{\mu_n}{\nu_n}$ has 2ℓ poles close to this circle, that is, twice as many poles as the approximated function! We formalize this as:

COROLLARY 2.5. *Let $q = e^{2\pi i \tau}, \tau$ be irrational. Assume that $r < R(q)$ and H_q has poles of total multiplicity ℓ on $\{z : |z| = r\}$. Let U be an open set containing this circle. Then there exists n_0 such that for $n \geq n_0$, μ_n/ν_n has poles of total multiplicity $\geq 2\ell$ in U.*

This is the first such example in the literature, in which *all* approximants of large order have more poles than the approximated function in a region of meromorphy. If we could show that there does not exist β for which the zero sets of $G_q(qz)$ and $G_q(\overline{\beta q z})$ are the same, this would establish a counterexample to the Baker-Gammel-Wills conjecture. For then, given any subsequence of the convergents, we can extract a further subsequence for which (2.14) holds for some β; that subsequence cannot converge uniformly in a compact set containing zeros of $G_q\left(\overline{\beta q z}\right)$ that are not zeros of $G_q(z)$. For special q, we shall do this in Section 7.

Another feature of Theorem 2.4 is that it describes what happens only in $|z| < R(q)$, yet the function H_q may have meaning in a much larger circle. If for example $R(q) < \frac{1}{4}$, then G_q is not defined in $R(q) < |z| < \frac{1}{4}$, but by Worpitzky, H_q is analytic in $|z| < \frac{1}{4}$. One might hope for an alternative formulation of (2.15) and (2.16) in this case. However this is not possible. Those separate limits guarantee normality and uniform boundedness of $\{\mu_n\}$ and $\{\nu_n\}$ in $|z| \leq r < R(q)$, but the following result shows that $\{\mu_n\}$ and $\{\nu_n\}$ cannot be uniformly bounded in $|z| \leq r$ for any $r > R(q)$.

THEOREM 2.6. *Let $q = e^{2\pi i \tau}$, τ irrational. Then for $0 < r < R(q)$,*

$$(2.19) \qquad \sup_{n \geq 1} \|\mu_n\|_{L_\infty(|z| \leq r)} < \infty; \ \sup_{n \geq 1} \|\nu_n\|_{L_\infty(|z| \leq r)} < \infty$$

and for $r > R(q)$,

$$(2.20) \qquad \sup_{n \geq 1} \|\mu_n\|_{L_\infty(|z| \leq r)} = \infty; \sup_{n \geq 1} \|\nu_n\|_{L_\infty(|z| \leq r)} = \infty.$$

Thus in the case $R(q) < \frac{1}{2+|1+q|}$, the numerators $\{\mu_n\}_{n=1}^{\infty}$ and denominators $\{\nu_n\}_{n=1}^{\infty}$ are normal in $\{z : |z| < R(q)\}$, while in $\left\{z : R(q) < |z| < \frac{1}{2+|1+q|}\right\}$, the numerators and denominators do not converge separately, nor can they be normal, yet their ratio converges to H_q.

3. Preliminaries

In this section, we gather some elementary identities from the theory of continued fractions, and also prove (2.12), (2.13) and some functional relations for G_q. For the reader's convenience, we include many of the proofs. Recall the notation $(a; q)_0 := 1$ and

$$(3.1) \qquad (a; q)_l := \prod_{j=1}^{\ell} (1 - aq^{j-1}), \qquad \ell \geq 1.$$

LEMMA 3.1. *Let μ_n and ν_n be given by (2.12), (2.13). Then*

$$\frac{\mu_n(z)}{\nu_n(z)} = 1 + \frac{qz|}{|1} + \frac{q^2z|}{|1} + \frac{q^3z|}{|1} + \cdots \frac{q^nz|}{|1}.$$

Proof. Fix $n \geq 1$. Following [1, p. 26], we set for $r \geq 0$,

$$F_r := \sum_{k=0}^{\infty} \frac{z^k q^{k(r+k)} (q; q)_{n-r-k+1}}{(q; q)_k (q; q)_{n-r-2k+1}} = \sum_{k=0}^{\infty} z^k q^{k(r+k)} \begin{bmatrix} n - r - k + 1 \\ k \end{bmatrix}.$$

Then

$$\begin{aligned}
F_r - F_{r+1} &= \sum_{k=0}^{\infty} \frac{z^k q^{k(r+k)} (q; q)_{n-r-k}}{(q; q)_k (q; q)_{n-r-2k}} \left[\frac{1 - q^{n-r-k+1}}{1 - q^{n-r-2k+1}} - q^k \right] \\
&= \sum_{k=1}^{\infty} \frac{z^k q^{k(r+k)} (q; q)_{n-r-k}}{(q; q)_k (q; q)_{n-r-2k+1}} (1 - q^k) \\
&= \sum_{j=0}^{\infty} \frac{z^{j+1} q^{(j+1)(r+j+1)} (q; q)_{n-r-j-1}}{(q; q)_j (q; q)_{n-r-2j-1}} = zq^{r+1} F_{r+2},
\end{aligned}$$

and hence

$$F_r / F_{r+1} = 1 + \frac{q^{r+1}z}{F_{r+1}/F_{r+2}}.$$

Moreover, we see easily using $\begin{bmatrix} m \\ l \end{bmatrix} = 0, m > l$, that $F_0 = \mu_n; F_1 = \nu_n; F_{n-1} = 1 + zq^n; F_n = 1$. So

$$\frac{\mu_n}{\nu_n} = F_0/F_1 = 1 + \frac{qz}{F_1/F_2}$$

$$= 1 + \frac{qz|}{|1} + \frac{q^2 z|}{|F_2/F_3} = \cdots$$

$$= 1 + \frac{qz|}{|1} + \frac{q^2 z|}{|1} + \frac{q^3 z|}{|1} + \cdots \frac{q^n z|}{|1}. \qquad \square$$

Next, we record the standard recurrence relations for the continued fraction numerators and denominators:

LEMMA 3.2.

(3.2) $$\mu_n(z) = \mu_{n-1}(z) + q^n z \mu_{n-2}(z);$$

(3.3) $$\nu_n(z) = \nu_{n-1}(z) + q^n z \nu_{n-2}(z);$$

(3.4) $$(\mu_n \nu_{n-1} - \mu_{n-1} \nu_n)(z) = (-1)^{n-1} z^n q^{\frac{n(n+1)}{2}}.$$

Proof. The first two are the standard recurrence relations for the numerator and denominator of a continued fraction [22, p. 20], [27, pp. 8–9] though they may also be easily proved from (2.12), (2.13) and the identity

$$\begin{bmatrix} m \\ l \end{bmatrix} = \begin{bmatrix} m-1 \\ l \end{bmatrix} + q^{m-l} \begin{bmatrix} m-1 \\ l-1 \end{bmatrix}.$$

The third is also a standard relation, and is an easy consequence of (3.2), (3.3). $\qquad \square$

Next, we record an error formula for the difference between H_q and the convergents to its c.f., making use of the functional equation in the process:

LEMMA 3.3.

(3.5) $$\left(H_q - \frac{\mu_n}{\nu_n} \right)(z) = \frac{(-1)^n z^{n+1} q^{(n+1)(n+2)/2}}{\nu_n(z)[\nu_{n+1}(z) + \nu_n(z)[H_q(q^{n+1}z) - 1]]}$$

(3.6) $$= \frac{(-1)^n z^{n+1} q^{(n+1)(n+2)/2}}{\nu_n(z)[\nu_n(z) H_q(q^{n+1}z) + q^{n+1} z \nu_{n-1}(z)]}.$$

Proof. We use the following elementary result in the theory of continued fractions: let $\{a_j\}, \{b_j\}$ be complex numbers and

$$\frac{A_k}{B_k} = b_0 + \frac{a_1|}{|b_1} + \cdots \frac{a_k|}{|b_k}, \qquad k \geq 0.$$

Then for $u \in \mathbb{C}$,

(3.7) $$b_0 + \frac{a_1|}{|b_1} + \cdots \frac{a_k|}{|b_k + u} = b_0 + \frac{a_1|}{|b_1} + \cdots \frac{a_k|}{|b_k} + \frac{u|}{|1} = \frac{A_k + A_{k-1} u}{B_k + B_{k-1} u}.$$

This follows immediately from the recurrence relations for the numerators and denominators of the continued fraction. See for example [22, p. 20], [27, p. 8]. Now in our situation, our iterated functional relation (1.5) for H_q gives

$$H_q(z) = 1 + \frac{qz|}{|1} + \frac{q^2 z|}{|1} + \cdots \frac{q^n z|}{|1 + u},$$

where $u := H_q(q^n z) - 1$. By (3.7),

$$(3.8) \qquad H_q(z) = \frac{\mu_n(z) + \mu_{n-1}(z)[H_q(q^n z) - 1]}{\nu_n(z) + \nu_{n-1}(z)[H_q(q^n z) - 1]}.$$

Then

$$\begin{aligned}
\left(H_q - \frac{\mu_{n-1}}{\nu_{n-1}}\right)(z) &= \frac{(\mu_n \nu_{n-1} - \mu_{n-1}\nu_n)(z)}{\nu_{n-1}(z)[\nu_n(z) + \nu_{n-1}(z)[H_q(q^n z) - 1]]} \\
&= \frac{(-1)^{n-1} z^n q^{n(n+1)/2}}{\nu_{n-1}(z)[\nu_n(z) + \nu_{n-1}(z)[H_q(q^n z) - 1]]}.
\end{aligned}$$

Replacing $n - 1$ by n gives the first identity (3.5) and then our recurrence relation (3.3) gives the second relation (3.6). $\square$

Next, we establish some functional relations for G_q:

LEMMA 3.4.   Let $q = e^{2\pi i \tau}$, with τ irrational. Then

$$(3.9) \qquad\qquad G_q(z) = G_q(qz) + qzG_q(q^2 z).$$

Moreover if $\ell \geq 1$,

$$(3.10) \qquad G_q\left(\frac{z}{q^\ell}\right) = G_q(qz)\mu_\ell\left(\frac{z}{q^\ell}\right) + qzG_q(q^2 z)\mu_{\ell-1}\left(\frac{z}{q^\ell}\right).$$

Proof. Firstly, (3.9) follows easily from the series definition of G_q. We prove (3.10) by induction on ℓ. If we define $\mu_{-1} := 1$, then (3.10) follows from (3.9) for $\ell = 0$. Assume now as an induction hypothesis that (3.10) is true for ℓ. Then using our recurrence relation for $\mu_{\ell+1}$,

$$\begin{aligned}
G_q(qz)&\mu_{\ell+1}\left(\frac{z}{q^{\ell+1}}\right) + qzG_q(q^2 z)\mu_\ell(\frac{z}{q^{\ell+1}}) \\
&= G_q(qz)\left[\mu_\ell\left(\frac{z}{q^{\ell+1}}\right) + z\mu_{\ell-1}\left(\frac{z}{q^{\ell+1}}\right)\right] + qzG_q(q^2 z)\mu_\ell\left(\frac{z}{q^{\ell+1}}\right) \\
&= G_q(z)\mu_\ell\left(\frac{z}{q^{\ell+1}}\right) + zG_q(qz)\mu_{\ell-1}\left(\frac{z}{q^{\ell+1}}\right) = G_q(\frac{z}{q^{\ell+1}})
\end{aligned}$$

by first (3.9) and then our induction hypothesis that (3.10) is true for ℓ. So we have the result for $\ell + 1$. $\square$

Note that if we define H_q by (1.3), then the functional relation (1.4) follows immediately from (3.9).

4. Roots of unity

In this section we prove Theorem 2.1 using the following result [37, Satz 2.38, p. 86]:

LEMMA 4.1. *Consider the* c.f.

$$(4.1) \qquad b_0 + \frac{a_1|}{|b_1|} + \cdots \frac{a_{l-1}|}{|b_{l-1}|} + \frac{a_l|}{|b_0|} + \frac{a_1|}{|b_1|} + \cdots \frac{a_{l-1}|}{|b_{l-1}|} + \frac{a_l|}{|b_0|} + \frac{a_1|}{|b_1|} + \cdots,$$

periodic of period ℓ. Let A_k/B_k denote the kth convergent, $k \geq 0$. Let $B_{\ell-1} \neq 0$, and x_1, x_2 denote the roots of the quadratic

$$(4.2) \qquad B_{\ell-1}x^2 + (a_\ell B_{\ell-2} - A_{\ell-1})x - a_\ell A_{\ell-2} = 0.$$

Assume either that (a) $x_1 = x_2$ *or* (b) $x_1 \neq x_2$ *and both the following hold*:

$$(4.3) \qquad |B_{\ell-1}x_1 + a_\ell B_{\ell-2}| > |B_{\ell-1}x_2 + a_\ell B_{\ell-2}|;$$

$$(4.4) \qquad A_k - x_2 B_k \neq 0, \qquad k = 0, 1, 2, \ldots \ell - 2.$$

Then

$$\lim_{k \to \infty} A_k/B_k = x_1.$$

Of course in our case $a_j = q^j z$; $b_j = 1$; $A_j = \mu_j$; $B_j = \nu_j$. Before applying the above result, we need

LEMMA 4.2. *Assume that q is a primitive ℓ^{th} root of unity. For our choice of $\{a_j\}, \{b_j\}$,*

$$(4.5) \qquad a_\ell B_{\ell-2} + A_{\ell-1} = z\nu_{\ell-2}(z) + \mu_{\ell-1}(z) = 1.$$

Proof. We shall use the explicit forms (2.12), (2.13) for μ_n, ν_n. We have

$$z\nu_{\ell-2}(z) + \mu_{\ell-1}(z) = \sum_{k=0}^{\left[\frac{\ell-2}{2}\right]} z^{k+1} q^{k(k+1)} \begin{bmatrix} \ell-2-k \\ k \end{bmatrix} + \sum_{k=0}^{\left[\frac{\ell}{2}\right]} z^k q^{k^2} \begin{bmatrix} \ell-k \\ k \end{bmatrix}$$

$$= 1 + \sum_{j=1}^{\left[\frac{\ell}{2}\right]} z^j q^{j(j-1)} \left(\begin{bmatrix} \ell-1-j \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} \ell-j \\ j \end{bmatrix} \right).$$

Now we see that

$$\begin{bmatrix} \ell-1-j \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} \ell-j \\ j \end{bmatrix} = \begin{bmatrix} \ell-1-j \\ j-1 \end{bmatrix} \left(1 + \frac{q^j(1-q^{\ell-j})}{1-q^j}\right)$$

$$= \begin{bmatrix} \ell-1-j \\ j-1 \end{bmatrix} \frac{1-q^\ell}{1-q^j} = 0$$

as q is a primitive ℓ^{th} root of unity. $\square$

We turn to the

Proof of Theorem 2.1. The quadratic (4.2) becomes (recall $q^\ell = 1$)

(4.6) $$\nu_{\ell-1}x^2 + (z\nu_{\ell-2}(z) - \mu_{\ell-1}(z))x - z\mu_{\ell-2}(z) = 0.$$

The discriminant of this quadratic is

$$
\begin{aligned}
D &:= (z\nu_{\ell-2}(z) - \mu_{\ell-1}(z))^2 + 4z\mu_{\ell-2}(z)\nu_{\ell-1}(z) \\
&= ([z\nu_{\ell-2}(z) + \mu_{\ell-1}(z)] - 2\mu_{\ell-1}(z))^2 + 4z\mu_{\ell-2}(z)\nu_{\ell-1}(z) \\
&= [z\nu_{\ell-2}(z) + \mu_{\ell-1}(z)]^2 - 4[z\nu_{\ell-2}(z) + \mu_{\ell-1}(z)]\mu_{\ell-1}(z) \\
&\quad + 4\mu_{\ell-1}(z)^2 + 4z\mu_{\ell-2}(z)\nu_{\ell-1}(z) \\
&= 1 - 4z(\mu_{\ell-1}\nu_{\ell-2} - \mu_{\ell-2}\nu_{\ell-1})(z) = 1 - 4z^\ell(-1)^{\ell-2}q^{\ell(\ell-1)/2},
\end{aligned}
$$

by first Lemma 4.2 and then (3.4). Next, we note that

(4.7) $$(-1)^{\ell-1}q^{\ell(\ell-1)/2} = 1.$$

Indeed for ℓ even, $q^{\ell/2} = -1$ (as q is a primitive ℓ^{th} root of unity), and for ℓ odd, $(\ell-1)/2$ is an integer. Thus

$$D = 1 + 4z^\ell$$

and the roots of the quadratic (4.6) are, by Lemma 4.2,

$$x_1 = \frac{-(z\nu_{\ell-2} - \mu_{\ell-1})(z) + \sqrt{D}}{2\nu_{\ell-1}(z)} = \frac{\mu_{\ell-1} - \frac{1}{2} + \sqrt{D/4}}{\nu_{\ell-1}};$$

$$x_2 = \frac{-(z\nu_{\ell-2} - \mu_{\ell-1})(z) - \sqrt{D}}{2\nu_{\ell-1}(z)} = \frac{\mu_{\ell-1} - \frac{1}{2} - \sqrt{D/4}}{\nu_{\ell-1}}.$$

(The branch of the $\sqrt{\ }$ is the principal one.) Now we examine (4.3) and (4.4). Firstly (4.3) becomes

$$|\nu_{\ell-1}x_1 + z\nu_{l-2}| > |\nu_{\ell-1}x_2 + z\nu_{l-2}|,$$

that is, in view of Lemma 4.2,

$$\left|\frac{1}{2} + \sqrt{D/4}\right| > \left|\frac{1}{2} - \sqrt{D/4}\right|.$$

Now by our choice of the principal branch of $\sqrt{\ }$, $\sqrt{D/4} = \alpha + i\beta$, where $\alpha > 0$, provided $D \notin (-\infty, 0]$. Then

$$\left|\frac{1}{2} + \sqrt{D/4}\right|^2 = \left(\frac{1}{2} + \alpha\right)^2 + \beta^2 > \left(\frac{1}{2} - \alpha\right)^2 + \beta^2 = \left|\frac{1}{2} - \sqrt{D/4}\right|^2.$$

So we have (4.3) provided

$$D = 1 + 4z^\ell \notin (-\infty, 0] \Leftrightarrow z^\ell \notin \left(-\infty, -\frac{1}{4}\right] \Leftrightarrow z \notin \mathcal{L}.$$

Next, we examine (4.4). We see that for $\nu_{\ell-1}(z) \neq 0$,

$$(4.8) \quad A_k - x_2 B_k \neq 0 \quad \Leftrightarrow \quad \mu_k - \frac{\nu_k}{\nu_{\ell-1}} \left[\mu_{\ell-1} - \frac{1}{2} - \sqrt{D/4} \right] \neq 0$$

$$\Leftarrow \quad \left(\mu_k \nu_{\ell-1} - \left[\mu_{\ell-1} - \frac{1}{2} \right] \nu_k \right)^2 - \frac{D}{4} \nu_k^2 \neq 0$$

$$\Leftrightarrow \quad J_k^2 + J_k \nu_k - z^\ell \nu_k^2 \neq 0,$$

where

$$J_k := \mu_k \nu_{\ell-1} - \mu_{\ell-1} \nu_k.$$

Now (3.4) shows that for each n, $\frac{\mu_n}{\nu_n} - \frac{\mu_{n-1}}{\nu_{n-1}}$ has a zero of order n at 0. Adding this for $n = k+1, k+2, \ldots, \ell-1$, shows that

$$\frac{J_k}{\nu_k \nu_{\ell-1}} = \frac{\mu_k}{\nu_k} - \frac{\mu_{\ell-1}}{\nu_{\ell-1}}$$

has a zero of order at least $k+1$ at 0. Thus

$$J_k = J_k(z) = z^{k+1} \pi_k(z),$$

where, as $\deg(\mu_k) \leq \frac{k+1}{2}$; $\deg(\nu_k) \leq \frac{k}{2}$, we see that π_k is a polynomial of degree at most $\frac{k+\ell}{2} - (k+1) = \frac{\ell-2-k}{2}$. So (4.8) becomes, after division by z^{k+1},

$$(4.9) \quad z^{k+1} \pi_k^2 + \pi_k \nu_k - z^{\ell-k-1} \nu_k^2 \neq 0.$$

(Recall that the c.f. converges at $z = 0$, so that point can be omitted.) The left-hand side of (4.9) is a polynomial of degree $\leq \ell-1$, so has at most $\ell-1$ zeros. Considering $k = 0, 1, 2, \ldots, \ell-2$, we obtain a set $\mathcal{P}$ of at most $(\ell-1)(\ell-1)$ exceptional points. Adding the at most $(\ell-1)/2$ zeros of $\nu_{\ell-1}$ gives a set $\mathcal{P}$ of at most $\frac{(\ell-1)(2\ell-1)}{2}$ points. $\qquad\square$

5. Proof of Theorems 2.2, 2.3

The proof of Theorem 2.2 requires three lemmas. The first is a special case of a theorem of Pólya:

LEMMA 5.1. *Let g be a function meromorphic in $|z| < \sigma$, and let g be analytic at 0, with Maclaurin series $\sum_{j=0}^{\infty} g_j z^j$. Let*

$$(5.1) \qquad D_n(g) := \det(g_{1+i+j})_{i,j=0}^{n-1}.$$

Then

$$(5.2) \qquad \limsup_{n\to\infty} |D_n(g)|^{1/n^2} \leq \sigma^{-1}.$$

Proof. This first appeared in [39]. A more accessible reference is [17, p. 305, Thm. 3], though the proof there is for analytic f. $\qquad\square$

Next, we record the well-known relation between the Hankel determinants $D_n(g)$ and the continued fraction coefficients of g. It was used for example in [2]:

LEMMA 5.2. *Assume that g is analytic near 0, and has (formal) continued fraction expansion*

$$c_0 + \frac{c_1 z|}{|1} + \frac{c_2 z|}{|1} + \frac{c_3 z|}{|1} + \cdots$$

with all $c_j \neq 0$. Then

$$(5.3) \qquad D_n(g) = c_1^n \prod_{j=1}^{n-1} (c_{2j} c_{2j+1})^{n-j}.$$

Proof. If g has Maclaurin series coefficients $\{g_j\}$ and we define

$$H_k^{(\ell)} := \det(g_{\ell+i+j})_{i,j=0}^{k-1}$$

then is it known [27, p. 257] that

$$c_{2k} = -\frac{H_{k-1}^{(1)} H_k^{(2)}}{H_k^{(1)} H_{k-1}^{(2)}}; c_{2k+1} = -\frac{H_{k+1}^{(1)} H_{k-1}^{(2)}}{H_k^{(1)} H_k^{(2)}}.$$

We deduce that

$$\prod_{k=1}^{\ell} (c_{2k} c_{2k+1}) = \prod_{k=1}^{\ell} \frac{H_{k-1}^{(1)} H_{k+1}^{(1)}}{(H_k^{(1)})^2} = \frac{H_{\ell+1}^{(1)}}{H_\ell^{(1)}} \Big/ \frac{H_1^{(1)}}{H_0^{(1)}} = \frac{H_{\ell+1}^{(1)}}{H_l^{(1)}} \Big/ c_1.$$

Multiplying this for $\ell = 1, 2, \ldots, n-1$ and noting that $D_n(g) = H_n^{(1)}$ and $H_1^{(1)} = c_1$ gives the result. $\qquad \square$

Next, we record one form of the *fundamental inequalities*, as a criterion for convergence of continued fractions:

LEMMA 5.3. *Assume that the c.f.*

$$\frac{1|}{|1} + \frac{a_2|}{|1} + \frac{a_3|}{|1} + \frac{a_4|}{|1} + \cdots$$

satisfies for some sequence $\{r_j\}_{j=1}^{\infty} \subset (0, \infty)$ the fundamental inequalities

$$(5.4) \qquad r_j |1 + a_j + a_{j+1}| \geq r_j r_{j-2} |a_j| + |a_{j+1}|, \qquad j \geq 1$$

where

$$a_1 := 0; \quad r_0 := 0; \quad r_{-1} := 0.$$

Let A_j/B_j denote the jth convergent to the c.f. above. Then $B_j \neq 0, j \geq 1$, and

$$(5.5) \qquad \left| \frac{A_{j+1}}{B_{j+1}} - \frac{A_j}{B_j} \right| \leq r_1 r_2 \cdots r_j, \qquad j \geq 1.$$

Proof. This is Theorem 9.1 in [48, p. 41] and inequality (9.4) in [48, p. 42].

<div align="right">□</div>

Now we turn to the

Proof of Theorem 2.2(a). We first show that the c.f. (1.6) converges to a function $H^*(z)$ analytic in $\{z : |z| < \frac{1}{2+|1+q|}\}$. This part works even if q is a root of unity. Let K be a compact subset of this ball. Choose $\varepsilon > 0$ such that

$$|z| < \frac{1 - \varepsilon}{2 + |1 + q|}, \qquad z \in K.$$

We apply the fundamental inequalities (5.4) with

$$
\begin{aligned}
a_j &:= q^j z, & j \geq 2; \\
r_j &:= 1 - \varepsilon, & j \geq 1.
\end{aligned}
$$

For $j = 1$, we see that

$$
\begin{aligned}
r_j \left|1 + a_j + a_{j+1}\right| &= (1 - \varepsilon)\left|1 + q^2 z\right| \\
&\geq \frac{1 - \varepsilon}{2} > |z| = r_1 r_{-1} |a_1| + |a_2|.
\end{aligned}
$$

For $j \geq 2$,

$$
\begin{aligned}
r_j \left|1 + a_j + a_{j+1}\right| &= (1 - \varepsilon)\left|1 + q^j z (1 + q)\right| \\
&\geq (1 - \varepsilon)\left(1 - \frac{|1 + q|}{2 + |1 + q|}\right) = \frac{2(1 - \varepsilon)}{2 + |1 + q|} \\
&> 2|z| \geq r_j r_{j-2}|a_j| + |a_{j+1}|.
\end{aligned}
$$

Thus the fundamental inequalities are satisfied. If $A_j(z)/B_j(z)$ denotes the jth convergent to the c.f.

$$\frac{1|}{|1} + \frac{q^2 z|}{|1} + \frac{q^3 z|}{|1} + \frac{q^4 z|}{|1} + \cdots$$

then Lemma 5.3 shows that $B_j(z) \neq 0$ for $j \geq 1$ and $z \in K$, and

$$\left|\frac{A_{j+1}(z)}{B_{j+1}(z)} - \frac{A_j(z)}{B_j(z)}\right| \leq (1 - \varepsilon)^j, \qquad j \geq 1.$$

Then $\{A_j/B_j\}_{j=1}^\infty$ converges uniformly in K and so the limit function is analytic in the interior of K. The same is then true for the c.f. H^* defined by (1.6). Thus H^* is analytic in $\{z : |z| < \frac{1}{2+|1+q|}\}$, so

$$\rho(q) \geq \frac{1}{2 + |1 + q|}.$$

Next, we note that if $R(q) > 0$, the function $H_q(z) := G_q(z)/G_q(qz)$ satisfies the same functional equation as does H^*, in view of the functional

equation (3.9) for G_q. Moreover, $H^*(0) = H_q(0) = 1$. Then H_q and H^* have the same c.f. expansion and hence the same Maclaurin series. This follows as the c.f. uniquely determines the corresponding Maclaurin series. Then $H_q(z) = G_q(z)/G_q(qz)$ provides a meromorphic continuation of H^* to $\{z : |z| < R(q)\}$. So we have the inequality

$$\rho(q) \geq \max\left\{\frac{1}{2 + |1 + q|}, R(q)\right\}.$$

To show $\rho(q) \leq 1$, we note from Lemma 5.2 that

$$|D_n(H_q)| = 1, n \geq 1$$

and from Lemma 5.1,

$$1 = \limsup_{n\to\infty} |D_n(H_q)|^{1/n^2} \leq \frac{1}{\rho(q)}.$$

Thus, we have $\rho(q) \leq 1$ and (2.7). To show that H_q has a natural boundary on $\{z : |z| = \rho(q)\}$, let us suppose that z_0 is a point of analyticity of H_q with $|z_0| = \rho(q)$. Then we can find a ball U centre z_0 in which H_q is analytic and hence meromorphic. The functional equation (1.4) in the form

$$H_q(qz) = \frac{qz}{H_q(z) - 1}$$

shows that $H_q(z)$ has a meromorphic continuation to the ball $qU = \{qz : q \in U\}$. Iteration of this argument shows that H_q has a meromorphic continuation to $q^jU = \{q^jz : z \in U\}, j \geq 1$. As finitely many such balls cover the circle $\{z : |z| = \rho(q)\}$, we obtain a meromorphic continuation of H_q to $\{z : |z| < \rho(q) + \varepsilon\}$, for some $\varepsilon > 0$, contradicting the definition of $\rho(q)$. So H_q must have a natural boundary on the circle $\{z : |z| = \rho(q)\}$. $\square$

In the proof of Theorem 2.2(b), we need part of a result of G. Petruska:

LEMMA 5.4.   Let $c \in [0, \infty]$.   There exists an irrational number τ with continued fraction

$$\tau = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots$$

(all a_i positive integers) such that if π_n/χ_n denotes the nth convergent to the c.f. of τ, then

(5.6) $$\lim_{n\to\infty} \frac{\log \chi_{n+1}}{\chi_n} = c.$$

Proof. For the case $0 < c < \infty$ this is part of Lemma 2 in [38, p. 354] and for the case $c = \infty$, this was noted in [13, p. 474, eqn. (1.17)]. Almost

every $\tau \in [0,1]$ satisfies (5.6) with $c = 0$. Indeed this follows from Khinchin's theorem [23] that for a.e. τ,

$$\lim_{n \to \infty} \frac{\log \chi_n}{n} = \frac{\pi^2}{12 \log 2}. \qquad \square$$

Proof of Theorem 2.2(b). Let us suppose that G_q is analytic at a point z_0 on the circle $|z| = R(q)$ and hence in some ball U centre z_0. We shall use the functional relation (3.10) in the form

$$G_q\left(\frac{u}{q^{\ell+1}}\right) = G_q(u)\mu_\ell\left(\frac{u}{q^{\ell+1}}\right) + G_q(qu)u\mu_{\ell-1}\left(\frac{u}{q^{\ell+1}}\right).$$

Let $1 > \varepsilon > 0$. Let us choose ℓ large with $\frac{z_0}{q^{\ell+1}} \in U$ and in fact such that the ball U_1 centre $\frac{z_0}{q^{\ell+1}}$ and $1 - \varepsilon$ times the radius of U lies inside U. Then the above identity shows that $G_q(qu)$ is meromorphic in U_1 and hence G_q is meromorphic in $qU_1 = \{qz : z \in U_1\}$. Since $\varepsilon > 0$ is arbitrary, we deduce that G_q is meromorphic in qU. By the same argument $G_q(u)$ is meromorphic in $q^j U$, $j \geq 1$. Finitely many such neighbourhoods cover the circle $\{z : |z| = R(q)\}$. Then G_q is analytic on this circle, except possibly for finitely many poles. Since there are only finitely many such poles, we can choose z_0 with $|z_0| = R(q)$ such that both z_0 and qz_0 are points of analyticity. Thus there exists an open ball B centre z_0, such that G_q is analytic in both B and qB. But the functional relation (3.9) shows that G_q is analytic in $q^2 B$, and hence also in $q^j B, j \geq 1$. Hence G_q is analytic on the whole circle $\{z : |z| = R(q)\}$, contradicting the definition of $R(q)$. Thus, G_q has a natural boundary on its circle of convergence.

Next we show that $R(q)$ may assume any value in $[0,1]$ by Petruska's Lemma 5.4 with $q = e^{2\pi i \tau}$. We shall show (recall (2.5)) that

$$R(q) = \liminf_{n \to \infty} \|j\tau\|^{1/j} = e^{-c}$$

and since e^{-c} may assume any value in $[0,1]$, the result follows. We recall some elementary facts from the theory of continued fraction expansions of real numbers: firstly if $\frac{j}{k}$ is not a convergent to the c.f. of τ, then [20, p. 153, Thm. 184]

$$\left|\tau - \frac{j}{k}\right| \geq \frac{1}{2k^2}$$

and so if k is not a denominator of some convergent,

$$\|k\tau\| \geq \frac{1}{2k}.$$

Hence if we let $\mathcal{S} := \{\chi_1, \chi_2, \chi_3, \ldots\}$, then

$$\lim_{k \to \infty, k \notin \mathcal{S}} \|k\tau\|^{1/k} = 1.$$

Next for convergents $\frac{\pi_n}{\chi_n}$, we have the inequalities

$$\frac{1}{2\chi_{n+1}} \leq |\chi_n\tau - \pi_n| < \frac{1}{\chi_{n+1}}.$$

See [20, p. 140] for the upper bound. The lower bound follows from the error formula (see for example [38, p. 354])

$$\chi_n\tau - \pi_n = \frac{(-1)^n}{\chi_{n+1} + \alpha_{n+1}\chi_n},$$

where $\alpha_{n+1} \in (0,1)$ and χ_n increases with n. Then we see that

$$\lim_{j\to\infty, j\in\mathcal{S}} \|j\tau\|^{1/j} = \lim_{n\to\infty} \|\chi_n\tau\|^{1/\chi_n} = \lim_{n\to\infty} \chi_{n+1}^{-1/\chi_n} = e^{-c},$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 2.2(c). When $R(q) = 1$, of course $\rho(q) = 1$ follows from (2.7), so that Theorem 2.2(c) follows immediately. Moreover, we noted in Section 2 that for a.e. q, $R(q) = 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 2.3(a). We recall that we showed in the proof of Theorem 2.2(a) that $H_q(z) \neq \infty, |z| < \frac{1}{2+|1+q|}$ even if q is a root of unity. Then our functional equation (1.4), in the form

(5.7) $$[H_q(z) - 1]H_q(qz) = qz$$

shows that $H_q(qz)$ cannot have any zeros in that ball (recall that $H_q(0) = 1$). So H_q does not assume the values $0, \infty$ there. By the same token, the functional relation shows that H_q cannot assume the value 1 in the punctured ball $\mathcal{B} := \{z : 0 < |z| < \frac{1}{2+|1+q|}\}$ (for if $H_q(z) = 1$, then $H_q(qz) = \infty$). Thus H_q omits the values $0, 1, \infty$ in that punctured ball. If $\{q_k\}_{k=1}^{\infty}$ satisfy (2.8), then in a given compact subset of $\mathcal{B}$, for large k, H_{q_k} omits the values $0, 1, \infty$ and by Montel's theorem [42, p. 54, p. 74], $\{H_{q_k}\}_{k=1}^{\infty}$ is normal there. Let H^* denote a limit function of some subsequence, so that H^* is either identically ∞ or is analytic in $\mathcal{B}$. It follows easily from (5.7) that H^* cannot be identically ∞, so is analytic in $\mathcal{B}$. In view of (5.7) and (2.8), we have the functional equation

(5.8) $$[H^*(z) - 1]H^*(qz) = qz, z \in \mathcal{B}.$$

Now for all q_k, the c.f. coefficients have absolute value 1, so that by Worpitzky's theorem [27, p. 35]

$$|H_{q_k}(z) - 1| \leq \frac{1}{2}, |z| \leq \frac{1}{4}$$

and so the same is true of H^*. It follows that 0 is a removable singularity of H^* and defining $H^*(0) = 1$, we obtain a function analytic in $|z| < \frac{1}{2+|1+q|}$ satisfying the same functional equation as H_q. Then both have the same c.f.,

both have the same set of convergents, and hence the same Maclaurin series, so that $H^* = H_q$. As H^* was the limit of any subsequence, the full sequence converges to H_q. □

Proof of Theorem 2.3(b). Let us assume that Ω_2 does not contain any pole of H_{q_k} for infinitely many k. By passing to a subsequence, we may assume that H_{q_k} has no poles in Ω_2 for all k. We may also assume that $z \in \Omega_1 \Rightarrow zq_k^{\pm 1} \in \Omega_2$ for all k (if necessary make Ω_1 a little smaller). Then our functional equation for H_{q_k} in the form

$$[H_{q_k}(z) - 1]H_{q_k}(q_k z) = q_k z$$

shows that as $H_{q_k}(q_k z) \neq \infty, z \in \Omega_1$, so that $H_{q_k}(z) \neq 1, z \in \Omega_1$. Similarly,

$$[H_{q_k}(z/q_k) - 1]H_{q_k}(z) = z$$

and $H_{q_k}(z/q_k) \neq \infty, z \in \Omega_1$ implies $H_{q_k}(z) \neq 0, z \in \Omega_1$. Thus H_{q_k} omit the values $0, 1, \infty$ in Ω_1 for all k, and so $\{H_{q_k}\}_{k=1}^{\infty}$ is a normal family there. Let H^* denote a subsequential limit. As above it is not identically ∞, so is analytic in Ω_1. Then we have the functional equation (5.8) for H^*. Iterating the functional relation leads to the same continued fraction as for H_q, periodic of period ℓ. Moreover, the error formula used in the proof of Lemma 3.3 gives (3.8) with $n = \ell$; that is,

$$H^* = \frac{\mu_\ell + \mu_{\ell-1}[H^* - 1]}{\nu_\ell + \nu_{\ell-1}[H^* - 1]}.$$

As before, $H^*(z)$ is one of the roots of the quadratic (4.6) (we also use (3.2), (3.3)). Then as in the proof of Theorem 2.1,

$$H^*(z) = \frac{\mu_{\ell-1}(z) - \frac{1}{2} \pm \sqrt{\frac{1}{4} + z^\ell}}{\nu_{\ell-1}(z)}.$$

It follows that we obtain a branch of $\sqrt{\frac{1}{4} + z^\ell}$ analytic in Ω_1, which is an open set containing one of the branchpoints of $\sqrt{\frac{1}{4} + z^\ell}$. This is of course impossible. So for large k, H_{q_k} must have a pole in Ω_1.

The above argument also works if ℓ is odd and we choose

$$\Omega_1 = \Omega_2 = \{z : r < |z| < r + \delta\}$$

for then we cannot find a branch of $\sqrt{\frac{1}{4} + z^\ell}$ analytic in Ω_1. (If ℓ is even, this argument fails as we may find a branch analytic in Ω_1.) □

6. Proof of Theorems 2.4, 2.6

As a preliminary to proving Theorem 2.4, we rearrange the expression (2.12) for μ_n; recall the notation (3.1).

LEMMA 6.1.   (a)

$$(6.1) \qquad \mu_n(z) = \sum_{j=0}^{[\frac{n+1}{2}]} \frac{q^{-j^2}}{(q^{-1};q^{-1})_j}(zq^{n+1})^j \sum_{\ell=0}^{[\frac{n+1}{2}]-j} \frac{z^\ell q^{\ell^2}}{(q;q)_\ell}.$$

(b)

$$(6.2) \qquad \nu_n(z) = \mu_{n-1}(zq).$$

Proof. We use the q-binomial theorem [16, p. 7, eqn. (1.3.2); p. 236, eq. (II.4)]

$$(6.3) \qquad (-uq;q)_k = \prod_{j=1}^{k}(1+q^j u) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{j(j+1)/2}u^j.$$

Now,

$$\begin{aligned}
\begin{bmatrix} n+1-k \\ k \end{bmatrix} &= \frac{1}{(q;q)_k}\prod_{j=1}^{k}(1-q^j(q^{n+1-2k})) \\
&= \frac{1}{(q;q)_k}\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix} q^{j(j+1)/2}(-q^{n+1-2k})^j \\
&= \sum_{j=0}^{k}\frac{1}{(q;q)_j(q;q)_{k-j}}q^{j(j+1)/2+(n+1-2k)j}(-1)^j
\end{aligned}$$

so that from (2.12),

$$\begin{aligned}
\mu_n(z) &= \sum_{k=0}^{[\frac{n+1}{2}]} z^k q^{k^2} \sum_{j=0}^{k} \frac{1}{(q;q)_j(q;q)_{k-j}} q^{j(j+1)/2+(n+1-2k)j}(-1)^j \\
&= \sum_{j=0}^{[\frac{n+1}{2}]} \frac{(-1)^j q^{j(j+1)/2+(n+1)j}}{(q;q)_j} z^j \sum_{k=j}^{[\frac{n+1}{2}]} \frac{q^{(k-j)^2-j^2}}{(q;q)_{k-j}} z^{k-j} \\
&= \sum_{j=0}^{[\frac{n+1}{2}]} \frac{q^{-j^2}}{(q^{-1};q^{-1})_j}(zq^{n+1})^j \sum_{\ell=0}^{[\frac{n+1}{2}]-j} \frac{q^{\ell^2}}{(q;q)_\ell} z^\ell.
\end{aligned}$$

In the last line, we used

$$(6.4) \qquad (q;q)_j = (-1)^j q^{j(j+1)/2}(q^{-1};q^{-1})_j.$$

(b) This follows directly from (2.13). $\square$

Next, we sketch from [35, p. 348], [19, p. 86] the proof of (2.4):

LEMMA 6.2. *Let $q = e^{2\pi i \tau}$, with τ irrational. The radius of convergence of G_q is*

(6.5) $$R(q) = \liminf_{n \to \infty} \left| \prod_{j=1}^{n} (1 - q^j) \right|^{1/n} = \liminf_{n \to \infty} |1 - q^n|^{1/n}.$$

Proof. Hardy and Littlewood [19, p. 86] established the remarkable identity

(6.6) $$\sum_{n=0}^{\infty} \frac{z^n}{\prod_{j=1}^{n}(1 - q^j)} = \exp \left(\sum_{n=1}^{\infty} \frac{z^n}{n(1 - q^n)} \right)$$

and noted that both power series in the last identity have the same radius of convergence even for our choice of q. (The identity was also known to earlier authors.) The Cauchy-Hadamard formula for the radius of convergence gives the result. $\square$

We turn to the

Proof of (2.15) *and* (2.16). Let $\mathcal{S}$ be an infinite sequence such that (2.14) holds. We use a section of the sum in the right-hand side of (6.1). For fixed positive integers m, and uniformly for $|z| \leq r < R(q)$,

$$\lim_{n \to \infty, n \in \mathcal{S}} \sum_{j=0}^{m} \frac{q^{-j^2}}{(q^{-1}; q^{-1})_j} (zq^{n+1})^j \sum_{\ell=0}^{[\frac{n+1}{2}]-j} \frac{q^{\ell^2}}{(q;q)_\ell} z^\ell$$

$$= \sum_{j=0}^{m} \frac{q^{-j^2}}{(q^{-1}; q^{-1})_j} (zq\beta)^j \sum_{\ell=0}^{\infty} \frac{q^{\ell^2}}{(q;q)_\ell} z^\ell = G_q(z) \overline{\sum_{j=0}^{m} \frac{q^{j^2}}{(q;q)_j} (\overline{zq\beta})^j}.$$

Moreover as $m \to \infty$, the last right-hand side approaches $G_q(z)\overline{G_q(\overline{zq\beta})}$. Moreover, for $|z| \leq r < R(q)$, and uniformly in $n \geq 2m+1$

$$\left| \sum_{j=m+1}^{[\frac{n+1}{2}]} \frac{q^{-j^2}}{(q^{-1}; q^{-1})_j} (zq^{n+1})^j \sum_{\ell=0}^{[\frac{n+1}{2}]-j} \frac{z^\ell q^{\ell^2}}{(q;q)_\ell} \right| \leq \left[\sum_{j=m+1}^{\infty} \frac{r^j}{|(q;q)_j|} \right] \left[\sum_{\ell=0}^{\infty} \frac{r^\ell}{|(q;q)_\ell|} \right].$$

The last right-hand side approaches 0 as $m \to \infty$. So we have (2.15). Then (2.16) follows from the identity (6.2); note that if $q^n \to \beta$, then $q^{n-1} \to \beta/q$. $\square$

In the proof of (2.17), we need an identity:

LEMMA 6.3. *Let $q = e^{2\pi i \tau}$, with τ irrational. Then*

(6.7) $$G_{\overline{q}}(qu)G_q(qu) + quG_{\overline{q}}(u)G_q(q^2 u) = \overline{G_q(\overline{qu})}G_q(qu)$$
$$+ qu\overline{G_q(\overline{u})}G_q(q^2 u) = 1.$$

Proof. We first establish the identity

$$(6.8) \qquad \Gamma := \sum_{j+k=\ell; j,k\geq 0} \frac{q^{j^2-k^2}}{(q;q)_j(\overline{q};\overline{q})_k} u^j = q^{\ell^2} u^\ell \frac{(q^{1-2\ell}/u;q)_\ell}{(q;q)_\ell}.$$

Using (6.4), we see that

$$\begin{aligned}
\Gamma &= u^\ell \sum_{j+k=\ell; j,k\geq 0} \frac{q^{j^2-k^2+k(k+1)/2}}{(q;q)_j(q;q)_k}(-u)^{-k} \\
&= \frac{q^{\ell^2} u^\ell}{(q;q)_\ell} \sum_{k=0}^{\ell} \left[\begin{array}{c} \ell \\ k \end{array}\right] q^{k(k+1)/2}(-q^{-2\ell}/u)^k \\
&= q^{\ell^2} u^\ell \frac{(q^{1-2\ell}/u;q)_\ell}{(q;q)_\ell},
\end{aligned}$$

by the *q*-binomial theorem (6.3). Then by (6.8),

$$\begin{aligned}
\Delta &:= G_{\overline{q}}(qu)G_q(qu) + quG_{\overline{q}}(u)G_q(q^2 u) - 1 \\
&= \sum_{\ell=1}^{\infty} (uq)^\ell \sum_{j+k=\ell; j,k\geq 0} \frac{q^{j^2-k^2}}{(q;q)_j(\overline{q};\overline{q})_k} + qu\sum_{\ell=0}^{\infty} u^\ell \sum_{j+k=\ell; j,k\geq 0} \frac{q^{j^2-k^2}q^{2j}}{(q;q)_j(\overline{q};\overline{q})_k} \\
&= \sum_{\ell=1}^{\infty} (uq)^\ell q^{\ell^2} \frac{(q^{1-2\ell};q)_\ell}{(q;q)_\ell} + qu\sum_{\ell=0}^{\infty} u^\ell q^{\ell^2} q^{2\ell} \frac{(q^{1-2\ell-2};q)_\ell}{(q;q)_\ell} \\
&= \sum_{\ell=1}^{\infty} u^\ell q^{\ell^2} \frac{(q^{1-2\ell};q)_{\ell-1}}{(q;q)_{\ell-1}} \left[q^\ell \frac{1-q^{-\ell}}{1-q^\ell} + 1\right] = 0. \qquad \square
\end{aligned}$$

We turn to the

Proof of (2.17). We assume that $\mathcal{S}$ satisfies (2.14). From (3.6),

$$\frac{H_q(z) - \frac{\mu_n(z)}{\nu_n(z)}}{(-1)^n z^{n+1} q^{(n+1)(n+2)/2}} = \frac{1}{\nu_n(z)[\nu_n(z)H_q(q^{n+1}z) + q^{n+1}z\nu_{n-1}(z)]}.$$

Here $H_q(q^{n+1}z) \to H_q(q\beta z)$ as $n \to \infty$ through $\mathcal{S}$, uniformly in compact subsets of $|z| < R(q)$ omitting zeros of $G_q(q^2\beta z)$. Using (2.16), we obtain uniformly in compact subsets of $|z| < R(q)$ omitting zeros of $G_q(q^2\beta z)$,

$$\lim_{n\to\infty, n\in\mathcal{S}} \nu_n(z) \left[\nu_n(z)H_q(q^{n+1}z) + q^{n+1}z\nu_{n-1}(z)\right]$$

$$= G_q(qz)\overline{G_q(\overline{\beta q z})} \left[G_q(qz)\overline{G_q(\overline{\beta q z})}H_q(q\beta z) + q\beta z G_q(qz)\overline{G_q(\overline{\beta z})}\right]$$

$$= \frac{G_q(qz)^2\overline{G_q(\overline{\beta}qz)}}{G_q(q^2\beta z)} \left[\overline{G_q(\overline{\beta}qz)}G_q(\beta qz) + q\beta z\overline{G_q(\overline{\beta}z)}G_q(q^2\beta z)\right]$$

$$= \frac{G_q(qz)^2\overline{G_q(\overline{\beta}qz)}}{G_q(q^2\beta z)}$$

by Lemma 6.3. The result then follows. □

Now we can turn to the

Proof of Corollary 2.5. Note first that μ_n, ν_n have no common zeros (see (3.4)), so every zero of ν_n is a pole of μ_n/ν_n. But the limit relation (2.16) shows that for the appropriate subsequence, if H_q has ℓ poles on $|z| = r$, that is $G_q(qz)$ has ℓ zeros there, then for large $n \in S$, ν_n has at least 2ℓ zeros counting multiplicity in any neighbourhood of that circle. As every subsequence of positive integers contains a subsequence satisfying (2.14), for some β, the result follows. □

Our proof of Theorem 2.6 is very similar to that of Theorem 2.4 in [35], but we provide most of the details.

Proof of Theorem 2.6. Let us set

$$H_j(u) := \prod_{k=1}^{j}(1 + q^{-k}u).$$

We see that

$$\begin{bmatrix} n \\ j \end{bmatrix} = H_j(-q^{n+1})/\prod_{k=1}^{j}(1 - q^k).$$

Since $\{-q^{n+1}\}_{n=j}^{\infty}$ is dense on the unit circle, we see that

$$\Gamma_j := \sup_{n \geq j}\left|\begin{bmatrix} n \\ j \end{bmatrix}\right| = \|H_j\|_{L_\infty(|z|=1)}/\left|\prod_{k=1}^{j}(1 - q^k)\right|.$$

Then we see that

(6.9) $$\|\mu_n\|_{L_\infty(|z|\leq r)} \leq \sum_{j=0}^{\infty}\Gamma_j r^j,$$

with a similar inequality for ν_n. It is shown in [35, p. 345] with the aid of the theory of uniform distribution (there q is replaced by q^{-1}) that

$$\lim_{j\to\infty}\|H_j\|_{L_\infty(|z|=1)}^{1/j} = 1$$

so that

$$\limsup_{j\to\infty}\Gamma_j^{1/j} = 1/R(q).$$

Thus the series in (6.9) converges if $r < R(q)$. Since the series is independent of n, we have the required uniform boundedness of $\{\mu_n\}, \{\nu_n\}$. Next if r is such that

$$C := \sup_{n \geq 1} \|\nu_n\|_{L_\infty(|z| \leq r)} < \infty,$$

Cauchy's inequalities give for $\left[\frac{n}{2}\right] \geq k$,

$$\left\| \begin{bmatrix} n-k \\ k \end{bmatrix} \right\| \leq \|\nu_n\|_{L_\infty(|z| \leq r)}/r^k \leq C/r^k.$$

As $n - k$ may assume any integral value $\geq k$ as n runs over integers with $\left[\frac{n}{2}\right] \geq k$, we obtain

$$\Gamma_k \leq C/r^k.$$

Taking k^{th} roots and letting $k \to \infty$ give

$$1/R(q) \leq 1/r$$

so that $r \leq R(q)$. Thus $\{\mu_n\}, \{\nu_n\}$ cannot be uniformly bounded in $|z| \leq r$ if $r > R(q)$.   □

7. Proof of Theorem 1.1

We shall assume throughout that we are dealing with q on the unit circle such that $R(q) = 1$. (Later on in this section, we shall specialize q to that given in Theorem 1.1.) Note that by Theorem 2.2, H_q is meromorphic in the unit ball, and G_q is analytic in the unit ball, and both have a natural boundary on the unit circle. We first outline the main steps in the proof of Theorem 1.1:

(I) We again show how a counterexample to the Baker-Gammel-Wills Conjecture follows if for all β on the unit circle,

(7.1) $\{z : G_q(qz) = 0\} \neq \left\{z : G_q\left(\overline{\beta q z}\right) = 0\right\}.$

(II) We show that to prove (7.1), instead of considering the full series for G_q, it suffices to show something like (7.1) for a partial sum

$$S_{m,q}(z) := \sum_{j=0}^{m} \frac{q^{j^2}}{(q;q)_j} z^j.$$

(We shall use $m = 50$.) This is achieved using Rouche's theorem and an estimate for the tail $G_q - S_{50,q}$.

(III) We use the principal of the argument to count the number of zeros of $S_{50,q}$ inside certain circles. To evaluate the integrals counting the zeros,

we use an elementary integration rule, and establish a rigorous estimate for the error. The calculation is performed using Matlab 6.0. Moreover, we use Matlab 6.0 and Mathematica 3.0 to estimate below the minimum modulus of $S_{50,q}$ on certain circles. Our numerical evaluations involve only evaluating $S_{50,q}$ and its first three derivatives at various explicit points, and sums involving it, or comparisons of its absolute value at a definite set of points. The calculation does not make any use of "black-box" zero finding routines, or numerical quadrature routines. These calculations show that the two zeros of $S_{50,q}$ of smallest modulus have the desired asymmetry, and we then deduce the same for G_q.

We now turn to these steps in detail:

(I) *It suffices to prove that* (7.1) *holds for every* $|\beta| = 1$. Let us suppose that (7.1) holds for every $|\beta| = 1$, but that some subsequence $\{\mu_n/\nu_n\}_{n \in \mathcal{S}}$ has

$$\lim_{n \to \infty, n \in \mathcal{S}} \mu_n/\nu_n = H_q$$

uniformly in compact subsets of the unit ball omitting poles of H_q. By extracting a further subsequence, we may assume that for some $|\beta| = 1$,

$$\lim_{n \to \infty, n \in \mathcal{S}} q^n = \beta.$$

Now let K be a closed ball in the unit ball, containing in its interior zeros of $G_q\left(\overline{\beta q z}\right)$ that are not zeros of $G_q(qz)$. We may also assume that K does not contain any zeros of $G_q(qz)$, that is, poles of H_q. For large $n \in \mathcal{S}$, μ_n/ν_n has no poles in K, because of the assumed uniform convergence. Since μ_n and ν_n are coprime polynomials (recall (3.4)), this forces ν_n not to have zeros in K for large n. But that contradicts the uniform convergence in (2.16), which by Hurwitz' Theorem, shows that each zero of $G_q\left(\overline{\beta q z}\right)$ in K must attract zeros of $\nu_n, n \in \mathcal{S}$.

Rather than working with (7.1), it will be easier to work with:

Reformulation of (7.1): *The zeros of* G_q *are not symmetric about any line through* 0. That is, after reflecting the set of zeros about any line through 0, we obtain a different set.

To see this, observe that if, for example, $\beta = 1$, (7.1) requires that the zeros of G_q do not occur in conjugate pairs. That is, the zeros of G_q are not symmetric about the real line. Of course the case of general β is similar.

(II) *Estimation of the tail* $G_q - S_{50,q}$. We estimate the tail for a special class of q including that in Theorem 1.1.

LEMMA 7.1.   *Let* $q = e^{2\pi i \tau}$, *where for some positive integer* $c \geq 2$,

(7.2) $$\tau := \frac{1|}{|c} + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|1} + \cdots = \frac{1}{c + \frac{1}{2}\left(\sqrt{5} - 1\right)}.$$

Then for $0 < r < s < 1$,

(7.3) $$\max_{|z| \leq r} |G_q(z) - S_{m,q}(z)|$$

$$\leq \frac{(r/s)^{m+1}}{1 - r/s} \exp\left(S_0 + \frac{\sqrt{5}}{4(1 - \alpha^{-8})} \frac{s^{2c+1}}{1 - s}\right) =: T,$$

where

(7.4) $$S_0 := \sum_{n=1}^{2c} \frac{s^n}{2n |\sin n\pi\tau|}; \quad \alpha := \frac{1}{2}\left(\sqrt{5} + 1\right).$$

Proof. From (6.6), and Cauchy's estimates, for $0 < s < 1$,

(7.5) $$\frac{1}{|(q;q)_n|} \leq s^{-n} \max_{|z|=s}\left|\exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n(1 - q^n)}\right)\right|$$

$$\leq s^{-n} \exp\left(\sum_{n=1}^{\infty} \frac{s^n}{2n |\sin n\pi\tau|}\right).$$

To estimate the last series, we use the classical continued fraction expansion (7.2) of τ. Let $\{\pi_j/\chi_j\}_{j=0}^{\infty}$ be the convergents in the continued fraction (7.2) of τ. With initial conditions

$$\begin{aligned}\pi_0 &= 0; & \pi_1 &= 1; \\ \chi_0 &= 1; & \chi_1 &= c,\end{aligned}$$

they satisfy the following recurrence relations for $n \geq 2$ ([20], [26]):

(7.6) $$\begin{aligned}\pi_n &= \pi_{n-1} + \pi_{n-2}; \\ \chi_n &= \chi_{n-1} + \chi_{n-2}.\end{aligned}$$

Of course this special form of the recurrence relation exists because all partial quotients in (7.2), other than the first, are unity. To solve this constant coefficient difference equation, one uses the characteristic equation

$$x^2 - x - 1 = 0,$$

with roots

$$\alpha := \frac{1}{2}\left(1 + \sqrt{5}\right); \qquad \beta := \frac{1}{2}\left(1 - \sqrt{5}\right).$$

Standard methods and the initial conditions give for $n \geq 0$,

(7.7) $$\pi_n = (\alpha^n - \beta^n)/\sqrt{5};$$

(7.8) $$\chi_n = ((c - \beta)\alpha^n + (\alpha - c)\beta^n)/\sqrt{5}.$$

Moreover, the form of τ and a simple calculation give

(7.9) $$\chi_n\tau - \pi_n = \frac{\beta^n}{c-\beta} = \frac{(-\alpha)^{-n}}{c-\beta},$$

so that

(7.10) $$\|\chi_n\tau\| = \frac{1}{(c-\beta)\,\alpha^n}.$$

Note that the last right-hand side is bounded above by

$$\frac{1}{(1-\beta)\,\alpha} = \frac{1}{\alpha^2} < \frac{1}{2},$$

so that (7.10) is true even for $n = 1$.

Next, we fix $j \geq 3$, and consider n such that

$$\chi_j \leq n < \chi_{j+1}.$$

By the best approximation property of continued fractions [26],

$$\|n\tau\| \geq \|\chi_j\tau\|$$

and hence

(7.11) $$n\|n\tau\| \geq \chi_j\|\chi_j\tau\| = \frac{1}{\sqrt{5}}\left(1 + \frac{\alpha - c}{c-\beta}\frac{(-1)^j}{\alpha^{2j}}\right),$$

by (7.8) and (7.10). Here

(7.12) $$0 < (-1)\frac{\alpha-c}{c-\beta} = \frac{c-\alpha}{c+\alpha^{-1}} < 1.$$

Thus if $j = 3$, the right-hand side of (7.11) exceeds $1/\sqrt{5}$. If $j \geq 4$, it exceeds

$$\frac{1}{\sqrt{5}}\left(1 - \alpha^{-8}\right).$$

Then for $n \geq \chi_3$,

$$\begin{aligned}
\frac{1}{2n\,|\sin n\pi\tau|} &= \frac{1}{2n\,(\sin\pi\|n\tau\|)} \\
&\leq \frac{1}{4n\|n\tau\|} \leq \frac{\sqrt{5}}{4\,(1-\alpha^{-8})}.
\end{aligned}$$

Since $\chi_3 = 2c+1$, we deduce from (7.5) that

$$\frac{1}{|(q;q)_n|} \leq s^{-n}\exp\left(S_0 + \frac{\sqrt{5}}{4\,(1-\alpha^{-8})}\frac{s^{2c+1}}{1-s}\right),$$

where S_0 is given by (7.4). Multiplying this by r^n and adding over $n \geq m+1$ gives (7.3). $\square$

We shall also need an immediate consequence of Rouche's theorem and Lemma 7.1:

LEMMA 7.2. *Assume the hypotheses and notation of Lemma* 7.1. *Let* γ *be a simple closed curve in* $\{z : |z| \leq r\}$ *such that*

$$(7.13) \qquad \min_{z \in \gamma} |S_{m,q}(z)| > T.$$

Then $S_{m,q}$ *and* G_q *have the same total multiplicity of zeros inside* γ.

(III) *Use of Mathematica* 3.0 *and Matlab* 6 *to verify what is needed for* $S_{50,q}$. In applying Lemma 7.2, we need to estimate the minimum modulus of $S_{m,q}$. Later on, we shall also need to estimate the maximum modulus of $S_{50,q}^{(j)}, j = 0, 1, 2, 3$. Since we prefer to evaluate $S_{m,q}$ only at a definite set of points, rather than relying on a "black-box" to find a minimum, we need:

LEMMA 7.3. *Let P be a polynomial of degree $\leq n$, let $m \geq 1$, and let γ be the circle $\{z : |z - a| = \varepsilon\}$. Then*

$$(7.14) \qquad \min_{z \in \gamma} |P(z)| \quad \geq \quad \min_{1 \leq j \leq m} \left| P\left(a + \varepsilon e^{2\pi i j / m}\right) \right|$$

$$-\frac{\pi n}{m} \max_{1 \leq j \leq 2n} \left| P\left(a + \varepsilon e^{2\pi i j / (2n)}\right) \right|;$$

$$(7.15) \qquad \max_{z \in \gamma} |P(z)| \quad \leq \quad \max_{1 \leq j \leq m} \left| P\left(a + \varepsilon e^{2\pi i j / m}\right) \right|$$

$$+\frac{\pi n}{m} \max_{1 \leq j \leq 2n} \left| P\left(a + \varepsilon e^{2\pi i j / (2n)}\right) \right|.$$

Proof. We may assume that $a = 0$ and $\varepsilon = 1$, so that γ is the unit circle. The general case follows by replacing $P(z)$ by $P(a + \varepsilon z)$. We use a Duffin-Schaefer type inequality due to Frappier, Rahman, and Ruscheweyh [15], [36, p. 690]:

$$(7.16) \qquad \|P'\|_{L_\infty(|z|=1)} \leq n \max_{1 \leq j \leq 2n} \left| P\left(e^{2\pi i j / (2n)}\right) \right|.$$

Let $z = e^{i\theta}$, $\theta \in [0, 2\pi]$, and choose $j \in \{0, 1, 2, \ldots, m\}$ such that $\left| \theta - \frac{2\pi j}{m} \right|$ is as small as is positive. Then

$$\left| P(z) - P\left(e^{2\pi i j / m}\right) \right| \quad = \quad \left| \int_{2\pi j / m}^{\theta} P'\left(e^{it}\right) i e^{it} dt \right|$$

$$\leq \quad \left| \theta - \frac{2\pi j}{m} \right| \|P'\|_{L_\infty(|z|=1)}$$

$$\leq \quad \frac{\pi}{m} \left(n \max_{1 \leq k \leq 2n} \left| P\left(e^{2\pi i k / (2n)}\right) \right| \right),$$

by (7.16). Then (7.14) and (7.15) follow. $\qquad \square$

To count the number of zeros of $S_{50,q}$, we use the principal of the argument: if γ is a circle $\{z : |z - a| = r\}$ on which $S_{50,q}$ has no zeros,

$$I(\gamma) := \frac{1}{2\pi i} \int_\gamma \frac{S'_{50,q}}{S_{50,q}} = \frac{r}{2\pi} \int_{-\pi}^{\pi} e^{i\theta} \frac{S'_{50,q}}{S_{50,q}} \left(a + re^{i\theta}\right) d\theta$$

is the total multiplicity of zeros of $S_{50,q}$ inside γ. We approximate $I(\gamma)$ by the simple rule

$$I_m(\gamma) = \frac{r}{m} \sum_{j=0}^{m-1} e^{2\pi ij/m} \frac{S'_{50,q}}{S_{50,q}} \left(a + re^{2\pi ij/m}\right).$$

Following is an elementary estimate for the error. It is by no means optimal, but suffices for our purposes:

LEMMA 7.4.

(7.17)
$$|I(\gamma) - I_m(\gamma)| \leq \frac{r\pi\sqrt{2}}{m^2} \Phi,$$

where

(7.18)

$$\Phi \quad := \quad \frac{\max_\gamma \left|S'_{50,q}\right|}{\min_\gamma |S_{50,q}|} + 3r \frac{\max_\gamma \left|S''_{50,q}\right|}{\min_\gamma |S_{50,q}|} + 3r \left(\frac{\max_\gamma \left|S'_{50,q}\right|}{\min_\gamma |S_{50,q}|}\right)^2 + r^2 \frac{\max_\gamma \left|S'''_{50,q}\right|}{\min_\gamma |S_{50,q}|}$$

$$+ \ 3r^2 \frac{\max_\gamma \left|S'_{50,q}\right| \max_\gamma \left|S''_{50,q}\right|}{(\min_\gamma |S_{50,q}|)^2} + 2r^2 \left(\frac{\max_\gamma \left|S'_{50,q}\right|}{\min_\gamma |S_{50,q}|}\right)^3.$$

Proof. First recall that if

$$R(z) = \sum_{j=-(m-1)}^{m-1} c_j z^j$$

is a trigonometric polynomial of degree $< m$, then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} R\left(e^{i\theta}\right) d\theta = \frac{1}{m} \sum_{j=0}^{m-1} R\left(e^{2\pi ij/m}\right).$$

Next, for any continuous complex-valued function g defined on the unit circle, we deduce that

$$E \quad := \quad \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} g\left(e^{i\theta}\right) d\theta - \frac{1}{m} \sum_{j=0}^{m-1} g\left(e^{2\pi ij/m}\right) \right|$$

$$\leq \quad 2 \inf \|g - R\|_{L_\infty(|z|=1)},$$

where the inf is taken over all trigonometric polynomials R of degree $< m$. To estimate this error of approximation, we use one case of the Favard estimates

[8, Thm. 4.3, p. 214]. This asserts that if h is a twice continuously differentiable real valued function defined on $[-\pi, \pi]$, there exists a trigonometric polynomial R of degree $< m$ such that

$$\|h - R\|_{L_\infty[-\pi,\pi]} \leq \frac{\pi}{2m^2}\|h''\|_{L_\infty[-\pi,\pi]}.$$

Applying this to $h(\theta) = \mathrm{Re}g\left(e^{i\theta}\right)$ and $h(\theta) = \mathrm{Im}g\left(e^{i\theta}\right)$ gives, if the second derivatives of g are continuous,

$$E \leq \frac{\sqrt{2}\pi}{m^2}\sup\left\{\left|\frac{d^2}{d\theta^2}g\left(e^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\}.$$

(The $\sqrt{2}$ can probably be dropped, as the Favard estimate probably also applies to complex-valued functions.) Applying this to

$$g\left(e^{i\theta}\right) := re^{i\theta}\frac{S'_{50,q}}{S_{50,q}}\left(a + re^{i\theta}\right),$$

we obtain

$$|I(\gamma) - I_m(\gamma)| \leq \frac{r\sqrt{2}\pi}{m^2}\sup\left\{\left|\frac{d^2}{d\theta^2}\left[e^{i\theta}\frac{S'_{50,q}}{S_{50,q}}\left(a + re^{i\theta}\right)\right]\right| : \theta \in [-\pi, \pi]\right\}.$$

An explicit calculation of this derivative in terms of the derivatives of $S_{50,q}$ and some elementary estimates then yield (7.18). $\qquad\square$

Now let us choose in Lemma 7.1,

(i) $c = m = 50$, so that τ of (7.2) becomes τ of Theorem 1.1.

(ii) $s = 0.9$ and $r = 0.46$.

A Mathematica 3.0 calculation shows that T of (7.3) is given by

$$T = 1.04093 \cdots \times 10^{-10}.$$

Thus

(7.19)
$$\max_{|z|\leq 0.46}|G_q - S_{50,q}|(z) \leq T < 1.04094 \times 10^{-10}.$$

Next, let

$$\begin{aligned}
z_1 &:= -0.299076 + 0.145052i; \\
z_2 &:= -0.269527 + 0.306036i
\end{aligned}$$

so that

$$\begin{aligned}
|z_1| &= 0.332395\cdots; \\
|z_2| &= 0.407802\cdots.
\end{aligned}$$

Also, let

$$\begin{aligned}
\gamma_0 &:= \{t : |t| = 0.46\}, \\
\gamma_j &:= \{t : |t - z_j| = 0.01\}, \qquad j = 1, 2.
\end{aligned}$$

Mathematica 3.0 calculated z_1 and z_2 as zeros of $S_{50,q}$, but we do not need in our proof to assume that these are (approximations to) zeros of $S_{50,q}$.

Now let us summarize what we need. Suppose that we can show

(A)

(7.20) $$\min_{\gamma_j} |S_{50,q}| > T, \qquad j = 0, 1, 2.$$

(B) For appropriate choices of $m_j, j = 0, 1, 2$,

(7.21) $$\left| I(\gamma_j) - I_{m_j}(\gamma_j) \right| < \frac{1}{2}, \qquad j = 0, 1, 2.$$

(C) The closest integer to $I_{m_j}(\gamma_j)$ is 2 for $j = 0$ and 1 for $j = 1, 2$.

Then (A) and (7.19) show that

$$\min_{\gamma_j} |S_{50,q}| > \max_{\gamma_j} |G_q - S_{50,q}|,$$

so that by Lemma 7.2, $S_{50,q}$ and G_q have the same number of zeros inside $\gamma_j, j = 0, 1, 2$. Next, (B) shows that the closest integer to $I_{m_j}(\gamma_j)$ is $I(\gamma_j), \quad j = 0, 1, 2$. Then (C) shows that

$$I(\gamma_0) = 2,$$

and

$$I(\gamma_1) = 1 = I(\gamma_2).$$

Then $S_{50,q}$ and hence G_q have zeros of total multiplicity 2 inside γ_0, and these must be the simple zeros inside $\gamma_j, j = 1, 2$. If we denote these zeros by $u_j, j = 1, 2$, then

$$|u_j - z_j| < 0.01, \qquad j = 1, 2,$$

so that

$$\begin{aligned}
0 &< |u_1| < |u_2|; \\
\arg(u_1) &\neq \arg(u_2).
\end{aligned}$$

The figure below contains the curves $\gamma_j, j = 0, 1, 2$. The asymmetry of the zeros z_1, z_2 and the circles containing the corresponding zeros u_1, u_2 of G_q is then clear. Thus the zeros of G_q cannot be symmetric about any line through 0, and in fact, this is true even for the two zeros u_1 and u_2 of smallest modulus. So we have satisfied our reformulation of (7.1) and H_q serves as a counterexample to the Baker-Gammel-Wills Conjecture.

Even more, it follows by the exact same argument from Step I above, that since the zeros u_1 and u_2 of smallest modulus cannot be symmetric about any line through 0, we cannot have a subsequence of $\{\mu_n/\nu_n\}_{n=1}^{\infty}$ converging uniformly in all compact subsets of $\{z : |z| < 0.46\}$. So we have completed the proof of Theorem 1.1.
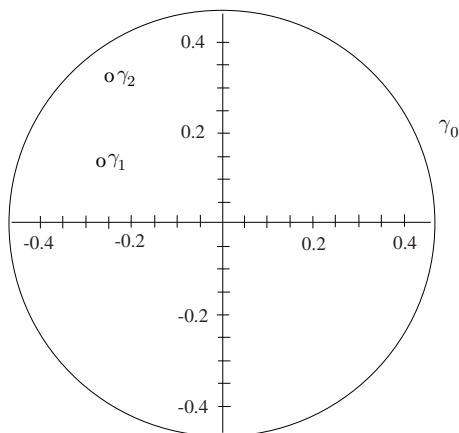
Smallest zeros of G_q

Now we turn to verifying (A), (B), (C). Initially calculations were performed using Mathematica 3.0, and later using Matlab 4. The calculations for the final version of this manuscript were performed using Matlab 6.

Proof of (A) *for $j = 0$.* A Matlab 6 calculation shows that

$$\max_{1 \le j \le 10^5} \left| S_{50,q} \left(0.46 e^{2\pi i j / 10^5} \right) \right| = 37.9643 \cdots;$$

$$\min_{1 \le j \le 2 \times 10^7} \left| S_{50,q} \left(0.46 e^{2\pi i j / \left(2 \times 10^7 \right)} \right) \right| = 0.01307 \cdots.$$

Then Lemma 7.3 with $m = 2 \times 10^7$ and $n = 50$ gives

$$\min_{|z|=0.46} |S_{50,q}(z)| \ge 0.01277 \cdots > T.$$

(Of course the maximum above exceeds that needed for Lemma 7.3).

Proof of (A) *for $j = 1$.* A Matlab 6 calculation shows that

$$\max_{1 \le j \le 10^5} \left| S_{50,q} \left(z_1 + 0.01 e^{2\pi i j / 10^5} \right) \right| = 0.04735 \cdots;$$

$$\min_{1 \le j \le 10^5} \left| S_{50,q} \left(z_1 + 0.01 e^{2\pi i j / 10^5} \right) \right| = 0.03581 \ldots.$$

Then Lemma 7.3 with $m = 10^5$ and $n = 50$ gives

$$\min_{|z - z_1| = 0.01} |S_{50,q}(z)| \ge 0.03580 \cdots > T.$$

Proof of (A) *for $j = 2$.* A Matlab 6 calculation shows that

$$\max_{1 \leq j \leq 10^5} \left| S_{50,q}\left(z_2 + 0.01 e^{2\pi i j/10^5}\right) \right| = 0.01516 \cdots;$$

$$\min_{1 \leq j \leq 10^5} \left| S_{50,q}\left(z_2 + 0.01 e^{2\pi i j/10^5}\right) \right| = 0.01205 \cdots.$$

Then Lemma 7.3 with $m = 10^5$ and $n = 50$ gives

$$\min_{|z - z_2| = 0.01} |S_{50,q}(z)| \geq 0.01203 \cdots > T.$$

Thus we have completed the proof of (A). Next, we turn to (B) and (C). We use Lemma 7.4, and so must estimate Φ from (7.18). To estimate $\max_{\gamma_j} \left| S_{50,q}^{(\ell)} \right|$, $\ell = 0, 1, 2, 3$, we used (7.15) of Lemma 7.3 with $n = 50$ and $m = 10^5$ in all cases, applied to $S_{50,q}^{(\ell)}$. The results appear in the table below:

j	$\min_{\gamma_j} |S_{50,q}|$	$\max_{\gamma_j} |S_{50,q}|$	$\max_{\gamma_j} \left|S'_{50,q}\right|$	$\max_{\gamma_j} \left|S''_{50,q}\right|$	$\max_{\gamma_j} \left|S'''_{50,q}\right|$
0	0.01277..	38.021..	311.06..	$2,562.2..$	$21,278.9..$
1	0.03580..	0.04742..	5.3911..	137.77..	2408.95..
2	0.01203..	0.01519..	1.7003..	39.4139..	941.25..

If we substitute these estimates in Lemma 7.4 with $m_j = 2 \times 10^7$ for $j = 0$, and $m_j = 10^5$ for $j = 1, 2$, we obtain the estimates in the third column of the following table.

j	m_j	$\left| I(\gamma_j) - I_{m_j}(\gamma_j) \right| \leq$	$I_m(\gamma_j)$
0	2×10^7	$3.17.. \times 10^{-2}$	$2 - \left(1.5 \times 10^{-13} + 10^{-14} i\right)$
1	10^5	$8.03.. \times 10^{-9}$	$1 + 10^{-14}$
2	10^5	$8.64.. \times 10^{-9}$	1

We see that the error in numerical integration is far less than $\frac{1}{2}$. Furthermore, to at least 12 decimals, $I_{m_j}(\gamma_j)$ is 2 for $j = 0$, and 1 for $j = 1, 2$. So we have completed (B), (C) and the proof of Theorem 1.1.   □

Remarks. (i) Obviously the accuracy of the calculation is fundamental in the above proof. We tested our calculation of $S_{50,q}$ in Mathematica 3.0 by calculating it in two different ways. The second was more careful, involving separating out real and imaginary, and then positive and negative parts of the summand. Adding positive parts separately and then negative parts separately to avoid roundoff, we obtained agreement with the simpler method of calculation to 12 decimals. Similar checking of other calculations indicated accuracy to 12 decimals, and independent calculations on Matlab gave the same results.

(ii) We again emphasize that the calculations above involved only evaluation of $S_{50,q}$ and its first three derivatives at translated roots of unity, together with their absolute values, and sums involving them. No "black-box" routines were involved. In the first version of this paper, the author used the "black-box" routines of Mathematica and Matlab to evaluate $I(\gamma_j)$, and did not provide an error estimate. The author must thank the referee for requesting an error estimate, which led to the inclusion of Lemma 7.4.

(iii) We used a rather crude method, namely the identity (6.6) to estimate the Maclaurin series coefficients of G_q. It is possible to obtain a far finer estimation of the coefficients of G_q. Indeed in [32], it was shown that for q such as in (7.2), we have

(7.22) $$|\log |(q;q)_n|| = O(\log n), \qquad n \to \infty,$$

instead of the geometric estimate (7.5). However the problem is that despite the slower growth in n, the size of the constant in the order term in (7.18) is so large as to render it useless except for large n.

8. Conclusions

While this paper answers one form of the Baker-Gammel-Wills Conjecture, by constructing an example in which all subsequences of $\{[n/n]\}_{n=1}^{\infty}$ have limit points of poles where the underlying function H_q does not, it suggests many more questions. Some of these are specific to H_q, but most have a more general flavour.

No matter which q we choose, Worpitzky's theorem ensures that even the full sequence $\{\mu_n/\nu_n\}_{n=1}^{\infty}$ converges uniformly in compact subsets of $\left\{z : |z| < \frac{1}{4}\right\}$. Can we always obtain uniform convergence of a subsequence near 0?

Problem 8.1. Let f be analytic in the unit ball.

(I) Does there exist a neighbourhood of 0 and a subsequence of $\{[n/n]\}_{n=1}^{\infty}$ converging uniformly to f some ball centre 0? If there is such a ball, can its radius be made independent of f?

(II) Does there exist for each point a in the unit ball, a neighbourhood of a, and a corresponding subsequence of $\{[n/n]\}_{n=1}^{\infty}$ converging uniformly to f in that neighbourhood?

We note that if there is such a ball centre 0 as in (I), with radius independent of f, then it would imply that for entire functions f, there is a subsequence converging uniformly in all compact subsets of the plane. This would follow by consideration of $f(z/\varepsilon)$ with ε small enough, rather than $f(z)$,

and use of invariance properties for Padé approximants. This idea was used by Buslaev, Gončar and Suetin in [7], in their resolution of the Baker-Graves-Morris conjecture for columns of the Padé table.

Another form of the Problem 8.1 involves a weaker form of convergence, namely convergence in capacity. For the definition and properties of logarithmic capacity (cap), see for example [25], [41]. It is known that for functions analytic in the unit ball, the full diagonal sequence need not converge in capacity, even in any neighbourhood of 0 [28], [29], [40]. Can a subsequence converge in capacity?

Problem 8.2. *Let f be meromorphic in the unit ball and analytic at 0.*

(I) *Does there exist a subsequence $\{[n/n]\}_{n \in \mathcal{S}}$ that converges in capacity to f? More precisely, given $0 < r < 1$ and $\varepsilon > 0$, we want*

$$\mathrm{cap}\{z : |z| \leq r \ and \ |f - [n/n]|\,(z) > \varepsilon\} \to 0, n \to \infty, n \in \mathcal{S}.$$

If this is not possible for each r, is it possible for some $r > 0$?

(II) *Does there exist a subsequence $\{[n/n]\}_{n \in \mathcal{S}}$ that converges at a given point to f? More precisely, we want*

$$\liminf_{n \to \infty} |[n/n]\,(z) - f\,(z)| = 0,$$

in the unit ball, except at poles of f.

We note that problem (I) was raised by H. Stahl in [46]. The author recalls that it was also raised by the Russian school of A. A. Gončar, but he cannot trace the reference.

Of course, problems about subsequences are difficult to resolve, for if there is failure, one has to prove this failure for every subsequence. This is even more difficult when a weak convergence concept, such as convergence in capacity, is involved.

Is there anything positive that can be said about full diagonal sequences $\{[n/n]\}_{n=1}^{\infty}$ for functions analytic in the unit ball? Can we find a property weaker than convergence in capacity, satisfied by the full diagonal sequence? A start in this direction was made in [33], where it is shown that for large n, $[n/n]$ provides good uniform approximation on at least $1/8$ of the circles centre 0 within the unit ball.

We now turn to questions concerning the Rogers-Ramanujan function

$$G_q\,(z) = \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q;q)_j} z^j.$$

Recall that the q-exponential functions are

$$e_q\,(z) \quad = \quad \sum_{j=0}^{\infty} \frac{1}{(q;q)_j} z^j;$$

$$E_q(z) \quad = \quad \sum_{j=0}^{\infty} \frac{q^{j(j-1)/2}}{(q;q)_j} z^j.$$

Now in $\{z : |z| < R(q)\}$, a direct calculation shows that

$$e_q(z) E_q(-z) = 1$$

and hence e_q and E_q have no zeros in that ball. In contrast, Theorem 2.3 shows that as q approaches a primitive ℓ^{th} root of unity, the total multiplicity of zeros of G_q in a neighbourhood of $(-1/4)^{-1/\ell}$ approaches ∞.

This suggests studying the structure of zeros of G_q:

Problem 8.3. (i) Investigate the structure of zeros of G_q when $|q| = 1$ and $R(q) > 0$.

(ii) Moreover, investigate whether - as seems likely - the asymmetry property (7.1) holds for every such q.

(iii) Obtain a proof of the asymmetry property (7.1) without the use of a numerical package.

One obvious question is whether our counterexample H_q can be used to provide a counterexample to the Baker-Gammel-Wills Conjecture restricted to functions f analytic in the unit ball. We can show that for q of (1.8), (1.9), $G_q(qz)$ has three simple zeros of smallest modulus, say,

$$0 < |z_1| < |z_2| < |z_3| < 1$$

and these are the poles of H_q of smallest modulus. Let

$$\begin{aligned} g(z) \quad &:= \quad H_q(z)(z - z_1)(z - z_2); \\ f(z) \quad &:= \quad g(zz_3). \end{aligned}$$

Then f is analytic in the unit ball.

Problem 8.4. Show that no subsequence of $\{[n/n]\}_{n=1}^{\infty}$ to f can converge uniformly in all compact subsets of the unit ball.

scholarship would be used to find a counterexample. That promise has been kept, some twenty years after their generous funding ended. Along the way research support has been provided by a Lady Davis Fellowship at the Technion in Haifa, and by the following institutions: the National Research Institute for Mathematical Sciences in Pretoria; the Centre for Constructive Mathematics of the University of South Florida; Witwatersrand University in Johannesburg; the National Research Foundation (formerly FRD) in Pretoria, and most recently, Georgia Institute of Technology.

Finally the author thanks the referee for requesting error estimates that led to Lemma 7.4, and for some historical insights.

Note added in proof. V. I. Buslaev has obtained an example of a function analytic in the unit ball for which the Baker-Gammel-Wills conjecture fails. His function is algebraic and also negates a conjecture of H. Stahl. See

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