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Quaternionic Cayley Transform

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In this paper we define a quaternionic Cayley transform for some linear operators acting in $\mathscr{H} \oplus \mathscr{H}$, where $\mathscr{H}$ is a Hilbert space, which permits the joint investigation of the pairs of symmetric operators. In particular, this leads to new criteria for the existence of commuting self-adjoint extensions of certain pairs of symmetric operators. © 1999 Academic Press

1. PRELIMINARIES

The problem of characterizing those densely defined symmetric operators that have self-adjoint extensions is related to the possibility of applying the well-known spectral theorem, valid for the latter class. And when such self-adjoint extensions do exist, it is interesting to describe their structure, as well as their connexions with the given symmetric operator. Among the classical contributions on this subject one should mention those of von Neumann [8] (see also [1, Chapt. XII] for further details and various ramifications).

An important device to find self-adjoint extensions for a symmetric operator is the Cayley transform, whose introduction is also due to von Neumann [8], which permits the reduction of this problem to the easier one of finding unitary extensions of a partial isometry.

A natural question concerning the unbounded self-adjoint operators is to define the commutation of two such operators in a reasonable way. The remarkable example of Nelson [2] showed that this problem could not be solved in simple terms. One usually says that two unbounded self-adjoint operators (strongly) commute if the corresponding spectral measures commute. A natural related problem requires the characterization of those pairs of unbounded symmetric operators that have self-adjoint extensions which, in addition, commute. A useful criterion in this sense also belongs to Nelson [2].

In this paper we propose a quaternionic approach to the study of pairs of unbounded symmetric operators in a Hilbert space, which allows us to

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regain most of the ingredients that occur in the classical theory: the Cayley transform, the description of all self-adjoint extensions via partial isometries, etc.

To present our results in a more specific manner, we need a certain notation.

Let $\mathscr{H}$ be a complex Hilbert space, whose scalar product (resp. norm) will be denoted by $\langle *, * \rangle$ (resp. $\|*\|$). Let also $\mathscr{H}^2 = \mathscr{H} \oplus \mathscr{H}$, whose scalar product (resp. norm) will be denoted by $\langle *, * \rangle_2$ (resp. $\|*\|_2$).

Let $\mathbf{1} = \mathbf{1}_{\mathscr{H}}$ be the identity of $\mathscr{H}$, and let

$$I = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \qquad J = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

acting on $\mathscr{H}^2$. We have the relations

$$\begin{aligned} &J^* = J, \qquad K^* = -K, \qquad L^* = L, \qquad J^2 = -K^2 = L^2 = I, \\ &JK = L = -KJ, \quad KL = J = -LK, \quad JL = K = -LJ. \end{aligned} \tag{1.1}$$

For every pair $z = (z_1, z_2) \in \mathbf{C}^2$ we define the matrix

$$Q(z) = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}. \tag{1.2}$$

The set $\{Q(z); z \in \mathbf{C}^2\}$, which can be identified with the algebra of quaternions, will be regarded as a subalgebra of the algebra of all bounded linear operators on $\mathscr{H}^2$.

Note the decomposition

$$Q(z) = (\mathrm{Re} z_1)\, I + i(\mathrm{Im} z_1)\, J + (\mathrm{Re} z_2)\, K + i(\mathrm{Im} z_2)\, L. \tag{1.3}$$

We also have $I = Q(1, 0)$, $iJ = Q(i, 0)$, $K = Q(0, 1)$, $iL = Q(0, i)$.

Remark 1.1. For all $z = (z_1, z_2) \in \mathbf{C}^2$ we have $\|Q(z)\, x\|_2 = \|z\|\, \|x\|_2$, $x \in \mathscr{H}^2$, where $\|z\|^2 = |z_1|^2 + |z_2|^2$.

LEMMA 1.2. *Let $D \subset \mathscr{H}^2$ be a linear subspace. The following conditions are equivalent*:

(a) *D is invariant under $Q(z)$ for all $z = (z_1, z_2) \in \mathbf{C}^2$*;

(b) *D is invariant under any two of the maps J, K, L*;

(c) *there exists a linear subspace $D_0 \subset \mathscr{H}$ such that $D = D_0 \oplus D_0$.*

If $E \subset \mathscr{H}^2$ is another linear subspace such that $D \subset E$, and $JD = E$, then $D = E$.

The easy details of the proof are left to the reader.

We now briefly describe the contents of this paper. We introduce in the next section a quaternionic Cayley transform (see Remark 2.7(2)), defined for some unbounded operators in $\mathscr{H}^2$, called here (J, L)-symmetric (see Definition 2.1), for which we recapture most of the properties of the classical Cayley transform (Theorem 2.14), as defined, for instance, in [4]. Then we characterize those (J, L)-symmetric operators that have (J, L)-self-adjoint extensions which are also normal (see Definition 2.1 and Theorem 2.21). We are particularly interested in those (J, L)-self-adjoint operators that are normal because of their connexion with the commuting self-adjoint pairs (see Theorem 3.7).

In the third section we investigate the (J, L)-symmetric operators which are associated to commuting pairs of symmetric operators. We propose new criteria to decide whether certain pairs of symmetric operators possess commuting self-adjoint extensions (Theorems 3.8 and 3.10), whose basic ideas are different from the classical ones (see [1, 2, 4] etc.). As a matter of fact, we obtain a complete description of all pairs of symmetric operators that have commuting self-adjoint extensions in terms of partial isometries (see Theorem 3.10).

In the same section we also introduce a quaternionic spectrum, defined for (J, L)-symmetric operators (Definition 3.11), in the spirit of [5]. This spectrum has interesting features for the case of (J, L)-self-adjoint normal operators, reminding us of the case of self-adjoint operators (see Theorems 3.14 and 3.15).

The efficiency of our methods is illustrated in the last part of this work, where a new proof of Nelson's criterion of (strong) commutativity for pairs of symmetric operators is given, and the existence of a representing measure for some (positive semi-definite) maps is discussed, in connection with the moment problem in the plane (see [3] for a thorough presentation of this subject).

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2. (J, L)-SYMMETRIC OPERATORS

By *operator* we mean a linear map T, defined on a linear subspace $D(T)$ of a Hilbert space, with values in a (possibly different) Hilbert space. The range of T will be denoted by $R(T)$, while $N(T)$ and $G(T)$ are the null space and the graph of T, respectively.

By *partial isometry* we mean an operator which is an isometry on its domain of definition.

By *projection* we always mean an orthogonal projection.

DEFINITION 2.1. An operator $S: D(S) \subset \mathscr{H}^2 \to \mathscr{H}^2$ is said to be (J, L)-*symmetric* if $D(S)$ is dense in $\mathscr{H}^2$ and $SJ \subset JS^*$, $SL \subset LS^*$, $JLD(S) \subset D(S)$.

An operator $S: D(S) \subset \mathscr{H}^2 \to \mathscr{H}^2$ is said to be (J, L)-*self-adjoint* if $D(S)$ is dense in $\mathscr{H}^2$ and $SJ = JS^*$, $SL = LS^*$.

Remark 2.2. (1) If S is (J, L)-symmetric, then $KS = SK$. Indeed, since $JL = K$, and $K^2 = -I$ by (1.1), we have, in fact, $KD(S) = D(S)$. Therefore, $x \in D(S)$ if and only if $Kx \in D(S)$. For such an x we have $Jx, Lx \in D(S^*)$, and thus

$$SKx = SJLx = JS^*Lx = JLSx = KSx,$$

by Definition 2.1.

(2) If S is (J, L)-self-adjoint, then S is (J, L)-symmetric. Indeed, in this case we have $SK = KS$ (and $S^*K = KS^*$ as well), and so $KD(S) \subset D(S)$.

(3) Throughout this text we use the notation

$$Q' = Q(i\sqrt{2}/2, i\sqrt{2}/2) = (i\sqrt{2}/2)(J + L).$$

We clearly have $Q'^* = -Q'$, $Q'^2 = -I$. In particular, Q' is unitary on the space $\mathscr{H}^2$.

Note also that a densely defined operator S is (J, L)-symmetric if and only if $SQ' \subset Q'S^*$, $KS = SK$. Indeed, since $J = -(i\sqrt{2}/2)\, Q'(I - K)$, $L = -(i\sqrt{2}/2)\, Q'(I + K)$, assuming $SQ' \subset Q'S^*$, $KS = SK$, we infer easily that $SJ \subset JS^*$, $SL \subset LS^*$, $KD(S) \subset D(S)$. The converse is clear, via (1).

LEMMA 2.3. *Let $S: D(S) \subset \mathscr{H}^2 \to \mathscr{H}^2$ be a (J, L)-symmetric operator. Then we have the following*:

(a) *The operator S is closable.*

(b) *The canonical closure of S is also (J, L)-symmetric.*

(c) *If S is (J, L)-self-adjoint, then S is closed.*

(d) *If $S^\# = S^* \mid Q'D(S)$, then $S^\#$ is (J, L)-symmetric.*

Proof. (a) Let $(x_k)_{k \geqslant 1}$ be a sequence in $D(S)$ such that $x_k \to 0$ and $Sx_k \to y$ as $k \to \infty$. Then $JSx_k = S^*Jx_k \to Jy$. Since $Jx_k \to 0$ and S^* is closed, it follows that $Jy = 0$, and so $y = 0$.

(b) Let $\overline{S}$ be the canonical closure of S. If $x \in D(\overline{S})$, then there exists a sequence $(x_k)_{k\geqslant 1}$ in $D(S)$ such that $x_k \to x$ and $Sx_k \to \overline{S}x$ $(k \to \infty)$.

Since $Jx = \lim_k Jx_k$ and $S^*Jx_k = JSx_k \to J\overline{S}x$, we have $Jx \in D(S^*)$, and $S^*Jx = J\overline{S}x$.

Similarly, $Lx \in D(S^*)$, and $S^*Lx = L\overline{S}x$.

Finally, as $Kx_k \to Kx$, and $SKx_k = KSx_k \to K\overline{S}x$, it follows that $Kx \in D(\overline{S})$.

(c) If S is (J, L)-self-adjoint, then $S = JS^*J$, and the operator JS^*J is clearly closed

(d) Let $x = Q'v$, with $v \in D(S)$. Thus

$$Q'S^{\#}x = Q'S^{\#}Q'v = -Sv = -S^{\#*}v = S^{\#*}Q'x,$$

because $S \subset S^{\#*}$. In addition, since $KS = SK$ implies $KS^* = S^*K$, we also have $KS^{\#} = S^{\#}K$. Consequently, $S^{\#}$ is (J, L)-symmetric, by Remark 2.2(3).

The structure of some (J, L)-symmetric operators is given by the following.

Proposition 2.4. *An operator $S: D(S) \subset \mathscr{H}^2 \to \mathscr{H}^2$ is (J, L)-symmetric, and $Q'D(S) \subset D(S)$, if and only if there are two symmetric operators S_1, S_2 in $\mathscr{H}$ such that the subspace $D(S_1) \cap D(S_2)$ is dense in $\mathscr{H}$, $D(S) = [D(S_1) \cap D(S_2)] \oplus [D(S_1) \cap D(S_2)]$, and $S = \begin{pmatrix} S_1 & S_2 \\ -S_2 & S_1 \end{pmatrix}$. In this case we also have $S^{\#} = \begin{pmatrix} S_1 & -S_2 \\ S_2 & S_1 \end{pmatrix}$.*

Proof. Assume first that S is a (J, L)-symmetric operator, and $Q'D(S) \subset D(S)$. Since $KD(S) \subset D(S)$, and therefore $JD(S) = Q'(I-K)D(S) \subset D(S)$, it follows from Lemma 1.2 that there exists a linear subspace $D_0 \subset \mathscr{H}$ such that $D(S) = D_0 \oplus D_0$. Then we can write

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

A simple calculation, based on the equality $S = -KSK$, shows that $S_{11} = S_{22}$, and $S_{12} = -S_{21}$. Therefore, we may take $S_1 = S_{11} = S_{22}$, $S_2 = S_{12} = -S_{21}$.

Let $x_1, y_1 \in D_0$. From the equality

$$\langle S(x_1 \oplus 0), y_1 \oplus 0\rangle_2 = \langle x_1 \oplus 0, S^*(y_1 \oplus 0)\rangle_2$$

we obtain $\langle S_1x_1, y_1\rangle = \langle x_1, S_1y_1\rangle$ (via the relation $JS \subset S^*J$), proving that S_1 is symmetric. Similarly, S_2 is symmetric. In addition, $D(S_1) = D(S_2) = D_0$, and hence $D(S_1) \cap D(S_2) = D_0$ is dense in $\mathscr{H}$.

Note that we actually have $D(S) = Q'D(S) = JD(S)$ (because of $-Q'^2 = J^2 = I$), showing that $S^\# = S^* \mid D(S)$ (with $S^\#$ defined in the previous lemma), and $S^\# = JSJ$. Therefore,

$$S^\# = \begin{pmatrix} S_1 & -S_2 \\ S_2 & S_1 \end{pmatrix}.$$

Conversely, if S_1, S_2 are as stated, then $KD(S) \subset D(S)$, and $Q'D(S) \subset D(S)$. Then, as one can easily see,

$$\langle JSJx, y \rangle_2 = \langle LSLx, y \rangle_2 = \langle x, Sy \rangle_2, \qquad x, y \in D(S).$$

This shows that $S^* \supset JSJ$, $S^* \supset LSL$, implying that S is (J, L)-symmetric.

Remark 2.5. If the operator $S: D(S) \subset \mathscr{H}^2 \to \mathscr{H}^2$ is (J, L)-self-adjoint, and $D(S) \subset D(S^*)$, then $D(S) = D(S^*)$, and $S^* = S^\#$. Indeed, since $JD(S) = D(S^*)$, $D(S) \subset D(S^*)$, it follows from Lemma 1.2 that $D(S) = D(S^*)$ (and so $S^* = S^\#$).

LEMMA 2.6. *Let S be (J, L)-symmetric. Then for every $z = (z_1, z_2) \in \mathbf{C}^2$ such that* $\mathrm{Re} z_1 = \mathrm{Re} z_2 = 0$ *we have the equality*

$$\|(S + Q(z))\, x\|_2^2 = \|Sx\|_2^2 + \|z\|^2\, \|x\|_2^2, \qquad x \in D(S). \tag{2.1}$$

Proof. Let $x \in D(S)$, and $z_1 = it_1$, $z_2 = it_2$ be fixed. Then $Q(z)\, x = i(t_1 J + t_2 L)\, x \in D(S^*)$, and

$$Q(z)^*\, Sx + S^* Q(z)\, x = -i(t_1 J + t_2 L)\, Sx + iS^*(t_1 J + t_2 L)\, x = 0$$

by virtue of (1.1), and the fact that S is (J, L)-symmetric. Therefore,

$$\begin{aligned}\|(S + Q(z))\, x\|_2^2 &= \|Sx\|_2^2 + \langle Sx, Q(z)\, x \rangle_2 + \langle Q(z)\, x, Sx \rangle_2 + \|Q(z)\, x\|_2^2 \\ &= \|Sx\|_2^2 + \|z\|^2\, \|x\|_2^2\end{aligned}$$

by the above calculation and Remark 1.1.

Remark 2.7 (1) Let S be (J, L)-symmetric, and let $x \in D(S)$ be such that $Q'x \in D(S)$. Then $x = -Q'^2 x = -(i\sqrt{2}/2)(J + L)\, Q'x \in D(S^*)$. If, moreover, $\|Sx\|_2 = \|S^*x\|_2$, then we have $\|(S^* \pm Q')\, x\|_2 = \|(S \pm Q')\, x\|_2$. Indeed, as in the proof of Lemma 2.6,

$$\begin{aligned}\|(S^* \pm Q')\, x\|_2^2 &= \|S^*x\|_2^2 + \langle S^*x, \pm Q'x \rangle_2 + \langle \pm Q'x, S^*x \rangle_2 + \|x\|_2^2 \\ &= \|Sx\|_2^2 + \|x\|_2^2 = \|(S \pm Q')\, x\|_2^2,\end{aligned}$$

since $\pm Q'S^*x \mp SQ'x = \pm Q'Sx \mp S^*Q'x = 0$.

(2) Let S be (J, L)-symmetric. It follows from (2.1) that

$$\|(S \pm Q')\, x\|_2^2 = \|Sx\|_2^2 + \|x\|_2^2, \qquad x \in D(S). \tag{2.1'}$$

This allows us to define the map

$$\begin{aligned} &V\colon R(S+Q') \to R(S-Q') \\ &V(S+Q')\, x = (S-Q')\, x, \qquad x \in D(S). \end{aligned} \tag{2.2}$$

The map V, which is a partial isometry, will be called the *quaternionic Cayley transform* (or, simply, the *Cayley transform*) of the (J, L)-symmetric operator S.

As a partial isometry, $V\colon R(S+Q') \to R(S-Q')$ is invertible, and its inverse $V^{-1} = R(S-Q') \to R(S+Q')$, which is also a partial isometry, is given by $V^{-1}(S-Q')\, x = (S+Q')\, x$, $x \in D(S)$.

LEMMA 2.8. *Let $S\colon D(S) \subset \mathscr{H}^2 \to \mathscr{H}^2$ to be (J, L)-symmetric, and let V be the Cayley transform of S. Then we have the following*:

(a) *The space $R(V-I)$ is dense in $\mathscr{H}^2$, and $V^{-1} = -KVK$.*

(b) *V is closed if and only if S is closed.*

(c) *The Cayley transform $V_\#$ of the operator $S^\# = S^* \mid Q'D(S)$ is given by the equation $V_\# = -Q'VQ'$.*

(d) *If S is (J, L)-self-adjoint, then V is unitary on $\mathscr{H}^2$.*

In addition, if $S_j\colon D(S_j) \subset \mathscr{H}^2 \to \mathscr{H}^2$ $(j=1, 2)$ are (J, L)-symmetric operators having the same Cayley transform, then $S_1 = S_2$.

Proof. (a) Let $x \in D(S)$, and let $y = (S+Q')\, x \in D(V)$. Thus by (2.2),

$$(I-V)\, y = (S+Q')\, x - (S-Q')\, x = 2Q'x. \tag{2.3}$$

Therefore, $R(V-I) = Q'(D(S))$, and the latter is dense since $D(S)$ is dense and Q' is unitary.

Let again $x \in D(S)$, and let $u = (S-Q')\, x \in D(V^{-1})$. Since $SKx = KSx$ by Remark 2.2(1), and $KQ' = -Q'K$, we have $Ku = (S+Q')\, Kx \in D(V)$, and $VKu = (S-Q')\, Kx$. Hence

$$KVKu = (S+Q')\, K^2 x = -(S+Q')\, x = -V^{-1}u,$$

showing that $V^{-1} \subset -KVK$.

Conversely, let v be a vector such that $Kv \in D(V)$. Then there exists a $w \in D(S)$ satisfying $Kv = (S + Q')\,w$. Then $v = -K^2 v = (S - Q)(-Kw) \in D(V^{-1})$ and

$$V^{-1} v = (S + Q')(-Kw) = -K(S - Q')\,w = -KVKv,$$

showing that $-KVK \subset V^{-1}$. This completes the proof of (a).

(b) Since the operator V is bounded, then V is closed if and only if $D(V) = R(S + Q')$ is closed. In addition, it follows from (2.1′) that the sequence $((S + Q')\,x_k)_{k \geqslant 1}$ is a Cauchy sequence if and only if both sequences $(x_k)_{k \geqslant 1}$ and $(Sx_k)_{k \geqslant 1}$ are Cauchy sequences, which implies readily our assertion.

(c) The Cayley transform $V_{\#}$ of the operator $S^{\#} = S^* \mid Q'D(S)$ exists by virtue of Lemma 2.3(d). Let $u = (S^{\#} + Q')\,x \in D(V^{\#})$, with $x = Q'v$, $v \in D(S)$. Thus we have

$$V_{\#} u = (S^{\#} - Q')\,x = Q'(S - Q')\,v = -Q'V(S + Q')\,Q'x = -Q'VQ'u.$$

(d) If S is (J, L)-self-adjoint, then S is closed (Lemma 2.3), and so the spaces $R(S \pm Q')$ are closed, by (b). We shall show that $R(S \pm Q') = \mathscr{H}^2$. Indeed, let $y \in \mathscr{H}^2$ be such that $\langle y, (S + Q')\,x \rangle_2 = 0$ for all $x \in D(S)$. Then $y \in D((S + Q')^*)$, and $(S + Q')^* y = 0$. But

$$(S + Q')^*\, y = (S^* - Q')\, y = J(S - Q'')\, Jy = 0,$$

where $Q'' = (i\sqrt{2}/2)(J - L) = Q(i\sqrt{2}/2, -i\sqrt{2}/2)$. Since $S - Q''$ is injective by (2.1), this implies that $Jy = y = 0$. Therefore, $R(S + Q') = \mathscr{H}^2$. Similarly, $R(S - Q') = \mathscr{H}^2$, showing that V is an isometry from $\mathscr{H}^2$ onto itself, i.e., V is a unitary operator on $\mathscr{H}^2$.

Now, let $S_j \colon D(S_j) \subset \mathscr{H}^2 \to \mathscr{H}^2 (j = 1, 2)$ be (J, L)-symmetric operators having the same Cayley transform V. Then for every $x_1 \in D(S_1)$ we can find an $x_2 \in D(S_2)$ such that $(S_1 + Q')\,x_1 = (S_2 + Q')\,x_2$, and $(S_1 - Q')\,x_1 = (S_2 - Q')\,x_2$. This implies $S_1 x_1 = S_2 x_2$, and $Q' x_1 = Q' x_2$. Therefore, $x_1 = x_2 \in D(S_2)$, and so $S_1 \subset S_2$. Similarly, $S_2 \subset S_1$, which completes the proof of the lemma.

Remark 2.9. Lemma 2.8(a) shows that the Cayley transform V of a (J, L)-symmetric operator is a partial isometry such that $R(V - I)$ is dense in $\mathscr{H}^2$.

Conversely, let V be a partial isometry in $\mathscr{H}^2$ such that $R(V - I)$ is dense. Then, it is known (see, for instance, [4, 13.18]) that $V - I$ is injective on

$D(V)$. (Indeed, if $(V-I)x=0$ for some $x\in D(V)$, then for all $y\in D(V)$ we have $\langle x, (V-I)y\rangle_2=\langle (V-I)x, Vy\rangle_2=0$, implying $x=0$.)

This allows us to define an operator $S: R(Q'(V-I))\to \mathscr{H}^2$ by the equality

$$S(Q'(V-I)x)=(V+I)x, \; x\in D(V). \tag{2.4}$$

The operator S, which is well defined for every partial isometry V with $V-I$ injective, will be called the *inverse quaternionic Cayley transform* (or, simply, the *inverse Cayley transform*) of the partial isometry V.

LEMMA 2.10. *Let $V: D(V)\subset \mathscr{H}^2\to\mathscr{H}^2$ be a partial isometry such that $R(V-I)$ is dense in $\mathscr{H}^2$, and $V^{-1}=-KVK$. Then the inverse Cayley transform S of V is (J, L)-symmetric.*

In addition, the Cayley transform of S coincides with V.

Proof. Let us prove that S is (J, L)-symmetric.

We show first that $JS\subset S^*J$. This means that if $u\in D(S)$, we have $Ju\in D(S^*)$, and $\langle Ju, Sv\rangle_2=\langle JSu, v\rangle_2$ for all $v\in D(S)$. In other words, if $u=Q'(V-I)x$, $v=Q'(V-I)y$, $x, y\in D(V)$, we must have

$$\langle JQ'(V-I)x, (V+I)y\rangle_2=\langle J(V+I)x, Q'(V-I)y\rangle_2, \tag{2.5}$$

which is equivalent to

$$\langle (K+I)(V-I)x, (V+I)y\rangle_2=\langle (K-I)(V+I)x, (V-I)y\rangle_2. \tag{2.5$'$}$$

Note that the left-hand side of (2.5′) can be written as

$$\begin{aligned}\langle (K+I)(V-I)x, (V+I)y\rangle_2 \\ &= -\langle Kx, y\rangle_2+\langle KVx, y\rangle_2+\langle Vx, y\rangle_2 \\ &\quad -\langle Kx, Vy\rangle_2+\langle KVx, Vy\rangle_2-\langle x, Vy\rangle_2,\end{aligned}$$

while the right-hand side of (2.5′) has the form

$$\begin{aligned}\langle (K-I)(V+I)x, (V-I)y\rangle_2 &= -\langle Kx, y\rangle_2-\langle KVx, y\rangle_2+\langle Vx, y\rangle_2 \\ &\quad +\langle Kx, Vy\rangle_2+\langle KVx, Vy\rangle_2-\langle x, Vy\rangle_2.\end{aligned}$$

This calculation shows that (2.5) holds if and only if

$$\langle KVx, y\rangle_2=\langle Kx, Vy\rangle_2, \qquad x, y\in D(V). \tag{2.6}$$

Using the equation $V^{-1}=-KVK$, which implies $KV=V^{-1}K$, we have

$$\langle KVx, y\rangle_2=\langle V^{-1}Kx, y\rangle_2=\langle V^{-1}Kx, V^{-1}Vy\rangle_2=\langle Kx, Vy\rangle_2,$$

proving that (2.6) holds. Therefore, $JS\subset S^*J$.

The relation $LS \subset S^*L$ is equivalent to

$$\langle LQ'(V-I)\,x, (V+I)\,y\rangle_2 = \langle L(V+I)\,x, Q'(V-I)\,y\rangle_2, \tag{2.7}$$

which in turn is equivalent to

$$\langle (K-I)(V-I)\,x, (V+I)\,y\rangle_2 = \langle (K+I)(V+I)\,x, (V-I)\,y\rangle_2 \tag{2.7'}$$

for all $x, y \in D(V)$.

If we compare, as for (2.5′), both sides of (2.7′), we infer that (2.7′) holds if and only if (2.6) holds, and the latter is fulfilled, as we have seen, a consequence of our hypothesis.

Now, let $u = Q'(V-I)\,x$ for a certain $x \in D(V)$. Thus

$$Ku = -Q'K(V-I)\,x = -Q'(V^{-1}K - K)\,x = Q'(V-I)\,V^{-1}Kx.$$

Therefore, $Ku \in D(S)$, showing that $KD(S) \subset D(S)$.

Consequently, S is (J, L)-symmetric.

Let W be the Cayley transform of S (which exists via the first part of the proof). Then we have $W(S+Q')\,x = (S-Q')\,x$, $x \in D(S)$. But $x = Q'(V-I)\,v$ for some $v \in D(V)$. Since $Sx = (V+I)\,v$, $Q'x = -(V-I)\,v$, it follows that $(S+Q')\,x = 2v$. In other words, $D(W) = R(S+Q') = D(V)$.

Finally, let $y = (S+Q')\,x = 2v$. Thus

$$Wy = (S-Q')\,x = (V+I)\,v + (V-I)\,v = 2Vv.$$

Therefore, $Wy = Vy$, and so $W = V$.

LEMMA 2.11. *Let $V: D(V) \subset \mathcal{H}^2 \to \mathcal{H}^2$ be a partial isometry such that $R(V-I)$ is dense in $\mathcal{H}^2$, with $V^{-1} = -KVK$, and let $W = -Q'VQ'$. Consequently the inverse Cayley transform T of W exists, it is (J, L)-symmetric, and $T = S^* \mid Q'D(S) = S^{\#}$, where S is the inverse Cayley transform of V.*

Proof. Clearly, W is a partial isometry on $D(W) = Q'D(V)$. Note that $R(W-I) = R(Q'(I-V)\,Q') = Q'R(V-I)$ is dense in $\mathcal{H}^2$. Moreover, $W^{-1} = -Q'V^{-1}Q' = -KWK$. Therefore, the inverse Cayley transform T of W is well defined, and (J, L)-symmetric, by Lemma 2.10. Moreover, $T(Q'(W-I)\,x) = (W+I)\,x$, $x \in D(W)$, as given by (2.4).

We prove now that $T \subset S^*$. Let $Q'(W-I)\,x \in D(T)$, $x \in D(W)$, and let $Q'(V-I)\,y \in D(S)$, $y \in D(V)$, be arbitrary. The relation $T \subset S^*$ is equivalent to

$$\langle (W+I)\,x, Q'(V-I)\,y\rangle_2 = \langle Q'(W-I)\,x, (V+I)\,y\rangle_2,$$

which, in turn, is equivalent to

$$-\langle (V+I)\, Q'x, (V-I)\, y\rangle_2 = \langle (V-I)\, Q'x, (V+I)\, y\rangle_2$$

for all $x \in D(W)$, $y \in D(V)$, and the latter is easily verified.

Finally, $D(T) = Q'R(W-I) = R(V-I) = Q'D(S)$, via (2.3).

LEMMA 2.12. *Let $V: D(V) \subset \mathcal{H}^2 \to \mathcal{H}^2$ be a partial isometry such that $R(V-I)$ is dense in $\mathcal{H}^2$ and $V^{-1} = -KVK$. If $D(V) = \mathcal{H}^2$, then V is a unitary operator on $\mathcal{H}^2$ and the inverse Cayley transform S of V is (J, L)-self-adjoint.*

Proof. Since $KD(V) = D(V^{-1}) = K\mathcal{H}^2 = \mathcal{H}^2$, the operator V is unitary on $\mathcal{H}^2$, and so $V^{-1} = V^*$. Let T be the inverse Cayley transform of $-Q'VQ'$, which is well defined and (J, L)-symmetric, by the previous lemma. Moreover, $T \subset S^*$.

We shall show the equality $T = S^*$. Let $u_* \in D(S^*)$, and let $v_* = S^* u_*$. Let also $u = Q'(V-I)\, x \in D(S)$, $x \in D(V)$, be arbitrary. Then we have the equality

$$\langle v_*, Q'(V-I)\, x\rangle_2 = \langle u_*, (V+I)\, x\rangle_2.$$

This equality can be written as

$$\begin{aligned}\langle Q'v_*, x\rangle_2 - \langle Q'v_*, Vx\rangle_2 - \langle u_*, x\rangle_2 - \langle u_*, Vx\rangle_2 \\ = \langle VQ'v_*, Vx\rangle_2 - \langle Q'v_*, Vx\rangle_2 - \langle Vu_*, Vx\rangle_2 - \langle u_*, Vx\rangle_2 = 0,\end{aligned}$$

which implies the relation

$$VQ'v_* - Q'v_* - Vu_* - u_* = 0. \tag{2.8}$$

From (2.8) we deduce the equation

$$u_* = (1/2)(V-I)(Q'v_* - u_*) = (1/2)\, Q'(W-I)(v_* + Q'u_*),$$

showing that $u_* \in D(T)$. Therefore, $T = S^*$.

Finally, since T is (J, L)-symmetric, and $S^{**} = S$ (because S is closed), we have $S^*J \subset JS^{**} = JS \subset S^*J$. Similarly, $S^*L = LS$, that is, S is (J, L)-self-adjoint.

The next result shows that the Cayley transform is an order preserving map.

LEMMA 2.13. *Let $S_j: D(S_j) \subset \mathcal{H}^2 \to \mathcal{H}^2$ be (J, L)-symmetric, and let V_j be the Cayley transform of S_j $(j = 1, 2)$. We have $S_1 \subset S_2$ if and only if $V_1 \subset V_2$.*

Proof. If $S_1 \subset S_2$, and if $y=(S_1+Q')x$, $x \in D(S_1) \subset D(S_2)$, then $V_1 y=(S_1-Q')x=(S_2-Q')x=V_2 y$, and so $V_1 \subset V_2$.

Conversely, if $V_1 \subset V_2$, and if $u \in D(S_1)$, then, by Lemmas 2.8 and 2.10, there exists $x \in D(V_1)$ such that $u=Q'(V_1-I)x=Q'(V_2-I)x \in D(S_2)$, and $S_2 u=(V_2+I)x=(V_1+I)x=S_1 u$, showing that $S_1 \subset S_2$.

The properties of the quaternionic Cayley transform are summarized in the following (see [4, 13.19] for the corresponding classical result).

THEOREM 2.14. *Let $\mathscr{H}$ be a given complex Hilbert space. The quaternionic Cayley transform induces an order preserving one-to-one map from the set of all (J, L)-symmetric operators in $\mathscr{H}^2$, onto the set of all partial isometries V in $\mathscr{H}^2$ having the properties $V^{-1}=-KVK$, and $R(V-I)$ dense in $\mathscr{H}^2$. Moreover, the (J, L)-symmetric operator S is closed (resp. (J, L)-self-adjoint) if and only if its Cayley transform V is closed (resp. unitary).*

The proof of Theorem 2.14 is a direct consequence of Lemmas 2.8, 2.10, 2.12, and 2.13.

COROLLARY 2.15. *Every (J, L)-symmetric operator has a (J, L)-self-adjoint extension.*

Proof. By virtue of Lemma 2.3, with no loss of generality, we may assume S closed.

Let S be (J, L)-symmetric and closed, and let V be the Cayley transform of S. Then $\mathscr{H}_1=D(V)$, $\mathscr{H}_2=R(V)$ are closed subspaces of $\mathscr{H}^2$, by Lemma 2.8(b). Moreover, $K\mathscr{H}_1=\mathscr{H}_2$. Since K is unitary, we also have $K\mathscr{H}_1^{\perp}=\mathscr{H}_2^{\perp}$.

Let $W: D(W)=\mathscr{H}_1^{\perp} \to \mathscr{H}_1^{\perp}$ be a partial isometry such that $W^{-1}=-KWK$, which always exists. Indeed, one may take $W=iK \mid \mathscr{H}_1^{\perp}$.

We define $U=V \oplus W$, which is clearly a unitary operator. Moreover, $U^{-1}=-KUK$, and $R(U-I) \supset R(V-I)$ is dense in $\mathscr{H}^2$.

By virtue of Lemma 2.12, the inverse Cayley transform A of U is a (J, L)-self-adjoint operator, which satisfies $A \supset S$, by Lemma 2.13. This completes the proof of the corollary.

We recall that a densely defined closed operator N in a Hilbert space is said to be *normal* if $D(N^*)=D(N)$ and $N^*N=NN^*$. It is known that N is normal if and only if $D(N^*)=D(N)$ and $\|N^*x\|=\|Nx\|$, $x \in D(N)$ (see, for instance, [1, XII.9.9–XII.9.12] for some details).

In order that a densely defined operator S has a normal extension it is necessary that $D(S) \subset D(S^*)$ and $\|Sx\|=\|S^*x\|$ for all $x \in D(S)$.

In the remainder of this section we shall investigate the class of (J, L)-symmetric operators that have (J, L)-self-adjoint normal extensions.

Remark 2.16. (1) Let V be a closed partial isometry in $\mathscr{H}^2$ such that $V^{-1} = -KVK$. Thus there exists a decomposition of the (Hilbert) space $D(V)$ of the form $D(V) = \mathscr{H}_V^+ \oplus \mathscr{H}_V^-$ such that $V \mid \mathscr{H}_V^\pm = \pm iK \mid \mathscr{H}_V^\pm$. Indeed, if $Z = -KV$, then Z is a unitary operator on the Hilbert space $D(V)$. Moreover, $Z^{-1} = -V^{-1}K^{-1} = KV = -Z$, and so $-Z^2$ is the identity on $D(V)$, showing that the spectrum of Z is contained in $\{i, -i\}$. Therefore, if $\mathscr{H}_V^\pm$ is the kernel of $Z \mp i$, then we have $D(V) = \mathscr{H}_V^+ \oplus \mathscr{H}_V^-$. Since $V = KZ$, we obtain the equality $V \mid \mathscr{H}_W^\pm = \pm iK \mid \mathscr{H}_V^\pm$. If we denote by $P_V^\pm$ the projection of $D(V)$ onto $\mathscr{H}_V^\pm$, we clearly have $V = iKP_V^+ - iKP_V^-$. The projections $P_V^\pm$ completely determine the operator V, and they will be called in the following the *K-projections* of V. An easy calculation, based on the relations $(i \pm Z)^* = -(i \pm Z)$ and $(i+Z)(i-Z) = 0$, shows that $P_V^\pm = (2i)^{-1} (i \mp KV)$.

(2) Let V be a closed partial isometry in $\mathscr{H}^2$ such that $V - I$ is injective. We have the equality $Q'R(V-I) = R(V-I)$ if and only if there exists a bounded operator $G: D(V) \to D(V)$ such that $Q'(V-I) = (V-I)\, G$. Indeed, if $Q'R(V-I) = R(V-I)$, then, since $V - I$ is injective, the operator G is well defined and closed and hence bounded on $D(V)$. The converse is obvious.

Note also that such an operator G has a bounded inverse on $D(V)$. In fact, $G^{-1} = -G$, as one can easily see.

Lemma 2.17. *Let V be a closed partial isometry in $\mathscr{H}^2$ such that $V^{-1} = -KVK$, with $V - I$ injective. Then $Q'R(V-I) = R(V-I)$ if and only if there is a constant $M \geqslant 0$ such that*

$$\|P_V^\pm P_V Q' P_I^\pm x\|_2 \leqslant M \, \|P_V^\mp P_V P_I^\pm x\|_2, \qquad x \in \mathscr{H}^2, \tag{2.9}$$

where $P_V^\pm$ are the K-projections of V, $P_I^\pm$ are the K-projections of I, and P_V is the projection of $\mathscr{H}^2$ onto $D(V)$.

Proof. Assume first that $Q'R(V-I) = R(V-I)$, and let G be as in Remark 2.16(2). Since $V - I = (iK - I)\, P_V^+ - (iK + I)\, P_V^-$, and

$$Q'(V-I) = (iK - I)\, Q'P_V^- - (iK + I)\, Q'P_V^+,$$

we obtain readily

$$(iK - I)(Q'P_V^- - P_V^+ G) - (iK + I)(Q'P_V^+ - P_V^- G) = 0. \tag{2.10}$$

Notice that $\mathscr{H}^2 = \mathscr{H}_I^+ \oplus \mathscr{H}_I^-$, where

$$\mathscr{H}_I^\pm = N(K \pm i) = N(iK \mp I) = R(iK \pm I)$$

is the decomposition of $\mathscr{H}^2$ given by Remark 2.16(1) for $V=I$ and $Z=-K$. Therefore, from (2.10) we deduce that

$$R(Q'P_V^{\pm}-P_V^{\mp}G)\subset N(K\mp i),$$

whence

$$P_I^{\pm}(Q'P_V^{\pm}-P_V^{\mp}G)=0.$$

Passing to adjoints, we obtain

$$G^*P_V^{\mp}P_VP_I^{\pm}=-P_V^{\pm}P_VQ'P_I^{\pm}. \tag{2.11}$$

From (2.11) we infer easily (2.9), with $M\geqslant\|G\|$.

Conversely, if (2.9) holds, we may define a bounded operator G_* on the space

$$D(G_*)=R(P_V^-P_VP_I^+)\oplus R(P_V^+P_VP_I^-)$$

by the formula

$$G_*P_V^{\mp}P_VP_I^{\pm}x=-P_V^{\pm}P_VQ'P_I^{\pm}x, \qquad x\in\mathscr{H}^2.$$

Note that the orthogonal complement of $D(G_*)$ in $D(V)$ is null. Indeed, if $x\in D(V)$ is orthogonal to $D(G_*)$, then we must have $P_I^{\pm}P_V^{\mp}x=0$, which implies $x\in N(V-I)=\{0\}$. Therefore, G_* is a bounded operator which has a (unique) bounded extension on $D(V)$ (via (2.9)), also denoted by G_*. If G is the adjoint of G_*, then G will satisfy the equation (2.11), which is equivalent to the equality $Q'(V-I)=(V-I)G$.

THEOREM 2.18. *Let S be a closed (J,L)-symmetric operator, and let V be the Cayley transform of S. Then we have $Q'D(S)\subset D(S)$, and $\|S^*x\|=\|Sx\|$, $x\in D(S)$, if and only if*

$$\|P_V^{\pm}P_VQ'P_I^{\pm}x\|_2=\|P_V^{\mp}P_VP_I^{\pm}x\|_2, \qquad x\in\mathscr{H}^2, \tag{2.12}$$

where $P_V^{\pm}$ are the K-projections of V, $P_I^{\pm}$ are the K-projections of I, and P_V is the projection of $\mathscr{H}^2$ onto $D(V)$.

Proof. Let S be a closed (J,L)-symmetric operator such that $Q'D(S)\subset D(S)$ (implying $Q'D(S)=D(S)\subset D(S^*)$) and $\|S^*x\|_2=\|Sx\|_2$, $x\in D(S)$. The equality $Q'D(S)=D(S)$ shows that

$$R(Q'(V-I))=Q'R(V-I)=D(S)=Q'D(S)=R(V-I),$$

via (2.3). Therefore, by Remark 2.16(2), there exists a bounded operator $G\colon D(V)\to D(V)$ such that $Q'(V-I)=(V-I)G$. We shall prove that G is unitary on the Hilbert space $D(V)$. Let $x \in D(S)$, and set $u = (S+Q')x \in D(V)$, $v = (S+Q')Q'x \in D(V)$. Note that

$$(V-I)Gu = Q'(V-I)u = -Q'(2Q'x) = 2x = (V-1)v,$$

by (2.3). Since $V-I$ is injective, we must have $Gu = v$, whence $G(S+Q')x = (S+Q')Q'x$, $x \in D(S)$. Moreover,

$$\begin{aligned}\|(S+Q')Q'x\|_2 &= \|Q'(S^*+Q')x\|_2 = \|(S^*+Q')x\|_2 \\ &= \|(S+Q')x\|_2, \qquad x \in D(S),\end{aligned} \tag{2.13}$$

since Q' is an isometry, and by Remark 2.7(1).

This shows that G is an (invertible) isometry, and hence a unitary operator. In addition, by virtue of (2.11), we should have $\|P_V^{\mp} P_V Q' P_I^{\pm} x\|_2 = \|P_V^{\mp} P_V P_I^{\pm} x\|_2$ for all $x \in \mathscr{H}^2$.

Conversely, if $\|P_V^{\pm} P_V Q' P_I^{\pm} x\|_2 = \|P_V^{\mp} P_V P_I^{\pm} x\|_2$ for all $x \in \mathscr{H}^2$, then, by the (proof of the) previous lemma, there exists a bounded operator $G: D(V) \to D(V)$, which is actually unitary, such that $(V-I)G = Q'(V-I)$. Therefore, $Q'D(S) = D(S)$ by (2.3). Moreover, as above, $G(S+Q')x = (S+Q')Q'x$, $x \in D(S)$. Because G is unitary, we deduce that (2.13) holds, whence we infer $\|S^*x\|_2 = \|Sx\|_2$, $x \in D(S)$ (see Remark 2.7(1)). This completes the proof of the theorem.

Remark 2.19. Let V be a closed partial isometry in $\mathscr{H}^2$ such that $V^{-1} = -KVK$. Then the operator $I - V$ is injective if and only if $P_I^+ P_V^- + P_I^- P_V^+$ is injective. Indeed, as in the proof of Lemma 2.17, $(V-I)x = 0$ if and only if $P_V^{\mp} x \in \mathscr{H}_I^{\mp}$ and, therefore, if and only if $(P_I^+ P_V^- + P_I^- P_V^+)x = 0$.

Note also that if V is everywhere defined, then $V - I$ is injective if and only if $R(I - V)$ is dense.

LEMMA 2.20. *Let S be a closed (J, L)-symmetric operator, let V be the Cayley transform of S, and let $P_V^{\pm}$ be the K-projections of V. Then S has a (J, L)-self-adjoint normal extension if and only if there exists a decomposition $\mathscr{H}^2 = \mathscr{H}^+ \oplus \mathscr{H}^-$ such that $P^{\pm} \mid D(V) = P_V^{\pm}$, $P_I^+ P^- + P_I^- P^+$ is injective, and*

$$\|P^{\pm} Q' P_I^{\pm} x\|_2 = \|P^{\mp} P_I^{\pm} x\|_2, \qquad x \in \mathscr{H}^2, \tag{2.14}$$

where $P^{\pm}$ are the projection of $\mathscr{H}^2$ onto $\mathscr{H}^{\pm}$, and $P_I^{\pm}$ are the K-projections of I.

Proof. Assume first that S has a (J, L)-self-adjoint normal extension A. If U is the Cayley transform of A, then U is a unitary operator on $\mathscr{H}^2$ that satisfies $U^{-1}=-KUK$. Let $P_U^{\pm}$ be the K-projections of U. Since $U\supset V$, we must have

$$iK(P_U^+-P_V^+)\,x-iK(P_U^- - P_V^-)\,x=0, \qquad x\in D(V),$$

implying $(P_U^+-P_U^-)\,|\,D(V)=P_V^+-P_V^-$. Let $x^+=P_V^+x^+\in D(V)$. Then $(P_U^+-P_U^-)\,x^+=x^+$, which implies $P_U^-x^+=0$, and so $P_U^+P_V^+=P_V^+$, $P_U^-P_V^+=0$. Similarly, $P_U^-P_V^-=P_V^-$, $P_U^+P_V^-=0$. Hence, $P_V^+\subset P_U^+$, and $P_V^-\subset P_U^-$.

Next, since $R(U-I)$ is dense, and thus $U-I$ is injective, we must have $P_I^+P_U^-+P_I^-P_U^+$ injective, by Remark 2.19. Moreover, as A is a (J,L)-self-adjoint normal extension of S, then we also have (2.14), via Theorem 2.18. Consequently, our assertion holds with $P^{\pm}=P_U^{\pm}$.

Conversely, assume that there exists a decomposition $\mathscr{H}^2=\mathscr{H}^+\oplus\mathscr{H}^-$ such that $P^{\pm}\,|\,D(V)=P_V^{\pm}$ and such that (2.14) holds.

Let U be the unitary operator iKP^+-iKP^-. Since $P^{\pm}\,|\,D(V)=P_V^{\pm}$, we clearly have $U\supset V$.

It is easily checked that $U^{-1}=-KUK$. In addition, $R(U-I)$ is dense, via Remark 2.19. Therefore, the inverse Cayley transform A of U exists, it is (J,L)-self-adjoint, and $A\supset S$, by Theorem 2.14. In addition, by virtue of Theorem 2.18, we obtain $D(A^*)=Q'D(A)=D(A)$, and $\|A^*x\|_2=\|Ax\|_2$, $x\in D(A)$. Therefore, the operator A is normal.

THEOREM 2.21. *Let S be a (J,L)-symmetric operator such that $Q'D(S)\subset D(S)$ and $\|S^*x\|=\|Sx\|$, $x\in D(S)$. Then S has a (J,L)-self-adjoint normal extension if and only if there are two projections $\tilde{P}^{\pm}$ in $\mathscr{H}^2$ such that $N(S^*-Q')=\tilde{P}^+\mathscr{H}^2\oplus\tilde{P}^-\mathscr{H}^2$ and*

$$\|\tilde{P}^{\pm}Q'P_I^{\pm}x\|_2=\|\tilde{P}^{\mp}P_I^{\pm}x\|_2, \qquad x\in\mathscr{H}^2, \tag{2.15}$$

where $P_I^{\pm}$ are the K-projections of I.

Proof. With no loss of generality we may assume S closed. Indeed, let $\bar{S}$ be the canonical closure of S, which exists and is (J,L)-symmetric, by Lemma 2.3. If $x\in D(\bar{S})$, then there exists a sequence $(x_k)_k$ in $D(S)$ such that $x_k\to x$, and $Sx_k\to\bar{S}x$, as $k\to\infty$. Note that $(S^*x_k)_k$ is a Cauchy sequence. Therefore, $x\in D(S^*)$ and $\|S^*x\|=\lim_{k\to\infty}\|Sx_k\|=\|\bar{S}x\|$. In addition, $Q'x_k\to Q'x$, and $SQ'x_k=Q'S^*x_k\to Q'S^*x$, as $k\to\infty$, implying $Q'x\in D(\bar{S})$.

Replacing S by $\bar{S}$ if necessary, we shall suppose that S is closed.

Assume first that S has a (J,L)-self-adjoint normal extension A, and let V, U be the Cayley transforms of S, A, respectively. Let also $P_V^{\pm}$, $P_U^{\pm}$ be the K-projections of V, U, respectively, let P_V be the projection of $\mathscr{H}^2$ onto

$D(V)$, and let $\tilde{P}_V^{\pm} = P_V^{\pm} P_V$. As in the proof of the previous lemma, since $V \subset U$, we have $P_V^{\pm} \subset P_U^{\pm}$, and therefore $\tilde{P}_V^{\pm} \leqslant P_U^{\pm}$.

We set $\tilde{P}^{\pm} = P_U^{\pm} - \tilde{P}_V^{\pm}$. Since $D(V) = \tilde{P}_V^{+} \mathscr{H}^2 \oplus \tilde{P}_V^{-} \mathscr{H}^2$ and $\mathscr{H}^2 = P_U^{+} \mathscr{H}^2 \oplus P_U^{-} \mathscr{H}^2$, we deduce that $D(V)^{\perp} = N(S^* - Q') = \tilde{P}^{+} \mathscr{H}^2 \oplus \tilde{P}^{-} \mathscr{H}^2$. Moreover, as we have $\|\tilde{P}_V^{\pm} Q' P_I^{\pm} x\|_2 = \|\tilde{P}_V^{\mp} P_I^{\pm} x\|_2$, $\|P_U^{\pm} Q' P_I^{\pm} x\|_2 = \|P_U^{\mp} P_I^{\pm} x\|_2$, $x \in \mathscr{H}^2$, by Theorem 2.18, we infer easily (2.15).

Conversely, assuming the existence of the pair $\tilde{P}^{\pm}$ with the stated properties, we set $P^{\pm} = \tilde{P}^{\pm} + \tilde{P}_V^{\pm}$ and define the operator $U = iK(P^+ - P^-)$ which is unitary and satisfies the equation $U^{-1} = -KUK$ (as in Lemma 2.20). Moreover, since $U \supset V$, the space $R(U - I)$ is dense in $\mathscr{H}^2$, and so $P_I^{+} P^- + P_I^{-} P^+$ is injective, by Remark 2.19. Since (2.15) and (2.12) hold, we infer that (2.14) also holds. Therefore, by Lemma 2.20, the operator S has a (J, L)-self-adjoint normal extension, which completes the proof of the theorem.

COROLLARY 2.22. *Let U be a unitary operator in $\mathscr{H}^2$ with the following properties: $U^{-1} = -KUK$, with $R(U - I)$ dense in $\mathscr{H}^2$, and $Q'U = UQ'$. Let also S be a closed (J, L)-symmetric operator and let V be the Cayley transform of S. If $V \subset U$, then S has a (J, L)-self-adjoint normal extension.*

Proof. The K-projections of U are given by $P_U^{\pm} = (2i)^{-1} (i \pm KU)$ (see Remark 2.16). This shows that $Q' P_U^{\pm} = P_U^{\mp} Q'$. Therefore, $\|P_U^{\pm} Q' P_I^{\pm} x\| = \|P_U^{\mp} P_I^{\pm} x\|$ for all $x \in \mathscr{H}^2$. By virtue of Theorems 2.16 and 2.18, the inverse Cayley transform A of U is (J, L)-self-adjoint and normal. Since $V \subset U$, the assertion follows.

3. PAIRS OF SYMMETRIC OPERATORS

Let $\mathscr{S} = (S_1, S_2)$ be a pair of symmetric operators in $\mathscr{H}$. We define the matrix

$$Q(\mathscr{S}) = \begin{pmatrix} S_1 & S_2 \\ -S_2 & S_1 \end{pmatrix} \tag{3.1}$$

on the subspace

$$D(Q(\mathscr{S})) = [D(S_1) \cap D(S_2)] \oplus [D(S_1) \cap D(S_2)] \subset \mathscr{H}^2.$$

Remark 3.1. If $\mathscr{S} = (S_1, S_2)$ is a pair of symmetric operators, and if $D(S_1) \cap D(S_2)$ is dense in $\mathscr{H}$, then $Q(\mathscr{S})$ is (J, L)-symmetric, by Proposition 2.4. If $\mathscr{S}^{\#} = (S_1 - S_2)$, we also have

$$\langle Q(\mathscr{S})\, x, y \rangle_2 = \langle x, Q(\mathscr{S}^{\#})\, y \rangle_2, \qquad x, y \in D(Q(\mathscr{S})), \tag{3.2}$$

as one can easily see (see the proof of Proposition 2.4), and so $Q(\mathscr{S}^{\#}) = Q(\mathscr{S})^{\#} = Q(\mathscr{S})^* \mid D(Q(\mathscr{S}))$.

LEMMA 3.2. *Let $\mathscr{S} = (S_1, S_2)$ be a pair of symmetric operators with $D(S_1) \cap D(S_2)$ dense in $\mathscr{H}^2$. Assume that*

$$\|Q(\mathscr{S})^{\#}\, x\|_2 = \|Q(\mathscr{S})\, x\|_2, \qquad x \in D(Q(\mathscr{S})). \tag{3.3}$$

If the operators S_1, S_2 are closed, then $Q(\mathscr{S})$ is closed.

Proof. To prove that $Q(\mathscr{S})$ is closed, let $y_k = y_k' \oplus y_k''$, $k \geqslant 1$ be a sequence from $D(Q(\mathscr{S}))$ such that $y_k \to y = y' \oplus y''$ and $Q(\mathscr{S})\, y_k \to u = u' \oplus u''$, as $k \to \infty$. Note that the sequence $(Q(\mathscr{S})^{\#}\, y_k)_{k \geqslant 1}$ is also convergent to a certain vector $v = v' \oplus v''$, as a consequence of (3.3). Hence, we have $y_k' \to y'$, $y_k'' \to y''$, $S_1 y_k' \to 2^{-1}(u' + v')$, $S_2 y_k' \to 2^{-1}(v'' - u'')$, $S_1 y_k'' \to 2^{-1}(u'' + v'')$, $S_2 y_k'' \to 2^{-1}(u' - v')$, as $k \to \infty$. Since both S_1, S_2 are closed, we infer that $y', y'' \in D(S_1) \cap D(S_2)$, $Q(\mathscr{S})\, y = u$ and so $Q(\mathscr{S})$ is closed.

LEMMA 3.3. *Let $\mathscr{A} = (A_1, A_2)$, where A_1, A_2 are commuting self-adjoint operators. Then the operator $Q(\mathscr{A})$ is normal.*

Proof. As we have already noticed in the previous section, it suffices to prove that $Q(\mathscr{A})$ is densely defined and closed, that $D(Q(\mathscr{A})^*) = D(Q(\mathscr{A}))$, and that $\|Q(\mathscr{A})^*\, x\|_2 = \|Q(\mathscr{A})\, x\|_2$, $x \in D(Q(\mathscr{A}))$.

Since A_1, A_2 commute, and therefore they have a joint spectral measure E (see, for instance, [6, IV.10]), the subspace $D(A_1) \cap D(A_2)$ is dense in $\mathscr{H}$, and so $D(Q(\mathscr{A}))$ is dense.

We prove first that (3.3) holds.

Let $x = x' \oplus x'' \in D(Q(\mathscr{A}))$ be fixed, and let $(\sigma_k)_{k \geqslant 1}$ be an increasing sequence of compact subsets of $\mathbf{R}^2$, such that $\bigcup_{k \geqslant 1} \sigma_k = \mathbf{R}^2$. Define $x_k = x_k' \oplus x_k'' = E(\sigma_k)\, x' \oplus E(\sigma_k)\, x''$, $k \geqslant 1$. Thus we have $x_k' \to x'$, $x_k'' \to x''$, $A_j x_k' \to A_j x'$, $A_j x_k'' \to A_j x''$, as $k \to \infty$, $j = 1, 2$. Thus $Q(\mathscr{A})\, x_k \to Q(\mathscr{A})\, x$, $Q(\mathscr{A})^{\#}\, x_k \to Q(\mathscr{A})^{\#} x$, as $k \to \infty$. Note also that

$$\begin{aligned}\|Q(\mathscr{A})\, x_k\|_2^2 &= \langle Q(\mathscr{A})^{\#}\, Q(\mathscr{A})\, x_k, x_k \rangle_2 = \langle Q(\mathscr{A})\, Q(\mathscr{A})^{\#}\, x_k, x_k \rangle_2 \\ &= \|Q(\mathscr{A})^{\#}\, x_k\|_2^2\end{aligned}$$

for all $k \geqslant 1$, by (3.2) and the fact that A_1, A_2 commute. Therefore,

$$\|Q(\mathscr{A})\, x\|_2 = \lim_{k \to \infty} \|Q(\mathscr{A})\, x_k\|_2 = \lim_{k \to \infty} \|Q(\mathscr{A})^{\#}\, x_k\|_2 = \|Q(\mathscr{A})^{\#}\, x\|_2,$$

which is precisely (3.3).

The fact that $Q(\mathscr{A})$ is closed now follows Lemma 3.2. Similarly, $Q(\mathscr{A})^{\#}$ is also closed.

We have only to prove that $Q(\mathscr{A})^* = Q(\mathscr{A})^{\#}$. By virtue of (3.2), it suffices to prove that $Q(\mathscr{A})^* \subset Q(\mathscr{A})^{\#}$. Let

$$x \oplus Q(\mathscr{A})^* x \in G(Q(\mathscr{A})^*) \ominus G(Q(\mathscr{A})^{\#}).$$

Then we have

$$\langle x, y \rangle_2 + \langle Q(\mathscr{A})^* x, Q(\mathscr{A})^{\#} y \rangle_2 = 0, \qquad y \in D(Q(\mathscr{A})^{\#}).$$

If we assume $y = y' \oplus y'' = E(\sigma) v' \oplus E(\sigma) v''$, with $\sigma \subset \mathbf{R}^2$ compact and v', v'' arbitrary in $\mathscr{H}$, then $Q(\mathscr{A})^{\#} y \in D(Q(\mathscr{A}))$. Hence, if $x = x' \oplus x''$, $1 = 1_{\mathscr{H}}$, we have

$$\langle x', (1 + A_1^2 + A_2^2) y' \rangle + \langle x'', (1 + A_1^2 + A_2^2) y'' \rangle = 0.$$

Since $1 + A_1^2 + A_2^2$ has a bounded inverse, we infer $x' = x'' = 0$. Therefore, $Q(\mathscr{A})^* = Q(A)^{\#}$, and thus $Q(\mathscr{A})$ is normal.

Remark 3.4. Let A be self-adjoint in the Hilbert space $\mathscr{H}$, and let $\mathscr{L}$ be a closed linear subspace of $\mathscr{H}$. Assume that the operator $B = A \mid (D(A) \cap \mathscr{L})$ is self-adjoint in $\mathscr{L}$. Then the operator $(i - A)^{-1}$ leaves invariant the subspace $\mathscr{L}$, and $(i - A)^{-1} \mid \mathscr{L} = (i - B)^{-1}$. Indeed, we have

$$(i - A)[(i - A)^{-1} x - (i - B)^{-1} x] = 0,$$

whence $(i - A)^{-1} x = (i - B)^{-1} x$ for all $x \in \mathscr{L}$.

LEMMA 3.5. *Let $\mathscr{S} = (S_1, S_2)$, where S_1, S_2 are symmetric operators, and assume that $Q(\mathscr{S})$ is normal. Then the subspace $D_0 = D(S_1) \cap D(S_2)$ is dense in $\mathscr{H}$, and the canonical closures of S_1, S_2 are commuting self-adjoint operators.*

Proof. Since $Q(\mathscr{S})$ is normal, and hence $D(Q(\mathscr{S}))$ is dense in $\mathscr{H}^2$, it follows that $D_0 = D(S_1) \cap D(S_2)$ is dense in $\mathscr{H}$.

With no loss of generality we may suppose $D_0 = D(S_1) = D(S_2)$. Indeed, if the canonical closure $\overline{S_j \mid D_0}$ of $S_j \mid D_0$ is self-adjoint, since $\bar{S}_j$ is symmetric, we must have $\overline{S_j \mid D_0} = \bar{S}_j$, $j = 1, 2$. Note also that $Q(\mathscr{S})^* = Q(\mathscr{S})^{\#}$, since $Q(\mathscr{S})^* \supset Q(\mathscr{S})^{\#}$ and $Q(\mathscr{S})$ is normal.

If B_1, B_2 are the canonical closures of $(1/2)(Q(\mathscr{S}) + Q(\mathscr{S})^*)$, $(1/2i)(Q(\mathscr{S})) - Q(\mathscr{S})^*)$, respectively, then B_1, B_2 are commuting self-adjoint operators (see [1, XII.9.11]). We obviously have

$$B_1 \mid D(Q(\mathscr{S})) = \begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix}, \qquad B_2 \mid D(Q(\mathscr{S})) = \begin{pmatrix} 0 & -iS_2 \\ iS_2 & 0 \end{pmatrix}.$$

Let A_j be the canonical closure of S_j, $j=1, 2$. Note that $x, y \in D(A_1)$ if and only if $x \oplus y \in D(B_1)$ and $B_1(x \oplus y) = A_1 x \oplus A_1 y$, which is easily checked.

Similarly, $x, y \in D(A_2)$ if and only if $x \oplus y \in D(B_2)$ and $B_2(x \oplus y) = (-iA_2 y) \oplus (iA_2)\, x$.

Next, we show that $A_1^* = A_1$. Let $u \in D(A_1^*)$, and let $v = A_1^* u$. Let also $x_1, x_2 \in D(A_1)$. Thus we have

$$\langle v \oplus 0, x_1 \oplus x_2 \rangle_2 = \langle u \oplus 0, B_1(x_1 \oplus x_2) \rangle_2.$$

This implies $u \oplus 0 \in D(B_1)$ and $B_1(u \oplus 0) = v \oplus 0$, and so $u \in D(A_1)$ and $A_1 u = v = A_1^* u$, proving that A_1 is self-adjoint.

A similar argument shows that A_2 is self-adjoint.

Let $\mathscr{L}_1 = \{x \oplus x; x \in \mathscr{H}\}$ and let $\theta_1 : \mathscr{H} \to \mathscr{L}_1$ be given by $\theta_1 x = (\sqrt{2}/2)(x \oplus x)$, $x \in \mathscr{H}$. Let also $\theta_2 : \mathscr{H} \to \mathscr{L}_2$ be given by $\theta_2 x$: $(\sqrt{2}/2)(x \oplus ix)$, $x \in \mathscr{H}$, where $\mathscr{L}_2 = \{x \oplus ix; x \in \mathscr{H}\}$. Then both θ_1, θ_2 are surjective isometries. We define $\tilde{A}_j = \theta_j A_j \theta_j^*$ on $\theta_j D(A_j)$, $j=1, 2$. Clearly, $\tilde{A}_1$, $\tilde{A}_2$ are self-adjoint operators in $\mathscr{L}_1$, $\mathscr{L}_2$, respectively. Moreover, $B_j \supset \tilde{A}_j$, $j=1, 2$, as one can easily check.

We want to show that the operators A_1, A_2 commute. To see this, it suffices to show that the bounded operators $(i-A_1)^{-1}$, $(i-A_2)^{-1}$ commute (see, for instance, [6, IV.10]).

We have

$$\begin{aligned}(i-A_2)^{-1}\,(i-A_1)^{-1} &= \theta_2^*(i-\tilde{A}_2)^{-1}\,\theta_2\theta_1^*(i-\tilde{A}_1)^{-1}\,\theta_1 \\ &= \theta_2^*(i-B_2)^{-1}\,\theta_2\theta_1^*(i-B_1)^{-1}\,\theta_1,\end{aligned}$$

since $(i-B_j)^{-1}$ extends $(i-\tilde{A}_j)^{-1}$ $(j=1, 2)$, by Remark 3.4.

Note that $\theta_2\theta_1^*$: $\mathscr{L}_1 \to \mathscr{L}_2$ acts as the matrix $M_1 = \left(\begin{smallmatrix}1 & 0\\ 0 & i\end{smallmatrix}\right)$. As we have $M_1 B_1 = B_1 M_1$ on $D_0 \oplus D_0$, and so $M_1 B_1 = B_1 M_1$ on $D(B_1)$, we infer $M_1(i-B_1)^{-1} = (i-B_1)^{-1}\,M_1$. Similarly, $\theta_1\theta_2^*$: $\mathscr{L}_2 \to \mathscr{L}_1$ acts as the matrix $M_2 = \left(\begin{smallmatrix}1 & 0\\ 0 & -i\end{smallmatrix}\right)$. As we also have $M_2 B_1 = B_1 M_2$ on $D(B_1)$, we infer $M_2(i-B_1)^{-1} = (i-B_1)^{-1}\,M_2$. Therefore,

$$\begin{aligned}\theta_2^*(i-B_2)^{-1}\,\theta_2\theta_1^*(i-B_1)^{-1}\,\theta_1 &= \theta_2^*(i-B_2)^{-1}\,(i-B_1)^{-1}\,\theta_2 \\ &= \theta_2^*(i-B_1)^{-1}\,(i-B_2)^{-1}\,\theta_2 \\ &= \theta_1^*(i-B_1)^{-1}\,\theta_1\theta_2^*(i-B_2)^{-1}\,\theta_2 \\ &= (i-A_1)^{-1}\,(i-A_2)^{-1},\end{aligned}$$

since B_1, B_2 commute. This completes the proof of the lemma.

Remark 3.6. With the notation of the previous proof, we have the equality

$$D(A_1) \cap D(A_2) = D(S_1) \cap D(S_2).$$

Indeed, if $x \in D(A_1) \cap D(A_2)$, then, as in the previous proof, we have $x \oplus x \in D(B_1) \cap D(B_2) = D_0 \oplus D_0$, where the last equality is true since $Q(\mathcal{S}) = B_1 + iB_2$ (see [1, XII.9.11]). Therefore, $x \in D_0$. The converse is obvious.

THEOREM 3.7. *Let $\mathcal{S} = (S_1, S_2)$ be a pair of symmetric operators in the Hilbert space $\mathcal{H}$. The following conditions are equivalent:*

(1) *The operator $Q(\mathcal{S})$ is normal.*

(2) *The space $D(S_1) \cap D(S_2)$ is dense in $\mathcal{H}$, the canonical closures A_j of the operators S_j, $j = 1, 2$ are commuting self-adjoint operators, and $D(S_1) \cap D(S_2) = D(A_1) \cap D(A_2)$.*

The proof of this theorem follows from Lemmas 3.3, 3.5, *and Remark* 3.6.

The next theorem is a spatial version of the well-known result which asserts that the canonical closure of a densely defined symmetric operator whose defect indices are null is self-adjoint. It also provides a criterion of commutativity for the canonical closures of some symmetric operators.

THEOREM 3.8. *Let $\mathcal{S} = (S_1, S_2)$ be a pair of symmetric operators in the Hilbert space $\mathcal{H}$. Suppose that there exists a linear subspace $D \subset D(S_1) \cap D(S_2)$, D is dense in $\mathcal{H}$, with the following properties:*

(1) $\|Q(\mathcal{S})\, x\| = \|Q(\mathcal{S})^{\#}\, x\|$, $x \in D \oplus D$.

(2) *The set $\{(Q(\mathcal{S}) + Q')\, x; x \in D \oplus D\}$ is dense in $\mathcal{H}^2$.*

Then the canonical closures of S_1, S_2 are commuting self-adjoint operators.

Proof. With no loss of generality we may assume $D(S_1) = D(S_2) = D$ (see the proof of Lemma 3.5). Notice that the operator $Q(\mathcal{S})$ is closable, and its canonical closure, say A, is (J, L)-symmetric, by Remark 3.1 and Lemma 2.3.

Next, we prove that $D(A) \subset D(A^*)$. Let $x, y \in D(A)$. Thus there are two sequences $(x_k)_k$, $(y_k)_k$ in $D \oplus D$ such that $x_k \to x$, $y_k \to y$, $Q(\mathcal{S})\, x_k \to Ax$,

$Q(\mathcal{S})\, y_k \to Ay$ as $k \to \infty$. Note that the sequence $(Q(\mathcal{S})^{\#}\, x_k)_k$ is also convergent, by (1), and let v be its limit. Then, by (3.2), we have

$$\langle v, y \rangle_2 = \lim_{k \to \infty} \langle Q(\mathcal{S})^{\#}\, x_k, y_k \rangle_2 = \lim_{k \to \infty} \langle x_k, Q(\mathcal{S})\, y_k \rangle_2 = \langle x, Ay \rangle_2,$$

showing that $x \in D(A^*)$, and $A^*x = \lim_{k \to \infty} Q(\mathcal{S})^{\#}\, x_k$.

Let V be the Cayley transform of A. It follows from (2) that $D(V) = \mathcal{H}^2$. Therefore, V is unitary and so A is actually (J, L)-self-adjoint, by Lemma 2.12. In particular, $JD(A) = D(A^*)$. As we also have $D(A) \subset D(A^*)$, we infer $D(A) = D(A^*)$, by Lemma 1.2.

Now we can see that operator A is, in fact, normal. Indeed, with the above notation, we have

$$\|A^*x\|_2 = \lim_{k \to \infty} \|Q(\mathcal{S})^{\#}\, x_k\|_2 = \lim_{k \to \infty} \|Q(\mathcal{S})\, x_k\|_2 = \|Ax\|_2,$$

via (1), for all $x \in D(A) = D(A^*)$.

We prove now that the canonical closure of $A + A^*$ coincides with the canonical closure of $Q(\mathcal{S}) + Q(\mathcal{S})^{\#}$. Let x be a vector with the property that there exists a sequence $(x_k)_k$ from $D(A + A^*) = D(A) = D(A^*)$ such that $x_k \to x$ and $(A + A^*)\, x_k \to y = (\overline{A + A^*})\, x$, as $k \to \infty$. Let $\varepsilon > 0$ be fixed. Then we can find a $k(\varepsilon)$ such that $k \geqslant k(\varepsilon)$ implies $\|x - x_k\|_2 < \varepsilon/2$, $\|y - (A + A^*)\, x_k\|_2 < \varepsilon/2$. Since $x_k \in D(A)$, we can find a sequence $(x_{km})_m$ in $D(Q(\mathcal{S}))$ such that $x_{km} \to x_k$, and $Q(\mathcal{S})\, x_{km} \to Ax_k$, as $m \to \infty$, for each $k \geqslant 1$. Thus as above, we also have $Q(\mathcal{S})^{\#}\, x_{km} \to A^*x_k$ as $m \to \infty$. For every $k \geqslant 1$ we may choose an index $m(k)$ such that $\|x_k - x_{km(k)}\|_2 < 1/k$, $\|Ax_k - Q(\mathcal{S})\, x_{km(k)}\|_2 < 1/2k$, $\|A^*x_k - Q(\mathcal{S})^{\#}\, x_{km(k)}\|_2 < 1/2k$. Assuming $k \geqslant \max\{k(\varepsilon), 2/\varepsilon\}$, we obtain $\|x - x_{km(k)}\|_2 < \varepsilon$, and $\|y - (Q(\mathcal{S}) + Q(\mathcal{S})^{\#})\, x_{km(k)}\|_2 < \varepsilon$. Consequently, the canonical closure of $A + A^*$ coincides with the canonical closure of $Q(\mathcal{S}) + Q(\mathcal{S})^{\#}$.

Note that $A = \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}$, with A_1, A_2 symmetric, and $A^* = A^{\#}$, by Proposition 2.4 and Remark 2.5. In addition, with no loss of generality we may assume that $D(A_1) = D(A_2) = D_0$, where $D_0 \oplus D_0 = D(A)$, and the canonical closures of A_1, A_2 are commuting self-adjoint operators, by Lemma 3.6. Since $A \supset Q(\mathcal{S})$, $A^* = A^{\#} \supset Q(\mathcal{S})^{\#}$, we have

$$(A_1 \oplus A_1) \mid D(A) = (1/2)(A + A^*) \supset (1/2)(Q(\mathcal{S}) + Q(\mathcal{S})^{\#}) = S_1 \oplus S_1.$$

Therefore, if $u \in D(\bar{A}_1)$, then

$$u \oplus 0 \in D(\overline{A + A^*}) = D(\overline{Q(\mathcal{S}) + Q(\mathcal{S})^{\#}}),$$

from which one deduces that the canonical closure of A_1 coincides with the canonical closure of S_1. Similarly, the canonical closure of A_2 coincides with the canonical closure of S_2, which completes the proof of the theorem.

COROLLARY 3.9. *Let $S_1, ..., S_n$ be symmetric operators in the Hilbert space $\mathscr{H}$. Suppose that there exists a linear subspace $D \subset D(S_1) \cap \cdots \cap D(S_n)$, where D is dense in $\mathscr{H}$, with the following properties:*

(1) $\|Q(\mathscr{S}_{jk})\, x\| = \|Q(\mathscr{S}_{jk})^{\#}\, x\|$, $x \in D \oplus D$ for all indices j, k with $1 \leqslant j < k \leqslant n$, where $\mathscr{S}_{jk} = (S_j, S_k)$.

(2) The set $\{(Q(\mathscr{S}_{jk}) + Q')\, x; x \in D \oplus D\}$, is dense in $\mathscr{H}^2$ for all indices j, k with $1 \leqslant j < k \leqslant n$.

Then $S_1, ..., S_n$ have commuting self-adjoint extensions.

Proof. We apply the previous theorem to each pair $\mathscr{S}_{jk} = (S_j, S_k)$, $1 \leqslant j < k \leqslant n$. The self-adjoint extension A_j of S_j is unambiguously defined because A_j is the canonical closure of S_j, $j = 1, ..., n$, via Theorem 3.8.

A general criterion concerning the existence of commuting self-adjoint extensions for some pairs of symmetric operators, and which corresponds to the case when the associated Cayley transform is not necessarily unitary, is given by the following.

THEOREM 3.10. *Let $\mathscr{S} = (S_1, S_2)$ be a pair of symmetric operators in the Hilbert space $\mathscr{H}$. Suppose that the linear subspace $D = D(S_1) \cap D(S_2)$ is dense in $\mathscr{H}$, and that $\|Q(\mathscr{S})\, x\| = \|Q(\mathscr{S})^{\#}\, x\|$, $x \in D \oplus D$. Then S_1, S_2 have self-adjoint commuting extensions if and only if there are two projections $\tilde{P}^{\pm}$ in $\mathscr{H}^2$ such that $N(Q(\mathscr{S})^* - Q') = \tilde{P}^+ \mathscr{H}^2 \oplus \tilde{P}^- \mathscr{H}^2$, and*

$$\|\tilde{P}^{\pm} Q' P_I^{\pm}\, x\|_2 = \|\tilde{P}^{\mp} P_I^{\pm}\, x\|_2, \qquad x \in \mathscr{H}^2, \tag{3.4}$$

where $P_I^{\pm}$ are the K-projections of I.

Proof. Assume first that (3.4) holds. If $S = Q(\mathscr{S})$, then $Q'D(S) \subset D(S)$ and $\|S^*x\| = \|Sx\|$ for all $x \in D(S)$, from the hypothesis. Since (3.4) is precisely (2.15), it follows from Theorem 2.21 that S has a (J, L)-self-adjoint normal extension A. We write $A = \left(\begin{smallmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{smallmatrix}\right)$, via Proposition 2.4, with A_1, A_2 symmetric. Since $A = Q(\mathscr{A})$ where $\mathscr{A} = (A_1, A_2)$, we obtain from Theorem 3.7 that the canonical closures of A_1, A_2, are commuting self-adjoint operators. Moreover, as $Q(\mathscr{A}) \supset Q(\mathscr{S})$, we deduce $A_j \supset S_j$, $j = 1, 2$, which implies our assertion.

Conversely, if S_1, S_2 have commuting self-adjoint extensions, say A_1, A_2, respectively, then $Q(\mathscr{A})$ is normal, with $\mathscr{A} = (A_1, A_2)$, again by Theorem 3.7. This implies $D(Q(\mathscr{A})^*) = D(Q(\mathscr{A}))$. Note also that the (J, L)-symmetric operator $Q(\mathscr{A})$ is actually (J, L)-self-adjoint. Indeed, as we have $Q(\mathscr{A})^{\#} = Q(\mathscr{A})^*$, it follows $Q(\mathscr{A})\,J = JQ(\mathscr{A})^*$, and $Q(\mathscr{A})\,L = LQ(\mathscr{A})^*$. Finally, since $Q(\mathscr{A})$ is a (J, L)-self-adjoint normal extension of $Q(\mathscr{S})$, it follows from Theorem 2.21 that (3.4) must be fulfilled.

In the remainder of this section we shall analyze a certain spectral behavior of (J, L)-self-adjoint operators, in the spirit of [5] (see also [7] for a different point of view).

DEFINITION 3.11. Let $S\colon D(S) \subset \mathscr{H}^2 \to \mathscr{H}^2$ be a closed operator. The set of those points $z \in \mathbf{C}^2$ such that the operator $Q(z) - S\colon D(S) \to \mathscr{H}^2$ is not bijective will be denoted by $\sigma_q(S)$ and called the *quaternionic spectrum* of S.

Clearly, if $z \notin \sigma_q(S)$, then $(Q(z) - S)^{-1}$ is (everywhere defined and) bounded on $\mathscr{H}^2$, by the closed graph theorem.

PROPOSITION 3.12. *Let S be (J, L)-self-adjoint. Consequently we have*

$$\sigma_q(S) \subset \{z = (z_1, z_2) \in \mathbf{C}^2;\ |\mathrm{Re} z_1|^2 + |\mathrm{Re} z_2|^2 \neq 0\} \cup \{0\}.$$

Proof. We shall show that if $z = (z_1, z_2) \in \mathbf{C}^2 \setminus \{0\}$ has the property $\mathrm{Re} z_1 = \mathrm{Re} z_2 = 0$, then the operator $Q(z) - S$ is invertible.

Since $z \neq 0$, it follows from (2.1) that $Q(z) - S$ is injective and has closed range. To prove that $Q(z) - S$ is also surjective, it suffices to show that $N((Q(z) - S)^*) = \{0\}$. Set $z_1 = it_1$, $z_2 = it_2$. Then $Q(z)^* = -i(t_1 J + t_2 L)$, and $JQ(z)^* = \tilde{Q}(z)\,J$, where $\tilde{Q}(z) = -i(t_1 J - t_2 L) = Q((-z_1, z_2))$. Thus we have

$$\begin{aligned} N(Q(z)^* - S^*) &= N(J(Q(z)^* - S^*)) = N((\tilde{Q}(z) - S)\,J) \\ &= N(\tilde{Q}(z) - S) = \{0\}, \end{aligned}$$

again by (2.1). Therefore $Q(z) - S$ is bijective, which implies our assertion.

For normal operators of the form $Q(\mathscr{A})$, with $\mathscr{A} = (A_1, A_2)$ a pair of symmetric operators, we shall obtain a more precise result.

LEMMA 3.13. *Let $\mathscr{A} = (A_1, A_2)$ be a pair of symmetric operators in $\mathscr{H}$, and let $x = x_1 \oplus x_2 \in D(Q(\mathscr{A}))$ be such that $\langle A_1 x_1, A_2 x_2 \rangle = \langle A_2 x_1, A_1 x_2 \rangle$. Then we have the estimate*

$$\|(Q(z) - Q(\mathscr{A}))\,x\|_2 \geqslant [(\mathrm{Im} z_1)^2 + (\mathrm{Im} z_2)^2]^{1/2}\,\|x\|_2. \tag{3.5}$$

Proof. A direct calculation, using the equality $\langle A_1 x_1, A_2 x_2 \rangle = \langle A_2 x_1, A_1 x_2 \rangle$, shows that

$$\|Q(\mathscr{A})\,x\|_2^2 = \|A_1 x_1\|^2 + \|A_1 x_2\|^2 + \|A_2 x_1\|^2 + \|A_2 x_2\|^2. \tag{3.6}$$

Note also that

$$Q(z)^* \, Q(\mathscr{A}) = \begin{pmatrix} \bar{z}_1 A_1 + z_2 A_2 & \bar{z} A_2 - z_2 A_1 \\ \bar{z}_2 A_1 - z_1 A_2 & \bar{z}_2 A_2 + z_1 A_1 \end{pmatrix},$$

$$Q(\mathscr{A})^{\#} \, Q(z) = \begin{pmatrix} z_1 A_1 + \bar{z}_2 A_2 & z_2 A_1 - \bar{z}_1 A_2 \\ z_1 A_2 - \bar{z}_2 A_1 & z_2 A_2 + \bar{z}_1 A_1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} &(Q(z)^* \, Q(\mathscr{A}) + Q(\mathscr{A})^{\#} \, Q(z)) \, x \\ &\qquad = 2((\mathrm{Re} z_1) \, A_1 \oplus A_1 + (\mathrm{Re} z_2) \, A_2 \oplus A_2) \, x. \end{aligned} \tag{3.7}$$

This implies the estimate

$$\begin{aligned} &\langle (Q(z)^* \, Q(\mathscr{A}) + Q(\mathscr{A})^{\#} \, Q(z)) \, x, x \rangle_2 \\ &\quad = 2 \, \mathrm{Re} z_1 (\langle A_1 x_1, x_1 \rangle + \langle A_1 x_2, x_2 \rangle) \\ &\qquad + 2 \, \mathrm{Re} z_2 (\langle A_2 x_1, x_1 \rangle + \langle A_2 x_2, x_2 \rangle) \\ &\quad \leqslant 2 \, |\mathrm{Re} z_1| \, (\|A_1 x_1\| \, \|x_1\| + \|A_1 x_2\| \, \|x_2\|) \\ &\qquad + 2 \, |\mathrm{Re} z_2| \, (\|A_2 x_1\| \, \|x_1\| + \|A_2 x_2\| \, \|x_2\|). \end{aligned}$$

Therefore, by (3.6), (3.7), and the above estimate as well, we have

$$\begin{aligned} &\|(Q(z) - Q(\mathscr{A})) \, x\|_2^2 \\ &\quad = \|Q(\mathscr{A}) \, x\|_2^2 + \|z\|^2 \, \|x\|_2^2 - \langle (Q(z)^* \, Q(\mathscr{A}) + Q(\mathscr{A})^{\#} \, Q(z)) \, x, x \rangle_2 \\ &\quad \geqslant \|A_1 x_1\|^2 + \|A_1 x_2\|^2 + \|A_2 x_1\|^2 + \|A_2 x_2\|^2 \\ &\qquad - 2 \, |\mathrm{Re} z_1| \, (\|A_1 x_1\| \, \|x_1\| + \|A_1 x_2\| \, \|x_2\|) \\ &\qquad - 2 \, |\mathrm{Re} z_2| \, (\|A_2 x_1\| \, \|x_1\| + \|A_2 x_2\| \, \|x_2\|) \\ &\qquad + (|z_1|^2 + |z_2|^2)(\|x_1\|^2 + \|x_2\|^2) \\ &\quad = (\|A_1 x_1\| - |\mathrm{Re} z_1| \, \|x_1\|)^2 + (\|A_1 x_2\| - |\mathrm{Re} z_1| \, \|x_2\|)^2 \\ &\qquad + (\|A_2 x_1\| - |\mathrm{Re} z_2| \, \|x_1\|)^2 + (\|A_2 x_2\| - |\mathrm{Re} z_2| \, \|x_2\|)^2 \\ &\qquad + ((\mathrm{Im} z_1)^2 + (\mathrm{Im} z_2)^2)(\|x_1\|^2 + \|x_2\|^2), \end{aligned}$$

from which we infer (3.5).

THEOREM 3.14. *Let $\mathscr{A} = (A_1, A_2)$, a pair of symmetric operators, such that $Q(\mathscr{A})$ is normal. Then $\sigma_q(Q(\mathscr{A})) \subset \mathbf{R}^2$.*

Proof. We shall show that the estimate

$$\|(Q(z)-Q(\mathscr{A}))\,x\|_2 \geqslant [(\mathrm{Im} z_1)^2+(\mathrm{Im} z_2)^2]^{1/2}\,\|x\|_2 \tag{3.8}$$

holds for all $x \in D(Q(\mathscr{A}))$.

Indeed, by virtue of Theorem 3.7, we may assume, with no loss of generality, that A_1, A_2 are commuting self-adjoint operators. Let E be the joint spectral measure of A_1, A_2, and let $D=\{E(\sigma)\,y;\ y \in \mathscr{H},\ \sigma \subset \mathbf{R}^2,\ \sigma$ compact$\}$. Thus for every $x \in D \oplus D$, the estimate (3.8) holds, by (3.5). Let us prove that (3.8) holds for all $x \in D(Q(\mathscr{A}))$. Indeed, as in the proof of Lemma 3.3, let $x=x' \oplus x'' \in D(Q(A))$ be fixed, and let $(\sigma_k)_{k \geqslant 1}$ be an increasing sequence of compact subsets of $\mathbf{R}^2$, such that $\bigcup_{k \geqslant 1} = \mathbf{R}^2$. Set $x_k = x'_k \oplus x''_k = E(\sigma_k)\,x' \oplus E(\sigma_k)\,x''$, $k \geqslant 1$. Then we have $x'_k \to x'$, $x''_k \to x''$, $A_j x'_k \to A_j x'$, $A_j x''_k \to A_j x''$ as $k \to \infty$, $j=1, 2$. Hence, $Q(\mathscr{A})\,x_k \to Q(\mathscr{A})\,x$, as $k \to \infty$, and so

$$\begin{aligned}
\|(Q(z)-Q(\mathscr{A}))\,x\|_2 &= \lim_{k\to\infty} \|(Q(z)-Q(\mathscr{A}))\,x_k\|_2 \\
&\geqslant \lim_{k\to\infty} [(\mathrm{Im} z_1)^2+(\mathrm{Im} z_2)^2]^{1/2}\|x_k\|_2 \\
&= [(\mathrm{Im} z_1)^2+(\mathrm{Im} z)^2]^{1/2}\|x\|_2.
\end{aligned}$$

If $z=(z_1, z_2) \in \mathbf{C}^2 \backslash \mathbf{R}^2$ is fixed, the operator $Q(z)-Q(\mathscr{A})$ is injective, by (3.8). The same estimate also shows that $R(Q(z)-Q(\mathscr{A}))$ is closed. To show that $Q(z)-Q(\mathscr{A})$ is surjective, it suffices to prove that $(Q(z)-Q(\mathscr{A}))^*$ is also injective. But $(Q(z)-Q(\mathscr{A}))^* = Q(z^{\#})-Q(\mathscr{A}^{\#})$, where $z^{\#}=(\bar{z}_1, -z_2)$, $\mathscr{A}^{\#}=(A_1, -A_2)$. Therefore, $(Q(z)-Q(A))^*$ is injective again by (3.8) (applied to $z^{\#}$ and $A^{\#}$). This completes the proof of the theorem.

THEOREM 3.15. *Let* $\mathscr{A}=(A_1, A_2)$, *a pair of positive symmetric operators, such that* $Q(A)$ *is normal. Consequently* $\sigma_q(Q(A)) \subset \mathbf{R}^2_+$.

Proof. By virtue of Theorem 3.14, and a change of notation if necessary, it suffices to prove that if $z=(t_1, t_2) \in \mathbf{C}^2$ is such that $t_1<0$, $t_2 \in \mathbf{R}$, then $z \notin \sigma_{sp}(Q(\mathscr{A}))$.

Fix a $z=(t_1, t_2)$ as above, and let $x=x_1 \oplus x_2$ be as in Lemma 3.13. We have

$$\begin{aligned}
&\langle (Q(z)^*\,Q(\mathscr{A})+Q(\mathscr{A})^{\#}\,Q(z))\,x, x\rangle_2 \\
&\quad = 2t_1(\langle A_1x_1, x_1\rangle+\langle A_1x_2, x_2\rangle)+2t_2(\langle A_2x_1, x_1\rangle+\langle A_2x_2, x_2\rangle).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|(Q(z)-Q(\mathscr{A}))\,x\|_2^2\\
&\quad=\|Q(\mathscr{A})\,x\|_2^2+\|z\|^2\|x\|_2^2-\langle(Q(z)^*\,Q(\mathscr{A})+Q(\mathscr{A})^*\,Q(z))\,x,x\rangle_2\\
&\quad=\|A_1x_1\|^2+\|A_1x_2\|^2+\|A_2x_1\|^2+\|A_2x_2\|^2\\
&\qquad-2t_1(\langle A_1x_1,x_1\rangle+\langle A_1x_2,x_2\rangle)-2t_2(\langle A_2x_1,x_1\rangle+\langle A_2x_2,x_2\rangle)\\
&\qquad+(t_1^2+t_2^2)(\|x_1\|^2+\|x_2\|^2)\\
&\quad\geqslant\|A_1x_1\|^2+\|A_1x_2\|^2-2t_1\langle A_1,x_1,x_1\rangle-2t_1\langle A_1x_2,x_2\rangle\\
&\qquad+(\|A_2x_1\|-t_2\|x_1\|)^2+(\|A_2x_2\|-t_2\|x_2\|)^2\\
&\qquad+t_1^2(\|x_1\|^2+\|x_2\|^2)\geqslant t_1^2\|x\|_2^2.
\end{aligned}$$

The above calculation, and the approximation from the proof of Theorem 3.14, lead to the estimate

$$\|(Q(z)-Q(\mathscr{A}))\,x\|_2^2\geqslant t_1^2\,\|x\|_2^2,\qquad x\in D(Q(\mathscr{A})).$$

This can be used (see the proof of Theorem 3.14) to deduce that the operator $Q(z)-Q(\mathscr{A})$ is bijective. We omit the details.

Remark. Let $\mathscr{A}=(A_1,A_2)$ be a pair of symmetric operators such that $Q(\mathscr{A})$ is normal. Thus for every $z\notin\sigma_q(Q(\mathscr{A}))$, the operator $(Q(z)-Q(\mathscr{A}))^{-1}$ is (bounded and) normal. Indeed, the operators $Q(z)-Q(\mathscr{A})$ and $Q(z)^*-Q(\mathscr{A})^*$ commute on $D(Q(\mathscr{A})\,Q(\mathscr{A})^*)$, whence we derive easily that the bounded operators $(Q(z)-Q(\mathscr{A}))^{-1}$, $(Q(z)^*-Q(\mathscr{A})^*)^{-1}$ commute.

4. FINAL REMARKS

First of all we want to show that our Theorem 3.8 implies the well-known criterion of commutativity for pairs of symmetric operators due to Nelson (see [2, Corollary 9.2]).

PROPOSITION 4.1. *Let S_1, S_2 be symmetric operators in $\mathscr{H}$, and let D be a dense subspace of D of $\mathscr{H}$ that is contained in the subspace $\bigcap_{1\leqslant j,k\leqslant 2} D(S_jS_k)$. Suppose $S_1S_2x=S_2S_1x$, $x\in D$. If the restriction of the operator $S_1^2+S_2^2$ to D is essentially self-adjoint, then S_1, S_2 are essentially self-adjoint and $\bar{S}_1$, $\bar{S}_2$ commute.*

Proof. If $\mathscr{S}=(S_1,S_2)$, let $Q(\mathscr{S})$ be defined on $D\oplus D$. We shall show that the conditions of Theorem 3.8 are fulfilled.

The property $\|Q(\mathscr{S})\,x\| = \|Q(\mathscr{S})^{\#}\,x\|$, $x \in D \oplus D$, follows by an easy calculation (as for (3.6)).

Note that $Q(\mathscr{S})\,Q(\mathscr{S})^{\#} = (S_1^2 + S_2^2) \oplus (S_1^2 + S_2^2)$ on $D \oplus D$. Thus

$$(Q(\mathscr{S}) + Q')(Q(\mathscr{S})^{\#} - Q') = (1 + S_1^2 + S_2^2) \oplus (1 + S_1^2 + S_2^2), \qquad (4.1)$$

on $D \oplus D$, via (3.7).

Let T be the canonical closure of $(S_1^2 + S_2^2) \mid D$, which is self-adjoint and positive. Therefore, the operator $(1 + T) \oplus (1 + T)$, which is an extension of (4.1), has a bounded inverse on $\mathscr{H}^2$. This implies that the operator $(1 + T_0) \oplus (1 + T_0)$ has a dense range, because its canonical closure is $(1 + T) \oplus (1 + T)$, where $T_0 = (S_1^2 + S_2^2) \mid D$. In particular, the range of $Q(\mathscr{S}) + Q'$ is dense in $\mathscr{H}^2$, via (4.1). Therefore, the second condition from Theorem 3.8 also holds, implying the desired conclusion.

Theorem 3.8 can be directly applied to obtain the commutativity of more concrete pairs of symmetric operators. The next result, which is an early version of [3, Theorem 2.5] (for $n = 2$), is an illustration of this assertion.

PROPOSITION 4.2. *Let $\theta(t) = (1 + t_1^2 + t_2^2)^{-1}$, $t = (t_1, t_2) \in \mathbf{R}^2$, and let $\mathscr{R}_\theta$ be the* **C***-algebra generated by all polynomial functions on* $\mathbf{R}^2$ *and by θ. Let also $\Lambda: \mathscr{R}_\theta \to \mathbf{C}$ be a linear map such that $\Lambda(|r|^2) \geqslant 0$, $r \in \mathscr{R}_\theta$. Consequently there exists a uniquely determined measure μ on $\mathbf{R}^2$ such that $\Lambda(r) = \int r\, d\mu$, $r \in \mathscr{R}_\theta$. Moreover, the algebra $\mathscr{R}_\theta$ is dense in $L^2(\mu)$.*

Sketch of proof. If Λ is as in the statement, we may define a sesquilinear form on $\mathscr{R}_\theta$ via the equation

$$\langle r_1, r_2 \rangle_\Lambda = \Lambda(r_1 \bar{r}_2), \qquad r_1, r_2 \in \mathscr{R}_\theta. \qquad (4.2)$$

Let $\mathscr{N} = \{r \in \mathscr{R}_\theta;\ \Lambda(r\bar{r}) = 0\}$. Then (4.2) induces a scalar product $\langle *, * \rangle$ on the quotient $\mathscr{R}_\theta / \mathscr{N}$, and let $\mathscr{H}$ be the completion of the quotient $\mathscr{R}_\theta / \mathscr{N}$ with respect to this scalar product.

We define in $\mathscr{H}$ the operators

$$T_j(r + \mathscr{N}) = t_j r + \mathscr{N}, \qquad r \in \mathscr{R}_\theta, \qquad j = 1, 2, \qquad (4.3)$$

which are symmetric and densely defined on $\mathscr{R}_\theta / \mathscr{N}$. If $D = \mathscr{R}_\theta / \mathscr{N} = D(T_j)$, we also consider the operator $Q(\mathscr{T}): D \oplus D \to \mathscr{H}$, given (3.1), where $\mathscr{T} = (T_1, T_2)$. We shall show that $\mathscr{T}$ satisfies the conditions of Theorem 3.8. Indeed, if $\xi \in D \oplus D$, then $\|Q(\mathscr{T})\,\xi\|_2^2 = \|Q(\mathscr{T})^{\#}\,\xi\|_2^2$, via (3.6).

Let us observe that the set $\{(Q(\mathscr{T}) + Q')\,\xi;\ \xi \in D \oplus D\}$ is dense in $\mathscr{H}^2$. We prove, in fact, that

$$(Q(\mathscr{T}) + Q')(D \oplus D) = D \oplus D.$$

Indeed, if $r_1, r_2 \in \mathscr{R}_\theta$ are arbitrary, the system of equations

$$(\alpha + t_1)\, u_1(t) + (\alpha + t_2)\, u_2(t) = r_1(t)$$
$$(\alpha - t_2)\, u_1(t) - (\alpha - t_1)\, u_2(t) = r_2(t),$$

with $\alpha = i\sqrt{2}/2$, has the solution

$$u_1(t) = (-\alpha + t_1)\, \theta(t)\, r_1(t) - (\alpha + t_2)\, \theta(t)\, r_2(t),$$
$$u_2(t) = (-\alpha + t_2)\, \theta(t)\, r_1(t) + (\alpha + t_1)\, \theta(t)\, r_2(t),$$

such that $u_1 \oplus u_2 \in \mathscr{R}_\theta \oplus \mathscr{R}_\theta$. This implies readily that the equation $(Q(\mathscr{T}) + Q')\, \eta = \xi$ has a solution $\eta \in D \oplus D$ for each $\xi \in D \oplus D$, showing that the second condition from Theorem 3.8 holds.

By virtue of Theorem 3.8, the symmetric operators T_1, T_2 have commuting self-adjoint extensions A_1, A_2, respectively. If E is the joint spectral measure of A_1, A_2, then $\mu(*) = \langle E(*)(1 + \mathscr{N}), 1 + \mathscr{N} \rangle$ has the stated property.

The remaining part of the proof, which is not relevant for our methods, will be omitted. We only mention that Proposition 4.2 can be used to obtain a solution to the Hamburger moment problem in two variables. For details, as well as for a substantial extension to several variables and unbounded semi-algebraic sets, see [3] (where one uses directly Nelson's criterion of strong commutativity of symmetric operators).

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