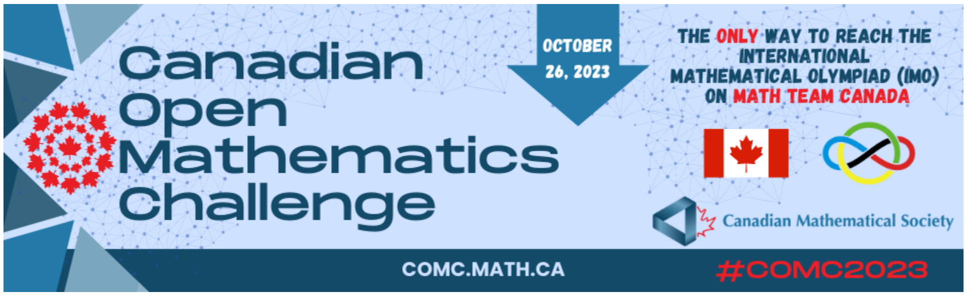
2023 Canadian Open Mathematics Challenge

Official Solutions



A competition of the Canadian Mathematical Society.

The COMC has three sections:

A. Short answer questions worth 4 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

B. Short answer questions worth 6 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.

C. Multi-part full solution questions worth 10 marks each. Solutions must be complete and clearly presented to receive full marks.

Section A

A1 Ty took a positive number, squared it, then divided it by 3, then cubed it, and finally divided it by 9. In the end he received the same number as he started with. What was the number?

Answer: 3

Solution:
Assume Ty started with the number x. After squaring it and dividing by 3, he will get the number $\frac{x^2}{3}$. After cubing this number and dividing by 9, he will get the number

$$\frac{\left(\frac{x^2}{3}\right)^3}{9} = \frac{\left(\frac{x^6}{3^3}\right)}{9} = \frac{x^6}{3^5}.$$

We thus get that $x = \frac{x^6}{3^5}$, and after dividing both sides by x and multiplying both sides by 3^5, we get $x^5 = 3^5$, and so $x = 3$.

A2 A point with coordinates $(a, 2a)$ lies in the 3rd quadrant and on the curve given by the equation $3x^2 + y^2 = 28$. Find a.

Answer: -2

Solution:
We plug in the point $(x, y) = (a, 2a)$ into the equation of the curve, to get

$$3a^2 + (2a)^2 = 28,$$

and now expanding and dividing both sides of the equation by 7, we get that $a^2 = 4$, and so $a = \pm 2$. Since the point lies in the third quadrant, we must choose $a = -2$.

A3 Tanya and Katya made an experiment and obtained two positive real numbers, each of which was between 4 and 100 inclusive. Tanya wrote the two numbers x and y and found their average while Katya wrote the second number twice and took the average of the three numbers x, y, and y. What is the maximum number by which their results may differ?

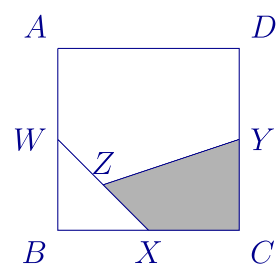
Answer: 16

Solution:
Tanya's calculation is $\dfrac{x+y}{2}$ and Katya's is $\dfrac{x+2y}{3}$. We want the positive difference of these two quantities; after taking the common denominator of 6, this is

$$\left|\frac{3x+3y-2x-4y}{6}\right| = \left|\frac{x-y}{6}\right|.$$

Since x and y are between 4 and 100, inclusive, the most they can differ is $100-4=96$; in this case, the difference between Tanya's calculation and Katya's calculation is $\frac{96}{6}=16$.

A4 Square $ABCD$ has side length 10 cm. Points W, X, Y, and Z are the midpoints of segments AB, BC, CD and WX, respectively. Determine the area of quadrilateral $XCYZ$ (in cm^2).



Answer: 25

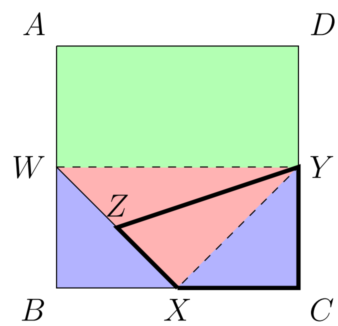
Solution 1:
We break up the quadrilateral $XCYZ$ into two triangles; triangle YZX and triangle YXC. As Z is on the midpoint of line WX, the areas of triangle YZX and YZW are equal; they are both half of the area of triangle WYX, which is a right isosceles triangle with bases of length $5\sqrt{2}$ cm. Thus, the area of triangle YZX is

$$\frac{1}{2}\times\frac{1}{2}(5\sqrt{2})^2 = \frac{25}{2}\,\text{cm}^2.$$

The area of triangle YXC is also $\frac{25}{2}$ cm^2, since it is a right isosceles triangle with bases of length 5 cm. Thus, the total area of quadrilateral $YZXC$ is 25 cm^2.

Solution 2:
As Z is the midpoint of WX, the areas of the triangles having equal bases and equal heights are equal: $[YWZ] = [YXZ]$. Clearly $\Delta WBX \cong \Delta YCX$, so $[WBX] = [YCX]$. Hence

$$[XCYZ] = \frac{1}{2}[WBCY] = \frac{1}{4}[ABCD] = 25\,\text{cm}^2.$$



Solution 3:
We place the points on a coordinate grid, with B being the origin, line BC being the x-axis, and line AB being the y-axis. We see that $X = (5, 0)$, $W = (0, 5)$, $C = (10, 0)$, and $Y = (10, 5)$. Thus, $Z = (\frac{5}{2}, \frac{5}{2})$. By the Shoelace Formula, the area of quadrilateral $XCYZ$ is equal to

$$\left|\frac{1}{2}\left(\left(5\cdot\frac{5}{2}+\frac{5}{2}\cdot 5+10\cdot 0+10\cdot 0\right)-\left(0\cdot\frac{5}{2}+\frac{5}{2}\cdot 10+5\cdot 10+0\cdot 5\right)\right)\right| = \left|\frac{1}{2}(25-75)\right| = 25.$$

Section B

B1 A bug moves in the coordinate plane, starting at $(0, 0)$. On the first turn, the bug moves one unit up, down, left, or right, each with equal probability. On subsequent turns the bug moves one unit up, down, left, or right, choosing with equal probability among the three directions other than that of its previous move. For example, if the first move was one unit up then the second move has to be either one unit down or one unit left or one unit right.

After four moves, what is the probability that the bug is at $(2, 2)$?

Answer: $\frac{1}{54}$

Solution 1:
The bug must alternate between moving up and right. On the first move, the probability the bug moves either up or right is $\frac{1}{2}$. On each subsequent move, the bug has three choices (down, left, and the other move that was not the bug's previous move); the probability the bug continues towards $(2, 2)$ is $\frac{1}{3}$. Thus, the total probability is $\frac{1}{2}\left(\frac{1}{3}\right)^3 = \frac{1}{54}$.

Solution 2: The required probability is equal to the number of ways to get from $(0, 0)$ to $(2, 2)$ in four moves divided by the number of all possible four-move paths starting from $(0, 0)$.

There are two ways to get from $(0, 0)$ to $(2, 2)$: either URUR or RURU. There are $4 \times 3 \times 3 \times 3$ four-move paths starting from $(0, 0)$. So, the probability is $\frac{2}{4\times3\times3\times3} = \frac{1}{54}$.

B2 This month, I spent 26 days exercising for 20 minutes or more, 24 days exercising 40 minutes or more, and 4 days of exercising 2 hours exactly. I never exercise for less than 20 minutes or for more than 2 hours.
What is the minimum number of hours I could have exercised this month?

Answer: 22

Solution: From the four days where I exercised 2 hours, I exercised a total of 8 hours. There are $24 - 4 = 20$ other days where I exercised at least 40 minutes; this contributes a minimum of 800 minutes, or 13 hours and 20 minutes. Finally, there are $26 - 24 = 2$ more days where I exercised at least 20 minutes; this contributes a minimum of 40 minutes, for a total of $8 + 13 + 1 = 22$ hours.

Alternative representations of the verbal solution:

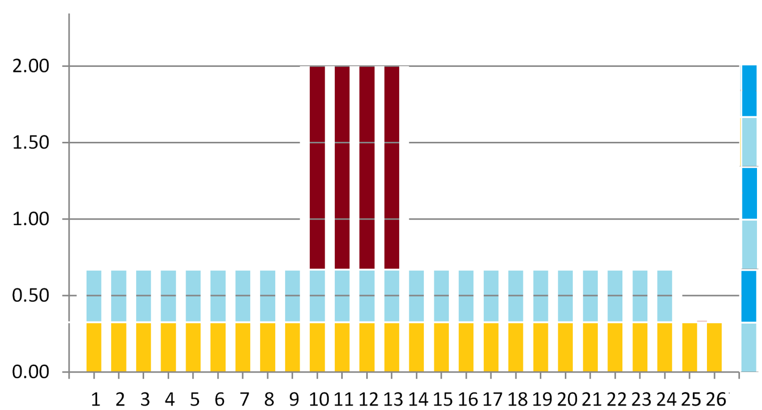
Calculation in minutes:

$$\frac{[(26-24)\times 20 + (24-4)\times 40 + 4\times 120] \text{ minutes}}{60 \text{ min / hr}} = \frac{840+480}{60} = 22 \text{ hours.}$$

Calculation in hours:

$$(26-24)\times\frac{1}{3} + (24-4)\times\frac{2}{3} + 4\times 2 = 14 + 8 = 22 \text{ hours.}$$

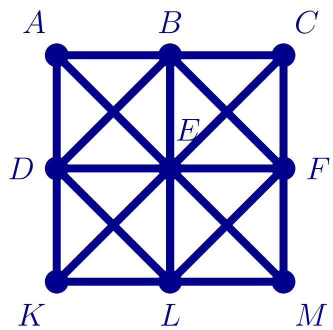
Graphical representation of minimum hours of exercise:



Reading off the graph of exercise horizontally, we have (in hours) $26 \times \frac{1}{3} + 24 \times \frac{1}{3} + 4 \times \frac{4}{3} = \frac{66}{3} = 22$ hours.

B3 A 3×3 grid of 9 dots labeled by A, B, C, D, E, F, K, L, and M is shown in the figure. There is one path connecting every pair of adjacent dots, either orthogonal (i.e. horizontal or vertical) or diagonal. A turtle walks on this grid, alternating between orthogonal and diagonal moves. One could describe any sequence of paths in terms of the letters $A, \cdots, M$. For example, $A - B - F$ describes a sequence of two paths AB and BF.

What is the maximum number of paths the turtle could traverse, given that it does not traverse any path more than once?



Answer: 17

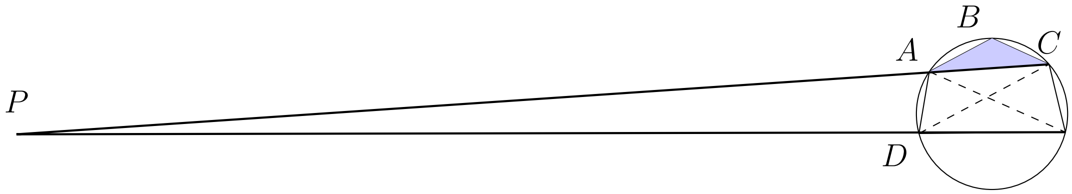
Solution: The answer is 17. This is an upper bound because there are 8 possible diagonal moves, hence at most 9 possible orthogonal moves. There are many ways to construct this.

One possibility for a 17-paths trip is $ABDEADLKEBFECFLMEL$.

B4 Consider triangle ABC with angles $\angle BAC = 24°$ and $\angle ACB = 28°$. Point D is constructed such that AB is parallel to CD, $AD = BC$, and AD and BC are not parallel. Similarly, point E is constructed such that AE is parallel to BC, $AB = CE$, and AB and CE are not parallel. Lines DE and AC intersect at point P. Determine angle $\angle CPE$ (in degrees).

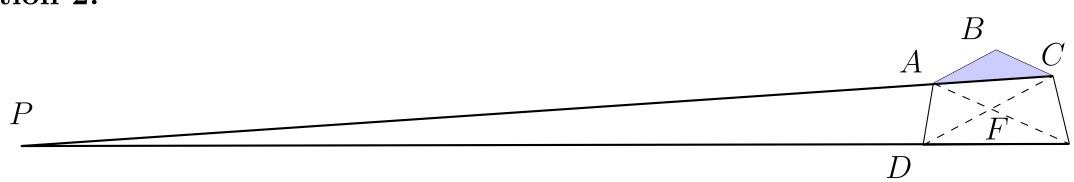
Answer: 4

Solution 1:



By construction, the points A, B, C, D, E are all concyclic because $ABCD$ and $ABCE$ are isosceles trapezoids.
From triangle ABC, $\angle ABC = 180° - 24° - 28° = 128°$;
From trapezoid $ABCE$, $\angle BCE = \angle ABC = 128°$;
$\angle ACE = \angle BCE - \angle BCA = 128° - 28° - 100°$;
$\angle DEA = \angle DCA = \angle BAC = 24°$;
From trapezoid $ABCE$, $\angle CEA = 180° - \angle BCE = 180° - 128° = 52°$;
$\angle DEC = \angle DEA + \angle AEC = 24° + 52° = 76°$;
Thus, from triangle CPE, $\angle CPE = 180° - 100° - 76° = 4°$.

Solution 2:



By construction, $ABCD$ and $ABCE$ are isosceles similar trapezoids. Let F be the point of intersection of AE and DC. Then $ABCF$ is a parallelogram.
From triangle ABC, $\angle ABC = 180° - 24° - 28° = 128°$;
From trapezoid $ABCE$, $\angle BCE = \angle ABC = 128°$;
$\angle ACE = \angle BCE - \angle BCA = 128° - 28° - 100°$;
From trapezoid $ABCE$, $\angle CEA = 180° - \angle BCE = 180° - 128° = 52°$;
Similarly, from trapezoid $DABC$, $\angle ADC = 52°$.
Since $\angle ADC = \angle CEA$ and $\angle AFD = \angle CFE$, triangles AFD and CFE are similar, so $\frac{DF}{EF} = \frac{AD}{CE}$. But $AD = BC$ and $CE = AB$, so $\frac{DF}{EF} = \frac{BC}{AB}$.
Since $\angle DFE = \angle AFC = \angle ABC$, triangles ABC and EFD are similar.
Thus, $\angle DEA = \angle BAC = 24°$;
Therefore, $\angle DEC = \angle DEA + \angle AEC = 24° + 52° = 76°$;
Thus, from triangle CPE, $\angle CPE = 180° - 100° - 76° = 4°$.

Section C

C1 Let F be a function which maps integers to integers by the following rules:
$F(n) = n - 3$ if $n \geq 1000$;
$F(n) = F(F(n+5))$ if $n < 1000$.

(a) Find $F(999)$.

(b) Show that $F(984) = F(F(F(1004)))$.

(c) Find $F(84)$.

(a) Answer: 998
(a) Solution: We must apply the second rule, since $999 < 1000$. We have that $F(999) = F(F(1004))$. Now, we can apply the first rule, to reduce this to $F(1001)$. Applying the first rule again gives us the result of 998.
$F(999) = F(F(1004)) = F(1001) = 998$.

(b) Solution 1: By direct evaluation, applying the second rule four times we have
$F(984) = F(F(989)) = F(F(F(994))) = F(F(F(F(999)))) = F(F(F(F(F(1004)))))$.

Now, applying the first rule twice, then the second rule, we obtain
$F(F(F(F(F(1004))))) = (F(F(F(F(1001)))) = (F(F(F(998))) = (F(F(F(F(1003))))$.

Now, applying the first rule twice, then the second rule, the first and the second we get
$(F(F(F(F(1003)))) = (F(F(F(1000)))) = F(F(997)) = F(F(F(1002))) = F(F(999)) = F(F(F(1004)))$.

Thus, $F(984) = F(F(F(1004)))$.

(b) Solution 2: For this and the next part, we try to write down a table of values for the function F. When $n \geq 1000$, $F(n) = n - 3$ is known, and in the previous part, we showed that $F(999) = 998$. Now,

$$F(998) = F(F(1003)) = F(1000) = 997,$$

$$F(997) = F(F(1002)) = F(999) = 998,$$

and

$$F(996) = F(F(1001)) = F(998) = 997.$$

We see that for $996 \leq n \leq 1001$ the results alternate between 998 and 997, namely $F(n) = 997$ if n is even, and $F(n) = 998$ if n is odd. We can prove this for $n \leq 995$ by mathematical induction.

Let $k \leq 995$. Assume that for all $k+1 \leq n \leq 1001$, we have that

$$F(n) = \begin{cases} 997 & \text{if } n \text{ is even,} \\ 998 & \text{if } n \text{ is odd} \end{cases}.$$

Then, we have

$$F(k) = F(F(k+5)) = \begin{cases} F(998) = 997 & \text{if } k \text{ is even,} \\ F(997) = 998 & \text{if } k \text{ is odd} \end{cases}.$$

(Here we use that $k+5$ is odd if k is even and $k+5$ is even if k is odd.)

This shows that $F(k)$ satisfies the same pattern, which completes the induction.

Equipped with our formula for F, we see that $F(984) = 997$ and $F(F(F(1004))) = F(F(1001)) = F(998) = 997$, as desired.

(c) Answer: 997
(c) Solution 1: By the formula proved above in Solution 2 for (b), we have $F(84) = 997$ (as 84 is even).

(c) Solution 2: By doing (a) we realize that $F^n(999) = F^{n-2}(999)$ for $n \geq 3$.

Here $F^n(x) = F(F(F...F(x)))$, where F occurs n times, e.g. $F^2(x) = F(F(x))$. Specifically,

$$F^3(999) = F^4(1004) = F^2(998) = F^3(1003) = F(997) = F^2(1002) = F(999).$$

Then $F(84) = F^{184}(84 + 5 * 183) = F^{184}(999) = F^2(999) = F(998) = 997$.

P.S. One can similarly use 1004 instead of 999. Since $F^n(1004) = F^{n-2}(1004)$ for $n \geq 3$:

$F(84) = F^{185}(84 + 5 * 184) = F^{185}(1004) = F^3(1004) = F^2(1001) = F(998) = 997$.

C2

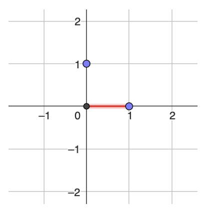
(a) Find the distance from the point $(1,0)$ to the line connecting the origin and the point $(0,1)$.

(b) Find the distance from the point $(1,0)$ to the line connecting the origin and the point $(1,1)$.

(c) Find the distance from the point $(1,0,0)$ to the line connecting the origin and the point $(1,1,1)$.

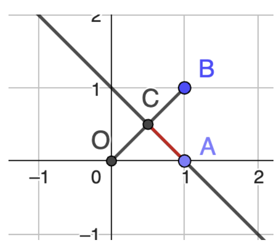
(a) Answer: 1

(a) Solution: The line connecting the origin and the point $(0,1)$ is the y-axis. We can see that the closest point to $(1,0)$ on the y-axis is the origin, so the distance from $(1,0)$ to the y-axis is 1.



(b) Answer: $\frac{\sqrt{2}}{2}$

(b) Solution: Let $A=(1,0)$, $B=(1,1)$, and O be the origin. Let C be the closest point on line BO to A. We know that AC and BO are perpendicular.



Since BO has slope 1, AC must have slope -1. Thus, the line equation for AC is $x+y=1$, and this intersects line BO, given by the equation $y=x$, at the point $(\frac{1}{2},\frac{1}{2})$. The distance AC is thus

$$\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}-1\right)^2}=\frac{\sqrt{2}}{2}.$$

Alternatively, note that the three given points are vertices of a unit square. The required distance is the distance between a vertex of a unit square and its centre (the point of intersection of its diagonals), that is a half of a diagonal's length, which is $\frac{\sqrt{2}}{2}$.

One may also notice that the line OB is the bisect of the first quadrant and that we have to find the leg of the right triangle with equal legs, the other angles of 45° with the hypotenuse equal to one. Then either the equation $2x^2=1$ or $x=\cos 45^\circ=\sin 45^\circ$ leading immediately to the correct answer.

(c) Answer: $\frac{\sqrt{6}}{3}$
(c) Solutions: Let $A = (1, 0, 0), B = (1, 1, 1)$, and O be the origin. Let C be the point on line BO closest to A.
Solution 1.1: We can parameterize the point C with the coordinates (t, t, t); then, lines AC and BO are perpendicular; thus, the dot product

$$(1, 1, 1) \cdot (t - 1, t, t) = 0.$$

Solving for t, we get that $t = \frac{1}{3}$, and hence $C = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and the distance AC is

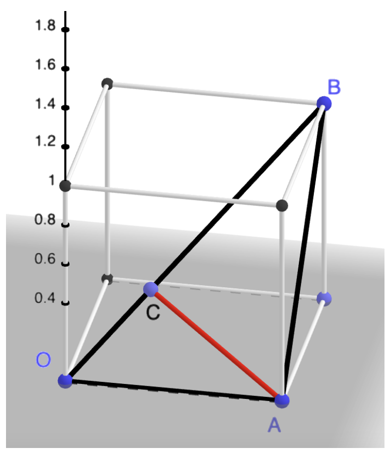
$$\sqrt{\left(\frac{1}{3} - 1\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{6}}{3}.$$

Solution 1.2: First, find the projection of the vector **OA** on the vector **OB** which is

$$\mathbf{OC} = \frac{\mathbf{OA} \cdot \mathbf{OB}}{\|\mathbf{OB}\|^2}\mathbf{OB} = \frac{1 \cdot 1 + 0 \cdot 1 + 0 \cdot 1}{1^2 + 1^2 + 1^2}(1, 1, 1) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

The end of the projection vector $C(1/3, 1/3, 1/3)$ is the closest point on the line to A, and the distance AC is found as in Solution 1.1.

Solution 2: We require the altitude from point A to line BO, hitting BO at C.



Note that triangle ABO is a right triangle with $AB = \sqrt{2}, BO = \sqrt{3}$, and $AO = 1$. Thus, we can write the area of triangle ABO in two different ways:

$$\frac{\sqrt{2}}{2} = \frac{AO \cdot AB}{2} = \frac{BO \cdot AC}{2} = \frac{\sqrt{3}}{2}AC,$$

and so we get

$$AC = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}.$$

C3 Alice and Bob are playing a game. There are initially $n \geq 1$ stones in a pile. Alice and Bob take turns, with Alice going first. On their turn, Alice or Bob roll a die with numbered faces 1, 1, 2, 2, 3, 3, and take at least one and at most that many stones from the pile as the rolled number on the dice. The person who takes the last stone wins.

(a) If $n = 2$, what is the probability that Alice wins?

(b) What is the smallest value of n for which Bob is more likely to win than Alice?

(c) Find all values n for which Bob is more likely to win than Alice.

(a) Answer: $\frac{2}{3}$
(a) Solution: If Alice rolls a 2 or a 3, then she may take both stones in the pile and win. Otherwise, she rolls a 1, and she must take exactly one stone; then, no matter what Bob rolls, he may take the other stone to win. Thus, Alice wins with probability $\frac{2}{3}$, when she rolls a 2 or a 3.

(b) Answer: 4
(b) Solution 1: Since Alice certainly wins the case $n = 1$, and from part (a) she is more likely to win the case $n = 2$, look first at the case $n = 3$ of three stones. If Alice throws 1, she must take one stone, leaving Bob with two stones, and from part (a) he has a 2/3 probability of winning. If Alice throws 2, she may either take one stone or two stones; but if she takes two stones, then Bob will certainly win by taking the last stone whatever he throws, so Alice should take only one stone, leaving Bob with again a 2/3 chance of winning. And of course, if Alice throws 3, she will take all three stones and win. Since each of these three throws has a 1/3 chance of occurring, Bob's probability of winning is $\frac{1}{3}(\frac{2}{3} + \frac{2}{3} + 0) = \frac{4}{9}$, thus Alice's probability of winning is $\frac{5}{9}$, greater than Bob's.

Similarly, for $n = 4$, Alice should take one stone regardless of whether she throws 1, 2 or 3, because this leaves Bob with three stones and probability 5/9 of winning, which is less than the probability 2/3 of his winning with two stones and certainly less than the certainty of his winning with only one stone. Thus Bob's probability of winning when $n = 4$ is $\frac{1}{3}(\frac{5}{9} + \frac{5}{9} + \frac{5}{9}) = \frac{5}{9}$), which is greater than 1/2, so Bob is more likely to win, and $n = 4$ is the answer to this part.

(b) Solution 2: This solution is similar to the first solution but is presented in a tabular format and uses some general notations.
Let us consider three events E_1, E_2, E_3: the event E_k happens when the die shows k. The probability of each of these events is 1/3.

Let P_n be the probability that a player wins if it is their turn, and there are currently n stones in the pile. Let $L_n = 1 - P_n$ be the probability that the player loses.

The following table shows Alice's best action and her probabilities to win P_n and to lose L_n.

n	event	action	Prob. to win	P_n	L_n	note
1	E_1 or E_2 or E_3	take 1	1	$\frac{1}{3}\times 1\times 3 = 1$	0	
2	E_1	take 1	0	$\frac{1}{3}\times 0+\frac{2}{3}\times 1=\frac{2}{3}$	$\frac{1}{3}$	
	E_2 or E_3	take 2	1			
3	E_1 or E_2	take 1	1/3	$\frac{2}{3}\times\frac{1}{3}+\frac{1}{3}\times 1=\frac{5}{9}$	$\frac{4}{9}$	$\frac{4}{9}>\frac{1}{3}$
	E_3	take 3	1			
4	E_1 or E_2 or E_3	take 1	4/9	$\frac{1}{3}\times\frac{4}{9}\times 3=\frac{4}{9}$	$\frac{5}{9}$	$\frac{5}{9}>\frac{4}{9}$

Since $L_4 = \frac{5}{9} > \frac{1}{2}$, we conclude that for $n = 4$ Bob, the second player, is more likely to win.

(b) Solution 3: For this and the next part, we introduce some general notation. Let P_n be the probability that a player wins if it is their turn, and there are currently n stones in the pile. For example, $P_1 = 1$, since the player can always remove the last stone; in the previous part, we calculated that $P_2 = \frac{2}{3}$. We wish to derive a recursive formula for P_n.

If the first player rolls a 1, then they must take one stone from the pile. The other player is then presented with $n-1$ stones, so they have a probability P_{n-1} of winning. If the first player rolls a 2, then they have a choice; they can either remove 1 or 2 stones, whichever is better for them; then, the second player has a probability $\min(P_{n-1}, P_{n-2})$ of winning. Similarly, if the first player rolls a 3, then the second player has a probability $\min(P_{n-1}, P_{n-2}, P_{n-3})$ of winning. In all of these cases, exactly one of the first or second player will win, so we get the following formula:

$$P_n = 1 - \frac{1}{3}\big(P_{n-1} + \min(P_{n-1}, P_{n-2}) + \min(P_{n-1}, P_{n-2}, P_{n-3})\big). \tag{1}$$

Note that here, we may take $P_0 = 0$, since if a player is presented with 0 stones, it means that the other player already won on their previous turn. We can thus calculate

$$P_3 = 1 - \frac{\frac{2}{3}+\frac{2}{3}+0}{3} = \frac{5}{9},$$

and

$$P_4 = 1 - \frac{\frac{5}{9}+\frac{5}{9}+\frac{5}{9}}{3} = \frac{4}{9}.$$

This completes part (b): when there are initially 4 stones, Alice has a $\frac{4}{9}$ probability of winning, which is less than Bob's $1 - \frac{4}{9} = \frac{5}{9}$ probability of winning.

(c) Answer: n is a multiple of 4
Note: The recursive formula (1) has to be derived in the beginning of a solution for part (c) unless it was already derived in part (b) as shown in Solution 3 above.

(c) Solution 1: We need to find the value of n for which $P_n < 1/2$. In parts (a) and (b) we showed that

$$P_4 < \frac{1}{2} < P_3 < P_2 < P_1, \tag{2}$$

and we derived an identity (1). Now we demonstrate that the same pattern as in (2) repeats itself for the next four values of P_n:

$$P_8 < \frac{1}{2} < P_7 < P_6 < P_5. \tag{3}$$

Denote $C = P_4$. From (2) and (1) we compute

$$P_5 = 1 - \frac{1}{3}\big(P_4 + \min(P_4, P_3) + \min(P_4, P_3, P_2)\big) = 1 - \frac{1}{3}\big(P_4 + P_4 + P_4\big) = 1 - C.$$

Note that $P_4 < 1/2 < P_5$ for $C < 1/2$. Next, we apply (1) with $n = 6$

$$P_6 = 1 - \frac{1}{3}\big(P_5 + \min(P_5, P_4) + \min(P_5, P_4, P_3)\big) = 1 - \frac{1}{3}\big(P_5 + P_4 + P_4\big) = \frac{2 - C}{3}.$$

Note that $P_4 < 1/2 < P_6 < P_5$ for $C < 1/2$ (since $1/2 < (2-C)/3 < 1-C$ for all $C < 1/2$). Again, we apply (1) with $n = 7$

$$P_7 = 1 - \frac{1}{3}\big(P_6 + \min(P_6, P_5) + \min(P_6, P_5, P_4)\big) = 1 - \frac{1}{3}\big(P_6 + P_6 + P_4\big) = \frac{5 - C}{9}.$$

Note that $1/2 < P_7 < P_6 < P_5$ for $C < 1/2$ (since $1/2 < (5 - C)/9 < (2 - C)/3$ for all $C < 1/2$). Finally, we compute

$$P_8 = 1 - \frac{1}{3}\big(P_7 + \min(P_7, P_6) + \min(P_7, P_6, P_5)\big) = 1 - \frac{1}{3}\big(P_7 + P_7 + P_7\big) = 1 - P_7.$$

This ends the proof of (3). Exactly the same computation as above (starting with $C = P_8$) will show that

$$P_{12} < \frac{1}{2} < P_{11} < P_{10} < P_9.$$

and, by induction, the same pattern must hold for values of P_n with $4k - 3 \le n \le 4k$ for any $k = 1, 2, 3, \ldots$. Thus $P_n < 1/2$ if and only if n is a multiple of 4.

(c) Solution 2: We can calculate the probabilities based on the recursive formula (1).

$$P_5 = 1 - \frac{1}{3}\left(\frac{4}{9} + \frac{4}{9} + \frac{4}{9}\right) = \frac{5}{9}, \qquad P_6 = 1 - \frac{1}{3}\left(\frac{5}{9} + \frac{4}{9} + \frac{4}{9}\right) = \frac{14}{27}.$$

$$P_7 = 1 - \frac{1}{3}\left(\frac{14}{27} + \frac{14}{27} + \frac{12}{27}\right) = \frac{41}{81}, \qquad P_8 = 1 - \frac{1}{3}\left(\frac{41}{81} + \frac{41}{81} + \frac{41}{81}\right) = \frac{40}{81}.$$

$$P_9 = 1 - \frac{1}{3}\left(\frac{40}{81} + \frac{40}{81} + \frac{40}{81}\right) = \frac{41}{81}, \qquad P_{10} = 1 - \frac{1}{3}\left(\frac{41}{81} + \frac{40}{81} + \frac{40}{81}\right) = \frac{122}{243}.$$

Similarly, $P_{11} = \dfrac{365}{729}$ and $P_{12} = \dfrac{364}{729}$.

We can see that $P_8 < \frac{1}{2}$, and $P_{12} < \frac{1}{2}$, so $n = 8$ and $n = 12$ are the next two cases when Bob is more likely to win than Alice. One may conjecture that n must be a multiple of 4. This conjecture needs to be proven.

In order to solve this problem, we will completely solve the recursion (1) for P_n. Seeing that our values accumulate around $\frac{1}{2}$, we substitute $Q_n = 2P_n - 1$. The question then asks us to classify all n for which Q_n is negative, and our recursion is given by the initial cases $Q_0 = -1, Q_1 = 1$, and $Q_2 = \frac{1}{3}$. Multiplying the recursive relation by 2, we get

$$2P_n = 2 - \frac{2}{3}(P_{n-1} + \min(P_{n-1}, P_{n-2}) + \min(P_{n-1}, P_{n-2}, P_{n-3})),$$

and subtracting 1 from both sides, we get

$$\begin{aligned} Q_n = 2P_n - 1 &= \frac{3 - 2P_{n-1} - 2\min(P_{n-1}, P_{n-2}) - 2\min(P_{n-1}, P_{n-2}, P_{n-3})}{3} \\ &= -\frac{Q_{n-1} + \min(Q_{n-1}, Q_{n-2}) + \min(Q_{n-1}, Q_{n-2}, Q_{n-3})}{3}. \end{aligned}$$

In essence, we have removed the constant term of the recursion by replacing P with Q. We now see that $Q_3 = \frac{1}{9}$ and $Q_4 = -\frac{1}{9}$, from the computations in part (b). Computing a few more values, we get

$$Q_5 = -\frac{-\frac{1}{9} - \frac{1}{9} - \frac{1}{9}}{3} = \frac{1}{9},$$

and

$$Q_6 = -\frac{\frac{1}{9} - \frac{1}{9} - \frac{1}{9}}{3} = \frac{1}{27}.$$

$$Q_7 = -\frac{\frac{1}{27} + \frac{1}{27} - \frac{1}{9}}{3} = \frac{1}{81}.$$

$$Q_8 = -\frac{\frac{1}{81} + \frac{1}{81} + \frac{1}{81}}{3} = -\frac{1}{81}.$$

Note that $Q_{n+4} = \frac{Q_n}{9}$ for $n = 0, 1, 2, 3, 4$.

We now prove that this is true for all n by induction. Assuming it is true for $n = 0, 1, \ldots, k$, we have

$$\begin{aligned} Q_{k+5} &= -\frac{Q_{k+4} + \min(Q_{k+4}, Q_{k+3}) + \min(Q_{k+4}, Q_{k+3}, Q_{k+2})}{3} \\ &= -\frac{1}{9}\frac{Q_k + \min(Q_k, Q_{k-1}) + \min(Q_k, Q_{k-1}, Q_{k-2})}{3} = \frac{1}{9}Q_{k+1}, \end{aligned}$$

as desired.

This provides us the full description of Q_n, and hence P_n. For the question at hand, we see that of the first 4 values Q_0, Q_1, Q_2, and Q_3, only Q_0 is negative; hence, Q_n is negative if and only if n is a multiple of 4.

(c) Solution 3: We can use equation (1) to generate values of P_n as in the beginning of Solution 2.

From this data, we can conjecture that for each integer $k \geq 1$,

$$P_{4k} = \frac{3^{2k}-1}{2 \cdot 3^{2k}} = \frac{1}{2} - \frac{1}{2 \cdot 3^{2k}},$$

$$P_{4k+1} = \frac{3^{2k}+1}{2 \cdot 3^{2k}} = \frac{1}{2} + \frac{1}{2 \cdot 3^{2k}},$$

$$P_{4k+2} = \frac{3^{2k+1}+1}{2 \cdot 3^{2k+1}} = \frac{1}{2} + \frac{1}{6 \cdot 3^{2k}},$$

and

$$P_{4k+3} = \frac{3^{2k+2}+1}{2 \cdot 3^{2k+2}} = \frac{1}{2} + \frac{1}{18 \cdot 3^{2k}}.$$

We now prove these four formulas by induction on k. Note that these formulas imply that

$$P_{4k} < \frac{1}{2} < P_{4k+3} < P_{4k+2} < P_{4k+1},$$

and thus $P_n < 1/2$ exactly when n is a multiple of 4.
The above formulas are true for $k = 1$ and $k = 2$, so assume they work for some positive integer k and look at the next integer $k + 1$. We get

$$P_{4(k+1)} = 1 - \frac{1}{3}(P_{4k+3} + \min(P_{4k+3}, P_{4k+2}) + \min(P_{4k+3}, P_{4k+2}, P_{4k+1})) =$$

$$= 1 - \frac{1}{3}(\frac{1}{2} + \frac{1}{18 \cdot 3^{2k}} + \frac{1}{2} + \frac{1}{18 \cdot 3^{2k}} + \frac{1}{2} + \frac{1}{18 \cdot 3^{2k}}) = \frac{1}{2} - \frac{1}{18 \cdot 3^{2k}} = \frac{1}{2} - \frac{1}{2 \cdot 3^{2(k+1)}},$$

so that $P_{4(k+1)} < 1/2$;

$$P_{4(k+1)+1} = 1 - \frac{1}{3}(P_{4(k+1)} + \min(P_{4(k+1)}, P_{4k+3}) + \min(P_{4(k+1)}, P_{4k+3}, P_{4k+2})) =$$

$$= 1 - \frac{1}{3}(\frac{1}{2} - \frac{1}{2 \cdot 3^{2(k+1)}} + \frac{1}{2} - \frac{1}{2 \cdot 3^{2(k+1)}} + \frac{1}{2} - \frac{1}{2 \cdot 3^{2(k+1)}}) = \frac{1}{2} + \frac{1}{2 \cdot 3^{2(k+1)}},$$

so that $1/2 < P_{4(k+1)+1}$;

$$P_{4(k+1)+2} = 1 - \frac{1}{3}(P_{4(k+1)+1} + \min(P_{4(k+1)+1}, P_{4(k+1)}) + \min(P_{4(k+1)+1}, P_{4(k+1)}, P_{4k+3})) =$$

$$= 1 - \frac{1}{3}(\frac{1}{2} + \frac{1}{2 \cdot 3^{2(k+1)}} + \frac{1}{2} - \frac{1}{2 \cdot 3^{2(k+1)}} + \frac{1}{2} - \frac{1}{2 \cdot 3^{2(k+1)}}) = \frac{1}{2} + \frac{1}{6 \cdot 3^{2(k+1)}},$$

so that $1/2 < P_{4(k+1)+2} < P_{4(k+1)+1}$; and

$$P_{4(k+1)+3} = 1 - \frac{1}{3}(P_{4(k+1)+2} + \min(P_{4(k+1)+2}, P_{4(k+1)+1}) + \min(P_{4(k+1)+2}, P_{4(k+1)+1}, P_{4(k+1)})) =$$

$$= 1 - \frac{1}{3}(\frac{1}{2} + \frac{1}{6 \cdot 3^{2(k+1)}} + \frac{1}{2} + \frac{1}{6 \cdot 3^{2(k+1)}} + \frac{1}{2} - \frac{1}{2 \cdot 3^{2(k+1)}}) = \frac{1}{2} + \frac{1}{18 \cdot 3^{2(k+1)}}.$$

Thus the formulas are true for the integer $k+1$, and by induction they are true for all positive integers k. Thus, we conclude that $P_n < 1/2$ exactly when n is a multiple of 4.

C4 For a positive integer n, let $\tau(n)$ be the sum of its divisors (including 1 and itself), and let $\phi(n)$ be the number of integers x, $1 \le x \le n$, such that x and n are relatively prime. For example, if $n = 18$, then $\tau(18) = 1+2+3+6+9+18 = 39$ and $\phi(18) = 6$ since the numbers 1, 5, 7, 11, 13, and 17 are relatively prime to 18.

(a) Prove that $\phi(n)\tau(n) < n^2$ for every positive integer $\boxed{n > 1.}$

(b) Determine all positive integers n such that $\phi(n)\tau(n) + 1 = n^2$.

(c) Prove that there are no positive integers n such that $\phi(n)\tau(n) + 2023 = n^2$.

Please note that the condition $\boxed{n > 1}$ was mistakenly omitted in the original statement of the problem in part (a). Our markers awarded full credit for proving the statement for $n > 1$ with or without any comments about the case $n = 1$ for which we have $\phi(n) = 1$ and $\tau(n) = 1$ and therefore $\phi(n)\tau(n) = 1 = n^2$.

Solution:
Let $n = p_1^{e_1} \dots p_k^{e_k}$, where $p_1, \dots p_k$ are distinct prime numbers and $e_1, \dots e_k$ are positive integers. Then,

$$\frac{\tau(n)}{n} = \prod_{i=1}^{k}\left(1 + \frac{1}{p_i} + \dots + \frac{1}{p_i^{e_i}}\right) = \prod_{i=1}^{k} \frac{(\frac{1}{p_i})^{e_i+1} - 1}{\frac{1}{p_i} - 1} = \prod_{i=1}^{k} \frac{(\frac{1}{p_i})^{e_i} - p_i}{1 - p_i} = \prod_{i=1}^{k}\left(\frac{p_i - {p_i}^{-e_i}}{p_i - 1}\right),$$

and

$$\frac{\phi(n)}{n} = \prod_{i=1}^{k}\left(\frac{p_i - 1}{p_i}\right).$$

Thus

$$\tau(n)\phi(n) = n^2 \prod_{i=1}^{k}\left(\frac{p_i - {p_i}^{-e_i}}{p_i}\right) = n^2 \prod_{i=1}^{k}\left(1 - \frac{1}{{p_i}^{e_i+1}}\right). \qquad (*)$$

For example, $18 = 2^1 3^2$ and $\tau(18)\phi(18) = 18^2(1 - \frac{1}{2^2})(1 - \frac{1}{3^3}) = 18 \times 18 \times \frac{3}{4} \times \frac{26}{27} = 6 \times 39$. For brevity, let us call the difference

$$\delta(n) = n^2 - \tau(n)\phi(n)$$

the *deficit*.

(a) From Eq.(*) it is clear that $\tau(n)\phi(n) < n^2$ because $\left(1 - \frac{1}{p_i^{e_i+1}}\right) < 1$ for all $1 \le i \le k$. Hence $\delta(n) > 0$.

(b) **Answer: n is prime**
By considering examples, one can conjecture that if n is prime then the equation is true. Indeed, in $n = p$ is prime then $\tau(n) = 1 + p$ and $\phi(n) = p - 1$, so $\tau(n)\phi(n) + 1 = p^2 - 1 + 1 = n^2$. We need to verify whether or not the equation holds for n being not prime.

Again, from Eq.(*) it follows that

$$\tau(n)\phi(n) = n^2 \prod_{i=1}^{k} \frac{(p_i^{e_i+1} - 1)}{p_i^{e_i+1}} = \prod_{i=1}^{k} p_i^{e_i-1}(p_i^{e_i+1} - 1) = \prod_{i=1}^{k} (p_i^{2e_i} - p_i^{e_i-1}).$$

Therefore, if for some $1 \le i \le k$, $e_i \ge 2$, then $p_i | \tau(n)\phi(n)$ and p_i divides n^2, hence p_i divides the deficit. If $\delta(n) = 1$, then all e_i must be equal to 1. Thus $n = p_1 p_2 \dots p_k$ for some k, and we need to have

$$(p_1^2 - 1)(p_2^2 - 1)\dots(p_k^2 - 1) + 1 = p_1^2 p_2^2 \dots p_k^2.$$

This can only happen when $k = 1$; that is, when n is prime.

(c) In both solutions shown below we assume that $\delta(n) = 2023$ for some positive integer n and find a contradiction. Thus, we prove the required statement.

Solution 1: We will use the following formula found above:

$$\tau(n)\phi(n) = \prod_{i=1}^{k} p_i^{e_i-1}(p_i^{e_i+1} - 1). \qquad (**)$$

(i) We first show that if all prime factors p_i are odd and at least one of the exponents e_i is *odd*, then $\delta(n) \equiv 1 \bmod 4$.

Indeed, in this case $p_j^{e_j+1}$ is the square of an odd integer, hence $p_j^{e_j+1} \equiv 1 \bmod 4$. Therefore $(p_j^{e_j+1} - 1) \equiv 0 \bmod 4$ and so $\tau(n)\phi(n) \equiv 0 \bmod 4$ by equation (**). Also, n is odd, hence $n^2 \equiv 1 \bmod 4$. It follows that $\delta(n) \equiv 1 \bmod 4$.

(ii) Suppose now that n is even, i.e. $p_1 = 2$. If $e_1 \ge 2$, then $2|\phi(n)\tau(n)$. If $e_1 = 1$ and $n > 2$, then there is an odd prime divisor p_2 of n, hence $p_2^{e_2+1} - 1$ is even and again $2|\phi(n)\tau(n)$ by equation (**). Thus, for any even $n > 2$ we have: n^2, $\phi(n)\tau(n)$, and hence $\delta(n)$, are even.

Applying the results (i) and (ii) to our problem, since $2023 \equiv 3 \bmod 4$, we conclude: any n for which $\delta(n) = 2023$ must be an odd square. (All exponents e_i must be even.)

In particular, all exponents $e_i \ge 2$. Then $\prod_{i=1}^{k} p_i^{e_i-1}$ is a common divisor of n^2 and $\tau(n)\phi(n)$, hence a divisor of $\delta(n)$. We can therefore determine prime divisors of n by observing those of $\delta(n)$.

In our case, since $2023 = 7 \cdot 17^2$, we must have $p_1 = 7$, $p_2 = 17$, and no other prime divisors. Thus $n = 7^{e_1} \cdot 17^{e_2}$. Then

$$\delta(n) = n^2 - \tau(n)\phi(n) = 7^{2e_1} \cdot 17^{2e_2} - 7^{e_1-1} \cdot 17^{e_2-1} \cdot (7^{e_1+1} - 1)(17^{e_2+1} - 1)$$

Therefore,

$$\delta(n) = 7^{e_1-1} \cdot 17^{e_2-1} \cdot A,$$

where

$$A = 7^{e_1+1} \cdot 17^{e_2+1} - (7^{e_1+1} - 1)(17^{e_2+1} - 1).$$

According to the assumptions of this case, $e_1 \geq 2$ and $e_2 \geq 2$ and are even. Assuming that $\delta(n) = 2023 = 7 \cdot 17^2$, we are left with only one possibility: $e_1 = 2$, $e_2 = 2$ and $A = 17$. At the same time we found that $A = 7^3 \cdot 17^3 - (7^3 - 1)(17^3 - 1)$, so $A \equiv 7^3 - 1 \equiv 7 \cdot (-2) - 1 \equiv -15 \bmod 17$. This provides a contradiction.

Solution: 2

Let us first show that n cannot be even. Indeed, if n would be even, then

$$\prod_{i=1}^{k} (p_i^{2e_i} - p_i^{e_i - 1}) = \phi(n)\tau(n) = n^2 - 2023$$

is odd. This implies that for all $1 \leq i \leq k, p_i^{2e_i} - p_i^{e_i-1}$ is odd. This means that $k = 1, p_1 = 2$ and $e_1 - 1 = 0$, which gives $n = 2$. But $\phi(2)\tau(2) + 2023 \neq 2^2$.

This shows that n must be odd, and hence $n^2 \equiv 1 \pmod 4$. It follows that

$$\prod_{i=1}^{k} (p_i^{2e_i} - p_i^{e_i - 1}) = \phi(n)\tau(n) = n^2 - 2023 \equiv 2 \pmod 4 .$$

Now, since n is odd, each term $p_i^{2e_i} - p_i^{e_i-1}$ is even. Since $\prod_{i=1}^{k} (p_i^{2e_i} - p_i^{e_i-1})$ is divisible by 2 but not by 4, we get $k = 1$ and hence $n = p_1^{e_1}$. It follows that

$$n^2 - 2023 = p_1^{2e_1} - p_1^{e_1-1} = n^2 - p_1^{e_1-1}$$

showing that 2023 is a power of a prime, a contradiction.

