MATHEMATICS

CONVERGING FACTORS FOR THE WEBER PARABOLIC CYLINDER FUNCTIONS OF COMPLEX ARGUMENT [1]), IA

BY

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INTRODUCTION

Numerical values of certain elementary functions (e.g. exp (x), sin (x), cos (x), ln (x)) are made available to digital computer users by means of programmed subroutines. The tendency will be to extend this list of "elementary" functions, and considerable interest therefore attaches to general and efficient methods for computing numerical values to great accuracy of the higher functions of Mathematical Physics. One such method is the application of the converging factor.

The Converging Factor

The converging factor is an important numerical device for hastening the convergence of slowly convergent series and increasing the accuracy obtainable by use of an asymptotic series. If the series is

$$S \sim u_0 + u_1 + u_2 + \dots \tag{1}$$

and the partial remainder R_n is

$$R_n \sim u_n + u_{n+1} + u_{n+2} + \dots \tag{2}$$

the converging factor C_n is defined by

$$R_n = u_n C_n. \tag{3}$$

The converging factor is most efficiently used, in the case of most applications to asymptotic series, with that value of n which corresponds to the term of smallest modulus in the series (1).

MILLER [1] has given a method for developing the converging factor C_n either as series of the form

$$C_n \sim \sum_{r=0}^{\infty} \beta_r z^{-r} \tag{4}$$

or as a series of the form

$$C_n \sim \sum_{r=0}^{\infty} \delta_r n^{-r} \tag{5}$$

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1) Communication MR62 of the Computation Department of the Mathematical Centre, Amsterdam.

for the cases in which either the function S satisfies a linear differential equation in z or the terms u_r satisfy a linear difference equation in r. He illustrated his method by obtaining converging factors for asymptotic series associated with the Weber parabolic cylinder functions.

In the paper referred to, real values only of the argument are considered. Here the computations are extended into the complex domain. Secondly a convenient recursive technique for obtaining the coefficients in the series for the converging factor is described.

Weber functions

The series which is to be studied is

$$(6)\quad \begin{cases} S_1(a;z) \sim e^{-z^2/4}\, z^{-a-1/2} \left\{ 1 - \dfrac{(a+1/2)(a+3/2)}{2.z^2} + \right. \\ \qquad\qquad \left. + \dfrac{(a+1/2)(a+3/2)(a+5/2)(a+7/2)}{2.4.z^4} \cdots \right\} \end{cases}$$

$$(7)\qquad \sim u_0 - u_1 + u_2 - \ldots.$$

It formally satisfies the differential equation

$$(8)\qquad \frac{d^2y}{dz^2} - (a+z^2/4)\, y = 0.$$

Two linearly independent situations of equation (8) are $S_1(a;z)$ and

$$(9)\qquad S_2(a;z) = S_1(-a;iz).$$

The terms u_r of the series (6) satisfy the recursion

$$(10)\qquad 2rz^2u_r = (a+2r-3/2)(a+2r-1/2)u_{r-1}.$$

We wish to determine that value n of r for which $|u_n|$ is a minimum. From (10) this is seen to occur when

$$(11)\qquad 2nx^2 \doteqdot (a+2n-3/2)(a+2n-1/2)$$

where

$$(12)\qquad z = xe^{i\theta}.$$

In order to derive an easily usable approximation we ignore the term

$$(13)\qquad \mu = (a-1/2)(a-3/2)$$

independent of n in (11), and obtain

$$(14)\qquad x^2 \doteqdot 2n+\lambda$$

where

$$(15)\qquad \lambda = 2(a-1)$$

or

$$(16)\qquad 2n = x^2-\lambda-k$$

where k is real and may always be chosen so that

$$-1 \leqslant k \leqslant 1. \tag{17}$$

The integer n having been determined, we define the remainder term R_n and converging factor Γ_n by

$$S_1(a; z) = \sum_{r=0}^{n-1} (-1)^r u_r + R_n \tag{18}$$

$$R_n = (-1)^n u_n \Gamma_n. \tag{19}$$

We shall obtain a series development of the form

$$\Gamma_n \sim \sum_{r=0}^{\infty} \frac{\beta_r(k)}{2^{r+1} x^{2r}} \tag{20}$$

when $\arg(z) \neq \pi/2$, first using the fact that Γ_n satisfies a differential equation in z and secondly the fact that Γ_n satisfies a recursion in n.

Differential Equation

The converging factor satisfies the differential equation

$$\left\{ \begin{aligned} & z^2 \frac{d^2 \Gamma_n}{dz^2} - z(z^2 + 2a + 4n + 1) \frac{d\Gamma_n}{dz} + (a + 2n + {}^1/_2)(a + 2n + {}^3/_2)\, \Gamma_n \\ & \qquad + 2nz^2 (\Gamma_n - 1) = 0. \end{aligned} \right. \tag{21}$$

This may quite crudely be verified by substituting the series

$$\left\{ \begin{aligned} \Gamma_n \sim 1 & - \frac{(a + 2n + {}^1/_2)(a + 2n + {}^3/_2)}{2(n+1)z^2} \\ & + \frac{(a + 2n + {}^1/_2)(a + 2n + {}^3/_2)(a + 2n + {}^5/_2)(a + 2n + {}^7/_2)}{4(n+1)(n+2)z^4} - \cdots \end{aligned} \right. \tag{22}$$

in (21). A constructive derivation, based on an idea which is clearly capable of general application to the construction of converging factors, has been given by Miller. He writes

$$u_n = a \text{ constant} \times e^{-{}^1/_4 z^2} z^{-a-2n-{}^1/_2} \tag{23}$$

so

$$\frac{\frac{d}{dz} u_n}{u_n} = \left(-\frac{1}{2} z - \frac{a + 2n + {}^1/_2}{z} \right) \tag{24}$$

and further

$$\left\{ \begin{aligned} \frac{\frac{d^2 u_n}{dz^2}}{u_n} & = \left(\frac{\frac{du_n}{dz}}{u_n} \right)^2 - \frac{1}{2} + \frac{(a + 2n + {}^1/_2)}{z^2} \\ & = \frac{1}{4} z^2 + a + 2n + \frac{(a + 2n + {}^1/_2)(a + 2n + {}^3/_2)}{z^2} \end{aligned} \right. \tag{25}$$

but

$$\frac{d^2R_n}{dz^2} - (a + {}^1/_4 z^2)\, R_n = (-1)^n 2n u_n \tag{26}$$

whence

$$\frac{d^2\Gamma_n}{dz^2}\, u_n + 2\left(\frac{d\Gamma_n}{dz}\right)\left(\frac{du_n}{dz}\right) + \Gamma_n\left(\frac{d^2u_n}{dz^2}\right) - \left(a + \frac{1}{4}\, z^2\right) \Gamma_n\, u_n = 2n u_n. \tag{27}$$

Removing u_n and its derivatives from (27) by way of (24) and (25), we arrive at (21).

In this section we shall suppose that a and n are fixed, so that z and k vary together. We have from equations (12) and (16)

$$dz = e^{i\theta} dx, \ \ 2x dx = dk. \tag{28}$$

By means of equations (12), (16) and (28) we may remove n from equation (21) and transform the result into a differential equation with k as the independent variable. We obtain, after some rearrangement

$$\begin{cases} x^4 \{4\Gamma_n'' - 2(\phi+2)\, \Gamma_n' + (\phi+1)\, \Gamma_n - \phi\} \\ \qquad + x^2 \{2(2k+\lambda-2)\, \Gamma_n + (4-\lambda-2k-\phi(\lambda+k))\, \Gamma_n + \phi(\lambda+k)\} \\ \qquad\qquad + \{k^2 + (\lambda-4)\, k + \mu - 2\lambda + 4\}\, \Gamma_n = 0, \end{cases} \tag{29}$$

where

$$\phi = e^{2i\theta} \tag{30}$$

and dashes denote differentiation with respect to k.

From (20) and (28) we have successively

$$\Gamma_n' \sim \frac{\beta_0'}{2} + \frac{\beta_1'}{2^2x^2} + \frac{\beta_2' - 2\beta_1}{2^3x^4} + \ldots + \frac{\beta_r' - 2(r-1)\beta_{r-1}}{2^{r+1}x^{2r}} \tag{31}$$

and

$$\Gamma_n'' \sim \frac{\beta_0''}{2} + \frac{\beta_1''}{2^2x^2} + \frac{\beta_2'' - 4\beta_1'}{2^3x^4} + \ldots + \frac{\beta_r'' - 4(r-1)\beta_{r-1}' + 4(r-1)(r-2)\beta_{r-2}}{2^{r+1}x^{2r}}. \tag{32}$$

Substituting the series (20) (31) and (32) in (29) and equating to zero the coefficients of the successive powers of x we obtain a recursion system between the functions $\beta_r(k)$. We have, in succession,

$$x^4\colon\ \ 4\beta_0'' - 2(\phi+2)\, \beta_0' + (\phi+1)\, \beta_0 = 2\phi \tag{33}$$

$$\begin{cases} x^2\colon\ \ 4\beta_1'' - 2(\phi+2)\, \beta_1' + (\phi+1)\, \beta_1 = -4(2k+\lambda-2)\, \beta_0' \\ \qquad\qquad -2\{4-\lambda-2k-\phi(\lambda+k)\}\, \beta_0 - 4(\lambda+k)\, \phi \end{cases} \tag{34}$$

and

$$\begin{cases} x^{-2r+4}\colon\ \ 4\beta_r'' - 2(\phi+2)\, \beta_r' + (\phi+1)\, \beta_r = 4\{4r-\lambda-2k-2\}\, \beta_{r-1}' \\ \qquad\qquad + 2\{k(\phi+2) + \lambda(\phi+1) - 2(r-1)\phi - 4r\}\, \beta_{r-1} \\ \qquad\qquad - 4\{k^2 + k(\lambda-4r+4)\, k + \mu - 2\lambda(r-1) + 4(r-1)^2\}\, \beta_{r-2} \\ \qquad\qquad\qquad (r = 2, 3, \ldots). \end{cases} \tag{35}$$

Inspection of equations (33), (34) and (35) reveals that they are formally satisfied by polynomials of the form

$$\beta_r(k) = \sum_{s=0}^{r} p_{r,s}k^s \qquad (r=0,\ 1,\ \ldots). \tag{36}$$

Equation (33) indicates that

$$\beta_0(k) = 2\phi/(\phi+1), \tag{37}$$

equation (34) yields

$$\beta_1(k) = \frac{4\phi}{(\phi+1)^2}\,k - \frac{8\phi^2}{(\phi+1)^3}, \tag{38}$$

and from equation (35) we may derive

$$\beta_2(k) = \frac{8\phi\, k^2}{(\phi+1)^3} - \frac{48\phi^2\, k}{(\phi+1)^4} - \left\{\frac{8\mu(\phi+1)^3+32\phi^2(1-2\phi)}{(\phi+1)^5}\right\} \tag{39}$$

and

$$\left\{\begin{aligned} \beta_3(k) = {} & \frac{16\phi\, k^3}{(\phi+1)^4} - \frac{192\phi^2\, k^2}{(\phi+1)^5} + \left\{\frac{-16\mu\phi(\phi+3)}{(\phi+1)^3} + \frac{64\phi^2(11\phi-4)}{(\phi+1)^6}\right\} \\ & - \frac{16\phi\mu\lambda}{(\phi+1)^2} + \frac{32\phi^2(\phi+4)\mu}{(\phi+1)^4} - \frac{128\phi^2(6\phi^2-8\phi+1)}{(\phi+1)^7}. \end{aligned}\right. \tag{40}$$

We wish, however, to devise some recursive process for determing the coefficients $p_{r,s}$. In principle this can be done since, knowing $\beta_{r-1}(k)$ and $\beta_{r-2}(k)$, $\beta_r(k)$ may be derived from equation (35). Let us examine how this may be accomplished in detail. Substituting polynomial expressions of the form (36) in (35) we obtain, after some rearrangement

$$\left\{\begin{aligned} & 4\sum_{s=0}^{r-2}(s+2)(s+1)p_{r,s+2}k^s - 2(\phi+2)\sum_{s=0}^{r-1}(s+1)p_{r,s+1}k^s + (\phi+1)\sum_{s=0}^{r}p_{r,s}k^s \\ & \qquad = 4\{4r-\lambda-2\}\sum_{s=0}^{r-2}(s+1)p_{r-1,s+1}\,k^s - 8\sum_{s=1}^{r-1}sp_{r-1,s}\,k^s \\ & + 2(\phi+2)\sum_{s=1}^{r}p_{r-1,s-1}\,k^s + 2\{\lambda(\phi+1)-2(r-1)\phi-4r\}\sum_{s=0}^{r-1}p_{r-1,s}\,k^s \\ & \qquad - 4\sum_{s=2}^{r}p_{r-2,s-2}\,k^s - 4(\lambda-4r+4)\sum_{s=1}^{r-1}p_{r-2,s-1}\,k^s \\ & \qquad - 4\{\mu-2\lambda(r-1)+4(r-1)^2\}\sum_{s=0}^{r-2}p_{r-2,s}\,k^s. \end{aligned}\right. \tag{41}$$

Equating to zero the coefficients of k in the order $s=r, r-1, \ldots, 0$, we obtain

$$p_{r,r} = \{2(\phi+2)p_{r-1,r-1} - 4p_{r-2,r-2}\}/(\phi+1), \tag{42}$$

$$\left\{\begin{aligned} p_{r,r-1} = {} & \{2(\phi+2)r\,p_{r,r} - 8(r-1)p_{r-1,r-1} \\ & + 2(\phi+2)p_{r-1,r-2} + 2\{\lambda(\phi+1)-2(r-1)\phi-4r\}p_{r-1,r-1} \\ & - 4p_{r-2,r-3} - 4(\lambda-4r+4)p_{r-2,r-2}\}/(\phi+1) \end{aligned}\right. \tag{43}$$

$$
(44)\quad \begin{cases} p_{r,s} = [2(s+1)(\phi+2)p_{r,s+1} - 4(s+1)(s+2)p_{r,s+2} \\ \quad + 4\{4r-\lambda-2\}(s+1)p_{r-1,s+1} - 8s\,p_{r-1,s} \\ \quad + 2(\phi+2)p_{r-1,s-1} + 2\{\lambda(\phi+1) - 2(r-1)\phi - 4r\}\,p_{r-1,s} \\ \quad - 4p_{r-2,s-2} - 4(\lambda - 4r + 4)p_{r-2,s-1} \\ \quad - 4\{\mu - 2\lambda(r-1) + 4(r-1)^2\}\,p_{r-2,s}]/(\phi+1) \\ \qquad\qquad (s = r-2, r-3, \ldots, 2) \end{cases}
$$

$$
(45)\quad \begin{cases} p_{r,1} = [4(\phi+2)p_{r,2} - 24p_{r,} + 8(4r-\lambda-2)p_{r-1,2} \\ \quad - 8p_{r-1,1} + 2(\phi+2)p_{r-1,0} + 2\{\lambda(\phi+1) - 2(r-1)\phi - 4r\}\,p_{r-1,1} \\ \quad - 4(\lambda - 4r + 4)p_{r-2,0} - 4\{\mu - 2\lambda(r-1) + 4(r-1)^2\}\,p_{r-2,2}]/(\phi+1) \end{cases}
$$

$$
(46)\quad \begin{cases} p_{r,0} = [2(\phi+2)p_{r,1} - 8p_{r,2} + 4(4r-\lambda-2)p_{r-1,1} \\ \quad + 2\{\lambda(\phi+1) - 2(r-1)\phi - 4r\}\,p_{r-1,0} \\ \quad - 4\{\mu - 2\lambda(r-1) + 4(r-1)^2\}p_{r-2,0}]/(\phi+1). \end{cases}
$$

Thus, if equations (37), (38) and (42)–(46) are used in that order the coefficients $p_{r,s}$ $(r=0, 1, \ldots; s=0, 1, \ldots, r)$ may always be expressed in terms of quantities which have previously been derived.

It will be observed, however, that equations (42)–(46) differ from one another according as to whether certain powers of k do or do not exist in the various sums in (41). This fact may also be expressed by the use of conditional statements, and thus an expression for $p_{r,s}$ which is generally true for $r \geqslant 2$ may be constructed. The special forms for $p_{0,0}$ and $p_{1,1}$, $p_{1,0}$ may also be incorporated in this expression, and thus we have

$$
(47)\quad \begin{cases} p_{r,s} = [\ \text{if } s<r \text{ then } 2(\phi+2)(s+1)p_{r,s+1} \\ \quad - \text{if } s<r-1 \text{ then } 4(s+2)(s+1)p_{r,s+2} \\ \quad + \text{if } s<r-1 \text{ then } 4(4r-\lambda+2)(s+1)p_{r-1,s+1} \\ \quad - \text{if } 0<s<r \text{ then } 8sp_{r-1,s} \\ \quad + \text{if } s>0 \text{ then } 2(\phi+2)p_{r-1,s-1} \\ \quad + \text{if } s<r \text{ then } 2\{\lambda(\phi+1) - 2(r-1)\phi - 4r\}\,p_{r-1,s} \\ \quad - \text{if } s>1 \text{ then } 4p_{r-2,s-2} \\ \quad - \text{if } 0<s<r \text{ then } 4(\lambda - 4r + 4)p_{r-2,s-1} \\ \quad - \text{if } s<r-1 \text{ then } 4\{\mu + 2(r-1)\{2(r-1) - \lambda\}\}\,p_{r-2,s} \\ \quad + \text{if } r=0 \text{ then } 2\phi \\ \quad - \text{if } r=1 \text{ then if } s=0 \text{ then } 4\lambda\phi \text{ and if } s=1 \text{ then } -4\phi]/(\phi+1). \end{cases}
$$

This definition is uniformly valid for $r=0, 1, \ldots$ and $s=r, r-1, \ldots, 0$. Its derivation does not, of course, represent an attempt at elegance for its own sake. It will be realised that there is considerable duplication in formulae (42)–(46), so that if we were to write down the formulae for $p_{r,s}$ in some algorithmic language for a digital computer based on formulae (42)–(46), we would in effect be wasting a large number of instructions in needless repetition. Use of formula (47) avoids this at the cost of a few conditional statements, which (in comparison with the complexity of the formulae used) is negligible

Difference Equations

In the notation of equation (18) we have

$$R_{n-1} - R_n = (-1)^{n-1} u_{n-1} \tag{48}$$

and since

$$R_n = (-1)^n u_n \Gamma_n \tag{49}$$

we have

$$u_{n-1}\Gamma_{n-1} + u_n \Gamma_n = u_{n-1} \tag{50}$$

or, using (10)

$$2nz^2(\Gamma_{n-1} - 1) + (a + 2n - {}^3/_2)(a + 2n - {}^1/_2)\Gamma_n = 0. \tag{51}$$

In this section we shall suppose that a and x are fixed, so that when n decreases to $n-1$, k becomes $k+2$; thus if

$$\Gamma_n \sim \frac{\beta_0(k)}{2} + \frac{\beta_1(k)}{2^2x^2} + \frac{\beta_2(k)}{2^3x^4} + \dots \tag{52}$$

then

$$\Gamma_{n-1} \sim \frac{\beta_0(k+2)}{2} + \frac{\beta_1(k+2)}{2^2x^2} + \frac{\beta_2(k+2)}{2^3x^4} + \dots . \tag{53}$$

In equation (51) we write $x^2\phi$ for z^2, substitute for $2n$ in terms of x and k, and insert the series (52) and (53), finally obtaining

$$\left\{ \begin{aligned} &\phi x^2\{x^2 - \lambda - k\} \left\{ \frac{\beta_0(k+2)}{2} - 1 + \frac{\beta_1(k+2)}{2^2x^2} + \frac{\beta_2(k+2)}{2^3x^4} + \dots \right\} \\ &\quad + \{x^4 - x^2(\lambda + 2k) + k^2 + \lambda k + \mu\} \left\{ \frac{\beta_0(k)}{2} + \frac{\beta_1(k)}{2^2x^2} + \frac{\beta_2(k)}{2^3x^4} + \dots \right\} = 0. \end{aligned} \right. \tag{54}$$

By equating to zero the coefficients of the successive powers of x in (54) we shall again obtain a system of recursions between the functions $\beta_r(k) r = 0, 1, \dots$. We have:

$$x^4: \quad \phi\beta_0(k+2) + \beta_0(k) = 2\phi, \tag{55}$$

$$\left\{ \begin{aligned} x^2: \quad \phi\beta_1(k+2) + \beta_1(k) = 2\{&\phi(\lambda + k)\,\beta_0(k+2) + (\lambda + 2k)\,\beta_0(k) \\ &- 2\phi(\lambda + k)\}. \end{aligned} \right. \tag{56}$$

$$\left\{ \begin{aligned} x^{-2r+4}: \quad \phi\beta_r(k+2) + \beta_r(k) = 2\{&\phi(\lambda + k)\,\beta_{r-1}(k+2) + (\lambda + 2k)\,\beta_{r-1}(k) \\ &- 2(k^2 + \lambda k + \mu)\,\beta_{r-2}(k)\}. \end{aligned} \right. \tag{57}$$

Before proceeding further we introduce factorial functions of the form

$$\underset{2}{k^{(s)}} = k(k-2)\,\dots\,(k - 2s + 2). \tag{58}$$

These quite clearly satisfy a recursion of the form

$$\underset{2}{k}^{(s+1)} = (k-2s)\,\underset{2}{k}^{(s)} \tag{59}$$

and thus

$$k\,\underset{2}{k}^{(s)} = \underset{2}{k}^{(s+1)} + 2s\,\underset{2}{k}^{(s)}. \tag{60}$$

Furthermore

$$\left\{\begin{aligned} k^2\,\underset{2}{k}^{(s)} &= k\,\underset{2}{k}^{(s+1)} + 2sk\,\underset{2}{k}^{(s)} \\ &= \underset{2}{k}^{(s+2)} + (4s+2)\,\underset{2}{k}^{(s+1)} + 4s^2\,\underset{2}{k}^{(s)}. \end{aligned}\right. \tag{61}$$

If the difference and displacement operators $\underset{2}{\Delta}$ and $\underset{2}{E}$ are defined by

$$\underset{2}{\Delta}\, g(k) \equiv g(k+2) - g(k),\ \ \underset{2}{E} \equiv 1 + \underset{2}{\Delta} \tag{62}$$

then

$$\underset{2}{\Delta}\,\underset{2}{k}^{(s)} = 2s\,\underset{2}{k}^{(s-1)} \tag{63}$$

and

$$\underset{2}{(k+2)}^{(s)} = \underset{2}{k}^{(s)} + 2s\,\underset{2}{k}^{(s-1)}. \tag{64}$$

Equipped with these formulae, we see that equations (55)–(57) are formally satisfied by expressions of the form

$$\beta_r(k) = \sum_{s=0}^{r} q_{r,s}\,\underset{2}{k}^{(s)}. \tag{65}$$

From (55) and (56) we have successively

$$\beta_0(k) = 2\phi/(\phi+1), \tag{66}$$

$$\beta_1(k) = \frac{4\phi}{(\phi+1)^2}\,\underset{2}{k}^{(1)} - \frac{8\phi^2}{(\phi+1)^3}, \tag{67}$$

and (57) may be rearranged to give

$$\left\{\begin{aligned} &(\phi+1)\sum_{s=0}^{r} q_{r,s}\,\underset{2}{k}^{(s)} + 2\phi\sum_{s=0}^{r-1}(s+1)\,q_{r,s+1}\,\underset{2}{k}^{(s)} \\ &\qquad = 2\Big\{(\phi+2)\sum_{s=1}^{r} q_{r-1,s-1}\,\underset{2}{k}^{(s)} + (\phi+1)\sum_{s=0}^{r-1}(4s+\lambda)\,q_{r-1,s}\,\underset{2}{k}^{(s)} \\ &+ 2\phi\sum_{s=0}^{r-2}(s+1)\,(\lambda+2s)\,q_{r-1,s+1}\,\underset{2}{k}^{(s)} - 2\sum_{s=2}^{r} q_{r-2,s-2}\,\underset{2}{k}^{(s)} \\ &\qquad - 2\sum_{s=1}^{r-1}(4s+\lambda-2)\,q_{r-2,s-1}\,\underset{2}{k}^{(s)} - 2\sum_{s=0}^{r-2}(4s^2+2\lambda s+\mu)\,q_{r-2,s}\,\underset{2}{k}^{(s)}. \end{aligned}\right. \tag{68}$$

Again a definition of $q_{r,s}$ which is uniformly valid for $r=0, 1, \ldots;$ $s=r, r-1, \ldots, 0;$ may be given:

$$
(69) \quad \left\{
\begin{aligned}
q_{r,s} = 2\{ &\text{ if } s<r \text{ then } -\phi(s+1)\, q_{r,s+1} \\
&+\text{if } s>0 \text{ then } (\phi+2)\, q_{r-1,s-1} \\
&+\text{if } s<r \text{ then } (\phi+1)\,(4s+\lambda)\, q_{r-1,s} \\
&+\text{if } s<r-1 \text{ then } 2\phi(s+1)\,(\lambda+2s)\, q_{r-1,s+1} \\
&-\text{if } s>1 \text{ then } 2q_{r-2,s-2} \\
&-\text{if } 0<s<r \text{ then } 2(4s+\lambda-2)\, q_{r-2,s-1} \\
&-\text{if } s<r-1 \text{ then } 2\{2s(2s+\lambda)+\mu\}\, q_{r-2,s} \\
&+\text{if } r=0 \text{ then } \phi \\
&-\text{if } r=1 \text{ then if } s=0 \text{ then } 2\lambda\phi \\
&\qquad\qquad \text{and if } s=1 \text{ then } 2\phi\}/(\phi+1).
\end{aligned}
\right.
$$

Comparison with the work of Miller and Airey

As mentioned at the beginning of this paper, Miller has derived relationships similar to equations (33)–(35) and (55)–(57) for the case in which z is real. Allowing for the difference in notation (Miller uses an auxilary variable b defined by $b=a-2$ as opposed to $\lambda=2(a-1)$, and derives sets of equations in which the unknown function is $\beta_{r+1}(k)$ and not $\beta_r(k)$), equations (33)–(35) and (55)–(57) reduce to Miller's equations when $\phi=1$. Miller derives explicit formulae for the initial $\beta_r(k)$ rather than a recursive definition of the coefficients $p_{r,s}$ and $q_{r,s}$; nevertheless, since we have derived expressions for $\beta_s(k)(s=0, 1, 2, 3)$ for the purpose of checking, we remark in passing that these expression reduce to those of Miller when $\phi=1$.

We now recall the work of Airey [2]. He is concerned with the asymptotic series

$$
(70) \qquad \frac{0!}{z'} - \frac{1!}{z'^2} + \frac{2!}{z'^3} - \ldots = \sum_{n=1}^{\infty} u_n
$$

where

$$
(71) \qquad u_n = (-1)^{n-1}\,(n-1)!\, z'^{-n},
$$

and writes (70) as

$$
(72) \qquad \sum_{r=1}^{\infty} u_r \sim \sum_{r=1}^{n-1} u_r + u_n\, C_n
$$

where

$$
(73) \qquad C_n \sim 1 - \frac{n}{z'} + \frac{n(n+1)}{z'^2} - \frac{n(n+1)(n+2)}{z'^3} + \ldots.
$$

He makes an auxiliary substitution

$$
(74) \qquad z' = xe^{i\theta'},\ \beta = e^{-i\theta'},\ x' = n+h,
$$

which is similar to our (16), and obtains

$$(75)\quad C_n \sim 1 - \frac{(x'-h)\beta}{x'} + \frac{(x'-h)(x'+1-h)\beta^2}{x'^2} - \frac{(x'-h)(x'+1-h)(x'+2-h)\beta^3}{x'^3} + \dots.$$

By formal expansion of each term of (75) in inverse powers of x', and regroupment, he obtains the expansion

$$(76)\quad \begin{cases} C_n \sim \dfrac{1}{1+\beta} + \dfrac{1}{x'}\left\{\dfrac{\beta^2}{(1+\beta)^3} + \dfrac{\beta}{(1+\beta)^2}h\right\} + \dfrac{1}{x'^2}\left\{\dfrac{-2\beta^3+\beta^4}{(1+\beta)^5} - \dfrac{\beta^2-2\beta^3}{(1+\beta)^4}h + \dfrac{\beta^2h^2}{(1+\beta)^3}\right\} \\ \quad + \dfrac{1}{x'^3}\left\{\dfrac{6\beta^4-8\beta^5+\beta^6}{(1+\beta)} + \dfrac{2\beta^3-10\beta^4+3\beta^5}{(1+\beta)^6}h + \dfrac{-3\beta^3+3\beta^2}{(1+\beta)^5}h^2 + \dfrac{\beta^3}{(1+\beta)^4}h^3\right\} + \dots. \end{cases}$$

Airey tabulated values of the coefficients of $x'^{-s}(s=0, 1, \dots)$ in this expression when $\beta=1$ and $h=1$. Miller noted that when $a={}^1/_2$ or ${}^3/_2$ the constant terms of the polynomial coefficients which he derived for the expansion of Γ_n, were the same as Airey's numbers. We shall later see that, allowing for the difference in notation, the coefficients of x'^{-s} $(s=0, 1, \dots)$ in (76) are in agreement with those given by (37)–(40).

At first sight this should seem to be more a cause for bewilderment than reassurance, for the asymptotic series

$$(77)\qquad 1 - \frac{(a+{}^1/_2)(a+{}^3/_2)}{2z^2} + \frac{(a+{}^1/_2)(a+{}^3/_2)(a+{}^5/_2)(a+{}^7/_2)}{2.4.z^4} - \dots$$

with which the Weber function may in some sense be associated, manifestly does not reduce to (70) when $a={}^1/_2$ or ${}^3/_2$. When $a={}^1/_2$ it becomes

$$(78)\qquad 1 - \frac{({}^1/_2)}{(z^2/2)} + \frac{({}^1/_2)({}^3/_2)}{(z^2/2)^2} - \dots$$

and when $a={}^3/_2$

$$(79)\qquad 1 - \frac{({}^3/_2)}{(z^2/2)} + \frac{({}^3/_2)({}^5/_2)}{(z^2/2)^2} - \dots.$$

In order to explain this curious agreement we must first establish the true significance of Airey's converging factor. We consider the asymptotic series development

$$(80)\qquad 1 - \frac{a}{z'} + \frac{a(a+1)}{z'^2} - \dots$$

which may be associated with the incomplete Γ-function.

We write this as

$$(81)\qquad \sum_{r=0}^{\infty} u_r = \sum_{r=0}^{n-1} u_r + u_n C_n$$

where

$$(82)\qquad u_r = (-1)^r \frac{a(a+1)\dots(a+r-1)}{z'^r}$$

and the converging factor C_n may be expanded as

$$C_n \sim 1 - \frac{(a+n)}{z'} + \frac{(a+n)(a+n+1)}{z'^2} - \dots \tag{83}$$

where n is so chosen that if

$$z' = x'e^{i\theta} \tag{84}$$

and

$$a+n = x'+h \tag{85}$$

then $0 \leqslant h \leqslant 1$.

Now C_n satisfies the differential equation

$$\frac{z'dC_n}{dz} - (a+n+z')\,C_n = -z'. \tag{86}$$

We may change the independent variable to h, and eliminate n form this equation by means of (85), and obtain

$$\frac{x'dC_n}{dh} + \{x'(1+e^{i\theta})+h\}\,C_n = x'e^{i\theta}. \tag{87}$$

We may substitute a series development of the form $C_n \sim \sum_{s=0}^{\infty} \beta_s(h)x'^{-s}$ in (87) and obtain a recursion system among the $\beta_s(h)$ $(s=0, 1, \dots)$ as done earlier in this paper. The point to notice about this system of recursions is that the functions $\beta_s(h)$ produced via equation (87), are independent of the parameter a, so that Airey's converging factor (75) is not only the converging factor for the exponential integral, but also for the incomplete Γ-function.

But the series (78) and (79) are special cases of (80). The only outstanding point is that the relationship between z' and h given by (85) is exact, but that that between z^2 and k, given by (16), was derived under the assumption that μ (given by (13)) was negligible compared with n^2. But when $a = {}^1/_2$ or ${}^3/_2$ μ is not only negligible but zero, and so the correspondence is complete, and the agreement referred to occurs.

It only remains to show how (37)–(40) reduces to (76) when $\mu = 0$. Replacing h by the complementary argument $h' = h-1$ in (76) we obtain

$$\left\{ \begin{aligned} &\frac{1}{1+\beta} + \left\{\frac{\beta}{(1+\beta)^2}h' - \frac{\beta}{(1+\beta)^3}\right\}\frac{1}{x'} + \left\{\frac{\beta^2}{(1+\beta)^3}h'^2 - \frac{3\beta^2}{(1+\beta)^4}h' + \frac{\beta^2(2\beta-1)}{(1+\beta)^5}\right\}\frac{1}{x'^2} \\ &\quad + \left\{\frac{\beta^3}{(1+\beta)^4}h'^3 - \frac{6\beta^3}{(1+\beta)^5}h'^2 + \frac{\beta^3(11-4\beta)}{(1+\beta)^6}h' - \frac{\beta^3(\beta^2-8\beta+6)}{(1+\beta)^7}\right\}\frac{1}{x'^3} + \dots . \end{aligned} \right. \tag{88}$$

If, in (88), we put $\phi = \beta^{-1}$, $k = 2h'$, and $x^2 = 2x'$ we arrive at the coefficients (37)–(40), and thus again Airey's work serves, to a certain extent, to check our own.

Singular Case

When z^2 is real and negative, $\phi = -1$: the formalism of the preceding two sections breaks down completely; we examine the problem afresh.

In the case being considered, equations (33)–(35), (55)–(57) become

$$2\beta_0'' - \beta_0' = -1 \tag{89}$$

$$2\beta_1'' - \beta_1' = -2(2k+\lambda-2)\beta_0' - (4-k)\beta_0 + 2(\lambda+k) \tag{90}$$

$$\left\{\begin{aligned} 2\beta_r'' - \beta_r' &= 2\{4r-\lambda-2k-2\}\beta'_{r-1} + \{k-2(r+1)\}\beta_{r-1} \\ &\quad - 2\{k^2 + k(\lambda-4r+4)k + \mu - 2\lambda(r-1) + 4(r-1)^2\}\beta_{r-2} \end{aligned}\right. \tag{91}$$

$$\underset{2}{\Delta}\beta_0(k) = 2 \tag{92}$$

$$\underset{2}{\Delta}\beta_1(k) = -2\{(\lambda+2k)\beta_0(k) - (\lambda+k)\beta_0(k+2) + 2(\lambda+k)\} \tag{93}$$

$$\underset{2}{\Delta}\beta_r(k) = -2\{(\lambda+2k)\beta_{r-1}(k) - (\lambda+k)\beta_{r-1}(k+2) - 2(k^2+\lambda k+\mu)\beta_{r-2}(k)\}. \tag{94}$$

Inspection of equations (89)–(94) reveals that at least the possibility exists that they are satisfied by polynomials of the form

$$\beta_r(k) = \sum_{s=0}^{2r+1} p_{r,s}\, k^s = \sum_{s=0}^{2r+1} q_{r,s}\, \underset{2}{k^{(s)}}. \tag{95}$$

But it is quite certain, at least, that equations (91) and (94) do not serve to determine $p_{r,0}$ and $q_{r,0}$ respectively, and since, for example $p_{r+1,2}$, $p_{r+1,1}$ are determined from $p_{r,0}$, it would appear that matters become progressively worse.

Let us, however, proceed upon the assumption that everything is known on the right hand sides of equations (91) and (94) except $p_{r-1,0}$ and $q_{r-1,0}$ respectively. Equations (89) and (92) give to begin with

$$p_{0,1} = q_{0,1} = 1. \tag{96}$$

Equation (91) may be rearranged as

$$\left\{\begin{aligned} &2\sum_{s=0}^{2r-1}(s+1)(s+2)\,p_{r,s+2}\,k^s - \sum_{s=0}^{2r}(s+1)\,p_{r,s+1}\,k^s \\ &= 2(4r-\lambda-2)\sum_{s=0}^{2r-2}(s+1)\,p_{r-1,s+1}\,k^s - 4\sum_{s=1}^{2r-1} s p_{r-1,s}\,k^s \\ &+ \sum_{s=1}^{2r} p_{r-1,s-1}\,k^s - 2(r+1)\sum_{s=0}^{2r-1} p_{r-1,s}\,k^s - 2\sum_{s=2}^{2r-1} p_{r-2,s-2}\,k^s \\ &- 2(\lambda-4r+4)\sum_{s=1}^{2r-2} p_{r-2,s-1}\,k^s - 2\{\mu-2\lambda(r-1)+4(r-1)^2\}\sum_{s=0}^{2r-3} p_{r-2,s}\,k^s. \end{aligned}\right. \tag{97}$$

This leads to

$$(98)\quad \begin{cases} p_{r,s} = -[p_{r-1,s-2} \\ \quad - \text{if } s \leqslant 2r \text{ then } 2\{s(s+1)\,p_{r,s+1} + (r+2s-1)\,p_{r-1,s-1} + p_{r-2,s-3}\} \\ \quad + \text{if } s \leqslant 2r-1 \text{ then } 2\{s(4r-\lambda-2)\,p_{r-1,s} - (\lambda-4r+4)\,p_{r-2,s-2}\} \\ \quad - \text{if } s \leqslant 2r-2 \text{ then } 2\{\mu - 2\lambda(r-1) + 4(r-1)^2\}\,p_{r-2,s-1}]/s, \\ \qquad\qquad (s = 2r+1(-1)3), \end{cases}$$

a relationship which may be used without difficulty.

Equation (94) may be rearranged to give

$$(99)\quad \begin{cases} \sum_{s=0}^{2r-1} (s+1)\,q_{r,s+1}\,\underset{2}{k^{(s)}} = \sum_{s=1}^{2r} q_{r-1,s-1}\,\underset{2}{k^{(s)}} - 2\sum_{s=0}^{2r-2} (s+1)(\lambda+2s)\,q_{r-1,s+1}\,\underset{2}{k^{(s)}} \\ \quad - 2\sum_{s=2}^{2r-1} q_{r-2,s-2}\,\underset{2}{k^{(s)}} - 2\sum_{s=1}^{2r-2} (4s+\lambda-2)\,q_{r-2,s-1}\,\underset{2}{k^{(s)}} \\ \quad - 2\sum_{s=0}^{2r-3} (4s^2+2\lambda s+\mu)\,q_{r-2,s}\,\underset{2}{k^{(s)}}. \end{cases}$$

This leads to

$$(100)\quad \begin{cases} q_{r,s} = -[q_{r-1,s-2} \\ \quad - \text{if } s \leqslant 2r \text{ then } 2q_{r-2,s-3} \\ \quad - \text{if } s \leqslant 2r-1 \text{ then } 2\{s(\lambda+2s-2)\,q_{r-1,s} + (4s+\lambda-6)\,q_{r-2,s-2}\} \\ \quad - \text{if } s \leqslant 2r-2 \text{ then } 2\{4(s-1)^2 + 2\lambda(s-1) + \mu\}\,q_{r-2,s-1}]/s \\ \qquad\qquad (s = 2r+1(-1)3). \end{cases}$$

The coefficients of k^1 and k^0 in (97) and $\underset{2}{k^{(1)}}$ and $\underset{2}{k^{(0)}}$ in (99) respectively give

$$(101)\quad \begin{cases} p_{r,2} = \{12p_{r,3} - 4(4r-\lambda-2)\,p_{r-1,2} - p_{r-1,0} + 2(r+3)\,p_{r-1,1} \\ \qquad + 2\{\mu - 2\lambda(r-1) + 4(r-1)^2\}\,p_{r-2,1} + 2(\lambda-4r+4)\,p_{r-2,0}\}/2, \end{cases}$$

$$(102)\quad \begin{cases} p_{r,1} = 4p_{r,2} - 2(4r-\lambda-2)\,p_{r-1,1} + 2(r+1)\,p_{r-1,0} \\ \qquad + 2\{\mu - 2\lambda(r-1) + 4(r-1)^2\}\,p_{r-2,0}, \end{cases}$$

$$(103)\quad q_{r,2} = -[q_{r-1,0} - 4(\lambda+2)\,q_{r-1,2} - 2(\lambda+2)\,q_{r-2,0} - 2(\mu+2\lambda+4)\,q_{r-2,1}]/2,$$

$$(104)\quad q_{r,1} = 2(\lambda q_{r-1,1} + \mu q_{r-2,0}).$$

Now so far we have used the facts that Γ_n satisfies a differential equation and a difference equation quite seperately and developed $\beta_r(k)$ as a polynomial and as a series of factorial functions quite independently. Now we must use these facts in conjunction.

Firstly

$$(105)\quad p_{r-1,0} = q_{r-1,0}$$

and secondly, as may easily be verified (c.f. equation (112) below)

$$(106)\quad q_{r,1} = p_{r,1} + 2p_{r,2} + 4p_{r,3} + \dots + 2^{2r}p_{r,2r+1}.$$

Equations (101), (102), (104), (105) and (106) may thus be used to derive $p_{r-1,0}=q_{r-1,0}$, and these may be substituted in (101), (102) and (103) to give $p_{r,2}$, $p_{r,1}$ and $q_{r,2}$. $q_{r,1}$ can of course be determined without computing $q_{r-1,0}$.

Writing

$$(107)\quad \begin{cases} P_r2 = 12p_{r,3}-4(4r-\lambda-2)\,p_{r-1,2}+2\{\mu-2\lambda(r-1)+4(r-1)^2\}\,p_{r-2,1} \\ \qquad\qquad + 2(r+3)\,p_{r+1,1}+2(\lambda-4r+4)\,p_{r-2,0} \end{cases}$$

and

$$(108)\quad Pr1 = 2\{\mu-2\lambda(r-1)+4(r-1)^2\}\,p_{r-2,0}-2(4r-\lambda-2)\,p_{r-1,1}$$

and using (106) we have

$$(109)\quad p_{r-1,0} = q_{r-1,0} = \{q_{r,1}-3Pr2-Pr1-4p_{r,3}-\ldots-2^{2r}p_{r,2r+1}\}/(2r-1).$$

Subsequently

$$(110)\quad p_{r,2} = \{Pr2-p_{r-1,0}\}/2$$

$$(111)\quad p_{r,1} = 4p_{r,2}+Pr1+2(r+1)\,p_{r-1,0};$$

$q_{r,2}$ is given by (103), and we may proceed to the next value of r. Use of conditional statements enables the anomalous equations (89), (90), (92) and (93) to be brought into this general scheme.

Checking

Since the expressions $\beta_r(k)$, whether derived as a polynomial or as a series of factorials, represent the same function, there exists the possibility of expressing one set of coefficients in terms of the other, and this may be used as a check.

In the non-singular case we have the matrix equations

$$(112)\quad (q_{r,s}) = (p_{r,s})L,\ (p_{r,s}) = (q_{r,s})L^{-1}$$

where

$$(113)\quad (p_{r,s}) = \begin{pmatrix} p_{0,0} & & \\ p_{1,0} & p_{1,1} & \\ p_{2,0} & p_{2,1} & p_{2,2} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix},\quad (q_{r,s}) = \begin{pmatrix} q_{0,0} & & \\ q_{1,0} & q_{1,1} & \\ q_{2,0} & q_{2,1} & q_{2,2} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

and

$$(114)\quad L = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 2 & 1 & & \\ 0 & 4 & 6 & 1 & \\ 0 & 8 & 28 & 12 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},\quad L^{-1} = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & -2 & 1 & & \\ 0 & 8 & -6 & 1 & \\ 0 & -48 & +44 & -12 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

If the elements in L are referred to as $l_{r,s}$ $(r,s=0, 1, \ldots)$ and those in L^{-1} as $l_{r,s}^{-1}$ $(r, s=0, 1, \ldots)$ then

$$l_{r,0} = 0,\ l_{r,s} = l_{r-1,s-1}+2sl_{r-1,s}, \qquad (r=1, 2, \ldots;\ s=1, 2, \ldots, r)$$

$$l_{r,0}^{-1} = 0,\ l_{r,s}^{-1} = l_{r-1,s-1}^{-1}-2(r-1)\, l_{r-1,s}^{-1} \qquad (r=1, 2, \ldots;\ s=1, 2, \ldots, r).$$

Use of these formulae (as we shall see in the ALGOL programme to be given) enables the matrix multiplications (112) to be replaced by a system of algebraic relationships.

Application of the ε-algorithm

We have now shown how the converging factor Γ_n may be expressed formally as the sum of a series. But it is a matter of numerical experience that in many cases a continued fraction which may in a certain sense be associated with a given power series converges far more rapidly than the series. We wish, therefore, to transform the series for Γ_n into such a continued fraction. This may conveniently be done by application of the ε-algorithm [3] the theory of which has been described elsewhere [4]; it will suffice have to state that if from the initial values

$$(115) \qquad \varepsilon_0{}^{(0)} = 0,\ \varepsilon_0{}^{(m)} = \sum_{r=0}^{m-1} \beta_r(k)n^{-r} \qquad (m=1, 2, \ldots)$$

$$(116) \qquad \varepsilon_1{}^{(m)} = n^m\{\beta_m(k)\}^{-1} \qquad (m=0, 1, \ldots)$$

further quantities $e_s{}^{(m)}$ $(m=0, 1, \ldots;\ s=2, 3, \ldots)$ are constructed by means of the relationship

$$(117) \qquad \varepsilon_s^{(m)} = \varepsilon_{s-2}^{(m+1)} + \frac{1}{\varepsilon_{s-1}^{(m+1)} - \varepsilon_{s-1}^{(m)}}$$

then the quantities $\varepsilon_{2s}^{(m)}$ are convergents of certain continued functions, and as such provide better estimates of the formal sum of the series whose partial sums are given by (115) than the partial sums. The quantities $\varepsilon_s{}^{(m)}$ may be displayed in the array

$$\begin{array}{cccc}
\varepsilon_0{}^{(0)} & & & \\
 & \varepsilon_1{}^{(0)} & & \\
\varepsilon_0{}^{(1)} & & \varepsilon_2{}^{(0)} & \\
 & \varepsilon_1{}^{(1)} & & \varepsilon_3{}^{(0)} \\
\varepsilon_0{}^{(2)} & & \varepsilon_2{}^{(1)} & \cdot \\
 & \varepsilon_1{}^{(2)} & \cdot & \\
\varepsilon_0{}^{(3)} & & & \cdot \\
 & \cdot & \cdot & \\
\cdot & & & \cdot \\
 & \cdot & \ddots & \\
\cdot & & & \\
 & \cdot & & \\
\cdot & & &
\end{array}$$

and it can be seen that the quantities in (117) occur at the vertices of a lozenge in this array. The various numbers of this array are most economically (with regard to storage space) computed by retaining a vector l which at a given stage contains the following quantities: $l_0 \equiv \varepsilon_0{}^{(m)}$, $l_1 \equiv \varepsilon_1{}^{(m-1)}$, $l_2 \equiv \varepsilon_2{}^{(m-2)}, \ldots, l_m \equiv \varepsilon_m{}^{(0)}$. (This corresponds to what, in a table of a function and its differences, would be a line of backward differences). We arrive with a new partial sum $\varepsilon_0{}^{(m+1)}$ and replace in succession

$$l_0 \text{ by } \varepsilon_0^{(m+1)},\ l_1 \text{ by } \varepsilon_1^{(m)},\ \ldots,\ l_m \text{ by } \varepsilon_m^{(1)}, \text{ and add } l_{m+1} \equiv \varepsilon_{m+1}^{(0)}.$$

The formation of these quantities is carried out by means of (117) and uses one working space and two auxiliary storage locations. In certain singular cases, as occur for example when a term is equal to zero, the latter procedure breaks down. This difficulty may be overcome by certain singular rules [7].

(To be continued)