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Entropy and the localization of eigenfunctions

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Abstract

We study the large eigenvalue limit for the eigenfunctions of the Laplacian, on a compact manifold of negative curvature – in fact, we only assume that the geodesic flow has the Anosov property. In the semi-classical limit, we prove that the Wigner measures associated to eigenfunctions have positive metric entropy. In particular, they cannot concentrate entirely on closed geodesics.

1. Introduction, statement of results

We consider a compact Riemannian manifold M of dimension $d \geq 2$, and assume that the geodesic flow $(g^t)_{t\in\mathbb{R}}$, acting on the unit tangent bundle of M, has a "chaotic" behaviour. This refers to the asymptotic properties of the flow when time t tends to infinity: ergodicity, mixing, hyperbolicity…: we assume here that the geodesic flow has the Anosov property, the main example being the case of negatively curved manifolds. The words "quantum chaos" express the intuitive idea that the chaotic features of the geodesic flow should imply certain special features for the corresponding quantum dynamical system: that is, according to Schrödinger, the unitary flow $\left(\exp(i\hbar t\frac{\Delta}{2})\right)_{t\in\mathbb{R}}$ acting on the Hilbert space $L^2(M)$, where Δ stands for the Laplacian on M and $\hbar$ is proportional to the Planck constant. Recall that the quantum flow converges, in a sense, to the classical flow (g^t) in the so-called semi-classical limit $\hbar \longrightarrow 0$; one can imagine that *for small values of* $\hbar$ the quantum system will inherit certain qualitative properties of the classical flow. One expects, for instance, a very different behaviour of eigenfunctions of the Laplacian, or the distribution of its eigenvalues, if the geodesic flow is Anosov or, in the other extreme, completely integrable (see [Sa95]).

The convergence of the quantum flow to the classical flow is stated in the Egorov theorem. Consider one of the usual quantization procedures $\mathrm{Op}_\hbar$, which associates an operator $\mathrm{Op}_\hbar(a)$ acting on $L^2(M)$ to every smooth compactly supported function $a \in C_c^\infty(T^*M)$ on the cotangent bundle T^*M. According to the Egorov theorem, we have for any fixed t

$$\left\|\exp\left(-it\frac{\hbar\Delta}{2}\right)\cdot \mathrm{Op}_\hbar(a)\cdot \exp\left(it\frac{\hbar\Delta}{2}\right) - \mathrm{Op}_\hbar(a\circ g^t)\right\|_{L^2(M)} \underset{\hbar\to 0}{=} O(\hbar)\,.$$

We study the behaviour of the eigenfunctions of the Laplacian,

$$-h^2\Delta\psi_h = \psi_h$$

in the limit $h \longrightarrow 0$ (we simply use the notation h instead of $\hbar$, and now $-\frac{1}{h^2}$ ranges over the spectrum of the Laplacian). We consider an orthonormal basis of eigenfunctions in $L^2(M) = L^2(M, d\mathrm{Vol})$ where Vol is the Riemannian volume. Each wave function ψ_h defines a probability measure on M:

$$|\psi_h(x)|^2 d\mathrm{Vol}(x),$$

that can be lifted to the cotangent bundle by considering the "microlocal lift",

$$\nu_h \colon a \in C_c^\infty(T^*M) \mapsto \langle \mathrm{Op}_h(a)\psi_h, \psi_h\rangle_{L^2(M)},$$

also called Wigner measure or Husimi measure (depending on the choice of the quantization $\mathrm{Op}_\hbar$) associated to the eigenfunction ψ_h. If the quantization procedure was chosen to be positive (see [Ze86, §3], or [Co85, 1.1]), then the distributions ν_hs are in fact probability measures on T^*M: it is possible to extract converging subsequences of the family $(\nu_h)_{h\to 0}$. Reflecting the fact that we considered eigenfunctions of energy 1 of the semi-classical Hamiltonian $-h^2\Delta$, any limit ν_0 is a probability measure carried by the unit cotangent bundle $S^*M \subset T^*M$. In addition, the Egorov theorem implies that ν_0 is invariant under the (classical) geodesic flow. We will call such a measure ν_0 a *semi-classical invariant measure*. The question of identifying all limits ν_0 arises naturally: the Snirelman theorem ([Sn74], [Ze87], [Co85], [HMR87]) shows that the Liouville measure is one of them, in fact it is a limit along a subsequence *of density one* of the family (ν_h), as soon as the geodesic flow acts ergodically on S^*M with respect to the Liouville measure. It is a widely open question to ask if there can be exceptional subsequences converging to other invariant measures, like, for instance, measures carried by closed geodesics. The *Quantum Unique Ergodicity* conjecture [RS94] predicts that the whole sequence should actually converges to the Liouville measure, if M has negative sectional curvature.

The problem was solved a few years ago by Lindenstrauss ([Li03]) in the case of an arithmetic surface of constant negative curvature, when the functions ψ_h are common eigenstates for the Laplacian and the Hecke operators; but little is known for other Riemann surfaces or for higher dimensions. In the setting of discrete time dynamical systems, and in the very particular case of linear Anosov diffeomorphisms of the torus, Faure, Nonnenmacher and De Bièvre found counterexamples to the conjecture: they constructed semi-classical invariant measures formed by a convex combination of the Lebesgue measure on the torus and of the measure carried by a closed orbit ([FNDB03]). However, it was shown in [BDB03] and [FN04], for the same toy model, that semi-classical invariant measures cannot be *entirely* carried on a closed orbit.

1.1. *Main results.* We work in the general context of Anosov geodesic flows, for (compact) manifolds of arbitrary dimension, and we will focus our attention on the entropy of semi-classical invariant measures. The Kolmogorov-Sinai entropy, also called metric entropy, of a (g^t)-invariant probability measure ν_0 is a nonnegative number $h_g(\nu_0)$ that measures, in some sense, the complexity of a ν_0-generic orbit of the flow. For instance, a measure carried on a closed geodesic has zero entropy. An upper bound on entropy is given by the Ruelle inequality: since the geodesic flow has the Anosov property, the unit tangent bundle S^1M is foliated into unstable manifolds of the flow, and for any invariant probability measure ν_0 one has

$$h_g(\nu_0) \leq \left| \int_{S^1M} \log J^u(v) d\nu_0(v) \right|, \tag{1.1.1}$$

where $J^u(v)$ is the unstable jacobian of the flow at v, defined as the jacobian of g^{-1} restricted to the unstable manifold of g^1v. In (1.1.1), equality holds if and only if ν_0 is the Liouville measure on S^1M ([LY85]). Thus, proving Quantum Unique Ergodicity is equivalent to proving that $h_g(\nu_0) = |\int_{S^1M} \log J_u d\nu_0|$ for any semi-classical invariant measure ν_0. But already a lower bound on the entropy of ν_0 would be useful. Remember that one of the ingredients of Elon Lindenstrauss' work [Li03] in the arithmetic situation was an estimate on the entropy of semi-classical measures, proven previously by Bourgain and Lindenstrauss [BLi03]. If the (ψ_h) form a common eigenbasis of the Laplacian and all the Hecke operators, they proved that *all the ergodic components* of ν_0 have positive entropy (which implies, in particular, that ν_0 cannot put any weight on a closed geodesic). In the general case, our Theorems 1.1.1, 1.1.2 do not reach so far. They say that many of the ergodic components have positive entropy, but components of zero entropy, like closed geodesics, are still allowed – as in the counterexample built in [FNDB03] for linear hyperbolic toral automorphisms (called "cat maps" thereafter). For the cat map, [BDB03] and [FN04] could prove directly – without using the notion of entropy – that a semi-classical measure cannot be entirely carried on closed orbits ([FN04] proves that if ν_0 has a pure point component then it must also have a Lebesgue component).

Denote

$$\Lambda = -\sup_{v \in S^1M} \log J^u(v) > 0.$$

For instance, for a d-dimensional manifold of constant sectional curvature -1, we find $\Lambda = d - 1$.

Theorem 1.1.1. *There exist a number $\bar{\kappa} > 0$ and two continuous decreasing functions $\tau\colon [0,1] \longrightarrow [0,1]$, $\vartheta\colon (0,1] \longrightarrow \mathbb{R}_+$ with $\tau(0) = 1$, $\vartheta(0) = +\infty$, such that: If ν_0 is a semi-classical invariant measure, and*

$$\nu_0 = \int_{S^1M} \nu_0^x d\nu_0(x)$$

is its decomposition in ergodic components, then, for all $\delta > 0$,

$$\nu_0\left(\{x, h_g(\nu_0^x) \geq \frac{\Lambda}{2}(1-\delta)\}\right) \geq \left(\frac{\bar{\kappa}}{\vartheta(\delta)}\right)^2 (1-\tau(\delta)).$$

This implies that $h_g(\nu_0) > 0$, *and gives a lower bound for the topological entropy of the support,* $h_{\rm top}(\text{supp } \nu_0) \geq \frac{\Lambda}{2}$.

What we prove is in fact a more general result about quasi-modes of order $h|\log h|^{-1}$:

THEOREM 1.1.2. *There are a number* $\bar{\kappa} > 0$ *and two continuous decreasing functions* $\tau \colon [0,1] \longrightarrow [0,1]$, $\vartheta \colon (0,1] \longrightarrow \mathbb{R}_+$ *with* $\tau(0) = 1$, $\vartheta(0) = +\infty$, *such that: If* (ψ_h) *is a sequence of normalized* L^2 *functions with*

$$\|(-h^2\Delta - 1)\psi_h\|_{L^2(M)} \leq ch|\log h|^{-1},$$

then for any semi-classical invariant measure ν_0 *associated to* (ψ_h), *for any* $\delta > 0$,

$$\nu_0\left(\{x, h_g(\nu_0^x) \geq \frac{\Lambda}{2}(1-\delta)\}\right) \geq (1-\tau(\delta))\left(\frac{\bar{\kappa}}{\vartheta(\delta)} - c\vartheta(\delta)\right)_+^2 - c\bar{\kappa}.$$

If c *is small enough, this implies that* ν_0 *has positive entropy.*

Remark 1.1.3. The proof gives an explicit expression of ϑ and τ as continuous decreasing functions of δ; they also depend on the instability exponents of the geodesic flow. I believe, however, that this is far from giving an optimal bound. In the case of a compact manifold of constant sectional curvature -1, an attempt to keep all constants optimal in the proof would probably lead to $\bar{\kappa} = 1$, τ is any number greater than $1 - \frac{\delta}{2}$, and $\vartheta = \big(2(\tau - (1-\delta/2))\big)^{-1}$ – which still does not seem optimal.

The main tool to prove Theorems 1.1.1 and 1.1.2 is an estimate given in Theorem 1.3.3, which will be stated after we have recalled the definition of entropy in subsection 1.2. The method only uses the Anosov property of the flow, and should work for very general Anosov symplectic dynamical systems. In [AN05], this is implemented (with considerable simplification) for the toy model of the (Walsh-quantized) "baker's map", for which Quantum Unique Ergodicity fails obviously. For that toy model we can also prove the following improvement of Theorem 1.1.1:

CONJECTURE 1.1.4. *For any semi-classical measure* ν_0,

$$h_g(\nu_0) \geq \frac{1}{2}\left|\int_{S^1M} \log J^u(v)\, d\nu_0(v)\right|.$$

We believe this holds for any Anosov symplectic system. Conjecture 1.1.4, if true, is optimal in the sense that the lower bound is reached for certain counterexamples to Quantum Unique Ergodicity (QUE) encountered for the baker's map or the cat map. In the same paper [AN05], we also show that Theorem 1.1.1 is optimal for the baker's map, in the sense that we can construct an ergodic semi-classical measure, with entropy $\Lambda/2$, whose support has topological entropy $\Lambda/2$. Thus, Theorem 1.1.1 should not be interpreted as a step in the direction of QUE, but rather as a general fact which holds even when QUE is known to fail.

It seems that an improvement of Theorem 1.1.1 would have to rely on a control of the multiplicities in the spectrum, which are expected to be much lower for eigenfunctions of the Laplacian than in the case of the cat map or the baker's map (where they are of order $(h|\log h|)^{-1}$ for certain eigenvalues). For a negatively curved d-dimensional manifold, the number of eigenvalues in the spectral interval $(h^{-2} - c(h|\log h|)^{-1}, h^{-2} + c(h|\log h|)^{-1})$ is bounded by $(2c + K)h^{d-1}|\log h|^{-1}$, where $2ch^{d-1}|\log h|^{-1}$ comes from the leading term in Weyl's law and $Kh^{d-1}|\log h|^{-1}$ is the remainder term obtained in [Be77]. The possible behaviour of quasi-modes of order $ch|\log h|^{-1}$ depends in a subtle way on the value of c, which controls the multiplicity and thus our degree of freedom in forming linear combinations of eigenfunctions. The theorem only proves the positive entropy of ν_0 when c is small enough. On the other hand, when c is not too close to 0, it should be possible to construct quasi-modes of order $ch|\log h|^{-1}$ for which ν_0 has positive entropy but nevertheless puts positive mass on a closed geodesic. For the cat map, we note that the counterexamples constructed in [FNDB03] concern eigenvalues of multiplicity $Ch|\log h|^{-1}$ for a very precise value of C (related to the Lyapunov exponent), and that the construction would not work for smaller values of C. For (genuine) eigenfunctions of the Laplacian, such counterexamples should not be expected if the multiplicity is really much lower than the general bound $h^{d-1}|\log h|^{-1}$ – however, just to improve the multiplicative constant in this bound requires a lot of work (see [Sa-hp] in arithmetic situations).

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In the next paragraph we recall the definition of metric entropy in the classical setting. Then, in paragraph 1.3, we try to adapt the construction on a semi-classical level; we construct "quantum cylinder sets" and try to evaluate their measures. Theorem 1.3.3 proves their exponential decay beyond the Ehrenfest time, and gives the key to Theorems 1.1.1, 1.1.2.

1.2. *Definition of entropy.* Let $S^1M = P_1 \sqcup \cdots \sqcup P_l$ be a finite measurable partition of the unit tangent bundle S^1M. The entropy of ν_0 with respect to the action of geodesic flow and to the partition P is defined by

$$\begin{aligned} h_g(\nu_0, P) &= \lim_{n \longrightarrow +\infty} -\frac{1}{n} \sum_{(\alpha_j)\in\{1,\ldots,l\}^{n+1}} \nu_0(P_{\alpha_0} \cap g^{-1}P_{\alpha_1} \cdots \cap g^{-n}P_{\alpha_n}) \\ &\qquad \times \log \nu_0(P_{\alpha_0} \cap g^{-1}P_{\alpha_1} \cdots \cap g^{-n}P_{\alpha_n}) \\ &= \inf_{n\in\mathbb{N}} -\frac{1}{n} \sum_{(\alpha_j)\in\{1,\ldots,l\}^{n+1}} \nu_0(P_{\alpha_0} \cap g^{-1}P_{\alpha_1} \cdots \cap g^{-n}P_{\alpha_n}) \\ &\qquad \times \log \nu_0(P_{\alpha_0} \cap g^{-1}P_{\alpha_1} \cdots \cap g^{-n}P_{\alpha_n}). \end{aligned}$$

The existence of the limit, and the fact that it coincides with the inf follow from a subadditivity argument. The entropy of ν_0 with respect to the action of the geodesic flow is defined as

$$h_g(\nu_0) = \sup_P h_g(\nu_0, P),$$

the supremum running over all finite measurable partitions P. For Anosov systems, this supremum is actually reached for a well-chosen partition P (in fact, as soon as the diameter of the P_is is small enough). In the proof of Theorem 1.1.2, we will use the Shannon-MacMillan theorem which gives the following interpretation of entropy: if ν_0 is ergodic, then for ν_0-almost all x, we have

$$\frac{1}{n} \log \nu_0\big(P^{\vee n}(x)\big) \underset{n\longrightarrow+\infty}{\longrightarrow} -h_g(\nu_0, P)$$

where $P^{\vee n}(x)$ denotes the unique set of the form $P_{\alpha_0} \cap g^{-1}P_{\alpha_1} \cdots \cap g^{-n}P_{\alpha_n}$ containing x. It follows that, for any $\varepsilon > 0$, we can find a set of ν_0-measure greater than $1-\varepsilon$ that can be covered by at most $e^{n(h_g(\nu_0,P)+\varepsilon)}$ sets of the form $P_{\alpha_0} \cap g^{-1}P_{\alpha_1} \cdots \cap g^{-n}P_{\alpha_n}$ (for all n large enough).

The entropy is nonnegative, and bounded *a priori* from above; on a compact d-dimensional riemannian manifold of constant sectional curvature -1, the entropy of any measure is smaller than $d-1$; more generally, for an Anosov geodesic flow, one has an *a priori* bound in terms of the unstable jacobian, called the Ruelle inequality (see [KH]): $h_g(\nu_0) \leq |\int_{S^1M} \log J^u d\nu_0|$, with equality if and only if ν_0 is the Liouville measure on S^1M ([LY85]).

For our purposes, we reformulate slightly the definition of entropy. The following definition, although equivalent to the usual one, looks a bit different, in that we only use partitions of the base M: the reason for doing so is that we prefer to work with multiplication operators in paragraph 1.3, instead of having to deal with more general pseudo-differential operators.

Let $P = (P_1, \dots P_l)$ be a finite measurable partition of M (instead of S^1M); we denote $\varepsilon/2$, $(\varepsilon > 0)$ an upper bound on the diameter of the P_is. We consider P as a partition of the tangent bundle, by lifting it to TM.

Let $\Sigma = \{1, \dots l\}^{\mathbb{Z}}$. To each tangent vector $v \in S^1M$ one can associate a unique element $I(v) = (\alpha_j)_{j\in\mathbb{Z}} \in \Sigma$, such that $g^j v \in P_{\alpha_j}$ for all integers j. Thus, we define a coding map $I\colon S^1M \longrightarrow \Sigma$. If we define the shift σ acting on Σ by

$$\sigma\big((\alpha_j)_{j\in\mathbb{Z}}\big) = (\alpha_{j+1})_{j\in\mathbb{Z}},$$

then $I \circ g^1 = \sigma \circ I$.

We introduce the probability measure μ_0 on Σ, the image of ν_0 under the coding map I. More explicitly, the finite-dimensional marginals of μ_0 are given by

$$\mu_0\big([\alpha_0, \dots, \alpha_{n-1}]\big) = \nu_0(P_{\alpha_0} \cap g^{-1}P_{\alpha_1} \cdots \cap g^{-n+1}P_{\alpha_{n-1}}),$$

where we have denoted $[\alpha_0, \dots, \alpha_{n-1}]$ the subset of Σ, formed of sequences in Σ beginning with the letters $(\alpha_0, \dots, \alpha_{n-1})$. Such a set is called a *cylinder set* (of length n). We will denote Σ_n the set of cylinder sets of length n: they form a partition of Σ.

Since ν_0 is carried by the unit tangent bundle, and is (g^t)-invariant, its image μ_0 is σ-invariant. The entropy of μ_0 with respect to the action of the shift σ is

$$h_\sigma(\mu_0) = \lim_{n\longrightarrow+\infty} -\frac{1}{n}\sum_{\mathcal{C}\in\Sigma_n} \mu_0(\mathcal{C})\log\mu_0(\mathcal{C}) \tag{1.2.1}$$

$$= \inf_n -\frac{1}{n}\sum_{\mathcal{C}\in\Sigma_n} \mu_0(\mathcal{C})\log\mu_0(\mathcal{C}) = h_g(\nu_0, P). \tag{1.2.2}$$

The fact that the limit exists and coincides with the inf comes from the remark that the sequence $(-\sum_{\mathcal{C}\in\Sigma_n}\mu_0(\mathcal{C})\log\mu_0(\mathcal{C}))_{n\in\mathbb{N}}$ is subadditive, which follows from the concavity of the log and the σ-invariance of μ_0 (see [KH]). We have decided to work with time 1 of the geodesic flow; it is harmless to consider partitions P depending only on the base, if the injectivity radius is greater than one – which we can always assume. If the diameter of the P_is is small enough, the partition P and its iterates under the flow generate the Borel σ-field, which implies that $h_g(\nu_0) = h_\sigma(\mu_0)$.

Note that the entropy (1.2.2) is an upper semi-continuous functional. In other words, when a sequence of (g^t)-invariant probability measures converges in the weak topology, lower bounds on entropy pass to the limit. The difficulty here is that we are in an unusual situation where we have a sequence of non-commutative dynamical systems converging to a commutative one: standard methods of dealing with entropy need to be adapted to this context.

1.3. *The semi-classical setting*; *exponential decay of the measures of cylinder sets.*

1.3.1. *The measure μ_h.* Since we will resort to microlocal analysis we have to replace characteristic functions $\mathbf{1}_{P_i}$ by smooth functions. We will assume that the P_i have smooth boundary, and will consider a smooth partition of unity obtained by smoothing the characteristic functions $\mathbf{1}_{P_i}$, that is, a finite family of C^∞ functions $A_i \geq 0$ ($i = 1, \ldots, l$), such that

$$\sum_{i=1}^{l} A_i = 1.$$

We can consider the A_is as functions on TM, depending only on the base point. For each i, denote Ω_i a set of diameter ε that contains the support of A_i in its interior.

In fact, the way we smooth the $\mathbf{1}_{P_i}$s to obtain A_i is rather crucial, and will be discussed in subsection 2.1. Let us only say, for the moment, that the A_i will depend on h in a way that

$$A_i^h \underset{h \longrightarrow 0}{\longrightarrow} 1 \tag{1.3.1}$$

uniformly in every compact subset in the interior of P_i, and

$$A_i^h \underset{h \longrightarrow 0}{\longrightarrow} 0 \tag{1.3.2}$$

uniformly in every compact subset outside P_i. We also assume that the smoothing is done at a scale h^κ ($\kappa \in [0, 1/2)$), so that the derivatives of A_i^h are controlled as

$$\|D^n A_i^h\| \leq C(n) h^{-n\kappa}.$$

This ensures that certain results of pseudo-differential calculus are still applicable to the functions A_i^h (see Appendix A1).

We now construct a functional μ_h defined on a certain class of functions on Σ. We see the functions A_i as multiplication operators on $L^2(M)$ and denote $A_i(t)$ their evolutions under the quantum flow:

$$A_i(t) = \exp\Big(-it\frac{h\Delta}{2}\Big) \circ A_i \circ \exp\Big(it\frac{h\Delta}{2}\Big).$$

We define the "measures" of cylinder sets under μ_h, by the expressions:

$$\mu_h\big([\alpha_0, \ldots, \alpha_n]\big) = \langle\, A_{\alpha_n}(n) \ldots . A_{\alpha_1}(1) A_{\alpha_0}(0)\, \psi_h, \psi_h \,\rangle_{L^2(M)} \tag{1.3.3}$$

$$= \langle\, e^{-in\frac{\hbar\Delta}{2}} A_{\alpha_n} e^{i\frac{\hbar\Delta}{2}} A_{\alpha_{n-1}} e^{i\frac{\hbar\Delta}{2}} \cdots e^{i\frac{\hbar\Delta}{2}} A_{\alpha_0}\, \psi_h, \psi_h \,\rangle_{L^2(M)}. \tag{1.3.4}$$

For $\mathcal{C} = [\alpha_0, \ldots, \alpha_{n-1}] \in \Sigma_n$, we will use the shorthand notation $\hat{\mathcal{C}}_h$ for the operator

$$\begin{aligned}\hat{\mathcal{C}}_h &= A_{\alpha_{n-1}}(n-1) \ldots . A_{\alpha_1}(1) A_{\alpha_0}(0)\\ &= e^{-i(n-1)\frac{\hbar\Delta}{2}} A_{\alpha_{n-1}} e^{i\frac{\hbar\Delta}{2}} A_{\alpha_{n-1}} e^{i\frac{\hbar\Delta}{2}} \cdots e^{i\frac{\hbar\Delta}{2}} A_{\alpha_0}.\end{aligned}$$

The functional μ_h is only defined on the vector space spanned by characteristic functions of cylinder sets. Note that μ_h is not a positive measure, because the operators $\hat{\mathcal{C}}_h$ are not positive. The first part of the following proposition is a compatibility condition; the second part says that μ_h is σ-invariant if ψ_h is an eigenfunction. The third condition holds if ψ_h is normalized in $L^2(M)$.

PROPOSITION 1.3.1. (i) *For every n, for every cylinder $[\alpha_0, \ldots, \alpha_{n-1}] \in \Sigma_n$,*

$$\sum_{\alpha_n} \mu_h\big([\alpha_0, \ldots, \alpha_n]\big) = \mu_h\big([\alpha_0, \ldots, \alpha_{n-1}]\big).$$

(ii) *If $\|(-h^2\Delta - 1)\psi_h\|_{L^2(M)} \leq ch|\log h|^{-1}$, then for every n, for every cylinder $\mathcal{C} = [\alpha_0, \ldots, \alpha_{n-1}] \in \Sigma_n$, and for any integer k,*

$$\begin{aligned}
&\Big|\mu_h(\sigma^{-k}\mathcal{C}) - \mu_h(\mathcal{C})\Big| \\
&\quad = \Bigg| \sum_{\alpha_{-1}, \cdots, \alpha_{-k}} \mu_h\big([\alpha_{-k}, \ldots \alpha_{-1}, \alpha_0, .., \alpha_{n-1}]\big) - \mu_h\big([\alpha_0, .., \alpha_{n-1}]\big)\Bigg| \\
&\quad \leq \frac{kc}{2|\log h|}\big(\|\hat{\mathcal{C}}_h\psi_h\| + \|\hat{\mathcal{C}}_h^* e^{\frac{ikh\Delta}{2}}\psi_h\|\big).
\end{aligned}$$

(iii) *For every $n \geq 0$,*

$$\sum_{[\alpha_0, \ldots, \alpha_{n-1}]} \mu_h\big([\alpha_0, \ldots, \alpha_{n-1}]\big) = 1.$$

We assume in the rest of the paper that we have extracted from the sequence $(\nu_h)_{-1/h^2 \in Sp(\Delta)}$ a sequence $(\nu_{h_k})_{k\in\mathbb{N}}$ that converges to ν_0 in the weak topology: $\langle \mathrm{Op}_{h_k}(a)\psi_{h_k}, \psi_{h_k}\rangle_{L^2(M)} \underset{k\longrightarrow+\infty}{\longrightarrow} \int_{S^1M} a d\nu_0$, for every $a \in C_c^\infty(TM)$. To simplify notations, we forget about the extraction, and simply consider that $\nu_h \underset{h\longrightarrow 0}{\longrightarrow} \nu_0$.

If the partition of unity (A_i) does not depend on h, the usual Egorov theorem shows that μ_h converges, as $h \longrightarrow 0$, to a σ-invariant probability measure defined by $\mu_0^{(A)}$ on Σ, defined by

$$\mu_0^{(A)}\big([\alpha_0, \ldots, \alpha_n]\big) = \nu_0\left(A_{\alpha_0}.A_{\alpha_1} \circ g^1 \ldots A_{\alpha_n} \circ g^n\right).$$

Convergence here means that the measure of each cylinder set converges. Now, suppose the partition of unity depends on h so as to satisfy (1.3.1), (1.3.2); we may, and will, also assume that ν_0 does not charge the boundary of P.

PROPOSITION 1.3.2. *The family (μ_h) converges to μ_0 as $h \longrightarrow 0$.*

Proof. Let $\mathcal{C} = [\alpha_0, \dots, \alpha_n]$ be a given cylinder set. By the Egorov theorem 4.2.3,

$$\|\hat{\mathcal{C}}_h - \mathrm{Op}_h\Big(A_{\alpha_0}\, A_{\alpha_1} \circ g^1 \dots A_{\alpha_{n-1}} \circ g^{n-1}\Big)\|_{L^2(M)} = O(h^{1-2\kappa}). \tag{1.3.5}$$

The function $A_{\alpha_0}\, A_{\alpha_1} \circ g \dots A_{\alpha_{n-1}} \circ g^{n-1}$ is nonnegative, and, as $h \longrightarrow 0$, it converges uniformly to 1 on every compact subset in the interior of $P_{\alpha_0} \cap g^{-1} P_{\alpha_1} \dots \cap g^{-n+1} P_{\alpha_{n-1}}$, since A_i converges uniformly to 1 on every compact subset in the interior of P_i (1.3.1). If we choose a positive quantization procedure Op_h, it follows from (1.3.5) that

$$\begin{aligned}
\liminf_{h\longrightarrow 0} \mu_h(\mathcal{C}) &= \liminf_{h\longrightarrow 0} \langle \mathrm{Op}_h\big(A_{\alpha_0}\, A_{\alpha_1} \circ g \dots A_{\alpha_{n-1}} \circ g^{n-1}\big)\psi_h, \psi_h\rangle \\
&\geq \liminf \nu_h\big(\mathrm{int}(P_{\alpha_0} \cap g^{-1} P_{\alpha_1} \dots \cap g^{-n+1} P_{\alpha_{n-1}})\big) \\
&\geq \nu_0\big(\mathrm{int}(P_{\alpha_0} \cap g^{-1} P_{\alpha_1} \dots \cap g^{-n+1} P_{\alpha_{n-1}})\big).
\end{aligned}$$

We have assumed that ν_0 does not charge the boundary of the P_is, and thus the last term coincides with $\nu_0\big(P_{\alpha_0} \cap g^{-1} P_{\alpha_1} \dots \cap g^{-n+1} P_{\alpha_{n-1}}\big)$. Similarly, using (1.3.2) one can prove that

$$\limsup_{h\longrightarrow 0} \mu_h(\mathcal{C}) \leq \nu_0\big(\overline{P_{\alpha_0} \cap g^{-1} P_{\alpha_1} \dots \cap g^{-n+1} P_{\alpha_{n-1}}}\big).$$

This ends the proof since we assumed ν_0 does not charge the boundary of the partition P. $\square$

The key technical result of this paper, proved in Section 3, is an upper bound on μ_h, valid for cylinder sets of large lengths.

1.3.2. *Decay of the measures of cylinder sets.* Because the geodesic flow is Anosov, each energy layer $S^\lambda M = \{v \in TM, \|v\| = \lambda\}$ ($\lambda > 0$) is foliated into strong unstable manifolds of the geodesic flow. The unstable jacobian $J^u(v)$ at $v \in TM$ is defined as the jacobian of g^{-1}, restricted to the unstable leaf at the point $g^1 v$. Given (α_0, α_1), we introduce the notation

$$\begin{aligned}
&J^u_n(\alpha_0, \alpha_1) \\
&\quad := \sup\left(\{J^u(v_0), v_0 \in P_{\alpha_0}, \|v_0\| \in [1-\varepsilon, 1+\varepsilon], g^1(v_0) \in P_{\alpha_1}\} \cup \{e^{-3\Lambda}\}\right).
\end{aligned}$$

Given a sequence $(\alpha_0, \dots, \alpha_n)$, we denote

$$J^u_n(\alpha_0, \dots, \alpha_n) = J^u_n(\alpha_0, \alpha_1) J^u_n(\alpha_1, \alpha_2) \cdots J^u_n(\alpha_{n-1}, \alpha_n).$$

THEOREM 1.3.3 (The main estimate). *Let $\chi \in C^\infty_c(T^*M)$ be compactly supported in a neighbourhood of the unit tangent bundle, $\{v \in T^*M, \|v\| \in [1-\frac{\varepsilon}{2}, 1+\frac{\varepsilon}{2}]\}$. Consider the operators $A_{\alpha_n}(n) A_{\alpha_{n-1}}(n-1) \dots A_{\alpha_0} \mathrm{Op}(\chi)$. For every $\mathcal{K} > 0$, there exists $h_{\mathcal{K}} > 0$ such that, uniformly for all $h < h_{\mathcal{K}}$, for all*

$n \leq \mathcal{K}|\log h|$,

$$\left\|A_{\alpha_n}(n)A_{\alpha_{n-1}}(n-1)\dots A_{\alpha_0}\mathrm{Op}(\chi)\right\|_{L^2(M)} \leq 2(2\pi h)^{-d/2}J_n^u(\alpha_0,\dots,\alpha_n)^{1/2}(1+O(\varepsilon))^n.$$

In our notation, remember that ε is also an upper bound on the diameter of the support of the A_is. It is fixed, but can be taken arbitrarily small.

Using Feynman's heuristics, the kernel of the operator

$$A_{\alpha_{n-1}}e^{i\frac{\hbar\Delta}{2}}A_{\alpha_{n-1}}e^{i\frac{\hbar\Delta}{2}}\cdots e^{i\frac{\hbar\Delta}{2}}A_{\alpha_0}$$

can be written as a paths integral,

$$K(n,x,y;\alpha_0,\dots,\alpha_n) = \sum_{\gamma(0)=x,\gamma(n)=y,\gamma(i)\in P_{\alpha_i},i=0,\dots,n} e^{\frac{i}{\hbar}\int_0^n \frac{\|\dot\gamma\|^2}{2}}.$$

It is known how to obtain a semi-classical expansion of this kernel in powers of h, for fixed n, if the flow has no conjugate points (which means that the critical points of the action $\int_0^n \frac{\|\dot\gamma\|^2}{2}$ are nondegenerate). As shown in [AMB92], the Anosov property implies that the inverse of the hessian of the action at critical points is bounded, *uniformly* with respect to time n. This explains how we are able to make a semi-classical expansion of $K(n,x,y;\alpha_0,\dots,\alpha_n)$ valid for large n. In a former version of this paper we proved Theorem 1.3.3 using this idea of paths integrals. This is, however, very delicate since it implies use of the stationary phase method on spaces of arbitrarily large dimension. The simpler proof presented here uses WKB methods, and was elaborated with Stéphane Nonnenmacher.

In Part 2 we state Theorem 1.3.3 to prove Theorems 1.1.1, 1.1.2. Theorem 1.3.3 is proved in part 3.

The paper has two appendices. In A1 we collect some facts about small scale pseudo-differential operators. In A2 we give details about the partition of unity A_i^h.

2. Proof of Theorem 1.1.1

We show how to prove Theorems 1.1.1 and 1.1.2, using Theorem 1.3.3. We prove, in fact, the following. Let $F \subset \Sigma$ be an invariant subset under the shift. We define the topological entropy $h_{\mathrm{top}}(F) \geq 0$ by saying that $h_{\mathrm{top}}(F) \leq \lambda$ if and only if, for every $\delta > 0$, there exists C such that F can be covered by at most $Ce^{n(\lambda+\delta)}$ cylinders of length n (for all n). We consider normalized quasi-eigenfunctions, $\|(-h^2\Delta - 1)\psi_h\|_{L^2(M)} \leq ch|\log h|^{-1}$, and we call μ_0 a semi-classical limit (transported on Σ by the coding map).

Proposition 2.0.4. *There exists a* $\bar{\kappa} > 0$ *such that, for all* $\delta > 0$, *we can find* $\vartheta > 0$ *and* $\tau \in (0, 1)$ *such that, for every set* $F \subset \Sigma$ *with* $h_{\rm top}(F) \leq \frac{\Lambda}{2}(1-\delta)$,

$$\mu_0(F) \leq (1-\tau)\Big(1 - \Big(\frac{\bar{\kappa}}{\vartheta} - c\vartheta\Big)_+^2\Big) + \tau + c\bar{\kappa}.$$

The proof gives τ and ϑ as continuous decreasing functions of δ. The proposition directly implies the main theorems: consider the invariant set $I_\delta = \{x, h_g(\mu_0^x) \leq \frac{\Lambda}{2}(1-\delta)\} \subset TM$. By the Shannon-McMillan theorem, if we are given any $\alpha > 0$, there exists a subset $I_\delta^\alpha \subset I_\delta$, with $\nu_0(I_\delta \setminus I_\delta^\alpha) \leq \alpha$, and such that I_δ^α (more precisely its image under the coding map) can be covered by $e^{n\left(\frac{\Lambda}{2}(1-\delta+\alpha)\right)}$ n-cylinders, for large n. Applying Proposition 2.0.4 for $\delta - \alpha$, we find that

$$\nu_0(I_\delta^\alpha) \leq (1-\tau(\delta-\alpha))\left(1 - \left(\frac{\bar{\kappa}}{\vartheta(\delta-\alpha)} - \vartheta(\delta-\alpha)c\right)_+^2\right) + \tau(\delta-\alpha) + c\bar{\kappa}$$

and, letting $\alpha \longrightarrow 0$,

$$\nu_0(I_\delta) \leq (1-\tau(\delta))\left(1 - \left(\frac{\bar{\kappa}}{\vartheta(\delta)} - \vartheta(\delta)c\right)_+^2\right) + \tau(\delta) + c\bar{\kappa};$$

in other words

$$\nu_0(S^1M \setminus I_\delta) \geq (1-\tau(\delta))\left(\frac{\bar{\kappa}}{\vartheta(\delta)} - \vartheta(\delta)c\right)_+^2 - c\bar{\kappa}.$$

The proof of Proposition 2.0.4 may be roughly explained as follows:

(a) Theorem 1.3.3 says that, for every cylinder $\mathcal{C} \in \Sigma_n$,

$$|\mu_h(\mathcal{C})| \leq 2\frac{e^{-n\Lambda/2}}{(2\pi h)^{d/2}}(1+O(\varepsilon))^n,$$

uniformly for $n \leq \mathcal{K}|\log h|$ and $h \leq h_{\mathcal{K}}$ ($\mathcal{K}$ can be taken arbitrarily large). Thus, for any $\theta \in (0, 1)$, a set of μ_h-measure greater than $(1-\theta)$ cannot be covered by less than $(1-\theta)\frac{(2\pi h)^{d/2}}{2}e^{n\Lambda/2}(1+O(\varepsilon))^{-n}$ cylinders of length n (see subsection 2.2).

(b) If $F \subset \Sigma$ is a σ-invariant set of topological entropy strictly less than $\frac{\Lambda}{2}(1-\delta)$, there exists C such that, for every $n \in N$, F can be covered by $Ce^{n\left(\frac{\Lambda}{2}(1-\delta/2)\right)}$ cylinder sets of length n (see subsection 2.3.)

The two observations (a) and (b) lead to the idea that it is difficult for the limit measure μ_0 to concentrate on a set of topological entropy less than $\Lambda/2$.

Sketch of the proof. We start with a variant of observation (b), proved in subsection 2.3:

(b′) Let $F \subset \Sigma$ be a σ-invariant set of topological entropy $h_{\text{top}}(F) \leq \frac{\Lambda}{2}(1-\delta)$. Then there exists a neighbourhood W_{n_1} of F, formed of cylinders of length n_1, such that, for N large enough, for every $\tau \in [0,1]$,

$$\sharp\Sigma_N(W_{n_1}, \tau) \leq e^{N\left(\frac{\Lambda}{2}(1-\delta/4)\right)} e^{(1-\tau)N(1+n_1)\log l},$$

where l is the number of elements of the partition P. We denoted $\Sigma_N(W_{n_1}, \tau)$ the set of N-cylinders $[\alpha_0, \ldots, \alpha_{N-1}]$ such that

$$\frac{\sharp\left\{j \in [0, N-n_1], [\alpha_j, \ldots, \alpha_{j+n_1-1}] \in W_{n_1}\right\}}{N - n_1 + 1} \geq \tau.$$

These correspond to orbits that spend a lot of time in the neighbourhood W_{n_1} of F.

If ε is small enough and τ is sufficiently close to 1, one can find ϑ such that, for $N \geq \vartheta|\log h|$,

$$(1-\theta)(2\pi h)^{d/2} e^{N\Lambda/2}(1+O(\varepsilon))^n > e^{N\left(\frac{\Lambda}{2}(1-\delta/4)\right)} e^{(1-\tau)N(1+n_1)\log l}.$$

It follows from (a) and (b') that

$$|\mu_h\big(\Sigma_N(W_{n_1}, \tau)\big)| \leq 1-\theta. \tag{2.0.1}$$

Then, using the σ-invariance of μ_h (say, in the case when the ψ_h are genuine eigenfunctions), we want to write, for $N = \vartheta|\log h|$,

$$|\mu_h\left(W_{n_1}\right)| = \Big|\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1} \mu_h\left(\sigma^{-k} W_{n_1}\right)\Big| \tag{2.0.2}$$

$$= \Big|\mu_h\Big(\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1} \mathbb{1}_{\sigma^{-k}W_{n_1}}\Big)\Big| \tag{2.0.3}$$

$$\leq \mu_h\big(\Sigma_N(W_{n_1},\tau)\big) + \tau\,\mu_h\big(\Sigma_N(W_{n_1},\tau)^c\big) \tag{2.0.4}$$

$$\leq (1-\tau)\mu_h\big(\Sigma_N(W_{n_1},\tau)\big) + \tau \tag{2.0.5}$$

$$\leq (1-\tau)(1-\theta) + \tau. \tag{2.0.6}$$

Passing to the limit $h \longrightarrow 0$, we get $\mu_0(W_{n_1}) \leq (1-\tau)(1-\theta)+\tau$; hence

$$\mu_0(F) \leq (1-\tau)(1-\theta) + \tau < 1.$$

For (2.0.4), we have used the fact that

$$\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1} \mathbb{1}_{\sigma^{-k}W_{n_1}} \leq 1$$

in general, and that

$$\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1} \mathbb{1}_{\sigma^{-k}W_{n_1}} \leq \tau$$

on $\Sigma_N(W_{n_1},\tau)^c$, the complement of $\Sigma_N(W_{n_1},\tau)$. *Unfortunately,* (2.0.4) *is not correct since μ_h is not a probability measure.*

We know however that μ_h converges weakly to a probability measure, and we may try to make this statement more quantitative. Semi-classical analysis tells us that μ_h is close to being a probability measure when restricted to the set of cylinders of length $N \leq \bar\kappa|\log h|$, for $\bar\kappa$ not too large. To sum up, the inequality (2.0.1) only holds for $N \geq \vartheta|\log h|$ whereas the lines (2.0.2)–(2.0.6) are valid for $N \leq \bar\kappa|\log h|$; one cannot expect ϑ to be smaller than $\bar\kappa$. To pass from one time-scale to the other, we use a sub-multiplicativity property stated in paragraph 2.2.

In paragraph 2.1 we give certain important facts about the partitions of unity we want to use. In 2.2, we come back to observation (a) and prove the crucial sub-multiplicativity lemma. Subsection 2.3 is dedicated to proving (b′). In subsection 2.4 we show that, until a certain time $\bar\kappa|\log h|$, the measure μ_h can be treated as a probability measure. Finally, we conclude as in (2.0.2)–(2.0.6).

2.1. *Partition of unity.* For our purposes, we need to be more specific about our partitions of unity (A_i). In order to apply semi-classical methods we need the A_i to be smooth, and on the other hand we would like the family A_i to behave almost like a family of orthogonal projectors: $A_i^2 \simeq A_i$, $A_iA_j \simeq 0$ for $i \neq j$.

Take a finite partition $M = P_1 \sqcup \cdots \sqcup P_l$ by sets of diameter less than $\varepsilon/2$. By modifying slightly the P_is we may assume that the semi-classical measure ν_0 does not charge the boundary of the partition. Our partition of unity will be defined by taking a convolution

$$\tilde A_i^h(x) = \frac{1}{h^\kappa}\mathbf{1}_{P_i} * \zeta\big(x/h^\kappa\big); \tag{2.1.1}$$

that is,

$$\tilde A_i^h(x) = \frac{1}{h^\kappa}\int \zeta\left(\frac{y}{h^\kappa}\right)\mathbf{1}_{P_i}(x-y)dy,$$

where ζ is a nonnegative, smooth compactly supported function, of integral 1; the convolution is to be unterstood in a local chart, and $\kappa \geq 0$ will be chosen later. Then, we take as a partition of unity the family

$$A_i = \frac{\tilde A_i^h}{\sum_{j=1}^l \tilde A_j^h}.$$

The partition of unity $(A_i)_{1\leq i\leq l}$ depends on h, and if $\kappa > 0$ it converges weakly to $(\mathbf{1}_{P_i})_{1\leq i\leq l}$ when $h \longrightarrow 0$. It has the following properties:

- $\overline{P_i} \subset \operatorname{supp} A_i \subset B(\overline{P_i},\varepsilon/2)$ for all i, for h small enough. In accordance with the notation of the previous sections, we denote $\Omega_i = B(\overline{P_i},\varepsilon/2)$.

- ${A_i}^2 = A_i$ except on a set of measure of order h^κ.
- For $i \neq j$, $A_i A_j = 0$ except on a set of measure of order h^κ.

We must choose κ so that semi-classical methods still work: that is, $\kappa < 1/2$ (see Appendix A1).

In addition, we need to assume that there exists some $p > 0$ such that

- For all i, $\|(A_i^2 - A_i)\psi_h\|_{L^2(M)} = O(h^{p/2})$.
- For $i \neq j$, $\|A_i A_j \psi_h\|_{L^2(M)} = O(h^{p/2})$.

In other words, the operators A_i act on ψ_h almost as a family of orthogonal projectors. Because $\|\psi_h\|_{L^2(M)} = 1$, it is always possible to construct the A_is in order to satisfy all the requirements above; this requires moving slightly the boundary of the partition P_i (of a distance $h^{\frac{1}{2}(\frac{1}{2}-p)}$) before applying the convolution (2.1.1). The construction is described in detail in Appendix A2.

2.2. *A sub-multiplicative property.* As already mentioned, we will have to face the problem that the inequality $|\mu_h(\mathcal{C})| \leq 2\frac{e^{-n\Lambda/2}}{(2\pi h)^{d/2}}(1+O(\varepsilon))^n$ is only useful when $2\frac{e^{-n\Lambda/2}}{(2\pi h)^{d/2}}(1+O(\varepsilon))^n < 1$, that is, $n \geq \vartheta|\log h|$ for a certain ϑ. On the other hand, observation (a) is only useful if μ_h is close to being a probability measure; semi-classical analysis tells us that this is the case on the set of cylinders of length $\leq \bar{\kappa}|\log h|$. *A priori*, $\bar{\kappa} < \vartheta$, and to reconcile the two regimes $n \leq \bar{\kappa}|\log h|$ and $n \geq \vartheta|\log h|$ we will need a certain sub-multiplicativity property (Lemma 2.2.3 and 2.2.4).

We introduce, as in Theorem 1.3.3, a cut-off function χ which is compactly supported in a neighbourhood of size $\varepsilon/2$ of the energy layer 1; and which is identically $\equiv 1$ on a smaller neighbourhood. It should be noted that, for such χ, we have $\|\mathrm{Op}_h(\chi)\psi_h - \psi_h\|_{L^2(M)} = O(ch|\log h|^{-1}) + O(h^\infty)$, as follows from the identity $\mathrm{Op}(1-\chi) = A(-h^2\Delta - 1) + R$ where A is a pseudo-differential operator of order 0 and R is a smoothing operator (see Appendix A1).

Definition 2.2.1. (i) Let W be a subset of Σ_n, the set of n-cylinders in Σ; we denote $W^c \subset \Sigma_n$ its complement. For a given $h > 0$ and $\theta \in [0,1]$, we say that W is an $(h,(1-\theta),n)$-*cover* of Σ if

$$\left\| \sum_{\mathcal{C}\in W^c} \hat{\mathcal{C}}_h \mathrm{Op}_h(\chi)\psi_h \right\|_{L^2(M)} \leq \theta. \tag{2.2.1}$$

(ii) We define

$$N_h(n,\theta) = \min\left\{\sharp W, W \text{ is a } (h,(1-\theta),n)\text{-cover of } \Sigma\right\},$$

the *minimal cardinality* of an $(h,(1-\theta),n)$-cover of Σ.

Remember the notation: for $\mathcal{C} = [\alpha_0, \dots, \alpha_{n-1}] \in \Sigma_n$, $\hat{\mathcal{C}}_h$ stands for the operator $\hat{\mathcal{C}}_h = A_{\alpha_{n-1}}(n-1) \dots A_{\alpha_1}(1) A_{\alpha_0}(0)$. In some sense, (2.2.1) means that the measure of the complement of W is small. Note that we consider the quantity $\| \sum_{\mathcal{C} \in W^c} \hat{\mathcal{C}}_h \mathrm{Op}_h(\chi) \psi_h \|_{L^2(M)}$, and not

$$|\sum_{\mathcal{C} \in W^c} \mu_h(\mathcal{C})| = |\sum_{\mathcal{C} \in W^c} \langle \hat{\mathcal{C}}_h \psi_h, \psi_h \rangle_{L^2(M)}|.$$

The reason is that we need a sub-multiplicative property of $N_h(n, \theta)$, stated below. We will need the following lemma, proved in Appendix A1:

LEMMA 2.2.2. *There exist* $\bar{\kappa}$ *and* $\alpha > 0$ *such that, for all* $n \leq \bar{\kappa} |\log h|$, *for every subset* $W \subset \Sigma_n$,

$$\left\| \sum_{\mathcal{C} \in W} \hat{\mathcal{C}}_h \mathrm{Op}_h(\chi) \right\|_{L^2(M)} \leq 1 + O(h^\alpha).$$

LEMMA 2.2.3 (Sub-multiplicativity 1). *Suppose the* (ψ_h) *are eigenfunctions*; *that is*, $(-h^2\Delta - 1)\psi_h = 0$.

If $\bar{\kappa}$ *and* α *are as in Lemma* 2.2.2, *then for every* $n \leq \bar{\kappa} |\log h|, k \in \mathbb{N}$ *and* $\theta \in (0, 1)$,

$$N_h\Big(kn, k\theta(1 + O(nh^\alpha))\Big) \leq N_h\big(n, \theta\big)^k.$$

The lemma can be adapted for approximate eigenfunctions:

LEMMA 2.2.4 (Sub-multiplicativity 2). *Suppose the* (ψ_h) *satisfy*

$$\|(-h^2\Delta - 1)\psi_h\|_{L^2(M)} \leq ch|\log h|^{-1}.$$

If $\bar{\kappa}$ *and* α *are as in Lemma* 2.2.2, *then for every* $n \leq \bar{\kappa} |\log h|, k \in \mathbb{N}$ *and* $\theta \in (0, 1)$,

$$N_h\Big(kn, \big(k\theta + k^2 n\, c|\log h|^{-1}\big)(1 + O(nh^\alpha))\Big) \leq N_h\big(n, \theta\big)^k.$$

Proof. Given an $(h, (1-\theta), n)$-cover of Σ, denoted W, we define $W^k \subset \Sigma_{kn}$ as the *set of* kn*-cylinders* $[\alpha_0, \dots, \alpha_{kn-1}]$ such that $[\alpha_{jn}, \dots, \alpha_{(j+1)n-1}] \in W$ for all $j \in [0, k-1]$, and we show that W^k is a $(h, 1-k\theta-k^2 n\, c|\log h|^{-1}, kn)$*-cover*:

Each $\mathcal{C} \in (W^k)^c$ may be decomposed into the concatenation of k cylinders of length n, $\mathcal{C} = \mathcal{C}^0 \mathcal{C}^1 \dots \mathcal{C}^{k-1}$, one of which is not in W. Thus, we have

(2.2.2)

$$\begin{aligned}
&\Big\| \sum_{\mathcal{C}\in (W^k)^c} \hat{\mathcal{C}}_h \mathrm{Op}_h(\chi)\psi_h \Big\|_{L^2(M)} \\
&= \Big\| \sum_{j=0}^{k-1} \sum_{\mathcal{C}^i\in W \text{ for } i>j, \mathcal{C}^j\in W^c, \mathcal{C}^i\in\Sigma_n \text{ for } i<j} \hat{\mathcal{C}}_h^{k-1}((k-1)n)\dots\hat{\mathcal{C}}_h^{j}(jn)\dots\hat{\mathcal{C}}_h^{0}\mathrm{Op}_h(\chi)\psi_h \Big\| \\
&= \Big\| \sum_{j=0}^{k-1} \sum_{\mathcal{C}^i\in W \text{ for } i<j, \mathcal{C}^j\in W^c} \hat{\mathcal{C}}_h^{k-1}((k-1)n)\dots\hat{\mathcal{C}}_h^{j}(jn)\mathrm{Op}_h(\chi)\psi_h \Big\|.
\end{aligned}$$

Using Lemma 2.2.2 to bound the norm of the operator

$$\sum_{\mathcal{C}^i\in W \text{ for } i>j} \hat{\mathcal{C}}_h^{k-1}((k-1)n)\dots\hat{\mathcal{C}}_h^{j-1}((j-1)n)\mathrm{Op}_h(\chi)$$

by $(1+O(h^\alpha))^{k-j}$, we see that (2.2.2) is less than

$$\begin{aligned}
&(1+O(h^\alpha))^n \sum_{j=0}^{k-1} \| \sum_{\mathcal{C}^j\in W^c} \hat{\mathcal{C}}_h^{j}(jn)\mathrm{Op}_h(\chi)\psi_h\| \\
&\quad = (1+O(h^\alpha))^n \sum_{j=0}^{k-1} \left(\| \sum_{\mathcal{C}^j\in W^c} \hat{\mathcal{C}}_h^{j}\mathrm{Op}_h(\chi)\psi_h\| + O(jn\ c|\log h|^{-1}) + 2O(ch|\log h|^{-1}) \right) \\
&\quad \le \left(k\theta + k^2 n\ c|\log h|^{-1}\right)(1+O(nh^\alpha)).
\end{aligned}$$

We used the fact that $\|\big(\exp(ith\Delta) - e^{\frac{it}{h}}\big)\psi_h\|_{L^2(M)} \le tc|\log h|^{-1}$ and the fact that $\|\mathrm{Op}_h(\chi)\psi_h - \psi_h\|_{L^2(M)} = O(ch|\log h|^{-1}) + O(h^\infty)$. □

The next proposition is just an expression of Observation (a).

Proposition 2.2.5. *For any $\mathcal{K} > 0$, there exists $h_{\mathcal{K}} > 0$ such that for $h \le h_{\mathcal{K}}$ and $N \le \mathcal{K}|\log h|$,*

$$N_h(N,\theta) \ge \frac{(1-\theta)}{2}(2\pi h)^{d/2} e^{N\frac{\Lambda}{2}}(1+O(\varepsilon))^{-N}.$$

Proof. Let W be an $(h,(1-\theta),N)$-cover of Σ. We have

$$|\sum_{\mathcal{C}\in W^c} \langle \hat{\mathcal{C}}_h \mathrm{Op}_h(\chi)\psi_h, \psi_h\rangle| \le \|\sum_{\mathcal{C}\in W^c} \hat{\mathcal{C}}_h \mathrm{Op}_h(\chi)\psi_h\| \le \theta.$$

Using the fact that

$$\sum_{\mathcal{C}\in\Sigma_N} \langle \hat{\mathcal{C}}_h \mathrm{Op}_h(\chi)\psi_h, \psi_h\rangle = \langle \mathrm{Op}_h(\chi)\psi_h, \psi_h\rangle = 1 + O(ch|\log h|^{-1}) + O(h^\infty),$$

we get

$$|\sum_{\mathcal{C}\in W}\langle \hat{\mathcal{C}}_h \mathrm{Op}_h(\chi)\psi_h,\psi_h\rangle| \geq 1-\theta+O(ch|\log h|^{-1}).$$

Thus,

$$1-\theta+O(ch|\log h|^{-1}) \leq \sum_{\mathcal{C}\in W}|\langle \hat{\mathcal{C}}_h \mathrm{Op}_h(\chi)\psi_h,\psi_h\rangle| \leq \sharp W\frac{2e^{-N\frac{\Lambda}{2}}}{(2\pi h)^{d/2}}(1+O(\varepsilon))^N,$$

where the last line comes from Theorem 1.3.3. □

This immediately implies:

LEMMA 2.2.6. *Given any $\delta>0$, we may choose ϑ large enough, and ε (the size of the partition) small enough, so that, for $N=\vartheta|\log h|$,*

$$N_h(N,\theta) > \big(1-\theta\big)e^{N\frac{\Lambda}{2}(1-\frac{\delta}{16})}.$$

As mentioned, semi-classical analysis is usually only valid until a certain time $\bar\kappa|\log h|$, in general with $\bar\kappa<\vartheta$. Lemma 2.2.4 is precisely the tool that will allow us to reduce the time scale: starting from Lemma 2.2.6, it tells us that, for $N=\bar\kappa|\log h|$, $0\leq\bar\kappa\leq\vartheta$,

$$N_h(N,\frac{\bar\kappa}{\vartheta}\theta-c\vartheta) \geq (1-\theta)^{\bar\kappa/\vartheta}e^{N\frac{\Lambda}{2}(1-\frac{\delta}{16})}. \tag{2.2.3}$$

2.3. *A combinatorial lemma.* Let us now put a precise statement behind observation (b). If F is a set of small topological entropy, Lemma 2.3.1 below says that the set of orbits spending a lot of time near F also has a small rate of exponential growth.

Let us consider an invariant subset $F\subset\Sigma$ of topological entropy $h_{\mathrm{top}}(F)\leq\frac{\Lambda}{2}(1-\delta)$. By definition, there exists n_0 such that F can be covered by (at most) $e^{n(h_{\mathrm{top}}(F)+\frac{\Lambda\delta}{4})}\leq e^{n\frac{\Lambda}{2}(1-\delta/2)}$ cylinders of length n, for all $n\geq n_0$. We denote $W_n\subset\Sigma_n$ a cover of minimal cardinality of F by n-cylinders. Given $N\in\mathbb{N}$, $n\leq N$ and $\tau\in[0,1]$, we denote $\Sigma_N(W_n,\tau)$ the set of N-cylinders $[\alpha_0,\ldots,\alpha_{N-1}]$ such that

$$\frac{\sharp\big\{j\in[0,N-n],[\alpha_j,\ldots,\alpha_{j+n-1}]\in W_n\big\}}{N-n+1}\geq\tau.$$

The next lemma bounds the cardinality of $\Sigma_N(W_n,\tau)$.

LEMMA 2.3.1 (Counting cylinder sets). *There exist $n_1\geq n_0$, and N_0 such that, for every $N\geq N_0$ and for every $\tau\in[0,1]$,*

$$\sharp\Sigma_N(W_n,\tau)\leq e^{N\frac{3\Lambda\delta}{8}}e^{Nh_{\mathrm{top}}(F)}e^{(1-\tau)N(1+n_1)\log l}.$$

Proof. Take $n_1 \geq n_0$ large enough so that

$$\lim_{N \longrightarrow +\infty} \frac{1}{N} \log \binom{N}{\lfloor \frac{N}{n_1} \rfloor} \leq \frac{\Lambda\delta}{100};$$

n_1 is now fixed.

Given a sequence $[\alpha_0 \ldots, \alpha_{N-1}] \in \Sigma_N$, define a sequence of "stopping times":

$$\begin{aligned}
\tau_0 &= \inf\left\{0 \leq j \leq N - n_1, [\alpha_j, \ldots, \alpha_{j+n_1-1}] \in W_{n_1}\right\},\\
\tau_0' &= \inf\left\{\tau_0 \leq j \leq N - n_1, [\alpha_j, \ldots, \alpha_{j+n_1-1}] \notin W_{n_1}\right\},\\
\tau_1 &= \inf\left\{\tau_0' - 1 + n_1 \leq j \leq N - n_1, [\alpha_j, \ldots, \alpha_{j+n_1-1}] \in W_{n_1}\right\},
\end{aligned}$$

and so on:

$$\begin{aligned}
\tau_{k+1} &= \inf\left\{\tau_k' - 1 + n_1 \leq j \leq N - n_1, [\alpha_j, \ldots, \alpha_{j+n_1-1}] \in W_{n_1}\right\},\\
\tau_{k+1}' &= \inf\left\{\tau_k \leq j \leq N - n_1, [\alpha_j, \ldots, \alpha_{j+n_1-1}] \notin W_{n_1}\right\}.
\end{aligned}$$

The sequence (τ_k) becomes stationary, equal to $N - n_1$, for $k \geq \left\lfloor \frac{N}{n_1} \right\rfloor$. Define the intervals $I_0 = [\tau_0, \tau_0' - 1 + n_1 - 1]$,…,$I_k = [\tau_k, \tau_k' - 1 + n_1 - 1]$. If $\mathcal{C} = [\alpha_0, \ldots, \alpha_{N-1}]$ is in $\Sigma_N(W_{n_1}, \tau)$, then the complement of $\cup I_k$ has cardinality less than $(1 - \tau)(N - n_1 + 1) + n_1 \leq (1 - \tau)N + n_1$.

A cylinder $\mathcal{C} = [\alpha_0, \ldots, \alpha_{N-1}] \in \Sigma_N(W_{n_1}, \tau)$ is completely determined by the following data:

(i) the intervals $(I_k)_{0 \leq k \leq \lfloor N/n_1 \rfloor}$,

(ii) the restriction of $\mathcal{C}$ to the union of the I_ks,

(iii) the values of $\mathcal{C}$ outside the I_ks.

Let us count in each case the number of possibilities:

(i) There are at most $\binom{N}{\lfloor N/n_1 \rfloor}^2$ possibilities, corresponding to the choices of the endpoints of the intervals I_k; by our choice of n_1, for N large enough this is less than $e^{N\frac{\Lambda\delta}{50}}$.

(ii) Each I_k can be split into a disjoint union of intervals of length n_1 and at most one interval of length less than n_1. The intervals of length (exactly) n_1 thus obtained are at most N/n_1, and they correspond to cylinders covering F: there are at most $(\sharp W_{n_1})^{N/n_1}$ possibilities. If $n_1 \geq n_0$ this is less than $\left(e^{n_1(h_{\rm top}(F) + \frac{\Lambda\delta}{4})}\right)^{N/n_1} = e^{N(h_{\rm top}(F) + \frac{\Lambda\delta}{4})}$. For the remaining intervals, of length strictly less than n_1, there can be at most $(1-\tau)N$ of them; this gives $l^{(1-\tau)Nn_1}$ possibilities.

(iii) For the values of α outside the I_ks, the number of possible choices is bounded by $l^{(1-\tau)N+n_1}$. Choose N_0 such that $l^{n_1} \leq e^{N_0\frac{\Lambda\delta}{50}}$.

This ends the proof of Lemma 2.3.1. $\square$

Remark 2.3.2. This estimate is very crude, since we argued as if all choices in (i), (ii) and (iii) were independent.

We can now choose $\tau \in (0,1)$ close enough to 1 so that

$$h_{\rm top}(F) + (1-\tau)N(1+n_1)\log l + \frac{3\Lambda\delta}{8} \leq \frac{\Lambda}{2}\left(1 - \frac{\delta}{8}\right);$$

now,

$$\sharp\Sigma_N(W_n, \tau) \leq e^{N\frac{\Lambda}{2}\left(1-\frac{\delta}{8}\right)}, \tag{2.3.1}$$

for all N large enough.

Comparing (2.3.1) with (2.2.3), for h small enough and $N = \bar\kappa|\log h|$, we have necessarily:

$$\left\| \sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\tau)^c} \hat{\mathcal{C}}_h \mathrm{Op}_h(\chi)\psi_h \right\|_{L^2} \geq \frac{\bar\kappa}{\vartheta}\theta - c\vartheta. \tag{2.3.2}$$

This is an attempt to say that the measure of the complement of $\Sigma_N(W_{n_1}, \tau)$ cannot be too small. We now have to relate (2.3.2) and

$$|\mu_h(\Sigma_N(W_{n_1},\tau)^c)| = \left| \sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\tau)^c} \langle \hat{\mathcal{C}}_h\psi_h, \psi_h\rangle \right|.$$

This is done in the next two paragraphs, and goes roughly as follows:

Imagine that the operators $\hat{\mathcal{C}}_h\mathrm{Op}_h(\chi)$ are orthogonal projectors, with orthogonal images for distinct cylinders $\mathcal{C}$. Ideally, this would be the case if:

– the operators A_i were a family of orthogonal projectors (that is, if the functions A_i were characteristic functions of disjoint sets);

– the operators $A_i(t)$ commuted with one another for all t. If so, we could write

$$\begin{aligned}\sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\tau)^c} \langle \hat{\mathcal{C}}_h\mathrm{Op}_h(\chi)\psi_h, \psi_h\rangle &= \sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\tau)^c} \|\hat{\mathcal{C}}_h\mathrm{Op}_h(\chi)\psi_h\|^2_{L^2} \\ &= \| \sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\tau)^c} \hat{\mathcal{C}}_h\mathrm{Op}_h(\chi)\psi_h\|^2_{L^2}\end{aligned} \tag{2.3.3}$$

so that (2.3.2) would imply the lower bound

$$|\mu_h(\Sigma_N(W_{n_1},\tau)^c)| \geq \left(\frac{\bar\kappa}{\vartheta}\theta - c\vartheta\right)_+^2.$$

The A_is, unfortunately, are not characteristic functions of disjoint sets; they form a smooth partition of unity; and the operators $A_i(t)$ do not commute. However,

- we have constructed the A_i so that they act on ψ_h almost as an orthogonal family of projectors.

– there exists $\bar\kappa > 0$ such that the operators $A_i(t)$ almost commute for $|t| \leq \bar\kappa|\log h|$:

PROPOSITION 2.3.3. *For all $\bar\kappa > 0$, for every $N \leq 2\bar\kappa|\log h|$, for every permutation τ of $\{0,\ldots,N\}$, for every sequence $t_0,\ldots,t_N$ such that $|t_i| \leq \bar\kappa|\log h|$, for every sequence $\alpha_0,\ldots,\alpha_N$,*

$$\begin{aligned}&\|\mathrm{Op}_h(\chi)^* A_{\alpha_N}(t_N)\ldots..A_{\alpha_1}(t_1)A_{\alpha_0}(t_0)\mathrm{Op}_h(\chi)\\ &-\mathrm{Op}_h(\chi)^*\mathrm{Op}_h(\chi)A_{\alpha_{\tau N}}(t_{\tau N})\ldots..A_{\alpha_{\tau 1}}(t_{\tau 1})A_{\alpha_{\tau 0}}(t_{\tau 0})\|_{L^2(M)} = O(h^{1-2\kappa-3\Lambda\bar\kappa}).\end{aligned}$$

The proof is given in Appendix A1. This gives hope to prove (2.3.3), at least up to a negligible remainder term:

2.4. *Relating $\|\sum\hat{\mathcal{C}}_h\mathrm{Op}_h(\chi)\psi_h\|$ and $\sum\langle\hat{\mathcal{C}}_h\psi_h,\psi_h\rangle$.* Remember that we constructed the partition of unity (A_i^h) in such a way that:

There exists $p > 0$ such that

$$\|(A_i^2 - A_i)\psi_h\|_{L^2(M)} = O(h^{p/2}) \text{ and } \|A_iA_j\psi_h\|_{L^2(M)} = O(h^{p/2}),$$

for all i and all $j \neq i$. Let us choose the parameter $\bar\kappa$ so that the conclusion of Proposition 2.3.3 holds. This ensures that there is no harm in treating the $\hat{\mathcal{C}}_h$ as orthogonal projectors in (2.3.2). Using Proposition 2.3.3, which allows commutation of the operators $A_i(t)$ and $\mathrm{Op}_h(\chi)$, for $|t| \leq \bar\kappa|\log h|$, we find that, for $N \leq \bar\kappa|\log h|$, for $\mathcal{C},\mathcal{C}' \in \Sigma_N, \mathcal{C} \neq \mathcal{C}'$,

$$|\langle\hat{\mathcal{C}}_h\mathrm{Op}_h(\chi)\psi_h, \hat{\mathcal{C}}'_h\mathrm{Op}_h(\chi)\psi_h\rangle| = O(h^{1-2\kappa-3\Lambda\bar\kappa}) + O(h^{p/2}),$$

and

$$\begin{aligned}&|\langle\hat{\mathcal{C}}_h\mathrm{Op}_h(\chi)\psi_h, \mathrm{Op}_h(\chi)\psi_h\rangle - \langle\hat{\mathcal{C}}_h\mathrm{Op}_h(\chi)\psi_h, \hat{\mathcal{C}}_h\mathrm{Op}_h(\chi)\psi_h\rangle|\\ &\qquad = N\big(O(h^{1-2\kappa-3\Lambda\bar\kappa}) + O(h^{p/2})\big).\end{aligned}$$

Then, for $N \leq \bar\kappa|\log h|$,

$$\sum_{C,C'\in\Sigma_N, C\neq C'} |\langle\hat{\mathcal{C}}_h\mathrm{Op}_h(\chi)\psi_h, \hat{\mathcal{C}}'_h\mathrm{Op}_h(\chi)\psi_h\rangle| = \big(O(h^{1-2\kappa-3\Lambda\bar\kappa}) + O(h^{p/2})\big)\sharp\Sigma_N^2$$

and

$$\begin{aligned}&\sum_{\mathcal{C}\in\Sigma_N} |\langle\hat{\mathcal{C}}_h\mathrm{Op}_h(\chi)\psi_h, \mathrm{Op}_h(\chi)\psi_h\rangle - \langle\hat{\mathcal{C}}_h\mathrm{Op}_h(\chi)\psi_h, \hat{\mathcal{C}}_h\mathrm{Op}_h(\chi)\psi_h\rangle|\\ &\qquad = N\big(O(h^{1-2\kappa-3\Lambda\bar\kappa}) + O(h^{p/2})\big)\sharp\Sigma_N.\end{aligned}$$

Since the cardinality of Σ_N grows exponentially, we take $\bar\kappa$ small enough so that, for $N \leq \bar\kappa|\log h|$,

$$\sum_{C,C'\in\Sigma_N, C\neq C'} |\langle\hat{\mathcal{C}}_h\mathrm{Op}_h(\chi)\psi_h, \hat{\mathcal{C}}'_h\mathrm{Op}_h(\chi)\psi_h\rangle| = O(h^{\bar\kappa})$$

and

$$\sum_{\mathcal{C}\in\Sigma_N} |\langle \hat{\mathcal{C}}_h \mathrm{Op}_h(\chi)\psi_h, \mathrm{Op}_h(\chi)\psi_h\rangle - \langle \hat{\mathcal{C}}_h \mathrm{Op}_h(\chi)\psi_h, \hat{\mathcal{C}}_h \mathrm{Op}_h(\chi)\psi_h\rangle| = O(h^{\bar\kappa}).$$

Remember also that $\|(\mathrm{Op}_h(\chi)-1)\psi_h\|_{L^2(M)} = O(ch) + O(h^\infty)$. For $\bar\kappa$ small enough and $N \leq \bar\kappa|\log h|$, we find for every subset $S \subset \Sigma_N$,

$$\sum_{\mathcal{C}\in S} |\mu_h(\mathcal{C})| = |\sum_{\mathcal{C}\in S} \mu_h(\mathcal{C})| + O(h^{\bar\kappa}) \tag{2.4.1}$$

$$= \sum_{\mathcal{C}\in S} \|\hat{\mathcal{C}}_h \mathrm{Op}_h(\chi)\psi_h\|^2 + O(h^{\bar\kappa}) \tag{2.4.2}$$

$$= \|\sum_{\mathcal{C}\in S} \hat{\mathcal{C}}_h \mathrm{Op}_h(\chi)\psi_h\|^2 + O(h^{\bar\kappa}). \tag{2.4.3}$$

The point is that, when working on cylinders of size $\bar\kappa|\log h|$, the measure μ_h is nonnegative, up to a negligible remainder term. The first line implies in particular that

$$\sum_{\mathcal{C}\in\Sigma_N} |\mu_h(\mathcal{C})| = 1 + O(h^{\bar\kappa}). \tag{2.4.4}$$

Coming back to (2.3.2), we get for $N = \bar\kappa|\log h|$, and n_1 as in Lemma 2.3.1,

$$\sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\tau)^c} |\mu_h(\mathcal{C})| \geq \left(\frac{\bar\kappa}{\vartheta}\theta - c\vartheta\right)_+^2 + O(h^{\bar\kappa})$$

and, because of (2.4.4),

$$\sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\tau)} |\mu_h(\mathcal{C})| \leq 1 - \left(\frac{\bar\kappa}{\vartheta}\theta - c\vartheta\right)_+^2 + O(h^{\bar\kappa}). \tag{2.4.5}$$

2.5. *End of the proof.* We use the σ-invariance of μ_h (Prop. 1.3.1 (ii)), and we get, for $N = \bar\kappa|\log h|$,

$$|\mu_h(W_{n_1})| \leq |\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1} \mu_h(\sigma^{-k}\Sigma(W_{n_1}))| + c\bar\kappa \tag{2.5.1}$$

$$= |\mu_h\Big(\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1} \mathbb{1}_{\sigma^{-k}\Sigma(W_{n_1})}\Big)| + c\bar\kappa \tag{2.5.2}$$

$$\leq \sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\tau)} |\mu_h(\mathcal{C})| + \tau \sum_{\mathcal{C}\notin\Sigma_N(W_{n_1},\tau)} |\mu_h(\mathcal{C})| + c\bar\kappa \tag{2.5.3}$$

$$\leq (1-\tau) \sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\tau)} |\mu_h(\mathcal{C})| + \tau + c\bar\kappa + O(h^{\bar\kappa}) \tag{2.5.4}$$

$$\leq (1-\tau)\left(1 - \left(\frac{\bar\kappa}{\vartheta}\theta - c\vartheta\right)_+^2\right) + \tau + c\bar\kappa + O(h^{\bar\kappa}). \tag{2.5.5}$$

For (2.5.3), we have used the fact that

$$\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1}\mathbb{1}_{\sigma^{-k}\Sigma(W_{n_1})}\leq 1,$$

in general, and that

$$\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1}\mathbb{1}_{\sigma^{-k}\Sigma(W_{n_1})}\leq \tau$$

on $\Sigma_N(W_{n_1},\tau)^c$. In the next line, we have used (2.4.4); and we conclude thanks to (2.4.5). Then, by Proposition 1.3.2, we can pass to the limit in (2.5.5), and obtain

$$\mu_0(W_{n_1})\leq(1-\tau)\left(1-\left(\frac{\bar{\kappa}}{\vartheta}\theta-c\vartheta\right)_+^2\right)+\tau+c\bar{\kappa}.$$

Since $F\subset W_{n_1}$, finally,

$$\mu_0(F)\leq(1-\tau)\left(1-\left(\frac{\bar{\kappa}}{\vartheta}\theta-c\vartheta\right)_+^2\right)+\tau+c\bar{\kappa}.$$

Noting that this last estimate holds for every $\theta<1$, we get

$$\mu_0(F)\leq(1-\tau)\left(1-\left(\frac{\bar{\kappa}}{\vartheta}-c\vartheta\right)_+^2\right)+\tau+c\bar{\kappa}$$

which proves Proposition 2.0.4.

3. The main estimate

We prove Theorem 1.3.3 about the norm of the operator

$$A_{\alpha_n}(n)\ldots A_{\alpha_0}\mathrm{Op}(\chi)=U^{-n}A_{\alpha_n}UA_{\alpha_{n-1}}\ldots UA_{\alpha_0}\mathrm{Op}(\chi)$$

(where we denote for simplicity $U^t=\exp(ith\frac{\Delta}{2})$ and $U=U^1$). Since U^t is unitary, the norm of this operator is the same as the norm of $A_{\alpha_n}UA_{\alpha_{n-1}}\ldots$ $\ldots UA_{\alpha_0}\mathrm{Op}(\chi)$.

The pseudo-differential operator $\mathrm{Op}(\chi)$ is defined as (see Appendix A1)

$$\mathrm{Op}(\chi)=\sum_l\varphi_l\ \mathrm{OP}(\chi)\ \varphi_l$$

where (φ_l) is an auxiliary partition of unity on M ($\sum_l\varphi_l(x)^2\equiv 1$) such that the support of each φ_l is endowed with local coordinates in $\mathbb{R}^d$. In local coordinates in the support of φ_l, $\mathrm{OP}(\chi)$ is then defined by the usual formula,

$$\mathrm{OP}(\chi)f(x)=(2\pi h)^{-d}\int f(z)e^{i\frac{\langle p,x-z\rangle}{h}}\chi(z,p)dzdp. \tag{3.0.1}$$

The function χ will be chosen in the form $\chi(z,p) = \chi_1(\|p\|_z)$ where χ_1 is a smooth function on $\mathbb{R}_+$ supported in $[1-\varepsilon/2, 1+\varepsilon/2]$ with $\chi_1 \equiv 1$ in a neighbourhood of 1. For $x \in \Omega_{\alpha_0}$, we can write

$$\mathrm{Op}(\chi)u(x) = \int u(z)\delta_z(x)dz, \tag{3.0.2}$$

where we denote δ_z the function

$$\delta_z(x) = \int e^{\frac{i\langle p, x-z\rangle}{h}} \chi(z,p) \frac{dp}{(2\pi h)^d}. \tag{3.0.3}$$

To be more precise, we should use an auxiliary partition of unity as in (3.0.1), and write $\mathrm{Op}(\chi)u(x) = \sum_l \varphi_l(x) \int \varphi_l(z)u(z)\delta_z(x)dz$; expressions such as (3.0.3) should then be understood in local coordinates in the support of each φ_l. For simplicity, and because these φ_l will play no role in the estimates, they will not appear any more in the formulae.

If we can estimate the norm of $A_{\alpha_n}UA_{\alpha_{n-1}}\dots UA_{\alpha_0}\delta_z$ for any z, we can use (3.0.2) to estimate the norm of $A_{\alpha_n}UA_{\alpha_{n-1}}\dots UA_{\alpha_0}\mathrm{Op}(\chi)u$ for arbitrary u. The estimates will be done by induction on n: we will propose an Ansatz – that is, an approximate expression – for $A_{\alpha_n}UA_{\alpha_{n-1}}\dots UA_{\alpha_0}\delta_z$, valid for "large" n.

3.1. *The Ansatz for $n=1$.* This first step is very standard, but we recall the main ideas in order to fix our notation. We look for an Ansatz for $U^t.A_{\alpha_0}\delta_z$, in the form

$$u(t,x,z) = \int e^{\frac{i\bar{S}_0(t,x,(z,p))}{h}} \left(\sum_{k=0}^{N-1} h^k a_k(t,x,(z,p))\right) \chi(z,p) \frac{dp}{(2\pi h)^d}. \tag{3.1.1}$$

If we want u to solve

$$\frac{\partial u}{\partial t} = ih\frac{\Delta_x u}{2}$$

up to order h^N, the unknown functions $\bar{S}_0$ and a_k should solve the partial differential equations

$$\begin{cases} \frac{\partial \bar{S}_0}{\partial t} + H(x, d_x\bar{S}_0) = 0 \text{ (Hamilton-Jacobi equation)} \\ \\ \frac{\partial a_k}{\partial t} = \frac{i\Delta a_{k-1}}{2} - \langle da_k, d\bar{S}_0\rangle - a_k\frac{\Delta \bar{S}_0}{2} \text{ (transport equation)} \end{cases} \tag{3.1.2}$$

with initial conditions

$$\begin{cases} \bar{S}_0(0,x,(z,p)) = \langle p, x-z\rangle \\ \\ a_0(0,x,(z,p)) = A_{\alpha_0}(x) \\ \\ a_k(0,x,(z,p)) = 0 \text{ for } k > 1. \end{cases}$$

The Hamiltonian is, of course, given by the Riemannian metric, $H(x,p) = \frac{\|p\|_x^2}{2}$.

Remark 3.1.1. Since the base point z is fixed in all the following calculations, we will omit it in the notation until Lemma 3.2.1.

Let us introduce the notation $T^{s,t}_{\bar{S}_0(p)}$ $(s \leq t)$ for the unitary "flow" giving the solutions of the time dependent equation

$$\frac{\partial a}{\partial t} = -\langle d_x a, d_x \bar{S}_0(t,x,p)\rangle - a\frac{\Delta_x \bar{S}_0(t,x,p)}{2}$$

with initial data $a(s)$. The explicit expression is

$$T^{s,t}_{\bar{S}_0(p)}(a)(x) = \frac{a \circ g^{-(t-s)}\big(x, d_x\bar{S}_0(t,x,p)\big)}{\sqrt{J^{s,t}_{\bar{S}_0(p)}(x)}} \tag{3.1.3}$$

where g^t is the geodesic flow; the function $J^{s,t}_{\bar{S}_0(p)}$ defined on M is itself the solution of

$$\frac{\partial_t J^{s,t}_{\bar{S}_0(p)}(x)}{J^{s,t}_{\bar{S}_0(p)}(x)} + \frac{\langle d_x J^{s,t}_{\bar{S}_0(p)}, d_x\bar{S}_0(t,x,p)\rangle}{J^{s,t}_{\bar{S}_0(p)}(x)} = \Delta_x \bar{S}_0(t,x,p)$$

with initial condition $J^{s,s}_{\bar{S}_0(p)} = 1$. The solution of this equation is
(3.1.4)

$$J^{s,t}_{\bar{S}_0(p)}\big(g^{t-s}(x, d_x\bar{S}_0(s,x,p))\big) = \exp\int_s^t \Delta\bar{S}_0\big(\tau, g^{\tau-s}(x, d_x\bar{S}_0(s,x,p))\big)d\tau;$$

the interpretation is that $J^{s,t}_{\bar{S}_0(p)}\big(g^{t-s}(x, d_x\bar{S}_0(s,x,p))\big)$ is the jacobian of the flow g^{t-s} restricted to the lagrangian submanifold generated by $\bar{S}_0(s,p)$, namely $\mathcal{L}_{\bar{S}_0(s,p)} = \{(x, d_x\bar{S}_0(s,x,p))\}$.

Remark 3.1.2. To give meaning to formulae such as (3.1.3) or (3.1.4), we see functions on M as functions on the cotangent bundle, depending only on the position. In other words, when we consider the function $x \mapsto g^t(x, dS(x))$ we actually have in mind $x \mapsto \pi g^t(x, dS(x))$, where π is the projection $T^*M \to M$.

We have

$$a_0(t,p) = T^{0,t}_{\bar{S}_0(p)} A_{\alpha_0},$$

and by the Duhamel formula,

$$a_k(t,p) = \int_0^t T^{s,t}_{\bar{S}_0(p)}\left(\frac{i\Delta a_{k-1}}{2}\right)(s,p)ds.$$

The function $u(t,x)$ (3.1.1) now satisfies the approximate equation

$$\frac{\partial u}{\partial t} = ih\frac{\Delta u}{2} - ih^N \int e^{\frac{i\bar{S}_0(t,x,p)}{h}} \frac{\Delta a_{N-1}}{2}(t,x,p)\chi(p)\frac{dp}{(2\pi h)^d};$$

the difference from the actual solution (with the same initial data) is bounded by $h^{N-d}\|\Delta a_{N-1}\|_{L^2} \leq h^{N-d}\|A_{\alpha_0}\|_{C^{2N}}$ in L^2 norm.

At this stage, the Ansatz reads

$$u(t,x)=\int e^{\frac{i\bar{S}_0(t,x,p)}{h}}\left(\sum_{k=0}^{N-1}h^k a_k(t,x,p)\right)\chi(p)\frac{dp}{(2\pi h)^d}. \tag{3.1.5}$$

For t away from 0, we can use the stationary phase method with respect to p and replace (3.1.5) by an expression of the form

$$u(t,x)=(2\pi h)^{-d/2}e^{\frac{iS_0(t,x)}{h}}\left(\sum_{k=0}^{N-1}h^k b_k^0(t,x)\right)$$

up to an error $O(h^N)\|A_{\alpha_0}\|_{C^{2N}}$, where

$$\begin{aligned}S_0(t,x)&=\bar{S}_0(t,x,p(t,x))\\ b_0^0(t,x)&=a_0(t,x,p(t,x))\chi(p(t,x));\end{aligned}$$

$p(t,x)$ is the vector based at z, which allows reaching x in time t (unique if we ask $\chi(p(t,x))\neq 0$). More generally, $b_k^0(t,x)$ is a linear combination of

$$\begin{gathered}D_p^{2k}a_0(t,x,p(t,x))\chi(p(t,x)),\\ D_p^{2(k-1)}a_1(t,x,p(t,x))\chi(p(t,x)),\ldots,a_k(t,x,p(t,x))\chi(p(t,x)),\end{gathered}$$

and hence involves $2k$ derivatives of A_{α_0}:

$$|d_x^m b_k^0(t,x)|\leq C(m+2k)\,h^{-\kappa(m+2k)} \tag{3.1.6}$$

(with $C(0)=1$).

Taking $t=1$, we find the expression

$$u(1,x)=(2\pi h)^{-d/2}e^{\frac{iS_0(1,x)}{h}}\left(\sum_{k=0}^{N-1}h^k b_k^0(1,x)\right) \tag{3.1.7}$$

as an approximation expression for $UA_{\alpha_0}\delta_z$, the difference from the actual solution being bounded by $h^{N-d}\|A_{\alpha_0}\|_{C^{2N}}$. Geometrically, the function $S_0(t,x)$ (when restricted to $x\in\Omega_{\alpha_1}$) is the generating function of the lagrangian manifold

$$\begin{aligned}\mathcal{L}_{S_0(t)}:=\{(x,d_xS_0(t,x)),x\in\Omega_{\alpha_1}\}&=\{(x,\xi)\in T^*M, x\in\Omega_{\alpha_1},\\ &\exists p\in T_z^*M \text{ s.t. } \|p\|_z\in[1-\varepsilon/2,1+\varepsilon/2],(x,\xi)=g^t(z,p)\},\end{aligned}$$

which is a union of "spheres" centered at z.

3.2. *The Ansatz for $n>1$.* By induction on n, we now propose an Ansatz for $U^tA_{\alpha_n}\ldots UA_{\alpha_1}UA_{\alpha_0}\delta_z$ $(0\leq t\leq 1)$. Starting from (3.1.7) we need to find an Ansatz for

$$U^t A_{\alpha_n} \dots A_{\alpha_2} U A_{\alpha_1} \left(e^{\frac{iS_0(1,x)}{h}} \left(\sum_{k=0}^{N-1} h^k b_k^0(1,x) \right) \right).$$

We will use $\Phi(x) = e^{\frac{iS_0(1,x)}{h}} (\sum_{k=0}^{N-1} h^k b_k^0(1,x))$ as a shorthand notation.

3.2.1. *The functions S_n.* The function $S_0(t)$ $(0 < t \leq 1)$ was defined in the previous paragraph. We define S_k by induction: Given $S_{k-1}(t)$ $(0 < t \leq 1)$, we define $S_k(t)$ as solution of the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H(x, d_x S) = 0$$

with initial data $S_k(0) = S_{k-1}(1)$; by the assumption about the injectivity radius, no caustics are met for $t \leq 1$, thus $S_k(t)$ is well defined as a smooth function on $\Omega_{\alpha_{k+1}}$. If we denote

$$\mathcal{L}_{S_k(t)} := \{(x, d_x S_k(t,x)), x \in \Omega_{\alpha_{k+1}}\}$$

the lagrangian manifold generated by $S_k(t)$, we have

$$\mathcal{L}_{S_k(t)} \subset g^t \mathcal{L}_{S_k(0)} = g^t \mathcal{L}_{S_{k-1}(1)}.$$

For an Anosov flow, the sphere bundle is transverse to the weak stable foliation. The lagrangian $\mathcal{L}_{S_0(1)} \subset T^*M$ is a union of (pieces of) spheres centered at z: as a consequence, $\mathcal{L}_{S_k(1)}$ becomes exponentially close, as $k \longrightarrow +\infty$, to a union of (pieces of) unstable leaves.

3.2.2. *The Ansatz.* By induction on n, we define a sequence of functions, $b_k^n(t,x)$ $(n \in \mathbb{N},\ k \leq N,\ x \in M,\ t \in [0,1])$ such that an Ansatz for $U^t A_{\alpha_n} \dots A_{\alpha_2} U A_{\alpha_1}.\Phi$ reads

$$U^t A_{\alpha_n} \dots A_{\alpha_2} U A_{\alpha_1}.\Phi \sim e^{\frac{iS_n(t,x)}{h}} \left(\sum_{k=0}^{N-1} h^k b_k^n(t,x) \right) + R_N^n(t,x)$$

with a remainder term of order h^N. We can make explicit the recurrence relation giving $(b_k^n)_{k=0,\dots,N-1}$ in terms of $(b_k^{n-1})_{k=0,\dots,N-1}$, as well as the remainder term R_N^n:

Suppose that the Ansatz found at the previous step gave the expression

$$U^t A_{\alpha_{n-1}} \dots A_{\alpha_2} U A_{\alpha_1}.\Phi = e^{\frac{iS_{n-1}(t,x)}{h}} \left(\sum_{k=0}^{N-1} h^k b_k^{n-1}(t,x) \right) + R_N^{n-1}(t,x)$$

where R_N^{n-1} is a remainder term which we know how to control in L^2 norm. Then

$$U^t A_{\alpha_n} \dots A_{\alpha_2} U A_{\alpha_1}.\Phi = U^t \left(e^{\frac{iS_{n-1}(1,x)}{h}} \left(\sum_{k=0}^{N-1} h^k c_k^{n-1}(1,x) \right) \right) + U^t A_{\alpha_n} \big(R_N^{n-1}\big)(x)$$

where $c_k^{n-1}(t,x) = A_{\alpha_n}(x) b_k^{n-1}(t,x)$. We now propose an Ansatz for

$$U^t\left(e^{\frac{iS_{n-1}(1,x)}{h}}\left(\sum_{k=0}^{N-1} h^k c_k^{n-1}(1,x)\right)\right)$$

in the form

$$v^n(t,x) = e^{\frac{iS_n(t,x)}{h}}\left(\sum_{k=0}^{N-1} h^k b_k^n(t,x)\right). \tag{3.2.1}$$

For v to be an approximate solution of $\partial_t v = ih\frac{\Delta v}{2}$ the coefficients should be solutions of

$$\begin{cases} \frac{\partial S_n}{\partial t} + H(x, d_x S_n) = 0 \\ \\ \frac{\partial b_k^n}{\partial t} = \frac{i\Delta b_{k-1}^n}{2} - \langle db_k^n, dS_n\rangle - \frac{b_k^n \Delta S_n}{2} \end{cases}$$

with initial conditions

$$\begin{cases} S_n(0,x) = S_{n-1}(1,x), \\ \\ b_k^n(0,x) = c_k^{n-1}(1,x). \end{cases}$$

Then v^n solves $\frac{\partial v^n}{\partial t}(t,x) = ih\frac{\Delta v^n}{2}(t,x) - ih^N e^{\frac{iS_n(t,x)}{h}} \frac{\Delta b_{N-1}^n(t,x)}{2}$. By the Duhamel formula,

$$\begin{aligned} U^t&\left(e^{\frac{iS_{n-1}(1,x)}{h}}\left(\sum_{k=0}^{N-1} h^k c_k^{n-1}(1,x)\right)\right) \\ &= v^n(t,x) + ih^N \int_0^t e^{\frac{i(t-s)h\Delta}{2}}\left(e^{\frac{iS_n(s,x)}{h}} \frac{\Delta b_{N-1}^n(s,x)}{2}\right) ds. \end{aligned}$$

We find the recurrence relation for the remainder terms:

$$R_N^n(t,x) = U^t A_{\alpha_n}\big(R_N^{n-1}\big)(x) + ih^N \int_0^t e^{\frac{i(t-s)h\Delta}{2}}\left(e^{\frac{iS_n(s,x)}{h}} \frac{\Delta b_{N-1}^n(s,x)}{2}\right).$$

This gives

$$\|R_N^n\|_{L^2(M)} \le \|R_N^{n-1}\|_{L^2(M)} + h^N \|\Delta b_{N-1}^n\|_2 \le \|R_N^{n-1}\|_{L^2(M)} + h^N \|\Delta b_{N-1}^n\|_\infty. \tag{3.2.2}$$

The recurrence relations for the coefficients b_k^n can be written

$$b_k^n(t,x) = T_{S_n}^{0,t} c_k^{n-1}(1,x) + \int_0^t T_{S_n}^{s,t}\left(\frac{i\Delta}{2} b_{k-1}^n(s,x)\right) ds \tag{3.2.3}$$

$$(c_k^{n-1} = A_{\alpha_n}.b_k^{n-1})$$

where

$$T_{S_n}^{s,t} a(x) = \frac{a \circ g^{-(t-s)}\big(x, d_x S_n(t,x)\big)}{\sqrt{J_{S_n}^{s,t}(x)}}$$

and J_{S_n} is the jacobian of the geodesic flow acting on the lagrangian manifold generated by S_n (defined as in (3.1.4) with $\bar{S}_0(p)$ replaced by S_n). Since a cut-off function A_{α_n} is inserted at each recurrence step, no caustics are ever met, and formula (3.2.1) defines a smooth function on M.

In vectorial notation, we can write $b^n = (b_0^n, \ldots, b_{N-1}^n) \in C^\infty([0,1], M)^N$. The recurrence relation becomes $(I - L_1^n)b^n = L_0^n b^{n-1}$ where L_0^n, L_1^n act on $C^\infty([0,1), M)^N$ as follows,

$$L_0^n = \begin{pmatrix} E^n & & & 0 \\ 0 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 0 & E^n \end{pmatrix} \text{ (a “diagonal” matrix)}$$

and

$$L_1^n = \begin{pmatrix} 0 & & & 0 \\ F^n\Delta & \ddots & & \\ & \ddots & \ddots & \\ 0 & & F^n\Delta & 0 \end{pmatrix} \text{ (a “nilpotent” operator),}$$

with

$$(E^n f)(t) = T_{S_n}^{0,t}(A_{\alpha_n} f(1)) \text{ and } (F^n f)(t) = \frac{i}{2}\int_0^t T_{S_n}^{s,t} f(s)ds.$$

The recurrence relation can be inverted,

$$b^n = \left(\sum_{k=0}^{N-1}[L_1^n]^k\right) L_0^n\, b^{n-1}.$$

It is easy to iterate this formula. We note that

$$[L_1^n]^{k_n} L_0^n [L_1^{n-1}]^{k_{n-1}} L_0^{n-1} \cdots [L_1^1]^{k_1} L_0^1 = 0$$

unless $k_1 + \cdots + k_n \leq N-1$. Thus, the formula expressing b^n in terms of b^0 is

$$b^n = \sum_{k_1+\cdots+k_n \leq N-1} [L_1^n]^{k_n} L_0^n [L_1^{n-1}]^{k_{n-1}} L_0^{n-1} \cdots [L_1^1]^{k_1} L_0^1\, b^0.$$

In the end, the formula expressing b_k^n in terms of b_k^0 $(k = 0, \ldots, N-1)$ is

$$b_k^n = \sum_{j=0,\ldots,N-1} \left(\sum_{k_1+\cdots+k_n = k-j \leq N} (F^n\Delta)^{k_n} E^n (F^{n-1}\Delta)^{k_{n-1}} E^{n-1} \cdots (F^2\Delta)^{k_1} E^1 \right) b_j^0.$$

We see in particular that the total number of derivatives of b^0 involved is never more than $2N$.

3.2.3. *Higher derivatives.* We need to control the derivatives $d^m b_k^n$ for $m \leq 2(N-k)$. Using the previous ideas we write a recurrence relation giving

$$\begin{aligned} B^n = B^n(t,x) := \big(& b_0^n(t,x), db_0^n(t,x), \ldots\ldots\ldots, d^{2N} b_0^n(t,x), \\ & b_1^n(t,x), db_1^n(t,x), \ldots, d^{2(N-1)} b_1^n(t,x), \\ & \ldots, \\ & b_{N-1}^n(t,x), \ldots, d^2 b_{N-1}^n(t,x) \big) \end{aligned}$$

in terms of B^{n-1} ($x \in M, t \in [0,1]$). In accordance with this disposition in array, we will denote $B_{(k,m)} = d^m b_k$ ($0 \leq k \leq N-1, m \leq 2(N-k)$).

Differentiating the recurrence relation (3.2.3) with respect to x, we get a relation of the form

$$\begin{aligned} d^m b_k^n(t,x) = & \sum_{j \leq m} T_{S_n}^{0,t} d^j c_k^{n-1}(1,x).\theta_{mj}^n(x) \\ & + \sum_{j \leq m} \int_0^t T_{S_n}^{s,t} d^{j+2} b_{k-1}^n(s,x).\alpha_{mj}^n(s,x) ds. \\ & (c_k^{n-1} = A_{\alpha_n} b_k^{n-1}). \end{aligned} \tag{3.2.4}$$

In this formula we denote

$$T_{S_n}^{s,t} d^j a(x) = \frac{d^j a_{g^{-(t-s)}(x, dS_n(t,x))}}{\sqrt{J_{S_n}^{s,t}(x)}},$$

$\theta_{mj}^n(x)$ is an m-linear form sending $(T_x M)^m$ to $(T_{g^{-t}(x,dS_n(t,x))} M)^j$, $\alpha_{mj}^n(s,x)$ is an m-linear form sending $(T_x M)^m$ to $(T_{g^{-(t-s)}(x,dS_n(t,x))} M)^{j+2}$.

The functions θ and α are uniformly bounded. We do not need to know their explicit expression, except for θ_{mm}^n: the latter can easily be shown to be

$$\theta_{mm}^n(x) = (dg_{(x,dS_n(t,x))}^{-t})^{\otimes m}.$$

In these formulae, $x \mapsto g_{(x,dS_n(t,x))}^{-t}$ is seen as a function from M to itself; see Remark 3.1.2.

In vectorial form, the recurrence relation (3.2.4) can be written as

$$(I - M_1^n) B^n = (M_{0,0}^n + M_{0,1}^n) B^{n-1},$$

where M_1^n is the nilpotent operator of order N,

$$M_1^n B^n = \left(\sum_{j \leq m} \int_0^t T_{S_n}^{s,t} d^{j+2} b_{k-1}^n(s,x).\alpha_{mj}^n(s,x) ds \right)_{0 \leq k \leq N-1,\, 0 \leq m \leq 2(N-k)},$$

$M_{0,0}^n$ is the diagonal operator

$$M_{0,0}^n B^{n-1} = \left(T_{S_n}^t d^m c_k^{n-1}(1,x).\theta_{mm}^n(x) \right)_{0 \leq k \leq N-1,\, 0 \leq m \leq 2(N-k)},$$

and $M_{0,1}$ is the nilpotent operator of order $\leq 2N$,

$$M_{0,1}B^{n-1} = \left(\sum_{j<m} T_n^{0,t} d^j c_k^{n-1}.\theta_{mj}^n\right)_{0\leq k\leq N-1,\ 0\leq m\leq 2(N-k)}.$$

As before, the recurrence relation can be inverted:

$$B^n = \left(\sum_{k\leq N} [M_1^n]^k\right)\left(M_{0,0}^n + M_{0,1}^n\right) B^{n-1},$$

then iterated,

$$B^n = \sum_{k_1,\dots,k_n,\epsilon_1,\dots,\epsilon_n} [M_1^n]^{k_n} M_{0,\epsilon_n}^n \dots [M_1^1]^{k_1} M_{0,\epsilon_1}^1 B^0. \tag{3.2.5}$$

Because of the special forms of M_1^n and $M_{0,1}^n$, the only terms that contribute to $B_{(k,m)}^n$ are those for which $\sum k_i \leq k$, $\sum \epsilon_i \leq m + 2(\sum k_i)$ (hence $\sum k_i \leq N$ and $\sum \epsilon_i \leq 2N$).

Call C a uniform bound for the differential forms θ and α. Remember that $c_k^{n-1} = A_{\alpha_n} b_k^{n-1}$, with $\|D^m A_{\alpha_n}\| \leq C(m)h^{-\kappa m}$. It follows easily that the operators of type M_1 and $M_{0,1}$ satisfy bounds of the form

$$\begin{aligned}&\sup_{t\in[0,1]} |(M_1 B(t,x))_{(k,m)}| \\ &\qquad\qquad \leq C \sup_{t\in[0,1]} \sup_{m'\leq m+2} |B(t,x)_{(k-1,m')}| C(m+2-m')h^{-\kappa(m+2-m')}\end{aligned}$$

and

$$\begin{aligned}&\sup_{t\in[0,1]} |(M_{0,1} B(t,x))_{(k,m)}| \\ &\qquad\qquad \leq C \sup_{t\in[0,1]} \sup_{m'\leq m-1} |B(t,x)_{(k,m')} C(m-1-m')| h^{-\kappa(m-1-m')}.\end{aligned}$$

For $M_{0,0}$, we have, for every (k,m) $(0\leq k\leq N-1, m\leq 2(N-k))$ and every x,

$$\left|(M_{0,0}^n B(t,x))_{(k,m)}\right| \leq \frac{1}{\sqrt{J_{S_n}^{0,n}(x)}} \left|B(1, g^{-t}(x, dS_n(t,x)))_{(k,m)}.(dg^{-t}_{(x,dS_n(t,x))})^{\otimes m}\right|.$$

If we put this estimate in (3.2.5), and use the composition property for the jacobian, $J_{S_k}^{0,t}(x) J_{S_{k-1}}^{s,1}(g^{-t}(x, dS_k(x,t)) = J_{S_{k-1}}^{s,t+1}(x)$, we find

$$(3.2.6)\quad |B^n(1,x)_{(k,m)}| \leq \tilde{C}(k,m)\frac{1}{\sqrt{J^{0,n}_{S_1}(x)}}\sum_{\sum k_i\leq k,\sum\epsilon_i\leq m+2k} C^{\sum k_i+\sum\epsilon_i}$$

$$\cdot\left(\sup_{(k'\leq k,m'\leq m+2k)} h^{-\kappa(m+2(k-k')-m')}\left|d^{m'}b^0_{k'}(0,g^{-n}(x,dS_n(1,x)))\right|\ \left|dg^{-n}_{(x,dS_n(1,x))}\right|^{m'}\right)$$

$$\leq \tilde{C}(k,m)\ \frac{1}{\sqrt{J^{0,n}_{S_1}(x)}}\Big(\sum_{\sum k_i\leq k,\sum\epsilon_i\leq m+2k} 1\Big)C^{m+3k}h^{-\kappa(m+2k)}\ \sup_{m'\leq m}\left|dg^{-n}_{(x,dS_n(1,x))}\right|^{m'}.$$

We used (3.1.6) in the last line. Although it does not really matter, we note that $C(0,0)=1$.

Let us inspect the behaviour of each term when n gets large. The term $J^{0,n}_{S_1}(x)$ is the same as the product

$$J^{0,1}_{S_n}(x)J^{0,1}_{S_{n-1}}\big(g^{-1}(x,dS_n(1,x))\big)$$
$$\dots J^{0,1}_{S_k}\big(g^{-(n-k)}(x,dS_n(1,x))\big)\dots J^{0,1}_{S_1}\big(g^{-(n-1)}(x,dS_n(1,x))\big).$$

Note that $\big(J^{0,n}_{S_1}(x)\big)^{-1}$ is the jacobian of g^{-n}, going from the lagrangian $\mathcal{L}_{S_n(1)}$ to $\mathcal{L}_{S_1(0)}$, evaluated at $(x,dS_n(1,x))$. As we saw, $\mathcal{L}_{S_k(1)}$ converges uniformly to a weak-unstable leaf as k gets large, so that $\big(J^{0,1}_{S_k}(g^{-(n-k)}(x,dS_n(1,x))\big)^{-1}$ converges to $J^u\big(g^{-(n-k+1)}(x,dS_n(1,x)\big)$ for large k. Taking Cesaro means, we have

$$\frac{1}{n}\log J^{0,n}_{S_1}(x)+\frac{1}{n}\sum_{k=1}^{n}\log J^u\big(g^{-(n-k+1)}(x,dS_n(1,x))\big)\longrightarrow 0.$$

It follows that we can bound $(J^n_{S_1}(x))^{-1/2}\leq J^u_n(\alpha_0,\dots,\alpha_n)^{1/2}(1+O(\varepsilon))^n$ for large n.

The next point is to note that $|dg^{-n}_{(x,dS_n(1,x))}|$ grows polynomially in n (uniformly in $x\in M$): if L is a d-dimensional submanifold, transversal to the strong stable foliation, then dg^{-n} is bounded on g^nL, independently of n. We apply this principle to $L=\mathcal{L}_{S_1(0)}$. There is a polynomial correction due to the fact that L is not contained in a fixed energy layer; the energy can vary in the interval $[1-\varepsilon/2,1+\varepsilon/2]$, so we also have to take into account the derivative of g^n with respect to energy, which grows linearly in n.

Finally we note that the number of terms $(\sum_{\sum k_i\leq k,\sum\epsilon_i\leq m+2k}1)$ in (3.2.6) is polynomial in n, it is at most $\tilde{C}(k,m)n^{m+3k}$. We have proved the following estimates (we reinsert the variable z that was omitted in the calculations; see Remark 3.1.1):

LEMMA 3.2.1. *For all $k\leq N$, for all $m\leq 2(N-k)$, for all n,*

$$|d^mb^n_k(1,x,z)|\leq C(k,m)n^{m+3k}J^u_n(\alpha_0,\dots,\alpha_n)^{1/2}(1+O(\varepsilon))^nh^{-\kappa(m+2k)},$$

if x is such that $g^{-k}(x,dS_n(1,x,z))\in\Omega_{\alpha_{n+1-k}}$ for all $k=0,\dots,n$. Otherwise, $b^n_k(1,x,z)\equiv 0$.

Comparing with (3.2.1), (3.2.2), we find

LEMMA 3.2.2.

$$|v^n(1,x,z)| \le J_n^u(\alpha_0,\dots,\alpha_n)^{1/2}(1+O(\varepsilon))^n \sum_{k=0}^{N-1} C(k,0) n^{3k} h^{k(1-2\kappa)};$$

$$\|R_N^n\|_{L^2(M)} \le nh^N \sup_{k\le N-1, m\le 2(N-k)} |d^m b_k^0| \le nh^N C(N) h^{-2\kappa N}.$$

We can now prove:

COROLLARY 3.2.3. *For any $\mathcal{K} > 0$, there exists $h_{\mathcal{K}}$ such that, for all $h < h_{\mathcal{K}}$,*

$$\begin{aligned}\|UA_{\alpha_n}UA_{\alpha_{n-1}}\dots UA_{\alpha_0}\mathrm{Op}(\chi)\delta_z\|_{L^2(M)} \\ \le 2(2\pi h)^{-d/2} J_n^u(\alpha_0,\dots,\alpha_n)^{1/2}(1+O(\varepsilon))^n\end{aligned}$$

uniformly for $n \le \mathcal{K}|\log h|$ and z in M.

Proof. We have

$$\begin{aligned}\|UA_{\alpha_n}UA_{\alpha_{n-1}}\dots UA_{\alpha_0}\mathrm{Op}(\chi)\delta_z - (2\pi h)^{-d/2} v^n(1,.,z)\|_{L^2(M)} \\ \le (2\pi h)^{-d/2}\|R_N^n\|_{L^2(M)}.\end{aligned}$$

Let $\mathcal{K} > 0$ be given. We can choose N large enough, and $h_{\mathcal{K}}$ such that

$$(2\pi h)^{-d/2} n C(N) h^{N(1-2\kappa)} << J_n^u(\alpha_0,\dots,\alpha_n)^{1/2}(1+O(\varepsilon))^n$$

for $n \le \mathcal{K}|\log h|$ and $h < h_{\mathcal{K}}$. This ensures that the remainder term R_N^n is negligible. We also choose $h_{\mathcal{K}}$ such that $\sum_{k=0}^{N-1} C(k,0) n^{3k} h^{k(1-2\kappa)} \le 2$ if $h < h_{\mathcal{K}}$. □

Theorem 1.3.3 is now a direct consequence of this corollary and of the decomposition (3.0.2).

4. Appendix A1: Small scale differential calculus

4.1. *Definition of* Op_h. Let $\Omega \subset \mathbb{R}^d$ be an open set, and $U = \Omega \times \Omega$. The space of symbols of order m is defined as:

$$\begin{aligned}\Sigma^m(U\times\mathbb{R}^d) := \big\{ & a \in C^\infty(U\times\mathbb{R}^d;\mathbb{C})/ \\ & \text{for every compact } K \subset U, \text{ for all } \alpha,\beta, \text{ there exists } C, \\ & |D_z^\alpha D_\xi^\beta a(z,\xi)| \le C(1+|\xi|)^{m-|\beta|} \text{ for all } (z,\xi) \in K\times\mathbb{R}^d\big\}.\end{aligned}$$

We denote $\Sigma^{-\infty} = \cap_{m\in\mathbb{Z}}\Sigma^m$ the space of regularizing symbols – it contains in particular the space of smooth compactly supported functions, $C_c^\infty(U\times\mathbb{R}^d)$.

Semi-classical symbols of order m and degree l (depending on a small parameter h) are defined as follows:

$$\Sigma^{m,l} = \{a_h(z,\xi) = h^l \sum_{j=0}^{\infty} h^j a_j(z,\xi),\ a_j \in \Sigma^{m-j}\}.$$

This means that $a_h(x,\xi)$ has an asymptotic development in powers of h, in the sense that

$$a - h^l \sum_{j=0}^{N-1} h^j a_j \in h^{l+N}\Sigma^{m-N}$$

uniformly in h. In this semiclassical context, the space of regularizing symbols is $\Sigma^{-\infty,+\infty} = \cap_{m\geq 0}\Sigma^{-m,m}$.

Let $a = a(x,y;\xi) \in \Sigma_c^{m,l}(\Omega\times\Omega\times\mathbb{R}^d)$. The subscript $_c$ means that the support of a in $\Omega\times\Omega$ is proper; in other words, for every compact $K\subset\Omega$, there exists a compact $K'\in\Omega$ such that $a(x,y,\xi)=0$ for $x\in K$ and $y\notin K'$. Define $\mathrm{OP}(a)u(x) = (2\pi h)^{-d}\int e^{\frac{i}{h}(x-y|\xi)}a(x,y,\xi)u(y)dyd\xi$, well defined if u is smooth. Denote $\Psi_c^{m,l}(\Omega)$ the space of these operators, called (proper) pseudo-differential operators of degree l and order m; $\Psi_c^{-\infty,\infty}(\Omega)$ consists of regularizing operators, which means here that the kernel is smooth and all its derivatives are $O(h^\infty)$ uniformly on compact sets. An operator in $\Psi_c^{0,0}(\Omega)$ acts continuously from $L^2(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$, uniformly in h. There exists an integer N_d depending on the dimension d such that, for all $a\in\Sigma_c^{0,0}$, for every compact set K, $\|\mathrm{OP}_h(a)\|_{L^2(K)} \leq (\|a\|_{0,K} + h^{1/2}\|Da\|_{0,K} + \cdots + h^{N_d/2}\|D^{N_d}a\|_{0,K})$.

Now let M be a smooth compact d-dimensional manifold. Choose a finite partition of unity φ_l ($\sum\varphi_l^2\equiv 1$), such that the support of each φ_l is endowed with local coordinates in $\mathbb{R}^d$; for $a\in\Sigma^{m,0}(T^*M)$, we define :

$$\mathrm{Op_h}(a) = \sum_l \mathrm{OP}(\varphi_l(x)\varphi_l(y)a(x,\xi))\ ,$$

where each term in the sum is defined in local coordinates thanks to the previous formula. The map $a\mapsto \mathrm{Op}_h(a)$ thus defined depends on the choice of the partition of unity, and of local coordinates. Its image, however, is well defined up to regularizing operators. The algebra $\Psi^{m,0}(M)$ of pseudodifferential operators on M is thus well defined, modulo regularizing operators.

4.2. *Small scale symbols.* We defined $\mathrm{Op}_h(a)$ when $a_h(x,\xi)$ is smooth and has a nice behaviour when $\xi\longrightarrow\infty$, $h\longrightarrow 0$. However, a more careful study shows that certain aspects of the theory are still valid if the derivatives of the symbols are allowed to explode at a reasonable rate, when $h\longrightarrow 0$. The theory is developed in detail in [DS]; we just point out a few facts that are useful in the paper. The main tool is the following variant of the stationary phase method.

LEMMA 4.2.1. *Let* $(a^{(h)})_{h\in(0,1]}$ *be a family of* C^∞ *functions on* $\mathbb{R}^d\times\mathbb{R}^d$, *with a fixed compact support, and satisfying the following estimates on the derivatives*:

$$\|D^n a^{(h)}\|_0 \leq C_n h^{-n\kappa}$$

for all $n\in\mathbb{N}$, *for some* $\kappa\in[0,1/2)$ *and some sequence of positive real numbers* (C_n).

(i) *The integral* $\int_{\mathbb{R}^d\times\mathbb{R}^d} a^{(h)}(x,\xi)e^{\frac{i\langle\xi,x\rangle}{2h}}dxd\xi$ *obeys the following asymptotics as* $h\longrightarrow 0$:

$$\frac{1}{(2\pi h)^d}\int_{\mathbb{R}^d\times\mathbb{R}^d} a^{(h)}(x,\xi)e^{\frac{i\langle\xi,x\rangle}{2h}}dxd\xi = a^{(h)}(0,0)+O(h^{1-2\kappa}).$$

One can even write an asymptotic development to all orders in powers of h.

(ii) *If, for all* h, $0\notin \operatorname{supp} a^{(h)}$, *then*

$$\frac{1}{(2\pi h)^d}\int_{\mathbb{R}^d\times\mathbb{R}^d} a^{(h)}(x,\xi)e^{\frac{i\langle\xi,x\rangle}{2h}}dxd\xi = O(h^\infty).$$

It follows that certain results of pseudo-differential calculus still hold if the derivatives of the symbols do not explode faster than powers of $h^{-\kappa}$ ($\kappa<1/2$). For instance:

THEOREM 4.2.2 (Calderon-Vaillancourt Theorem). *On a* d*-dimensional compact manifold, there exists an integer* N_d *such that, for all* $a\in C^\infty(T^*M)$,

$$\|\mathrm{Op}_h(a)\|_{L^2(M)} \leq (\|a\|_0 + h^{1/2}\|Da\|_0+\cdots+h^{N_d/2}\|D^{N_d}a\|_0).$$

In particular, if $a^{(h)}$ *depends on* h *in a way that*

$$\|D^n a^{(h)}\|_0 \leq C_n h^{-n\kappa}$$

for all $n\in\mathbb{N}$, *for some* $\kappa\in[0,1/2)$ *and some sequence of real numbers* (C_n), *then the operators* $\mathrm{Op}_h(a^{(h)})$ *are uniformly bounded in* $L^2(M)$.

One can then show:

THEOREM 4.2.3. *Let* $(a^{(h)})$ *and* $(b^{(h)})$ *be two families of* C^∞ *functions on* T^*M, *with a common compact support, and satisfying estimates of the form*

$$\|D^n a^{(h)}\|_0 \leq C_n h^{-n\kappa}$$

and

$$\|D^n b^{(h)}\|_0 \leq C_n h^{-n\kappa}.$$

Then

(i) $$\|\mathrm{Op}_h(a^{(h)})\mathrm{Op}_h(b^{(h)}) - \mathrm{Op}_h(a^{(h)}b^{(h)})\|_{L^2(M)} = O(h^{1-2\kappa}).$$

(ii) $$\| [\mathrm{Op}_h(a^{(h)}), \mathrm{Op}_h(b^{(h)})] \|_{L^2(M)} = O(h^{1-2\kappa}).$$

(iii) *If, for all* h, $\mathrm{supp}\, a^{(h)} \cap \mathrm{supp}\, b^{(h)} = \emptyset$, *then*

$$\|\mathrm{Op}_h(a^{(h)})\mathrm{Op}_h(b^{(h)})\|_{L^2(M)} = O(h^\infty).$$

(iv) (*Egorov Theorem*) *For any given* t,

$$\|\mathrm{Op}_h(a^{(h)})(t) - \mathrm{Op}_h(a^{(h)} \circ g^t)\|_{L^2(M)} = O(h^{1-2\kappa}).$$

Remember the notation: $A(t) = e^{\frac{-ith\Delta}{2}} A e^{\frac{ith\Delta}{2}}$, for any operator A.

We will also need a result about the range of validity of the Egorov theorem.

THEOREM 4.2.4 (Ehrenfest time for the evolution of observables, from [BR02]). *There exists* $\Lambda > 0$ *such that, for every* $\kappa \in [0, 1/2)$, *if* $(a^{(h)})$ *is a family of* C^∞ *functions on* T^*M, *with a common compact support, satisfying estimates of the form*

$$\|D^n a^{(h)}\|_0 \le C_n h^{-n\kappa},$$

then for $\bar\kappa > 0$

$$\sup_{|t| \le \bar\kappa |\log h|} \|\mathrm{Op}_h(a^{(h)})(t) - \mathrm{Op}_h(a^{(h)} \circ g^t)\|_{L^2(M)} = O(h^{1-2\kappa-2\Lambda\bar\kappa}),$$

for all $h \in (0, 1]$.

This follows directly from the arguments in [BR02]; the assumptions that the symbol $a^{(h)}$ and its derivatives are bounded can be relaxed to $\|D^n a^{(h)}\|_0 \le C_n h^{-n\kappa}$. For the number Λ we can take an upper bound for the Lyapunov exponents of the geodesic flow.

Putting together Theorem 4.2.3 and Theorem 4.2.4, we obtain:

COROLLARY 4.2.5. *For every* $\kappa \in [0, 1/2)$, *if* $(a^{(h)})$, $(b^{(h)})$ *are families of* C^∞ *functions on* TM, *with a common compact support, and satisfying estimates of the form*

$$\|D^n a^{(h)}\|_0 \le C_n h^{-n\kappa},$$

$$\|D^n b^{(h)}\|_0 \le C_n h^{-n\kappa},$$

then there exists a constant C *such that*

$$\| [\mathrm{Op}_h(a^{(h)})(t), \mathrm{Op}_h(b^{(h)})] \|_{L^2(M)} \le C h^{1-2\kappa-2\Lambda\bar\kappa}$$

for all $\bar\kappa > 0$ *and all* $|t| \le \bar\kappa |\log h|$.

We can prove Lemma 2.3.3:

COROLLARY 4.2.6. *Let χ be an energy cut-off, supported in a neighbourhood of the energy layer $\{\|v\| = 1\}$. For all $\bar\kappa > 0$ for every $N \leq 2\bar\kappa|\log h|$, for every permutation τ of $\{0, \ldots, N\}$, for every sequence $t_0, \ldots, t_N$ such that $|t_i| \leq \bar\kappa|\log h|$, for every sequence $\alpha_0, \ldots, \alpha_N$,*

$$\begin{aligned}&\|\mathrm{Op}_h(\chi)^* A_{\alpha_N}(t_N) \ldots .. A_{\alpha_1}(t_1) A_{\alpha_0}(t_0) \mathrm{Op}_h(\chi)\\ &- \mathrm{Op}_h(\chi)^* \mathrm{Op}_h(\chi) A_{\alpha_{\tau N}}(t_{\tau N}) \ldots .. A_{\alpha_{\tau 1}}(t_{\tau 1}) A_{\alpha_{\tau 0}}(t_{\tau 0})\|_{L^2(M)} = O(h^{1-2\kappa-3\Lambda\bar\kappa}).\end{aligned}$$

Proof. In the case when τ is a transposition of two consecutive integers, the proposition follows directly from Corollary 4.2.5, since the functions A_α satisfy $\|D^n A_\alpha\|_0 \leq C(n) h^{-\kappa n}$.

Otherwise, the result can be proved noting that one can write any permutation of $\{0, \ldots, N\}$ as the product of at most $(N+1)^2$ such transpositions. □

As a corollary we can prove Lemma 2.2.2:

COROLLARY 4.2.7. *Let χ be an energy cut-off, supported in a neighbourhood of the energy layer $\{\|v\| = 1\}$. There exist $\bar\kappa$ and $\alpha > 0$ such that, for all $n \leq \bar\kappa|\log h|$, for every subset $W \subset \Sigma_n$,*

$$\|\sum_{\mathcal{C}\in W} \hat{\mathcal{C}}_h \mathrm{Op}_h(\chi)\|_{L^2(M)} \leq 1 + O(h^\alpha).$$

Proof. Define $B_i = \sqrt{A_i}$. By Corollary 4.2.6, we have

$$\begin{aligned}\sum_{\mathcal{C}\in W} \langle \hat{\mathcal{C}} \mathrm{Op}_h(\chi)\psi, \psi\rangle = &\sum_{[\alpha_0,\ldots,\alpha_n]\in W} \|B_{\alpha_n}(n) \ldots B_{\alpha_0} \mathrm{Op}_h(\chi^{1/2})\psi\|^2_{L^2(M)}\\ &+ \sharp W . O(h^{1-2\kappa-3\Lambda\bar\kappa}).\end{aligned}$$

We see that each operator $\hat{\mathcal{C}}\mathrm{Op}_h(\chi)$ is close to being a positive operator, and we know their sum has norm less than $1+O(h)$. Of course we should choose $\bar\kappa$ small enough so that the remainder term remains small, i.e. $\sharp W . O(h^{1-2\kappa-3\Lambda\bar\kappa}) = O(h^\alpha)$ – this is possible since $\sharp W$ grows exponentially with n. □

5. Appendix A2: construction of the partition of unity (A_i^h)

The purpose of this appendix is to show how to construct the A_i so as to satisfy the requirements of subsection 2.1.

Of course, this holds if we have the property: There exists $p > 0$ such that

$$\int_B |\psi_h(x)|^2 d\,\mathrm{Vol}(x) = O(h^p),$$

where B is the tubular neighbourhood of size h^κ of the boundary of the partition P. Thus, we try to modify very slightly the partition P so that its boundary is piecewise smooth, and the smooth hypersurfaces $(S_k)_{k=1,\ldots,L}$ forming the boundary satisfy

$$\int_{V_k(h^\kappa)} |\psi_h(x)|^2 d\,\mathrm{Vol}(x) = O(h^p) \tag{5.0.1}$$

where $V_k(h^\kappa)$ is a tubular neighbourhood of S_k of size h^κ.

Starting with an initial partition $P(0) = P$ whose boundary consists of a finite number of smooth hypersurfaces $(S^k(0))_{k=1,\dots,L}$, we can deform it slightly to a partition $P(h)$, with boundary components $(S^k(h))_{k=1,\dots,L}$ that satisfy (5.0.1). The new partition will depend on h, but in a way that does not affect the proof of Theorem 1.1.1: in our construction the boundary components $(S^k(h))_{k=1,\dots,L}$ will converge to the original $(S^k(0))_{k=1,\dots,L}$.

We start with a simple remark. Consider an open subset $U \subset M$ equipped with a chart $\Phi : U \longrightarrow \mathbb{R}^d$ that sends U to the cube $(-2,2)^d$. Let $\tilde{S} \subset [-1,1]^{d-1}$, $\tilde{S}(0) = \tilde{S}\times\{0\} \subset (-2,2)^d$, and $S(0) = \Phi^{-1}(\tilde{S})$. And more generally, given $0 < \varepsilon < 1$ and $0 < s < 1/4$, we define

$$\begin{aligned}
\tilde{S}_\varepsilon &= \{x \in (-2,2)^{d-1},\ d(x,\tilde{S}) \le \varepsilon\} \subset (-2,2)^{d-1},\\
\tilde{S}_\varepsilon(m,h) &= \tilde{S}_\varepsilon \times \{mh^{1/2-s}\},\\
\tilde{V}_\varepsilon(m,h) &= \tilde{S}_\varepsilon \times [(m-1/2)h^{1/2-s}, (m+1/2)h^{1/2-s}
\end{aligned}$$

and, finally,

$$\begin{aligned}
S_\varepsilon(m,h) &= \Phi^{-1}(\tilde{S}_\varepsilon(m,h)),\\
V_\varepsilon(m,h) &= \Phi^{-1}(\tilde{V}_\varepsilon(m,h))
\end{aligned}$$

(the latter is a tubular neighbourhood of size $h^{1/2-s}$ of the former); m is an integer in $[-h^{-1/2+2s}, h^{-1/2+2s}]$. Since

$$\sum_{m\in[-h^{-1/2+2s},h^{-1/2+2s}]} \int_{V_\varepsilon(m,h)} |\psi_h(x)|^2 d\,\mathrm{Vol}(x) \le 1$$

there must exist an $m_0 \in [-h^{-1/2+2s}, h^{-1/2+2s}]$ (depending on h) such that

$$\int_{V_\varepsilon(m_0,h)} |\psi_h(x)|^2 d\,\mathrm{Vol}(x) \le h^{1/2-2s}.$$

This means that $S_\varepsilon(m_0,h)$ satisfies (5.0.1) with $\kappa = 1/2 - s$ and $p = 1/2 - 2s$ (which is even better than what we need). Besides, $S_\varepsilon(m_0,h)$ is at distance h^s from $S_\varepsilon(0)$.

We conclude that there is a hypersurface h^s-close to $S(0)$ that satisfies (5.0.1).

Let us now consider a partition $P(0)$, with boundary components $(S^k(0))_{k=1,\dots,L}$. For every k, we know that there exists a hypersurface $S^k_\varepsilon(h)$ h^s-close to $S^k_\varepsilon(0)$ that satisfies (5.0.1) with $p = 1/2 - 2s$. We need to show, in addition, that for each k, there exists $S^k(h) \in S^k_\varepsilon(h)$ such that the $S_k(h)$s form the boundary of a new partition.

Although this is probably always true for general partitions with piecewise smooth boundary, we will avoid a tedious combinatorial argument by considering only special “cubic” partitions that we describe below: In the universal cover $\widetilde{M}$, consider a polyhedral fundamental domain $D(0)$ for the action of $\Gamma = \pi_1(M)$, whose boundary is piecewise smooth; consider also an open, relatively compact subset $U \subset M$, containing $D(0)$, and equipped with a chart $\Phi : U \longrightarrow \mathbb{R}^d$ that sends U to the cube $(-2,2)^d$. Given $\alpha > 0$, one has a partition of $(-2,2)^d$ into cubes of size ε, delimited by the hypersurfaces $\tilde{S}^{k,m}(0) = \{x_k = m\varepsilon\}$ ($k = 1, \ldots, d$, $m \in \mathbb{Z}$, $|m| \leq 2/\varepsilon$). This partition gives a partition of U which, restricted to the fundamental domain $D(0)$, gives our partition $P(0)$ of M. More precisely, the boundary of $P(0)$ is formed by the image in M of

– parts of the $S^{k,m}(0) = \Phi^{-1}(\tilde{S}^{k,m}(0))$;

– the boundary of $D(0)$.

Most elements of $P(0)$ are sent to cubes by the chart Φ, except for those intersecting the boundary of the fundamental domain, which look like a cube cut by a smooth hypersurface.

The boundary of the “polyhedra” $D(0)$ consists of a finite number of smooth hypersurfaces $S^k(0)$; applying the previous procedure, we can find some $S^k_\varepsilon(h)$ satisfying (5.0.1) and such that

– for each k, we can find a subset $S^k(h) \subset S^k_\varepsilon(h)$ such that the $S^k(h)$s form the boundary of a new fundamental domain $D(h)$.

– $S^k(h)$ is at distance h^s from $S^k(0)$.

In the cube $(-2,2)^d$, always by the same procedure, we can move the $\tilde{S}^{k,m}(0)$s to

$$\tilde{S}^{k,m}(h) = \{x_k = m\alpha + m_0(k,m)h^{1/2-s}\}$$

($m_0(k,m) \in [-h^{-1/2+2s}, h^{-1/2+2s}]$ as previously) so that

$$S^{k,m}(h) := \Phi^{-1}(\tilde{S}^{k,m}(h))$$

satisfies (5.0.1), for every k, m. Besides, the $\tilde{S}^{k,m}(h)$ still delimit a partition of $(-2,2)^d$ into cubes and thus the $S^{k,m}(h)$ delimit a partition of the open set $U \in M$.

This partition of U, restricted to the fundamental domain $D(h)$, gives our partition $P(h)$ of M. More precisely, the boundary of $P(h)$ is formed by the image in M of

– parts of the $S^{k,m}(h) = \Phi^{-1}(\tilde{S}^{k,m}(0))$;

– the boundary of $D(h)$.

The boundary of the new partition $P(h)$ satisfies (5.0.1) and converges to the boundary of $P(0)$, in the C^∞ topology, when $h \longrightarrow 0$. The characteristic function of $P_i(h)$ converges to the characteristic function of $P_i(0)$, uniformly on every compact set not intersecting the boundary of $P_i(0)$ (for every $i = 1, \ldots, l$).

We finally construct the smooth partition of unity A_i^h by applying the convolution (2.1.1) to $P_i(h)$ instead of P_i.

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