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New and old facts about entropy in uniform spaces and topological groups

Dikran Dikranjan [a,*], Manuel Sanchis [b,1], Simone Virili [c]

[a] *Dipartimento di Matematica e Informatica, Università di Udine, Via delle Scienze, 206-33100 Udine, Italy*
[b] *Institut Universitari de Matemàtiques i Aplicacions de Castelló (IMAC), Universitat Jaume I, Campus Riu Sec s/n, 8029 AP Castelló, Spain*
[c] *Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C-08193 Bellaterra, Barcelona, Spain*

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Dedicated to Professor Luigi Salce on the occasion of his 65th birthday

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ABSTRACT

In 1965 Adler, Konheim and McAndrew defined the topological entropy of a continuous self-map of a compact space. In 1971 Bowen extended this notion to uniformly continuous self-maps of (not necessarily compact) metric spaces and this approach was pushed further to uniform spaces and topological groups by many authors, giving rise to various versions of the topological entropy function. In 1981 Peters proposed a completely different (algebraic) entropy function for continuous automorphisms of non-compact LCA groups. The aim of this paper is to discuss some of these notions and their properties, trying to describe the relations among the various entropies and to correct some errors appearing in the literature.

1. Introduction

The notion of entropy is ubiquitous. The first appearance of this concept has been in the context of thermodynamics in the first half of the XIX century. In the 1930s Shannon recovered this notion and defined a notion of entropy that turned out to be a useful tool in the context of information theory. The measure entropy defined by Kolmogorov and Sinai played a fundamental role in the ergodic theory of dynamical systems since the 1950s. In 1965 Adler, Konheim and McAndrew [1] defined the topological entropy of a continuous self-map of a compact space, extended and deeply studied by many authors in the sequel. In the final part of [1], the authors suggested also a different notion of entropy of endomorphisms of discrete Abelian groups, called algebraic entropy. This kind of entropy received surprisingly less attention than topological entropy. Recently, the algebraic entropy and its extensions to different contexts were deeply studied [9–17,23–26,40–42,2,48,50,57]. The aim of this survey is to discuss the various notions of entropy on a topological space or a group defined in [1,7,37,38,11,48] and describe the relations among them, correcting some errors appearing in the literature (see Examples 4.8, 4.20 and Claim A.2).

The topological entropy on compact spaces and the uniform entropy on uniform spaces are exposed in Section 2. In particular, in Section 2.1 we recall the first definition of topological entropy for *continuous self-maps of compact spaces* from [1]. Some basic properties and examples are discussed here as well.

In his celebrated paper [7], Bowen gave a definition of entropy for *uniformly continuous self-maps on metric spaces*. Later on, Hood [31] extended Bowen's entropy to *uniformly continuous self-maps on uniform spaces*. This entropy function $h_U(-)$ will be recalled and discussed in Section 2.2.

* Corresponding author.
E-mail addresses: dikran.dikranjan@dimi.uniud.it (D. Dikranjan), sanchis@mat.uji.es (M. Sanchis), simone@mat.uab.cat (S. Virili).

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At this point, it appears natural to ask whether the topological and uniform entropy coincide whenever both make sense. In particular, given a compact space X and a continuous self-map $\phi : X \to X$, is it true that the topological entropy $h_T(\phi)$ of ϕ and the uniform entropy $h_U(\phi)$ of ϕ with respect to the unique admissible uniform structure $\mathscr{U}$ on X coincide? An affirmative answer to this question is known in the case of a compact metric space X (see for example [52, Theorem 6.12]). In Section 2.3 we extend this coincidence result to the general case (see Corollary 2.14). This is why, we prefer to avoid using two terms (topological and uniform) and adopt the more general term *uniform entropy* also for self-maps of compact spaces, where the uniform entropy coincides with the (widely used) topological entropy.

In the last two Sections 2.4–2.5 we discuss some notions of entropy on non-compact spaces. In particular, the first part of Section 2.4 presents some results contained in [29], while the second part proposes a new notion of entropy on non-compact spaces leaving some open questions. In Section 2.5 we discuss an alternative approach to uniform entropy of a uniformly continuous self-map ϕ of a *locally compact space* X, endowed with a Borel measure μ satisfying a specific compatibility condition with respect to ϕ.

In Section 3 some known and some new results about uniform entropy and algebraic entropy on locally compact (LC) and locally compact Abelian (LCA) groups respectively are given. In Sections 3.1–3.2 we recall the definition of Peters' entropy and of algebraic entropy (which we denote by $h_A(-)$). In particular, Section 3.1 consists mainly of definitions, while in Section 3.2 we discuss some deep results about the algebraic entropy on *discrete Abelian* groups. In fact, in this context $h_A(-)$ is deeply understood and it is possible to establish a uniqueness theorem for the algebraic entropy, that characterizes $h_A(-)$ as the unique function satisfying five axioms (see Theorem 3.5). Also some important dynamical aspects are considered. The results in these subsections are based on the recent papers [9–11,26,17]. Then, in Section 3.3 we study the uniform entropy on *compact* groups. A uniqueness theorem for this entropy, imposing a set of eight axioms, was obtained by Stojanov [43] (see Theorem 3.12). We discuss these axioms and the possibility to remove or weaken some of them. The reader more familiar with the measure entropy can find right after the theorem the important fact about the coincidence of the measure entropy with the uniform entropy for continuous endomorphisms of compact groups equipped with their Haar measure. In the final part of the subsection, mainly based on [39,43,55,56], we focus on some classical dynamical aspects.

In Section 4 we discuss two claims appearing in [38]. In particular, Section 4.1 examines the possibility to connect algebraic and uniform entropy on LCA groups using the Pontryagin–Van Kampen duality, while Section 4.2 deals with the so-called Addition Theorem.

In the final part of the paper, having a merely algebraic setting, we discuss the so-called L-entropy which generalizes to module theory the concept of entropy. This part is based on [42,41,46]. In particular, following [41], we give a uniqueness theorem for the L-entropy similar to the one discussed in Section 3.2 for the algebraic entropy on discrete Abelian groups (see Theorem A.7). Then we use a connection, established in [41], between L-entropy and multiplicity (in the sense of Northcott [36]) in order to correct a claim appearing in [56].

Notation and terminology

Our notation and terminology is standard. For instance, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{Q}$ and $\mathbb{R}$ respectively stand for the set of integers, the set of natural numbers, the set of positive integers, the set of rationals and the set of reals. When considered as a topological space $\mathbb{R}$ is equipped with the Euclidean topology.

We denote by e_G or simply e the neutral element of a generic group G, the Abelian groups will be written additively and their neutral element will be denoted by 0. For a group G and $a \in G$ we denote by φ_a the inner automorphism (conjugation) $x \mapsto x^a := a^{-1}xa$ of G. Let F and I be sets. We denote by F^I the Cartesian product $\prod_{i\in I} F$. If F is a group, then F^I is a group and we denote by $F^{(I)}$ the direct sum $\bigoplus_{i\in I} F$, namely the subgroup of all elements of F^I with finite support.

Unless otherwise stated, all topological spaces are assumed to be Tychonoff. The closure of a subset A of a space X will be denoted by $\overline{A}$ (or $\overline{A}^X$ or $cl_X(A)$ to prevent a possibility of confusion). The restriction of a function f to a subset A will be denoted by $f\restriction_A$.

The Pontryagin dual of a topological Abelian group G is denoted by $\widehat{G}$.

For undefined notions and terminology see [18,19,21,28,30,51].

2. Topological entropy versus uniform entropy

2.1. Topological entropy

We now recall the definition of topological entropy given in [1]. Let X be a compact topological space and let

$$\mathcal{O}(X) = \{\mathcal{U}\colon\ \mathcal{U} \text{ is an open cover of } X\}.$$

We allow open covers to have empty members. Given $\mathcal{U}, \mathcal{V} \in \mathcal{O}(X)$, the *join* of $\mathcal{U}$ and $\mathcal{V}$ is the open cover

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V\colon\ U \in \mathcal{U},\ V \in \mathcal{V}\}.$$

For $\mathcal{U} \in \mathcal{O}(X)$, let $N(\mathcal{U})$ denote the number of elements of a subcover of $\mathcal{U}$ with minimal cardinality.

If $\phi : X \to X$ is a continuous self-map, then the *topological entropy of ϕ respect to a cover $\mathcal{U}$* is defined as

$$h_T(\phi, \mathcal{U}) = \lim_{n\to\infty} \frac{\log N(\mathcal{U} \vee \phi^{-1}(\mathcal{U}) \vee \phi^{-2}(\mathcal{U}) \vee \cdots \vee \phi^{-n+1}(\mathcal{U}))}{n},$$

where, as usual, $\phi^{-1}(\mathcal{U})$ stands for the open cover $\{\phi^{-1}(U)\colon U \in \mathcal{U}\}$.

The *topological entropy* of ϕ is

$$h_T(\phi) = \sup\{h_T(\phi, \mathcal{U})\colon \mathcal{U} \in \mathcal{O}(X)\}. \tag{1}$$

2.1.1. Basic properties and examples

In this subsection we recall the very basic properties of topological entropy and we give the main examples in the context of entropy, namely the Bernoulli shifts. Proofs of these properties and a comprehensive outline of the standard facts about $h_T(-)$ can be found in [1,51,52].

The first property to notice is the *monotonicity* of $h_T(-)$ under continuous images. That is, given two continuous self-maps $\phi_i : X_i \to X_i$ $(i = 1, 2)$ of the compact spaces X_1, X_2, if there exists a continuous and surjective map $\alpha : X_1 \to X_2$, with $\alpha\phi_1 = \phi_2\alpha$, then

$$h_T(\phi_2) \leqslant h_T(\phi_1).$$

An easy consequence of monotonicity under continuous images is the *invariance* of $h_T(-)$ under topological conjugation. More precisely, for compact spaces X and Y, a continuous self-map $\phi : X \to X$ and a homeomorphism $\alpha : Y \to X$

$$h_T(\phi) = h_T(\alpha^{-1}\phi\alpha)$$

[1, Theorem 1]. The last relevant property of $h_T(-)$ that we want to state explicitly is the so-called *logarithmic law*. For a continuous self-map $\phi : X \to X$ of a compact space X,

$$h_T(\phi^n) = n \cdot h_T(\phi) \tag{2}$$

for every $n \in \mathbb{N}$, moreover $h_T(\phi^{-1}) = h_T(\phi)$ if ϕ is a homeomorphism [1, Theorem 2 and its corollary].

Now we come to an important reduction that allows us to simplify the computation of topological entropy in many cases:

Remark 2.1 *(Reduction to surjective self-maps).* Let X be a compact metric space and $\phi : X \to X$ a continuous self-map. Consider the subspace

$$E_\phi(X) := \bigcap \phi^n(X). \tag{3}$$

It is easy to see that $E_\phi(X)$ is closed, $\phi(E_\phi(X)) = E_\phi(X)$ (i.e., $E_\phi(X)$ is ϕ-invariant and the induced self-map $\phi\restriction_{E_\phi(X)}$ of $E_\phi(X)$ is surjective) and $h_T(\phi) = h_T(\phi\restriction_{E_\phi(X)})$ [51, Corollary 8.6.1]. Moreover $E_\phi(X)$ is the largest subspace of X with these properties.

The Bernoulli shifts introduced in the next definition provide a fundamental source of examples for the computation of (not only topological) entropy.

Definition 2.2. Let F be a set.

(i) We denote by $\lambda_F : F^{\mathbb{N}} \to F^{\mathbb{N}}$ the *left Bernoulli shift* defined by $(x_n)_{n\in\mathbb{N}} \mapsto (x_{n+1})_{n\in\mathbb{N}}$, i.e.,

$$(x_0, x_1, \ldots, x_n, \ldots) \mapsto (x_1, x_2, \ldots, x_{n+1}, \ldots).$$

(ii) Furthermore, denote by $\beta_F : F^{\mathbb{Z}} \to F^{\mathbb{Z}}$ the *two-sided Bernoulli shift* defined by

$$(x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}}.$$

(iii) If a specific point $a \in F$ is fixed, one can also define the *right Bernoulli shift* $\rho_F : F^{\mathbb{N}} \to F^{\mathbb{N}}$ by letting

$$(x_0, x_1, \ldots, x_n, \ldots) \mapsto (a, x_0, \ldots, x_{n-1}, \ldots).$$

If in the above definition F is a group, then β_F (resp., λ_F) is a group automorphism (resp., endomorphism). In this case, we *always* take the specific point $a \in F$ of part (iii) to be the neutral element e of F. With this choice, also ρ_F is a group endomorphism. Sometimes, we briefly call the two-sided Bernoulli shift β_F a *Bernoulli automorphism* (note that neither λ_F

nor ρ_F can be automorphisms). Finally, a Bernoulli shift (no matter if right, left or two-sided) is called *simple* if F is a simple group.

In case F is a topological space (or just a finite set considered as a discrete compact space), $F^{\mathbb{Z}}$ and $F^{\mathbb{N}}$ will carry the product topology. Then all shifts are continuous. If $|F| > 1$, the left Bernoulli shift is surjective but not injective, while the right Bernoulli shift is injective but not surjective. This difference explains why historically there has been less interest in the right Bernoulli shifts (being non-surjective, they are not *measure preserving* when $F^{\mathbb{N}}$ is endowed with its standard measure). We shall see in the sequel the prominent role of the right Bernoulli shift ρ_F when F is a finite Abelian group (and especially the restriction of ρ_F to the direct sum $F^{(\mathbb{N})}$) in the case of algebraic entropy (see Theorem 3.5).

Example 2.3. Let F denote a finite non-empty set, then

(i) $h_T(\beta_F) = \log|F|$ (see for example [43] or [56]);
(ii) as $E_{\rho_F}(F^{\mathbb{N}})$ reduces to a single point, one has $h_T(\rho_F) = 0$, by Remark 2.1.

2.2. *Uniform entropy*

Topological entropy on compact spaces was extended by Bowen [7] to uniformly continuous self-maps of a metric space (X, d) by using the metric uniformity $\mathscr{U}_d$. Later on, Hood [31] adapted Bowen's definition to uniformly continuous self-maps on a uniform space. To recall the definition of the uniform entropy $h_U(-)$ let $(X, \mathscr{U})$ be a uniform space and let $\phi : X \to X$ be a uniformly continuous self-map. Set

$$\mathcal{K}(X) = \{K\colon\ K \text{ is a compact subset of } X\}.$$

Given an entourage of the diagonal $V \in \mathscr{U}$, and a natural number n,

– a subset $F \subseteq X$ is said to (n, V)*-span* a compact $K \in \mathcal{K}(X)$ *with respect to* ϕ, if for every $x \in K$ there is $y \in F$ such that $(\phi^j(x), \phi^j(y)) \in V$ for each $0 \leqslant j < n$;
– a subset $F \subseteq X$ is said to be an (n, V)*-separated set with respect to* ϕ, if for each pair of distinct points $x, y \in F$ there exists j such that $0 \leqslant j < n$ and $(\phi^j(x), \phi^j(y)) \notin V$.

For $V \in \mathscr{U}$, $K \in \mathcal{K}(X)$ and $n \in \mathbb{N}$, set

– $r_n(V, K, \phi) = \min\{|F|\colon\ F\ (n, V)\text{-spans } K \text{ with respect to } \phi\}$;
– $s_n(V, K, \phi) = \max\{|F|\colon\ F \subseteq K \text{ and } F \text{ is } (n, V)\text{-separated with respect to } \phi\}$.

The numbers $r_n(V, K, \phi)$ and $s_n(V, K, \phi)$ are finite and well defined as K is compact. Let us list some useful relations between them:

Fact 2.4. ([31, Lemma 1]) Let ϕ be a uniformly continuous self-map on the uniform space $(X, \mathscr{U})$. If $K \in \mathcal{K}(X)$ and $W, V \in \mathscr{U}$ with $W \subseteq V$, then

(i) $\sigma_n(V, K, \phi) \leqslant \sigma_n(W, K, \phi)$, $n \geqslant 0$;
(ii) $r_n(V, K, \phi) \leqslant s_n(W, K, \phi)$, $n \geqslant 0$;

where σ_n stands either for r_n or for s_n.

Given $K \in \mathcal{K}(X)$, $n \in \mathbb{N}$, and letting $\sigma = r, s$, define

$$\sigma(V, K, \phi) = \limsup_{n \to \infty} \frac{\log \sigma_n(V, K, \phi)}{n},$$

for every $V \in \mathscr{U}$. Furthermore, let

$$h_\sigma(K, \phi) = \sup\bigl\{\sigma(V, K, \phi)\colon\ V \in \mathscr{U}\bigr\}.$$

The following result permits us to work either with $h_r(K, \phi)$ or with $h_s(K, \phi)$.

Fact 2.5. ([31, Lemma 1]) Let $(X, \mathscr{U})$ be a uniform space and let $\phi : X \to X$ be a uniformly continuous self-map. If $K \in \mathcal{K}(X)$, we have $h_r(K, \phi) = h_s(K, \phi)$.

This gives the notion of *uniform entropy* or *Bowen's entropy* $h_U(\phi)$ of ϕ (with respect to the uniform structure $\mathcal{U}$):

$$h_U(\phi) = \sup\{h_r(X,\phi):\ K \in \mathcal{K}(X)\} = \sup\{h_s(X,\phi):\ K \in \mathcal{K}(X)\}. \tag{4}$$

It is not difficult to prove that $h_U(-)$ is monotone under continuous images, invariant under topological conjugation, and satisfies the logarithmic law. Furthermore its values on the three Bernoulli shifts coincide with the ones of $h_T(-)$. For these and other basic properties we refer to [52].

2.3. Uniform entropy meets topological entropy

The goal of this section is to present an approach to the uniform entropy similar to the definition of topological entropy, that is, by means of covers (see (7)). The first in using this formulation has been Hofer [29]. However, no links were established in [29] between this (a priori) different notion of entropy and uniform entropy. Here we prove that the two approaches are equivalent. As an application we shall show that the topological entropy and the uniform entropy coincide in the case of a continuous self-map on a compact space. The necessary results and definitions are borrowed from [3].

Given a uniform space $(X,\mathcal{U})$, consider the uniform covers $\mathcal{C}_{\mathcal{U}} = \{\mathcal{C}(V):\ V \in \mathcal{U}\}$ defined as

$$\mathcal{C}(V) = \{V(x):\ x \in X\}, \quad V \in \mathcal{U},$$

where as usual $V(x) = \{y:\ (x,y) \in V\}$.

Definition 2.6. Let $(X,\mathcal{U})$ be a uniform space. If $\mathcal{A}$ is a uniform cover of X and K is a compact subset of X, then we define the number $N(K,\mathcal{A}) = \min\{|\mathcal{B}_K|:\ \mathcal{B}_K \subset \mathcal{A} \text{ and } K \subseteq \bigcup \mathcal{B}_K\}$.

The compactness of K guarantees that the number $N(K,\mathcal{A})$ is finite and well defined.

The next definition connects the notion of uniform entropy and topological entropy.

Definition 2.7. Let $(X,\mathcal{U})$ be a uniform space and let $\phi : X \to X$ be a uniformly continuous self-map. For each compact subset K of X and each $V \in \mathcal{U}$, we define $c_n(V,K,\phi) = N(K, \bigvee_{i=0}^{n-1} \phi^{-i}(\mathcal{C}(V)))$.

The following lemma is an easy consequence of the previous definition.

Lemma 2.8. *Let $(X,\mathcal{U})$ be a uniform space and let $\phi : X \to X$ be a uniformly continuous self-map. If $V, W \in \mathcal{U}$ with $W \subseteq V$, $n \in \mathbb{N}$ and $K \in \mathcal{K}(X)$, then $c_n(V,K,\phi) \leqslant c_n(W,K,\phi)$.*

The next lemma establishes a useful connection between $s_n(V,K,\phi)$, $r_n(V,K,\phi)$ and $c_n(V,K,\phi)$.

Lemma 2.9. *Let $(X,\mathcal{U})$ be a uniform space and let $\phi : X \to X$ be a uniformly continuous self-map. If $V \in \mathcal{U}$, $K \in \mathcal{K}(X)$ and $n \in \mathbb{N}$, then we have*

(i) $s_n(V,K,\phi) \leqslant c_n(W,K,\phi)$ *for each $W \in \mathcal{U}$ with $W \circ W \subseteq V$;*
(ii) $c_n(V,K,\phi) \leqslant r_n(V,K,\phi)$.

Proof. (i) By Lemma 2.8, we only need to prove that K contains no $(n, W \circ W, \phi)$-separated subsets of size $> c_n(W,K,\phi)$. Indeed, suppose there is a subset $E \subseteq K$ such that $|E| > c_n(W,K,\phi)$ and choose a subcover $\mathcal{B}_K \subseteq \bigvee_{i=0}^{n-1} \phi^{-i}(\mathcal{C}(W))$ of K of cardinality $c_n(W,K,\phi)$. Since $|E| > c_n(W,K,\phi)$, we can find two distinct elements $x_1, x_2 \in E$ and $y_1, y_2, \ldots, y_n \in K$ such that

$$x_1, x_2 \in W(y_1) \cap \phi^{-1}(W(y_2)) \cap \cdots \cap \phi^{-n+1}(W(y_n)) \in \mathcal{B}_K. \tag{5}$$

Now, if $0 \leqslant j < n$, (5) tells us that $x_1, x_2 \in W(y_{j+1})$ which implies that $(\phi^j(x_1), \phi^j(x_2)) \in W \circ W$. Thus, the subset E of K is not $(n, W \circ W, \phi)$-separated.

(ii) Let $E = \{x_i\}_{i=1}^s$ be a subset of K of minimal cardinality which (n,V,ϕ)-spans K. By definition, given $k \in K$, there exists $x_i \in E$ such that $(\phi^j(k), \phi^j(x_i)) \in V$ for each $j = 0, 1, \ldots, n-1$, that is,

$$k \in \phi^{-j}(V(\phi^j(x_i))) \tag{6}$$

for all $j = 0, 1, \ldots, n-1$. Hence the family $\{\phi^{-j}(V(\phi^j(x_i)))\}_{i=1}^s$ covers K for all $j = 0, 1, \ldots, n-1$.

Now, for each $i = 1, 2, \ldots, s$, consider the set B_i defined as

$$B_i = \bigcap_{j=0}^{n-1} \phi^{-j}(V(\phi^j(x_i))).$$

Notice that $x_i \in B_i$ ($i = 1, 2, \ldots, s$), so that (6) tells us that the family $\{B_i\}_{i=1}^{s}$ is a subcover of $\bigvee_{i=0}^{n-1} \alpha^{-i}(\mathcal{C}(V)) \cap K$ of cardinality s. Thus, by the definition of $c_n(V, K, \phi)$, we have $c_n(V, K, \phi) \leqslant s$. This completes the proof. □

Lemma 2.9 permits us to give the following

Definition 2.10. Let $(X, \mathscr{U})$ be a uniform space and let $\phi : X \to X$ be a uniformly continuous self-map. If $K \in \mathcal{K}(X)$ and $V \in \mathscr{U}$, then we define

$$c(V, K, \phi) = \limsup_{n \to \infty} \frac{\log c_n(V, K, \phi)}{n} \quad \text{for each } n \geqslant 0.$$

As a straightforward consequence of Lemma 2.9 we obtain the following

Corollary 2.11. *Let $(X, \mathscr{U})$ be a uniform space and let $\phi : X \to X$ be a uniformly continuous self-map. If $K \in \mathcal{K}(X)$, then*

(i) *$s(V, K, \phi) \leqslant c(W, K, \phi)$ for all $V, W \in \mathscr{U}$ with $W \circ W \subseteq V$;*
(ii) *$c(V, K, \phi) \leqslant r(V, K, \phi)$ for all $V \in \mathscr{U}$.*

The previous corollary allows us to consider the number $h_{UC}(\phi) \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ defined as

$$h_{UC}(\phi) = \sup\bigl\{h_{UC}(K, \phi): K \in \mathcal{K}(X)\bigr\} \tag{7}$$

where $h_{UC}(K, \phi) = \sup\{c(V, K, \phi): V \in \mathscr{U}\}$.

Remark 2.12. If the uniform space $(X, \mathscr{U})$ is a compact, then $c(V, X, \phi) = h_T(\phi, \mathcal{C}(V))$ for every $V \in \mathscr{U}$, where $h_T(\phi, \mathcal{C}(V))$ is computed on the underlying topological space. Since every open cover of the compact topological space X is uniform, i.e., admits a refinement of the form $\mathcal{C}(V)$ for some $V \in \mathscr{U}$ (see, for instance, [19, Exercise 8.1.H]), we deduce that

$$h_{UC}(\phi) = h_T(\phi) \tag{8}$$

from the definition of topological entropy.

The following theorem points out that for a uniformly continuous self-map ϕ of a uniform space the uniform entropy of ϕ coincides with $h_{UC}(\phi)$, that is, uniform entropy can be obtained in a similar way as the topological entropy.

Theorem 2.13. *If $\phi : X \to X$ is a uniformly continuous self-map of a uniform space $(X, \mathscr{U})$, then $h_U(\phi) = h_{UC}(\phi)$.*

Proof. Let $K \in \mathcal{K}(X)$. By Corollary 2.11, we have $h_s(K, \phi) \leqslant h_{UC}(K, \phi) \leqslant h_r(K, \phi)$. The result now follows from Fact 2.5. □

Now we can prove that in compact spaces the topological entropy of the continuous self-maps coincides with the uniform entropy computed with respect to the unique admissible uniform structure.

Corollary 2.14. *Let X be a compact space and let $\mathscr{U}$ be the unique admissible uniform structure on X. Then $h_T(\phi) = h_U(\phi)$ for every continuous self-map $\phi : X \to X$.*

Proof. The self-map $\phi : (X, \mathscr{U}) \to (X, \mathscr{U})$ is uniformly continuous, so Remark 2.12 implies $h_{UC}(\phi) = h_T(\phi)$. The result now follows from Theorem 2.13. □

We conclude this subsection with the following

Example 2.15. If a uniform space X has no infinite compact subsets, then $h_U(\phi) = 0$ for every uniformly continuous self-map $\phi : X \to X$, since $h_{UC}(K, \phi) = 0$ for every compact subset K of X as $|N(K, \mathcal{V})| \leqslant |K|$ for every open cover $\mathcal{V}$ of X.

2.4. Topological entropy in the non-compact case

This subsection is addressed to analyze some concepts of entropy on non-compact spaces. We begin with two notions introduced by Hofer [29]. In closing this subsection we take up a new notion of entropy related to the finest uniformity on a Tychonoff space.

The first entropy function defined by Hofer is a natural adjustment of the concept of topological entropy when the space X is non-compact. The idea is the following: given a continuous self-map $\phi : X \to X$ on a topological space X, we make the same procedure as in the definition of Adler, Konheim and McAndrew (see Section 2.1) but using *finite open covers* in place of *open covers*. Then we obtain an entropy function which is denoted by $h_{T_f}(-)$.

The second kind of Hofer's entropy is defined for Tychonoff spaces X as follows. As usual, βX stands for the Stone–Čech compactification of a space X. It is a well-known fact that every continuous map ϕ from X to a compact space K has a continuous extension ϕ^β to βX.

Definition 2.16. Let $\phi : X \to X$ be a continuous self-map of a Tychonoff space X. Then the entropy $h_\beta(\phi)$ is defined as the topological entropy of the self-map $\phi^\beta : \beta X \to \beta X$.

Obviously, $h_{T_f}(\phi) = h_\beta(\phi) = h_T(\phi)$ for compact spaces.

For $h_\beta(-)$ the next theorem follows from the properties of the topological entropy (see Section 2.1.1). In the case of $h_{T_f}(-)$, the proof runs along lines similar to the ones used in the classical case (see [29]).

Theorem 2.17. *Let $\phi : X \to X$ be a continuous self-map. If $h^\star(\phi)$ is either $h_{T_f}(\phi)$ or $h_\beta(\phi)$, then the following properties hold:*

(i) *$h^\star(\phi^n) = nh^\star(\phi)$ for every $n \in \mathbb{N}$. Moreover, if ϕ is a homeomorphism, then $h^\star(\phi^n) = |n|h^\star(\phi)$ for every $n \in \mathbb{Z}$.*
(ii) *If Y is a closed ϕ-invariant subset of X, then $h^\star(\phi\restriction_Y) \leqslant h^\star(\phi)$.*
(iii) *Let $\rho : Y \to Y$ be a continuous self-map. If $f : Y \to X$ is a continuous surjection such that $f\rho = \phi f$, then $h^\star(\phi) \leqslant h^\star(\rho)$. In particular, the entropy $h^\star$ is invariant under topological conjugacy.*

Given a continuous self-map $\phi : X \to X$, it is natural to ask the following

Question 2.18. ([29, remark, p. 239]) Let $\phi : X \to X$ be a continuous self-map. Is $h_{T_f}(\phi) = h_\beta(\phi)$?

In this direction, the main result of [29] states

Theorem 2.19. *Let X be a normal space. If $\phi : X \to X$ is a homeomorphism, then $h_\beta(\phi) = h_{T_f}(\phi)$.*

The proof of Theorem 2.19 heavily relies on the fact that the Stone–Čech compactification of a normal space is the Wallman compactification defined by means of ultrafilters of closed sets: the point is that $\overline{F}^{\beta X} \cap \overline{G}^{\beta X} = \emptyset$ whenever F and G are pairwise disjoint closed subsets of X. This feature can be applied in a straightforward way to obtain *a one-to-one correspondence* between finite open covers of X and basic open covers of βX.

In the case of a non-normal Tychonoff space X, this procedure strongly fails: βX is now the Wallman compactification defined by using z-ultrafilters and we can only guarantee that $\overline{F}^{\beta X} \cap \overline{G}^{\beta X} = \emptyset$ for pairwise disjoint zero sets F and G, and hence one can only obtain *a one-to-one correspondence* between *finite cozero covers* of X and basic open covers of βX. As a consequence one has

Theorem 2.20. *If $\phi : X \to X$ is a continuous self-map, then $h_\beta(\phi) \leqslant h_{T_f}(\phi)$.*

Now we offer a new notion of entropy for non-compact spaces. Let X be a space and let $\mathscr{F}$ denote the universal uniform structure on X. Taking into account that every continuous map from X to a uniform space is uniformly continuous for the universal uniform structure $\mathscr{F}$ (see, for instance, [22, 15G, (5)]) the following definition makes sense.

Definition 2.21. Let $\phi : X \to X$ be a continuous self-map. The entropy $h_{\mathscr{F}}(\phi)$ of ϕ is defined as the uniform entropy of ϕ considered as a uniformly continuous self-map on $(X, \mathscr{F})$.

We now establish some basic facts concerning this new entropy function.

Fact 2.22.

(i) Since a compact space admits a unique admissible uniform structure, Corollary 2.14 entails that $h_{\mathscr{F}}(-)$ coincides with $h_U(-)$ for compact spaces.
(ii) Since $h_{\mathscr{F}}(-)$ is defined by means of the uniform entropy, it satisfies conditions (i)–(iii) of Theorem 2.17.
(iii) Let $\mathscr{U}$ be an admissible uniform structure on a space X. If $\phi : (X, \mathscr{U}) \to (X, \mathscr{U})$ is a uniformly continuous self-map, then $h_U(\phi) \leqslant h_{\mathscr{F}}(\phi)$: notice that the identity map $id : (X, \mathscr{F}) \to (X, \mathscr{U})$ is uniformly continuous so that every uniform cover for $\mathscr{U}$ is a uniform cover for $\mathscr{F}$.

Let us see now that both $h_\beta(-)$ and $h_{T_f}(-)$ differ from $h_{\mathscr{F}}(-)$.

Example 2.23. Let G be a non-torsion discrete Abelian group and σ the endomorphism $x \overset{\sigma}{\mapsto} kx$, $k > 1$. By Example 2.15, $h_{\mathcal{F}}(\sigma) = 0$. We show that $h_\beta(\sigma) = h_{T_f}(\sigma) = \infty$. By Theorem 2.20, it suffices to check $h_\beta(\sigma) = \infty$. Let $G^\#$ denote the group G equipped with its Bohr topology. Then $\sigma : G^\# \to G^\#$ is continuous, and its extension $b\sigma$ to the Bohr compactification bG of $G^\#$ (i.e., the completion of $G^\#$) has $h_U(b\sigma) = \infty$, by [4, Theorem 3.3, Case 1]. The continuous extension $j^\beta : \beta G \to bG$ of the inclusion map $j : G \to bG$ satisfies $j^\beta \sigma^\beta = b\sigma j^\beta$. Hence, we have $\infty = h_U(b\sigma) \leqslant h_U(\sigma^\beta) = h_\beta(\sigma)$.

This example shows that the global inequality $h_{\mathcal{F}}(-) \geqslant h_\beta(-)$ fails due to examples with very special properties: the space $G^\#$ is normal precisely when G is countable [44], while the self-map σ is a continuous endomorphism, hence uniformly continuous with the respect to the unique compatible uniformity of the topological group $G^\#$.

The previous example leaves open the following question:

Question 2.24. Does the inequality $h_{\mathcal{F}}(\phi) \leqslant h_\beta(\phi)$ hold for every continuous self-map $\phi : X \to X$ of a Tychonoff space X?

2.5. Bowen's entropy in locally compact spaces

Let (X, d) be a *locally compact* metric space endowed with a Borel measure μ and let $\phi : X \to X$ be a uniformly continuous self-map. Bowen introduced in [7, Section 2] another entropy function $k(-)$, in case μ satisfies a specific compatibility condition with respect to ϕ (see (i)–(iii) below).

Indeed, for $x \in X$, $n \in \mathbb{N}_+$ and $\varepsilon > 0$ define the *Bowen's ball*

$$D_n(x, \varepsilon, \phi) = \bigcap_{k=0}^{n-1} \phi^{-1}\big(B_\varepsilon\big(\phi^k(x)\big)\big),$$

where $B_\varepsilon(x)$ denotes the ball given by the metric d. Call the Borel measure μ on X *ϕ-homogeneous*, if

(i) $\mu(K) < \infty$ for all compact sets K in X;
(ii) $\mu(K_0) > 0$ for some compact subset K_0 of X;
(iii) for each $\varepsilon > 0$ there exist $\delta > 0$ and $c > 0$ such that

$$\mu\big(D_n(y, \delta, \phi)\big) \leqslant c\mu\big(D_n(x, \varepsilon, \phi)\big),$$

for all $n \in \mathbb{N}_+$ and all $x, y \in X$.

It follows from (iii) that the value

$$k(\phi, x, \varepsilon) = \limsup_{n\to\infty} -\frac{\log \mu(D_n(x, \varepsilon, \phi))}{n}, \tag{9}$$

does not depend on the particular choice of $x \in X$. Now we can define the *rate of decay* of the measure of Bowen's ball

$$k(\mu, \phi) = \lim_{\varepsilon\to\infty} k(\phi, x, \varepsilon).$$

Bowen proved that

$$h_s(K, \phi) = k(\mu, \phi) \tag{10}$$

for every compact $K \subseteq X$ with $\mu(K) > 0$, in particular, $h_U(\phi) = k(\mu, \phi)$, in view of (ii). Note that the more precise equality (10) implies that *Bowen's entropy $h_s(K, \phi)$ does not depend on the compact set K as far as $\mu(K) > 0$*. This allows for a simpler computation of h_U by choosing an appropriate compact set with $\mu(K) > 0$.

Hood [31] noticed that the definition of Bowen's entropy $k(-)$ can be extended to a uniformly continuous self-map $\phi : X \to X$ of a uniform locally compact space $(X, \mathcal{U})$ as follows. For an entourage V of the uniform structure, $x \in X$ and $n \in \mathbb{N}_+$, one can define the counterpart of the Bowen's ball with respect to U by $D_n(x, V, \phi) = \bigcap_{k=0}^{n-1} \phi^{-1}(V(\phi^k(x)))$. Using this notion, one can define ϕ-homogeneous Borel measures μ on X as above. When X is endowed with such a measure μ, one obtains again $h_U(\phi) = k(\mu, \phi)$ (see Proposition 13 from [31]).

In what follows we discuss Hood's generalization of Bowen's entropy $k(-)$ for a locally compact group G, endowed with a left Haar measure μ and considered in its left uniform structure. Let

$$\mathcal{C}(G) = \{K\colon\ K \text{ is a compact neighborhood of } e_G \text{ in } G\}.$$

For a positive integer n, the *n-th ϕ-cotrajectory* of a fixed $V \in \mathcal{C}(G)$ is

$$C_n(\phi, V) = V \cap \phi^{-1}V \cap \cdots \cap \phi^{-n+1}V. \tag{11}$$

Now, for $x \in G$, $n \in \mathbb{N}_+$ and $V \in \mathscr{C}(G)$, one has $D_n(x, V, \phi) = xC_n(\phi, U)$. Therefore, the left invariance of μ yields that μ is ϕ-homogeneous for every continuous endomorphism ϕ of G and $\mu(D_n(x, V, \phi)) = \mu(C_n(\phi, V))$. Now Bowen's entropy $k(\phi, V)$, with $V \in \mathscr{C}(G)$ (generalizing (9) from the metric case) is

$$k(\phi, V) = \limsup_{n\to\infty} -\frac{\log \mu(C_n(\phi, V))}{n}, \tag{12}$$

and Bowen's entropy of ϕ is

$$k(\phi) = \sup\{k(\phi, V):\ V \in \mathscr{C}(G)\}. \tag{13}$$

As we mentioned, a reformulation in terms of LC groups of the results from [31] gives, for an LC group G and a continuous endomorphism $\phi : G \to G$,

$$k(\phi) = h_U(\phi).$$

In view of this equality, we will use the symbol $h_U(-)$ also to denote $k(-)$. We conclude this section with a remark that simplifies the computation of the supremum (13) in many cases.

Remark 2.25.

(a) One can see that the value of $k(\phi, V)$ computed in (12) does not depend on the choice of the Haar measure μ on G [48, Lemma 2.1].

(b) Obviously, for $V, V' \in \mathscr{C}(G)$, with $V \subseteq V'$, $\mu(V) \leqslant 1$ and $\mu(V') \leqslant 1$, one has $C_n(\phi, V) \subseteq C_n(\phi, V')$, so $k(\phi, V) \geqslant k(\phi, V')$. In this sense, $k(\phi, V)$ is monotone with respect to V. Therefore, in (13) one can take the neighborhoods V from a base of $\mathscr{C}(G)$.

(c) It is well known that when G is totally disconnected, then $\mathscr{C}(G)$ has a base of clopen compact subgroups (see [28]). Therefore, one can take in (13) clopen compact subgroups V in this case. This allows to eliminate the Haar measure μ completely. Indeed, let $u := \mu(V) > 0$. The open subgroup $C_n(\phi, V)$ of V has a finite index $[V : C_n(\phi, V)]$, so $\mu(C_n(\phi, V)) = \frac{u}{[V:C_n(\phi,V)]}$. Hence, (12) can be rewritten in the following measure-free form

$$k(\phi, V) = \limsup_{n\to\infty} \frac{\log[V : C_n(\phi, V)] - u}{n} = \limsup_{n\to\infty} \frac{\log[V : C_n(\phi, V)]}{n}, \tag{14}$$

as $\lim \frac{\log u}{n} = 0$. Moreover, when G is compact and totally disconnected, this formula can be improved as follows. From $[G : C_n(\phi, V)] = [G : V][V : C_n(\phi, V)]$ and the equality $\lim_{n\to\infty} \frac{\log[G:V]}{n} = 0$ one can easily deduce

$$\limsup_{n\to\infty} \frac{\log[G : V][V : C_n(\phi, V)]}{n} = \limsup_{n\to\infty} \frac{\log[V : C_n(\phi, V)]}{n}.$$

This gives

$$k(\phi, V) = \limsup_{n\to\infty} \frac{\log[G : C_n(\phi, V)]}{n}. \tag{15}$$

(d) A topological group G is said to be *SIN* (abbreviation for *small invariant neighborhoods*), if G has a local base at e_G consisting of invariant neighborhoods V (i.e., $V^x = V$ for all $x \in G$). It is easy to see that in a locally compact SIN group G every inner automorphism $\varphi_a : x \mapsto x^a$ has uniform entropy zero (by item (b) it suffices to note that $k(V, \varphi_a) = 0$ for every compact invariant neighborhood V of e_G). In particular, every inner automorphism of a compact group has uniform entropy zero. Easy examples show that this property fails in locally compact groups that are not SIN, e.g. the group $SL_2(\mathbb{R})$.

3. Algebraic and uniform entropy on topological groups

In this section we discuss the uniform entropy and the algebraic entropy on some categories of topological groups. Since few results can be proved for the whole category of topological groups, we limit our interest mainly to some full subcategories of the category $\mathfrak{LC}$ of all locally compact (LC) groups, denoted in the sequel as follows:

- $\mathfrak{C}$ (resp., $\mathfrak{CA}$) denotes the category of compact (resp., compact Abelian) groups;
- $\mathfrak{LCA}$ denotes the category of locally compact Abelian (LCA) groups;
- $\mathfrak{DA}$ (resp., $\mathfrak{TA}$) denotes the category of discrete Abelian (resp., torsion discrete Abelian) groups;
- vec($\mathbb{R}$) denotes the category of finite-dimensional real vector spaces;
- $\mathfrak{Pro}$ denotes the category of profinite Abelian groups.

The usual point of view is to consider entropy (no matter if topological, uniform, algebraic, measure-theoretic, etc.) as a map from the set of endomorphisms of a fixed object in a category $\mathfrak{D}$ to $\mathbb{R}_{\geqslant 0} \cup \{\infty\}$. This point of view can be made more rigorous as follows.

Given a category $\mathfrak{D}$, let $\mathrm{Flow}(\mathfrak{D})$ denote the category of *flows* of $\mathfrak{D}$ whose objects are the endomorphisms $\phi : G \to G$, where G is an object of $\mathfrak{D}$. Sometimes we need to make explicit the domain of an object $\phi : G \to G$ of $\mathrm{Flow}(\mathfrak{D})$. In these cases we shall refer to an object of $\mathrm{Flow}(\mathfrak{D})$ as a pair (G, ϕ).

A morphism $\alpha : (G, \phi) \to (H, \psi)$ in $\mathrm{Flow}(\mathfrak{D})$ is a commutative square

$$\begin{array}{ccc} G & \xrightarrow{\ \phi\ } & G \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ H & \xrightarrow{\ \psi\ } & H. \end{array}$$

For a full subcategory $\mathfrak{D}$ of $\mathfrak{LC}$ and two functions h, h' from $\mathrm{Flow}(\mathfrak{D})$ to $\mathbb{R}_{\geqslant 0} \cup \{\infty\}$ we use the following terminology:

(1) h *is invariant under conjugation* if for every pair of isomorphic objects $\phi, \psi \in \mathrm{Flow}(\mathfrak{D})$ one has $h(\phi) = h(\psi)$;
(2) h *satisfies the logarithmic law* if for every $\phi \in \mathrm{Flow}(\mathfrak{D})$ we have that $h(\phi^n) = n \cdot h(\phi)$, for all $n \in \mathbb{N}$;
(3) if $\mathfrak{D}$ is closed under taking closed normal subgroups and quotients over closed normal subgroups, then h *satisfies the Addition Theorem* (*briefly AT*) if, for every $(G, \phi) \in \mathrm{Flow}(\mathfrak{D})$, and every closed ϕ-invariant normal subgroup $H \subseteq G$, we have

$$h(\phi) = h(\phi \restriction_H) + h(\bar{\phi}),$$

where $\bar{\phi} : G/H \to G/H$ is the morphism induced on the quotient by ϕ;
(4) if $\mathfrak{D}$ is closed under taking quotients over closed normal subgroups, then h *is continuous on inverse limits* if, for every $(G, \phi) \in \mathrm{Flow}(\mathfrak{D})$, and every family $\{H_i\}_i$ of closed ϕ-invariant normal subgroups $H_i \subseteq G$, we have

$$h(\bar{\phi}) = \sup_i h(\bar{\phi}_i),$$

where $H = \bigcap_i H_i$, and $\bar{\phi} : G/H \to G/H$, $\bar{\phi}_i : G/H_i \to G/H_i$ are the maps induced on the quotients by ϕ;
(5) if $\mathfrak{D}$ is closed under taking closed subgroups, then h *is continuous on direct limits* if, for every $(G, \phi) \in \mathrm{Flow}(\mathfrak{D})$, and every family $\{H_i\}_i$ of closed ϕ-invariant subgroups $H_i \subseteq G$, we have

$$h(\phi_H) = \sup_i h(\phi_i),$$

where $H = cl_G(\sum_i H_i)$, and $\phi_H : H \to H$, $\phi_i : H_i \to H_i$ are the maps induced by ϕ;
(6) if $\mathfrak{D}$ consists of *Abelian* groups, we say that the (ordered) pair (h, h') *satisfies the Bridge Theorem* (*briefly BT*) on $\mathrm{Flow}(\mathfrak{D})$ if, for every $\phi \in \mathrm{Flow}(\mathfrak{D})$

$$h(\phi) = h'(\widehat{\phi}),$$

where $\widehat{\phi} : \widehat{G} \to \widehat{G}$ is the Pontryagin–Van Kampen dual of $\phi : G \to G$.

The category $\mathrm{Casc}(\mathfrak{D})$ of *cascades* of $\mathfrak{D}$ is the full subcategory of $\mathrm{Flow}(\mathfrak{D})$ with objects all automorphisms in $\mathfrak{D}$. The properties (1)–(3) and (6) will be considered, with the obvious modifications, for $\mathrm{Casc}(\mathfrak{D})$ as well. We refer to [29] for the notion of cascade and to [10, Section 3], [47] for the category of flows.

It is relevant to know whether (some subset of) the properties (1)–(6) can be used to uniquely characterize an entropy function. We refer to results of this kind as *Uniqueness Theorems* (see Theorems 3.5–3.12–4.12–A.7). It turns out that in addition to the properties listed above we also need normalizations. A first relevant normalization is given by the values taken on the Bernoulli shifts (see Definition 2.2). Let us introduce a second tool to normalize the values of an entropy function, namely the *Mahler measure*.

Let $n \in \mathbb{N}$, denote by $\mathbb{Q}$ the discrete group of rationals and by $\mathbb{K}$ the Pontryagin–Van Kampen dual of $\mathbb{Q}$. An endomorphism ϕ of $\mathbb{Q}^n$ (resp., $\mathbb{K}^n$) is described via an $n \times n$ rational matrix $A \in \mathrm{Mat}_n(\mathbb{Q})$. Let $p(X)$ be the (monic) characteristic polynomial of A over $\mathbb{Q}$ and $s \in \mathbb{N}_+$ be the least common multiple of the denominators of the coefficients of $p(X)$. The characteristic polynomial of A taken over $\mathbb{Z}$ is the primitive polynomial

$$p_1(X) = sp(X) = sX^n + a_1 X^{n-1} + \cdots + a_n \in \mathbb{Z}[X].$$

Note that $p_1(X)$ and the eigenvalues λ_i, $i = 1, 2, \ldots, n$, of A depend only on ϕ. The number

$$m\bigl(p_1(X)\bigr) = \log s + \sum_{|\lambda_i|>1} \log|\lambda_i|,$$

is known as (*logarithmic*) *Mahler measure* of $p_1(X)$ [20]. We shall call it also *Mahler measure of* ϕ and denote it by $m(\phi)$.

3.1. Algebraic entropies

In the two papers [37] and [38], of 1979 and 1981 respectively, Peters defined a new entropy function with the aim to dualize in an appropriate sense the concept of Bowen's entropy defined by (13). Here we recall Peters's definition as well as the alternative notion of algebraic entropy proposed in [11,48]. Both notions are based on the concept of Haar measure so we fix an LCA group G, a continuous endomorphism $\phi : G \to G$ and a Haar measure μ on G. We recall that the *modulus* is a group homomorphism

$$mod_G : \mathrm{Aut}(G) \to \mathbb{R}_{>0}, \quad \alpha \mapsto mod_G(\alpha)$$

such that $\mu(\alpha(E)) = mod_G(\alpha)\mu(E)$, where E is a measurable subset of G, and $\mathrm{Aut}(G)$ is the group of topological isomorphisms from G onto G (see [28, (15.26), p. 208] for the existence of such a function).

Example 3.1. The following examples are well known.

(i) If G is compact or discrete, then $mod_G(\alpha) = 1$ for every $\alpha \in \mathrm{Aut}(G)$.
(ii) If $G = \mathbb{R}^n$ for some positive integer n, then $mod_G(\alpha) = |\det_{\mathbb{R}}(\phi)|$ for every $\alpha \in \mathrm{Aut}(G)$.

Given a positive integer n and $K \in \mathscr{C}(G)$, the key idea of Peters is to replace the n-th ϕ-cotrajectory in (11) by the following sum

$$P_n(\phi, K) = K + \phi^{-1}K + \cdots + \phi^{-n+1}K,$$

where ϕ is a topological isomorphism. Then *Peters's entropy of ϕ with respect to K* is defined by

$$H_\infty(\phi, K) = \limsup_{n\to\infty} \frac{\log \mu(P_n(\phi, K))}{n},$$

and *Peters's entropy* of ϕ is $h_\infty(\phi) = \sup\{H_\infty(\phi, K)\colon\ K \in \mathscr{C}(G)\}$. Note that $h_\infty(\phi) = 0$ whenever $G \in \mathscr{C}(G)$ is compact.

The main reason to introduce the entropy $h_\infty(-)$ can be found in [38, Section 9] where the following fascinating claim appears:

Claim 3.2. *([38]) The pair (h_∞, h_U) satisfies BT on* $\mathrm{Casc}(\mathfrak{LCA})$.

Roughly speaking $h_\infty(-)$ is meant to be "dual" to $h_U(-)$. This works for $\mathrm{Casc}(\mathfrak{DA})$ (see [38] or Theorem 4.9), taking into account Fact 3.4(ii). It works (trivially) for $\mathrm{Casc}(\mathfrak{CA})$ as well, since $h_\infty(-)$ vanishes on automorphisms of compact groups, while $h_U(-)$ vanishes on self-maps of discrete spaces (see Example 2.15). Unfortunately, Claim 3.2 strongly fails even for automorphisms of the reals (see Example 4.8). Because of the failure of Claim 3.2 and of the fact that this definition covers only topological isomorphisms, the entropy $h_\infty(-)$ seems to be unsatisfactory. This is why the first named author with Giordano Bruno in [11], and the third named author in [48], modified Peters's definition respectively in the discrete and in the general case. As in the case of Peters's entropy, the key idea is to find the right notion of "trajectory" in order to replace the cotrajectory (11).

For an Abelian group G, an endomorphism $\phi : G \to G$ and a subset $E \subseteq G$, call the subset

$$T_n(\phi, E) = E + \phi E + \cdots + \phi^{n-1}E$$

the *n-th ϕ-trajectory* of K. If E is finite (resp., compact, a subgroup), then $T_n(\phi, E)$ is finite (resp., compact, a subgroup).

For an LCA group G, a continuous endomorphism $\phi : G \to G$, $n \in \mathbb{N}_+$ and $K \in \mathscr{C}(G)$, the *algebraic entropy of ϕ with respect to K* is

$$H_A(\phi, K) = \limsup_{n\to\infty} \frac{\log \mu(T_n(\phi, K))}{n}. \tag{16}$$

The *algebraic entropy* of ϕ is $h_A(\phi) = \sup\{H(\phi, K)\colon\ K \in \mathscr{C}(G)\}$.

Remark 3.3. The name "algebraic entropy" is justified by the explicit use of the *algebraic operation* "+" of the group in the definition of the trajectory (unlike the cotrajectory (11) in the definition of Bowen's entropy in $\mathfrak{LC}$).

The following fact establishes a precise relation between $h_A(-)$ and $h_\infty(-)$:

Fact 3.4. ([48, Remark 2.8]) Let G be an LCA group and $\phi : G \to G$ a topological isomorphism. Then

(i) $h_\infty(\phi) = h_A(\phi) - \log(mod_G(\phi))$;
(ii) in particular, $h_\infty(-)$ and $h_A(-)$ coincide on topological isomorphisms of *discrete* or *compact* Abelian groups.

In the forthcoming Sections 3.2–3.3, we study $h_A(-)$ and $h_U(-)$ on discrete Abelian groups and on compact groups respectively. In particular we give the fundamental properties that enable one to completely characterize these two entropy functions. Then we state some dynamical consequences.

3.2. Algebraic entropy on discrete Abelian groups

We are giving first the case of discrete groups, although chronologically the compact cases came first. This is motivated by the various applications we give of the results in the discrete case.

In the sequel G will be a discrete Abelian group. The Haar measure for G is given by $\mu(E) = |E|$, the size of a subset E of G (by letting $|E| = \infty$ when E is infinite). Now $\mathscr{C}(G)$ is the family of finite subsets of G containing the 0 element. For $F \in \mathscr{C}(G)$ the value of $H_A(\phi, F)$ in (16) is obtained as a limit in [11]. In case G is torsion, $h_A(\phi)$ coincides with the function (that can be defined also in the general case)

$$\mathrm{ent}(\phi) = \sup\bigl\{H_A(\phi, F)\colon\ F \text{ is a finite } \textit{subgroup} \text{ of } G\bigr\}. \tag{17}$$

The first definition of ent$(-)$ was given in [1] and some of its basic properties were investigated in [53]. For a deep study of ent$(-)$ we refer to [16] (for alternative defining formulae see Appendix A).

3.2.1. Uniqueness of the algebraic entropy for the discrete Abelian groups

The following uniqueness theorem for the algebraic entropy in $\mathfrak{DA}$ is proved in [11, Theorem 1.5]. It shows that the function $h_A(-)$ is completely determined by a set of five axioms.

Theorem 3.5 *(Uniqueness Theorem). The algebraic entropy* $h_A : \mathrm{Flow}(\mathfrak{DA}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *is the unique function such that*

(DA1) *$h_A(-)$ is invariant under conjugation*;
(DA2) *$h_A(-)$ is continuous on direct limits*;
(DA3) *$h_A(-)$ satisfies AT*;
(DA4) *if K is a finite Abelian group and ρ_K is the right Bernoulli shift of $K^{\mathbb{N}}$, then $h_A(\rho_K \restriction_{K^{(\mathbb{N})}}) = \log|K|$*;
(DA5) *for every automorphism ϕ of $\mathbb{Q}^n$ the algebraic entropy of ϕ coincides with the Mahler measure of ϕ, in symbols $h_A(\phi) = m(\phi)$.*

If we restrict $h_A(-)$ to the subcategory $\mathfrak{TA}$ of $\mathfrak{DA}$, condition (DA5) can be ignored and the axiom (DA1)–(DA4) suffice to guarantee uniqueness (see [16], where also the logarithmic law is given as a fifth axiom, but one can prove as in [40, Proposition 2.1] that it is not necessary).

The verification of the properties (DA1), (DA2) and (DA4) are quite standard (see [11]), unlike the highly non-trivial property (DA5), called *Algebraic Yuzvinski Formula*. Its proof is the main result of [26]. Finally, property (DA3), the Addition Theorem, is proved in [11] making substantial use of (DA5). Giordano Bruno [23] proved that $h_A(\rho_K) = \infty$ whenever K is a non-trivial finite Abelian group. This explains the necessity to use the restriction $\rho_K \restriction_{K^{(\mathbb{N})}}$ in (DA4), rather then the right shift ρ_K itself.

Given a positive integer n and an endomorphism $\phi : \mathbb{Q} \to \mathbb{Q}$, an elementary proof of the weaker form of (DA5),

$$h_A(\phi) \geqslant \max\bigl\{\log s, \log|\lambda_1|, \ldots, \log|\lambda_n|\bigr\},$$

where $\{\lambda_1, \ldots, \lambda_n\}$ is the family of complex eigenvalues of ϕ and s is the least common multiple of the denominators of the coefficients of the (monic) characteristic polynomial of ϕ over $\mathbb{Q}$, using only basic facts from linear algebra was given in [17]. This fact provides an alternative proof of the forthcoming Theorem 3.8 for the case of $\mathbb{Q}^n$, making no use of the stronger (DA5).

Using the Addition Theorem one can reduce the computation of the algebraic entropy to the case of automorphisms. This can be obtained in two steps. Firstly, for $\phi \in \mathrm{End}(G)$ consider the *hyperkernel*

$$\ker_\infty \phi = \bigcup_{n=0}^{\infty} \ker \phi^n$$

of ϕ. Obviously, the induced endomorphism $\bar{\phi} : G/\ker_\infty \phi \to G/\ker_\infty \phi$ is injective and $h_A(\bar{\phi}) = h_A(\phi)$, as $h_A(\phi \restriction_{\ker_\infty \phi}) = 0$ (see Example 3.7(b)). Secondly, for an injective $\phi \in \mathrm{End}(G)$ one can consider the ϕ-invariant subgroup $E_\phi(G) := \bigcap \phi^n(G)$. Then the restriction $\phi \restriction_{E_\phi(G)}\colon E_\phi(G) \to E_\phi(G)$ is an automorphism, while the induced endomorphism $\bar{\phi} : G/E_\phi(G) \to G/E_\phi(G)$ satisfies $E_{\bar{\phi}}(G/E_\phi(G)) = 0$ and $h_A(\phi) = h_A(\bar{\phi}) + h_A(\phi \restriction_{E_\phi(G)})$. Note that this step need not preserve the entropy, as $h_A(\bar{\phi})$ need not vanish.

3.2.2. Dynamical aspects: Pinsker subgroup and (quasi-)periodicity

All results and notions in this part come from [9], where the reader can find major details. Let G be a discrete Abelian group and $\phi \in \mathrm{End}(G)$. For a finite $F \subseteq G$ and $n \in \mathbb{N}_+$ let

$$\tau_{F,\phi}(n) = \left|T_n(\phi, F)\right|.$$

Now, $\tau_{F,\phi}(n) \leqslant |F|^n$, so $\tau_{F,\phi}(n)$ has at most exponential growth. Motivated by this, one says that

- $\tau_{F,\phi}(n)$ has *exponential growth*, if there exists $b > 1$, such that $\tau_{F,\phi}(n) \geqslant b^n$ for all $n \in \mathbb{N}_+$;
- $\tau_{F,\phi}(n)$ has *polynomial growth*, if there exists a polynomial $P(X) \in \mathbb{Z}[X]$, such that $\tau_{F,\phi}(n) \leqslant P(n)$ for all $n \in \mathbb{N}_+$.

One has the following important and somewhat surprising fact:

Theorem 3.6. *([9, Dichotomy Theorem]) The function $\tau_{F,\phi}(n)$ has either exponential or polynomial growth for every non-empty finite subset F of G and every $\phi \in \mathrm{End}(G)$.*

This Dichotomy Theorem captures a typical behavior of the Abelian groups. Indeed, one can similarly define algebraic entropy for endomorphisms of arbitrary (not necessarily Abelian) groups (see [12]). It is a well-known fact that such a dichotomy is not available even for the identity endomorphism $\phi = id_G$ if the group G is not Abelian. This question was raised by Milnor [34], a negative answer was given by Grigorchuk [27], who constructed a series of groups with intermediate growth (see [12] for more details on the connection between the algebraic entropy and the growth rate).

For an Abelian group G and $\phi \in \mathrm{End}(G)$ one says that

- ϕ *has polynomial growth*, if $\tau_{F,\phi}(n)$ has polynomial growth for every finite subset F of G;
- ϕ *has exponential growth*, if $\tau_{F,\phi}(n)$ has exponential growth for some finite subset F of G.

Obviously, Theorem 3.6 implies the following *Global Dichotomy Theorem*: every $\phi \in \mathrm{End}(G)$ has either a polynomial growth or an exponential growth.

An element $x \in G$ is said to be a *quasi-periodic point of* ϕ if there exist $m \neq n \in \mathbb{N}$ such that $\phi^n(x) = \phi^m(x)$.

Example 3.7. Let G be an Abelian group, $\phi \in \mathrm{End}(G)$ and F be a finite subset of G.

(i) If all elements of F are quasi-periodic, then $\tau_{F,\phi}(n)$ has polynomial growth. Thus, $H_A(\phi, F) = 0$.
(ii) Consequently,
  - $h_A(\phi) = 0$, if ϕ has polynomial growth.
  - $h_A(\phi \restriction_{\ker_\infty(\phi)}) = 0$, as all the elements in $\ker_\infty(\phi)$ are trivially quasi-periodic.

Every endomorphism ϕ of an Abelian group G admits:

- a greatest ϕ-invariant subgroup $\mathbf{P}(G, \phi)$ of G (named *Pinsker subgroup* of ϕ) such that $h_A(\phi \restriction_{\mathbf{P}(G,\phi)}) = 0$;
- a greatest ϕ-invariant subgroup $Pol(G, \phi)$ of G such that $\phi \restriction_{Pol(G,\phi)}$ has polynomial growth;
- a smallest ϕ-invariant subgroup $\mathfrak{Q}(G, \phi)$ of G such that the induced endomorphism of $G/\mathfrak{Q}(G, \phi)$ has no quasi-periodic points.

Using Example 3.7, one can immediately see that $\mathbf{P}(G, \phi) \supseteq Pol(G, \phi)$ and both $Pol(G, \phi)$ and $\mathbf{P}(G, \phi)$ contain all quasi-periodic points. The last fact, combined with an easy iterated application of the Addition Theorem [9] gives $\mathbf{P}(G, \phi) \supseteq \mathfrak{Q}(G, \phi)$. Another iterated argument, used in [9] proves the less trivial inclusion $Pol(G, \phi) \supseteq \mathfrak{Q}(G, \phi)$, which gives the chain of inclusions

$$\mathbf{P}(G, \phi) \supseteq Pol(G, \phi) \supseteq \mathfrak{Q}(G, \phi). \tag{18}$$

The inclusion $\mathbf{P}(G, \phi) \subseteq \mathfrak{Q}(G, \phi)$ follows from the next theorem, proved in [9] with a substantial use of the Algebraic Yuzvinski Formula:

Theorem 3.8. *Let G be a non-trivial Abelian group and $\phi \in \mathrm{End}(G)$. If $h_A(\phi) = 0$ then ϕ has non-trivial quasi-periodic points.*

Using the equality $\mathbf{P}(G, \phi) = \mathfrak{Q}(G, \phi)$ and the chain of inclusions (18), one obtains:

Theorem 3.9. *([9, Main Theorem]) $\mathfrak{Q}(G, \phi) = Pol(G, \phi) = \mathbf{P}(G, \phi)$ for every $(G, \phi) \in \mathrm{Flow}(\mathfrak{D}\mathfrak{A})$.*

Following [9], we say that $\phi \in \mathrm{End}(G)$

– is *algebraically ergodic*, if $\mathfrak{Q}(G,\phi) = \{0\}$;
– has *completely positive algebraic entropy* (and we write $h_A(\phi) \gg 0$), if $h_A(\phi\restriction_H) = 0$ yields $H = \{0\}$ for every ϕ-invariant subgroup H of G.

Clearly, $h_A(\phi) \gg 0$ if and only if $\mathbf{P}(G,\phi) = \{0\}$. So Theorem 3.6 implies the following algebraic version of Rohlin's Theorem [39] concerning the uniform entropy:

Theorem 3.10. *$\phi \in \mathrm{End}(G)$ is algebraically ergodic if and only if ϕ has completely positive algebraic entropy.*

Actually, the following more precise statement can be deduced from the previous results:

Corollary 3.11. *Let $(G,\phi) \in \mathrm{Flow}(\mathfrak{DA})$ and denote by $\bar\phi : G/\mathbf{P}(G,\phi) \to G/\mathbf{P}(G,\phi)$ the induced endomorphism. Then:*

(i) $h_A(\bar\phi) \gg 0$;
(ii) $\mathbf{P}(G/\mathbf{P}(G,\phi), \bar\phi) = \{0\}$;
(iii) $h_A(\bar\phi) = h_A(\phi)$.

3.3. Uniform entropy on compact groups

The uniform entropy in the realm of topological groups has been studied mainly for compact groups where it is quite well understood (for uniform entropy of continuous endomorphisms on a totally bounded topological group see [4]). In this section we try to give an overview of the main results.

3.3.1. Uniqueness of the uniform entropy for compact groups

The following uniqueness theorem for the entropy $h_U(-)$ on compact groups has been proved by Stojanov [43]. It provides a complete characterization of $h_U(-)$ with a set of eight axioms.

Theorem 3.12 *(Uniqueness Theorem). ([43]) The uniform entropy $h_U : \mathrm{Flow}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ is the unique function such that*

(C1) *$h_U(-)$ is monotone, that is, if $(G,\phi), (H,\psi) \in \mathrm{Flow}(\mathfrak{C})$ and there exists a continuous monomorphism $\alpha : G \to H$ such that $\alpha\phi = \psi\alpha$, then $h_U(\phi) \leqslant h_U(\psi)$;*
(C2) *if $\phi \in \mathrm{Flow}(\mathfrak{C})$, then $h_U(\phi) = h_U(\phi\restriction_{E_\phi(G)})$, with $E_\phi(G)$ as in Remark 2.1;*
(C3) *$h_U(-)$ satisfies the logarithmic law;*
(C4) *$h_U(\varphi_a) = 0$ for every inner automorphism φ_a of G;*
(C5) *$h_U(-)$ is continuous on inverse limits;*
(C6) *$h_U(-)$ satisfies AT;*
(C7) *if $G = F^{\mathbb{Z}}$ for some finite group F and $\phi = \beta_F$ is the two-sided Bernoulli shift, then $h_U(\phi) = \log|F|$;*
(C8) *if $G = \mathbb{K}^n$ for some positive integer n, then $h_U(\phi) = m(\phi)$ is the Mahler measure of ϕ.*

This theorem provides another way to prove that the uniform entropy of a surjective continuous endomorphism of a compact group coincides with its Kolmogorov–Sinai entropy $h_{mes}(-)$ (possessing the same properties). For the coincidence of $h_U(-)$ and $h_{mes}(-)$ we refer to [5,6] (or [54] in the totally disconnected case).

We pass now to discuss the various axioms. It is not difficult to see that $h_U(-)$ satisfies (C1), (C3), (C4), and (C7), for a proof we refer to [52] (for (C4) see also Remark 2.25(b)). We conjecture that axiom (C3) can be replaced by the following weaker one:

(C3*) if $\phi \in \mathrm{Flow}(\mathfrak{C})$ has $h_U(\phi^n) = 0$ for some $n \in \mathbb{N}_+$, then $h_U(\phi) = 0$.

We are not aware if there exists a function $h' : \mathrm{Flow}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ satisfying axioms (C1)–(C2) and all axioms (C4)–(C8), such that (C3*) fails.

The proof of (C5) can be found in [43, Proposition 2.1]. Furthermore, a partially extended form for LCA groups of the continuity on inverse limits is given in [37, Corollary 11]. Nevertheless the proof heavily uses the non-proved Addition Theorem for LCA groups (see Section 4.2). Thus it is fair to say that the validity of this extension is still open. According to [43, Lemma 4.5], property (C5) allows for a reduction of the computation of the uniform entropy to the case of metrizable compact groups.

The Addition Theorem (C6) has been proved by Yuzvinski [56] in the metrizable case, but can be derived also from a more general result of Bowen [7] always in the metric case. The case of arbitrary compact groups, can be easily deduced from the metric case using (C5).

Axiom (C2) is simply a reduction to *surjective* endomorphisms (an application of the Addition Theorem (C6) and the continuity on inverse limits (C5) shows that (C2) holds for $h_U(-)$). To the effect of this property one can note that if F is a finite group, then the right Bernoulli shift $\rho_F : F^{\mathbb{N}} \to F^{\mathbb{N}}$ is injective, but not surjective and $E_{\rho_F}(F^{\mathbb{N}}) = 0$, hence $h_U(\rho_F) = 0$. Furthermore, it turns out that axiom (C2) can be replaced by a much weaker one:

Theorem 3.13. *([15]) Theorem* 3.12 *remains true whenever axiom* (C2) *is replaced by*:

(C2*) $h_U(\zeta_L) = 0$ *for all simple compact connected Lie groups* L *and the trivial endomorphism* $\zeta_L : L \to L$.

Moreover, there exists a function $h' : \mathrm{Flow}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *satisfying axiom* (C1) *and all axioms* (C3)–(C8)*, such that* $h'(\zeta_L) = \infty$ *for all simple compact connected Lie groups* L.

The proof of (C8) firstly appeared in [56], a different proof can be found in [32]. It can be shown that if in (C8) one takes only endomorphisms of $\mathbb{T}^n$ (instead of $\mathbb{K}^n$), then uniqueness fails [43].

In the rest of this subsection we show that the computation of the uniform entropy of a continuous endomorphism ϕ of a compact group G can always be reduced to the case when ϕ is an automorphism (compare with the last part of Section 3.2.1). Firstly one reduces to *surjective* endomorphisms according to (C2). Secondly one reduces to *automorphisms*. Namely, let G be a compact group and $\phi : G \to G$ a continuous surjective endomorphism. Then $K_\phi(G) := \overline{\ker_\infty \phi}$ is a ϕ-invariant closed normal subgroup of G with $\phi^{-1}(K_\phi(G)) = K_\phi(G)$, in other words the induced endomorphism $\bar{\phi} : G/K_\phi(G) \to G/K_\phi(G)$ is an automorphism. In particular, $K_{\bar{\phi}}(G/K_\phi(G)) = \{e\}$, while the restriction $\phi_1 := \phi \restriction_{K_\phi(G)}: K_\phi(G) \to K_\phi(G)$ is surjective and has dense hyperkernel, i.e., $K_{\phi_1}(K_\phi(G)) = K_\phi(G)$. The next example shows that $h_U(\phi_1) > 0$ (i.e., $h_U(\phi) > h_U(\bar{\phi})$ due to the Addition Theorem) unless ϕ_1 is trivial. In other words, this reduction step *is not supposed to preserve the entropy*. Yet it is useful since the formula $h_U(\phi) = h_U(\bar{\phi}) + h_U(\phi_1)$ allows for a reduction to separate computation of the entropy in these two cases.

Example 3.14.

(i) Let F be a finite non-trivial group. It is easy to see that the left Bernoulli shift $\lambda_F : F^{\mathbb{N}} \to F^{\mathbb{N}}$ has dense hyperkernel. Since $h_U(\lambda_F) = \log|F| > 0$, we deduce that a surjective endomorphism of a compact group with dense hyperkernel need not necessarily have zero uniform entropy.

(ii) This phenomenon can be formulated in a more precise form as follows: *If* G *is a non-trivial compact totally disconnected group and* $\phi : G \to G$ *is a continuous surjective endomorphism with* $K_\phi(G) = G$*, then* $h_U(\phi) > 0$. This follows from the following more precise property established by Stojanov [43, Corollary 3.3] in this situation. Namely, for every open normal subgroup M of G such that G/M is simple there exists a closed normal subgroup H of G with $\phi(H) = H$ such that the induced endomorphism $\bar{\phi} : G/H \to G/H$ is conjugated to $\lambda_{G/M}$. Since $h_U(\phi) \geqslant h_U(\bar{\phi}) = \log|G/M|$, the above claim becomes obvious. On the other hand, such an open normal subgroup M of G obviously exists, as G is totally disconnected.

3.3.2. Dynamical aspects: ergodic automorphisms

Definition 3.15. A continuous surjective endomorphism $\phi : G \to G$ of a compact group G, equipped with its Haar measure μ, is said to be *ergodic*, if one of the following two equivalent properties is satisfied:

(i) $\phi^{-1}B = B$ implies $\mu(B) = 0$ or $\mu(B) = 1$ for every measurable set B;

(ii) for every pair A, B of measurable subsets of G with $\mu(A) > 0$ and $\mu(B) > 0$ there exists $n \in \mathbb{N}$ such that $\mu(\phi^{-n}(A) \cap B) > 0$.

If $\phi : G \to G$ is ergodic, then for every ϕ-invariant closed normal subgroup N of G the induced endomorphism of G/N is ergodic as well.

Let us recall the following criterion, due to Halmos and Rohlin [51, Theorem 1.10]: *an automorphism* ϕ *of a compact Abelian group* G *is ergodic if and only if its Pontryagin dual* $\widehat{\phi}$ *has no periodic points beyond* 0. Therefore, one obtains:

Fact 3.16. ([9]) A continuous surjective endomorphism ϕ of a compact Abelian group G is ergodic if and only if its Pontryagin dual $\widehat{\phi}$ is algebraically ergodic, i.e., $\mathfrak{Q}(\widehat{\phi}, \widehat{G}) = 0$.

The Abelian case of the next example follows from Halmos–Rohlin's criterion:

Example 3.17. ([43, Lemma 3.5]) A continuous automorphism ϕ of a compact totally disconnected group G is ergodic if and only if there exists no proper open normal ϕ-invariant subgroup of G.

One can show that every continuous automorphism ϕ of a compact totally disconnected group G admits a largest closed ϕ-invariant subgroup $\mathscr{E}(\phi)$ such that the restriction $\phi \restriction_{\mathscr{E}(\phi)}$ is ergodic:

Proposition 3.18. *([43, Proposition 3.4]) Let G be a compact totally disconnected group and let $\phi : G \to G$ be a continuous automorphism. There exists a transfinite sequence $\{N_\alpha\colon \alpha \leqslant \tau\}$ of closed ϕ-invariant subgroups such that*

(i) *$N_0 = G$ and $N_{\alpha+1}$ is open in N_α for $\alpha < \tau$;*
(ii) *$N_\beta = \bigcap_{\alpha<\beta} N_\alpha$ for all limit ordinals $\beta \leqslant \tau$;*
(iii) *$\phi\restriction_{N_\tau}$ is ergodic (i.e., $\mathscr{E}(\phi) = N_\tau$).*

A proof in the Abelian case can be obtained using the forthcoming Bridge Theorem 4.10 and [16, Proposition 3.13]. For a version of this theorem in the Abelian case, where total disconnectedness is completely removed, see Theorem 4.11.

The structure of the ergodic restriction $\phi\restriction_{\mathscr{E}(\phi)}$ is described by the following theorem of Yuzvinski:

Theorem 3.19. *([56, Theorem 11.7]) If ϕ is an ergodic automorphism of a totally disconnected compact metrizable group G, then there exists a decreasing sequence $G = G_0 \supseteq G_1 \supseteq \cdots$ of ϕ-invariant closed subgroups such that*

(i) *$\bigcap_n G_n = \{e\}$;*
(ii) *G_{n+1} is a normal subgroup of G_n for every $n \in \mathbb{N}$;*
(iii) *the automorphism of G_n/G_{n+1} induced by ϕ is a simple Bernoulli automorphism.*

As a consequence of Theorem 3.19, one can see that every ergodic automorphism ϕ of a totally disconnected compact group G is densely periodic (i.e., the set of periodic points of ϕ is dense in G) [56, 11.8].

Example 3.20. One may notice that ergodic automorphisms of non-Abelian connected compact groups did not appear so far. Indeed, Wu [54] proved that a finite-dimensional compact connected group admitting ergodic automorphisms must necessarily be Abelian. On the other hand, this is not true in the infinite-dimensional case: the two-sided Bernoulli shift $\beta_K : K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ of any compact connected group K is an ergodic automorphism, yet $K^{\mathbb{Z}}$ may fail to be Abelian (just take K to be a non-Abelian compact connected group).

4. The Bridge Theorem and the Addition Theorem

As already mentioned, the pair (h_∞, h_U) does not satisfy the Bridge Theorem on Casc($\mathfrak{LCA}$). We actually do not know if the counterpart for $h_A(-)$ does hold. We state this question more precisely:

Question 4.1. Does the pair (h_A, h_U) satisfy BT on Flow($\mathfrak{LCA}$)?

The forthcoming Section 4.1 is dedicated to the discussion of the above question. In particular, Question 4.1 is known to have positive answer for the subcategories Flow(vec($\mathbb{R}$)), Flow($\mathfrak{TA}$), and Flow($\mathfrak{DA}$). Clearly the proof for Flow($\mathfrak{DA}$) generalizes the case of Flow($\mathfrak{TA}$). The latter can be easily derived directly from the definitions of the two entropies involved (see Theorem 4.10). The proof in the former case can be found in [11], it is more complicated and makes recourse to the properties (DA1)–(DA5) of the algebraic entropy. There is also a second proof of the BT for Flow($\mathfrak{DA}$), based on the Uniqueness Theorems for $h_A(-)$ and $h_U(-)$ [11]. The Bridge Theorem is very useful as it converts results about $h_U(-)$ into results about $h_A(-)$ and vice versa.

In [38], Peters introduces a new entropy function $h_*(\phi)$ for a topological isomorphism ϕ from an LCA group G onto itself, using the formula

$$h_*(\alpha) = h_U(\alpha) + h_\infty(\alpha). \tag{19}$$

The following claim can be found in [38, Theorem 10]:

Claim 4.2. *([38])*

(i) *$h_U(-)$ satisfies the AT on* Casc($\mathfrak{LCA}$);
(ii) *$h_\infty(-)$ satisfies the AT on* Casc($\mathfrak{LCA}$);
(iii) *$h_*(-)$ satisfies the AT on* Casc($\mathfrak{LCA}$).

Only a proof of part (i) of the above claim is given in [38, Theorem 10] (but it contains various gaps, see Section 4.2.2), then (the false) Claim 3.2 is used to deduce part (ii) and so (iii). Even if the proof of Claim 4.2 does not work, we do not know whether these statements are true or not:

Question 4.3. Does $h_U(-)$ (resp., $h_A(-)$, $h_\infty(-)$) satisfy AT on Flow($\mathfrak{LCA}$)?

In the proof of [32, Lemma 4.5] one can find: “Bowen's definition of entropy is not in general additive over products [51, p. 176]”. But at that place in Walters' book [51] one finds an example of direct product of two self-homeomorphisms of $\mathbb{R}$ that are shifts $x \mapsto x+1$, so notoriously non-homomorphisms. Hence, to the best of our knowledge, Question 4.3 seems to be open for LCA groups even in the case of direct products.

Some cases of the above questions are studied in Section 4.2. First of all, in Section 4.2.1, we use the concept of “*compact-covering*” map to give some general result in the spirit of Question 4.3. Then we analyze the proof of Claim 4.2.

The new entropy function $h_{A+U}(-)$ in the next question is motivated by Peters' definition (19). Due to the problems related to $h_\infty(-)$ we hope that $h_{A+U}(-)$ can be the correct substitute for $h_*(-)$.

Question 4.4. Let $h_{A+U} : \mathrm{Flow}(\mathfrak{LCA}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be defined as $h_{A+U}(\phi) = h_A(\phi) + h_U(\phi)$ for every $\phi \in \mathrm{Flow}(\mathfrak{LCA})$. Is it possible to find a set of axioms to characterize $h_{A+U}(-)$ with some Uniqueness Theorem? In particular, does $h_{A+U}(-)$ satisfy the AT?

Taking into account the role of the duality theory for Abelian (locally) compact groups in the previous results, the question arises whether the previous results can be improved in the realm of non-Abelian compact groups. The first step is to choose a suitable duality framework with a suitable notion of entropy for the *dual objects*.

Problem 4.5. Find a counterpart of BT in the case of $\mathrm{Flow}(\mathfrak{C})$ using Tannaka–Kreĭn duality for compact not necessarily Abelian groups.

4.1. The Bridge Theorem for some LCA groups

In this section we discuss Question 4.1 in the case of flows on finite-dimensional real vector spaces and on discrete Abelian groups. Furthermore we apply these results to show the failure of Claim 3.2 (see Example 4.8).

Let n be a positive integer, and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous (i.e., $\mathbb{R}$-linear) endomorphism. We start recalling the formulae to compute $h_U(\phi)$ and $h_A(\phi)$. The first equality was proved in [7, Theorem 15], the second in [48, Fact A]:

Theorem 4.6. *Let n be a positive integer, and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous endomorphism. Then we have*

$$h_U(\phi) = \sum_{|\lambda_i|>1} \log|\lambda_i| = h_A(\phi)$$

where $\{\lambda_i\colon i = 1, \ldots, n\} \subseteq \mathbb{C}$ is the family of the eigenvalues of ϕ.

In order to prove that (h_A, h_U) satisfies BT on $\mathrm{Flow}(\mathrm{vec}(\mathbb{R}))$ it suffices to show that, given a continuous endomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ (with $n \in \mathbb{N}_+$), ϕ and $\widehat{\phi}$ have the same eigenvalues. With this reduction and an application of Theorem 4.6, we get the following

Corollary 4.7. (h_A, h_U) *satisfies BT on* $\mathrm{Flow}(\mathrm{vec}(\mathbb{R}))$.

Example 4.8. Given a real number r, we denote by ϕ_r the endomorphism of $\mathbb{R}$ such that $\phi_r(x) = r \cdot x$. Notice that ϕ_r is a topological isomorphism of $\mathbb{R}$ whenever $r \neq 0$. We claim that

$$h_\infty(\phi_r) = h_U(\widehat{\phi_r}) \quad \text{if and only if} \quad r = \pm 1, \quad \text{while } h_A(\phi_r) = h_U(\widehat{\phi_r}) \text{ for every } r \in \mathbb{R}.$$

Indeed, applying Corollary 4.7 we get

$$h_U(\widehat{\phi_r}) = h_A(\phi_r) = \begin{cases} \log|r| & \text{if } |r| > 1; \\ 0 & \text{otherwise;} \end{cases}$$

and this gives the part of our claim regarding (h_A, h_U). To see the first part of the claim we need to recall that

$$h_\infty(\phi_r) = h_A(\phi) - \log\big|\det_{\mathbb{R}}(\phi_r)\big| = h_A(\phi) - \log|r|,$$

this comes by Fact 3.4(i) and Example 3.1(ii). Thus we get

$$h_\infty(\phi_r) = \begin{cases} \log|r| - \log|r| = 0 \neq \log|r| = h_A(\phi) = h_U(\widehat{\phi}) & \text{if } |r| > 1; \\ 0 - \log|r| = 0 = h_A(\phi) = h_U(\widehat{\phi}) & \text{if } |r| = 1; \\ 0 - \log|r| = -\log|r| \neq 0 = h_A(\phi) = h_U(\widehat{\phi}) & \text{otherwise.} \end{cases}$$

Therefore, for the group $\mathbb{R}$, Claim 3.2 fails for all automorphisms beyond $\pm id_{\mathbb{R}}$.

A similar example can be given in the field $\mathbb{Q}_p$ of p-adic numbers. Here the automorphism $\phi : \mathbb{Q}_p \to \mathbb{Q}_p$ defined by $\phi(x) = x/p$ for $x \in \mathbb{Q}_p$ has $h_U(\phi) = h_A(\phi) = \log p$, while $h_\infty(\phi)$. The relevance of this equalities to the above examples comes from the fact that $\widehat{\mathbb{Q}_p} = \mathbb{Q}_p$ and $\widehat{\phi} = \phi$, thus $\log p = h_U(\widehat{\phi}) \neq h_\infty(\phi) = 0$.

The proof of the next consequence of Theorem 3.5 uses the fact that the restriction of $h_U(-)$ to $\mathrm{Flow}(\mathfrak{CA})$ has various strong properties. In particular one needs to know that this restriction of $h_U(-)$ satisfies all the axioms stated in the forthcoming Theorem 4.12.

Theorem 4.9. *([10]) (h_A, h_U) satisfies BT on* $\mathrm{Flow}(\mathfrak{DA})$.

The particular case of this theorem for discrete torsion Abelian groups (and dually for profinite Abelian groups) is due to Weiss [53]. For the reader's convenience we give a short direct proof using Pontryagin duality:

Theorem 4.10. *([53]) (h_A, h_U) satisfies BT on* $\mathrm{Flow}(\mathfrak{TA})$.

Proof. Let G be a discrete torsion Abelian group and $\phi : G \to G$ an endomorphism. Consider also the Pontryagin–Van Kampen dual $\widehat{\phi} : \widehat{G} \to \widehat{G}$. It is well known that $\widehat{G}$ is a profinite Abelian group, in particular it is a compact Abelian group. According to Remark 2.25, the uniform entropy of $\widehat{\phi}$ is

$$h_U(\widehat{\phi}) = \sup\{k(\widehat{\phi}, U):\ U \text{ is an open subgroup of } \widehat{G}\}. \tag{20}$$

Moreover, by (15)

$$k(\widehat{\phi}, U) = \limsup_{n\to\infty} \frac{\log[G : C_n(\widehat{\phi}, U)]}{n}. \tag{21}$$

Furthermore we already noticed that

$$h_A(\phi) = \mathrm{ent}(\phi) = \sup\{H_A(\phi, F):\ F \text{ is a finite subgroup of } G\}. \tag{22}$$

The annihilator $F = U^\perp$ is a finite subgroup of G for every open subgroup U of $\widehat{G}$. Furthermore,

$$T_n(\phi, F) = F + \cdots + \phi^{n-1}(F) = C_n(\widehat{\phi}, U)^\perp \quad \text{and} \quad T_n(\phi, F) \cong \widehat{G}/C_n(\widehat{\phi}, U).$$

Hence $|T_n(\phi, F)| = |G : C_n(\widehat{\phi}, U)|$ for all $n \in \mathbb{N}_+$. After taking logarithms, dividing by n and taking limits, we get $H_A(\phi, F) = k(\widehat{\phi}, U)$. To finish the proof it suffices to note that when U runs over the family of all open subgroups the subgroup $F = U^\perp$ runs over the family of finite subgroup of G (as $U = F^\perp$) and apply (20), (21) and (22). □

Let G be a compact group, an endomorphism $\phi : G \to G$ has *completely positive uniform entropy* if every non-trivial factor of the endomorphism has positive uniform entropy. Rohlin [39] proved that ergodic automorphisms of compact metric Abelian groups have completely positive uniform entropy. This was extended to arbitrary compact metric groups by Yuzvinski [56]. Indeed, the case of totally disconnected groups easily follows from Theorem 3.19 (since both properties are preserved under taking quotients, it suffices to see that all ergodic automorphisms have positive uniform entropy).

Now, using the Bridge Theorem 4.9, one can see that an automorphism ϕ of a compact Abelian group G has completely positive uniform entropy if and only if its Pontryagin–Van Kampen dual $\widehat{\phi}$ has completely positive algebraic entropy. Therefore, using again Theorem 4.9, we deduce from Theorem 3.10 an alternative proof of Rohlin's Theorem [39].

From Theorem 4.9 and Corollary 3.11 we get the following more precise result:

Theorem 4.11. *([9, Theorem 7.4, Corollaries 7.6, 7.7]) For every continuous automorphism ϕ of a compact* Abelian *group G there exists a closed ϕ-invariant subgroup $\mathscr{E}(\phi)$ of G such that*:

(i) *the restriction $\phi \restriction_{\mathscr{E}(\phi)}$ is ergodic*;
(ii) *if for some closed ϕ-invariant subgroup H of G the restriction $\phi \restriction_H$ is ergodic, then $H \subseteq \mathscr{E}(\phi)$*;
(iii) *the induced endomorphism $\bar{\phi} : K/\mathscr{E}(\phi) \to K/\mathscr{E}(\phi)$ has uniform entropy* 0;
(iv) *if for some closed ϕ-invariant subgroup N of G the induced endomorphism $\bar{\phi} : G/N \to G/N$ has uniform entropy* 0, *then N contains $\mathscr{E}(\phi)$.*

Using the Bridge Theorem 4.9 one obtains from Theorem 3.5 the following admittedly simpler counter-part of Theorem 3.12 in the Abelian case:

Theorem 4.12. *The uniform entropy h_U* : $\mathrm{Flow}(\mathfrak{CA}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *is the unique function such that*:

(CA1) *h_U is invariant under conjugation*;

(CA2) *h_U is continuous on inverse limits*;
(CA3) *h_U satisfies AT*;
(CA4) *if $G = F^{\mathbb{Z}}$ for some finite group F, then $h_U(\beta_F) = \log|F|$*;
(CA5) *if $G = \mathbb{K}^n$ for some positive integer n, then $h_U(\phi) = m(\phi)$ is the Mahler measure of ϕ.*

Let us note that the last four axioms (CA2)–(CA5) coincide with the last four axioms (C5)–(C8) from the Uniqueness Theorem 3.12. Axiom (CA1) is a weaker form of the first axiom (C1) (monomorphisms replaced by isomorphisms). Finally we remark that, while the redundancy of axiom (C4) (as well as the stronger form (C1)) in the Abelian case is obvious, it is less obvious that axioms (C2) and (C3) are obsolete in the Abelian case.

4.2. The Addition Theorem

In this section we discuss Question 4.3. In particular, in Section 4.2.1 we use the notion of "compact-covering" map in order to establish monotonicity of $h_U(-)$ under taking quotients and restrictions to closed invariant normal subgroups. In Section 4.2.2 we discuss Peters's proof of AT for $h_*(-)$ on $\mathrm{Casc}(\mathfrak{L})$.

We start with a direct proof of the Addition Theorem for profinite and discrete torsion Abelian groups:

Theorem 4.13. *$h_A(-)$ satisfies AT on* $\mathrm{Flow}(\mathfrak{TA})$ *and $h_U(-)$ satisfies AT on* $\mathrm{Flow}(\mathfrak{Pro})$.

Proof. The first assertion can be easily deduced by [16, Theorem 3.1] which states that ent satisfies AT on $\mathrm{Flow}(\mathfrak{TA})$, using the fact that $\mathrm{ent}(\phi) = h_A(\phi)$ for every endomorphism ϕ of a discrete torsion Abelian group. Hence we only need to prove the second assertion. We will obtain such proof using Weiss's Theorem 4.10. Let G be a profinite Abelian group, $\phi : G \to G$ be a continuous endomorphism, and H a closed ϕ-invariant subgroup of G. We have to prove that $h_U(\phi) = h_U(\phi\restriction_H) + h_U(\bar{\phi})$, where $\bar{\phi} : G/H \to G/H$ is the map induced by ϕ on the quotient.

Let $X = \widehat{G}$ and $\psi = \widehat{\phi}$. By the Pontryagin–Van Kampen duality, $Y = H^\perp$ is a ψ-invariant subgroup of X, and $Y^\perp = H^{\perp\perp} = H$ (being H closed) and $\widehat{G/H} \cong H^\perp$. Moreover, we have the following commutative diagrams:

$$\begin{array}{ccccc} H & \hookrightarrow & G & \twoheadrightarrow & G/H \\ \downarrow{\scriptstyle \phi\restriction_H} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \bar{\phi}} \\ H & \hookrightarrow & G & \twoheadrightarrow & G/H \end{array} \qquad\qquad \begin{array}{ccccc} X/Y & \twoheadleftarrow & X & \hookleftarrow & Y \\ \uparrow{\scriptstyle \overline{\psi}} & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \psi\restriction_Y} \\ X/Y & \twoheadleftarrow & X & \hookleftarrow & Y. \end{array}$$

The second diagram is obtained by the first one applying the Pontryagin–Van Kampen duality functor. Applying Weiss's Theorem 4.10 we obtain

$$h_U(\phi) = \mathrm{ent}(\psi), \qquad h_U(\phi\restriction_H) = \mathrm{ent}(\overline{\psi}) \quad \text{and} \quad h_U(\bar{\phi}) = \mathrm{ent}(\psi\restriction_Y). \tag{23}$$

By the already proved AT in the discrete case (see also [16, Theorem 3.1]) one has

$$\mathrm{ent}(\psi) = \mathrm{ent}(\overline{\psi}) + \mathrm{ent}(\psi\restriction_Y). \tag{24}$$

Hence (23)–(24) together give the desired equality $h_U(\phi) = h_U(\phi\restriction_H) + h_U(\bar{\phi})$. □

4.2.1. Homomorphisms allowing lifting of compact sets

In this subsection we recall the concept of *compact-covering* map introduced in [33]. For applications of this concept to uniform entropy on spaces we refer to [31].

Definition 4.14. ([33]) A mapping $\phi : E \to F$ of topological spaces is *compact-covering* if each compact subset of F is the image of some compact subset of E.

Example 4.15. Here are three natural examples of compact-covering homomorphisms.

(i) Let G, G_1 be LCA groups. Then every continuous, open and surjective homomorphism $\phi : G \to G_1$ is compact-covering. Indeed, this seems to be widely known (see for example [18]).
(ii) Let G be a topological group and H an open normal subgroup of G. Then the natural projection $\pi : G \to G/H$ is compact-covering.
(iii) Let G be a topological group isomorphic to the direct product of two closed subgroups G_1 and G_2. Then both projections $\pi_i : G \to G_i$, $i = 1, 2$, are compact-covering.

In the following lemma we collect some monotonicity properties owned by the uniform entropy. We remark that part (iv) of Lemma 4.16 remains true also for arbitrary compact-covering uniformly continuous maps between uniform spaces, where the concept of "quotient over a subgroup" has to be substituted by "quotient over a suitable relation", for this kind of approach we refer to [31, Section 3].

Lemma 4.16. *Let G be a topological group and let $\phi : G \to G$ be a continuous endomorphism of G. Assume that H is a ϕ-invariant closed normal subgroup of G. Then*:

(i) *if $K \in \mathcal{K}(H)$ and U is a neighborhood of the identity in G, then every finite subset F of K that is $(n, U \cap H)$-separated with respect to $\phi \restriction_H$ is also (n, U)-separated with respect to ϕ;*
(ii) $h_U(\phi) \geqslant h_U(\phi \restriction_H)$.

Suppose now that the natural projection $\pi : G \to G/H$ is compact-covering and denote by $\bar{\phi} : G/H \to G/H$ the map induced by ϕ. Then:

(iii) *if $\bar{K} \in \mathcal{K}(G/H)$, $\bar{U}$ is a neighborhood of the identity in G/H, and $\bar{F}$ is a finite subset of $\bar{K}$ that is $(n, \bar{U})$-separated with respect to $\bar{\phi}$, then there exist $K \in \mathcal{K}(G)$, a neighborhood U of the identity in G, and a finite subset F of K such that $\pi(K) = \bar{K}$, $\pi(U) \subseteq \bar{U}$, $\pi(F) = \bar{F}$ and F is (n, U)-separated with respect to ϕ;*
(iv) $h_U(\phi) \geqslant h_U(\bar{\phi})$.

Proof. (i) Suppose that for $x, y \in F$ one has $y^{-1}x \notin \alpha^{-j}(U \cap H)$ for some j. Then $y^{-1}x \notin \alpha^{-j}(U) \cap H$, as H is invariant. Since $y^{-1}x \in H$, this yields $y^{-1}x \notin \alpha^{-j}(U)$.

(ii) easily follows from (i).

(iii) Since π is compact covering, there exists $K \in \mathcal{K}(G)$ such that $\pi(K) = \bar{K}$, and we can always find a neighborhood U of the identity in G such that $\pi(U) \subseteq \bar{U}$. Now choose a finite subset $F \subseteq G$ such that $\pi(F) = \bar{F}$ and $\pi \restriction_F$ is injective. We need to prove that F is (n, U)-separated. Choose $x, y \in F$, and assume $\phi^j(x)^{-1} \cdot \phi^j(y) \in U$ for every $0 \leqslant j < n$. Then clearly

$$\bar{\phi}^j\big(\pi(x)\big)^{-1} \cdot \bar{\phi}^j\big(\pi(y)\big) = \bar{\phi}^j \pi\big(x^{-1} \cdot y\big) = \pi \phi^j\big(x^{-1} \cdot y\big) \in \pi(U) \subseteq \bar{U} \quad \text{for every } 0 \leqslant j < n.$$

Since $\bar{F}$ is $(n, \bar{U})$-separated, we deduce that $\pi(x) = \pi(y)$. Hence $x = y$ by our choice of F.

(iv) easily follows from (iii). □

Now we give a more precise version of (ii) when the subgroup H is open:

Corollary 4.17. *Let G be topological group and let $\phi : G \to G$ be a continuous endomorphism of G. Then for every ϕ-invariant open subgroup H of G one has $h_U(\phi) = h_U(\phi \restriction_H)$ and $h_U(\bar{\phi}) = 0$ for the induced map $\bar{\phi} : G/H \to G/H$. Hence the Addition Theorem holds in this case.*

Proof. By Example 4.15(ii) and Lemma 4.16(ii) we get $h_U(\phi) \geqslant h_U(\phi \restriction_H)$; we have to prove the converse inequality. Notice that for every $K \in \mathcal{K}(G)$ there exists $C \in \mathcal{K}(H)$ and a finite subset F of G such that $K \subseteq F \cdot C$ (just use the discreteness of G/H to prove that $\pi(K)$ has to be finite). Let us see that

$$r(U, F \cdot C, \phi) \leqslant r(U \cap H, C, \phi \restriction_H). \tag{25}$$

Let $n \in \mathbb{N}$ and assume that F_1 is an $(n, U \cap H)$-spanning subset of C. Then $F_2 = F \cdot F_1$ is obviously an (n, U)-spanning subset of $F \cdot C$. Hence $r_n(U, F \cdot C, \phi) \leqslant |F| \cdot r_n(U \cap H, C, \phi \restriction_H)$. Taking logarithms, dividing by n and taking the lim sup with respect to n we get (25).

Since G/H is discrete, $h_U(\bar{\phi}) = 0$ follows from Example 2.15. The last assertion is the obvious equality $h_U(\phi) = h_U(\phi \restriction_H) + h_U(\bar{\phi})$. □

Applying Example 4.15(i) to Lemma 4.16(iv) we easily get the following

Corollary 4.18. *Let G be an LCA group, $\phi : G \to G$ a continuous endomorphism, and H a ϕ-invariant closed subgroup. Then $h_U(\phi) \geqslant h_U(\bar{\phi})$, where $\bar{\phi} : G/H \to G/H$ is the map induced on the quotient by ϕ.*

The following question remains open:

Question 4.19. Is Corollary 4.18 true for arbitrary topological (Abelian) groups?

4.2.2. The proof of Claim 4.2

Here we discuss the proof of Claim 4.2 (given as Theorem 10 in [38, Section 10]). Only a proof of part (i) is given in [38], then (the erroneous) Claim 3.2 is used to deduce part (ii) and so (iii).

The proof of part (i) itself contains various gaps that we are going to analyze using the notation of [38]. Here is the statement of part (i) in that notation. If G is an LCA group, $\alpha : G \to G$ is a topological isomorphism of G and H is a closed subgroup of G with $\alpha(H) = H$, then one has

$$h_U(\alpha, G) = h_U(\alpha_1, H) + h_U(\alpha_2, G/H) \tag{26}$$

where α_1 is the restriction of α to H and α_2 is the map induced by α on the quotient G/H.

Let $\phi : G \to G/H$ be the projection. The first error comes in the proof of the inequality '$\geqslant$' of (26). In fact the argument suffers the same problem as the erroneous proof of Proposition 4 from [7] (see [8]). In fact, at the end of p. 484 in [38], the author deduces $h_U(\alpha, G) \geqslant h_U(\alpha_1, H) + h_U(\alpha_2, G/H)$ from the correct inequality $s_n(U, K + C) \geqslant s_n(U \cap H, C)s_n(\phi(U), \phi(K))$. This deduction cannot be done in general since lim sup does not respect addition.

We now pass to the second inequality '$\leqslant$' in (26). First of all, if a finite set F (n, U)-spans a compact subset K, then one has to consider images (as in the definition of (n, U)-spanning), not inverse images. So the correct writing in [38, line 2, p. 485] must be

$$\text{for every } y \in K \text{ there exists } x \in F \text{ with } \alpha^j(y) - \alpha^j(x) \in U \text{ for } 0 \leqslant j < n, \text{ i.e. } K \subseteq F + \bigcap_{j=0}^{n-1} \alpha^{-j}(U),$$

while the wrong differences $\alpha^{-j}(y) - \alpha^{-j}(x)$ for $0 \leqslant j < n$ appear in the paper. Nevertheless, since α is an automorphism, one can formally argue also with α^{-j} in place of α^j, using the fact that the next statement is invariant under this change. On [38, lines 2–3, p. 485] one can find the claim that

$$\alpha^{-j}(y) - \alpha^{-j}(x) \in H + U \quad \text{for } 0 \leqslant j < n, \text{ is equivalent to } y - x \in H + \bigcap_{j=0}^{n-1} \alpha^j(U). \tag{27}$$

In other words, $\bigcap_{j=0}^{n-1}(H + \alpha^j(U)) = H + \bigcap_{j=0}^{n-1} \alpha^j(U)$. This equality is wrong since these two sets need not coincide, a counter-example is available already when both G and H are compact (so one can take $K = G$ and $C = H$ in [38, Section 10]) and $n = 2$. To further simplify the situation, take U to be an open subgroup of G (so that $U + U = U$). Using the (failing) equivalence (27), the author deduces that whenever F_1, F_2 are finite sets of G such that

- H is (n, U)-spanned by F_1; and
- G/H is $(n, \phi(U))$-spanned by $\phi(F_2)$ with respect to α_2;

then

$$F = F_2 + F_1, \quad (2, U)\text{-spans } G. \tag{28}$$

Finally, from the last claim the inequality

$$r_n(U, G) \leqslant r_n\big(\phi(U), G/H\big)r_n(U \cap H, H) \tag{29}$$

is deduced [38, line 17, p. 485]. The next example shows that (27), (28) and (29) are all wrong.

Example 4.20. Let $K = \mathbb{Z}/2\mathbb{Z}$, $G = K^{\mathbb{Z}}$ and let $\alpha = \beta_K : G \to G$ be the two-sided Bernoulli shift of Definition 2.2(ii). Let $U = \{x = (x_i) \in G\colon\ x_0 = 0\}$ and let

$$H = \big\{x = (x_i) \in G\colon\ x_i = x_j \text{ for all } i, j \in \mathbb{Z}\big\}$$

be the diagonal subgroup of G. Then $[G : U] = [G : \alpha(U)] = |H| = 2$, while $r_2(U, G) = [G : U \cap \alpha(U)] = 4$. Moreover,

$$H \cap U = H \cap \alpha(U) = H \cap \alpha^{-1}(U) = \{0\} \quad \text{and} \quad U + H = G = \alpha(U) + H = \alpha^{-1}(U) + H,$$

so these sums are direct. In particular, $\phi(U) = G/H$ and $\alpha_2 = id_{G/H}$, as $\beta_K(x) - x \in H$ for all $x \in G$. Hence, $r_2(\phi(U), G/H) = 1$, while $r_2(U \cap H, H) = 2$. So (29) fails for $n = 2$. To see that (27) fails as well, observe that

$$H + \bigcap_{j=0}^{1} \alpha^j(U) = H + \big(U \cap \alpha(U)\big) \neq G = \bigcap_{j=0}^{1}\big(H + \alpha^j(U)\big) = (H + U) \cap \big(H + \alpha(U)\big). \tag{30}$$

Finally, for the subset $F_2 = \{0\}$ of G the set $\phi(F_2) = \{0\}$ $(2, \pi(U))$-spans G/H, as $\phi(U) \cap \alpha_2^{-1}(\phi(U)) = G/H$. Let $F_1 = H$. Then obviously F_1 $(2, U \cap H)$-spans H. Nevertheless, (30) implies that $F = F_2 + F_1 = H$ does not $(2, U)$-span G and so also (28) fails.

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Appendix A. A claim of Yuzvinski, algebraic *L*-entropy and multiplicity

For the entropy function ent(−) defined in (17), Yuzvinski claims at the end of his paper [56], that for every torsion Abelian group G and every $\phi \in \mathrm{End}(G)$ one has

$$\mathrm{ent}(\phi) = \sup\left\{\log\left|\frac{T(\phi,F)}{\phi(T(\phi,F))}\right|:\ F \text{ a finite subgroup of } G\right\}, \tag{A.1}$$

where, as usual, $T(\phi, F)$ denotes the smallest ϕ-invariant subgroup of G containing F. This formula is *false* without the assumption that ϕ is injective, as the following example shows.

Example A.1. Let G be a non-zero torsion Abelian group, and $\phi : G \to G$ be the zero endomorphism. Take any non-zero finite subgroup F; then $T(\phi, F) = F$ and $\phi(T(\phi, F)) = 0$. So that the right-hand side of (A.1) is greater than or equal to $\log|T(\phi, F)/\phi(T(\phi, F))| = \log|F|$, while $\mathrm{ent}(\phi) = 0$. In particular, when G is an *infinite* torsion Abelian group, for every $n \in \mathbb{N}_+$ we can pick a finite subgroup F_n of G of size $\geqslant n$. Then the right-hand side of (A.1) is $\geqslant \sup_n \log n = \infty$. So one has $0 = \mathrm{ent}(\phi) \neq \infty$ in (A.1).

Obviously, the gap in Example A.1 is due to the (drastic) choice of the zero endomorphism. In fact, Yuzvinski's claim is true for *injective* endomorphisms:

Claim A.2 *(Yuzvinski). Let G be a torsion Abelian group and let ϕ be an injective endomorphism of G. Then* $\mathrm{ent}(\phi)$ *can be computed using the formula in* (A.1)*.*

We now obtain a proof of this claim by showing that this is only a very particular case of a result in [41] connecting a general notion of algebraic entropy to the *multiplicity* of an endomorphism introduced in [35] and generalized in [48]. Indeed, the above Claim A.2 is a particular case of the forthcoming Corollary A.14.

In the sequel let R be a unitary and associative ring, let $\mathrm{Mod}(R)$ be the category of unitary right R-modules and let $\mathscr{F}_R$ be the class of all finitely generated right R-modules. We often say R-module or simply module to mean right R-module. For $M \in \mathrm{Mod}(R)$ let $\mathscr{F}_R(M)$ (or simply $\mathscr{F}(M)$, when there is no place for consfusion) denote the family of all finitely generated R-submodules of M. In this context there is a first useful simplification:

Theorem A.3. *([10,41]) The category* $\mathrm{Flow}(\mathrm{Mod}(R))$ *is isomorphic to* $\mathrm{Mod}(R[X])$*.*

The above isomorphism is given by $(M, \phi) \mapsto M_\phi \in \mathrm{Mod}(R[X])$, where M_ϕ as an R-module is just M and X acts on M_ϕ via ϕ. Also a morphism in $\mathrm{Flow}(\mathrm{Mod}(R))$ is easily seen to commute with the action of X and thus it becomes an $R[X]$-homomorphism. In what follows we will identify the categories $\mathrm{Flow}(\mathrm{Mod}(R))$ and $\mathrm{Mod}(R[X])$, denoting by M_ϕ a generic object. We shall sometimes abuse notation, denoting a ϕ-invariant submodule N of M with the structure of $R[X]$-module induced by the restriction of ϕ to N simply by N_ϕ.

Finally we remark that given a full subcategory $\mathfrak{D}$ of $\mathrm{Mod}(R)$ and denoting by $\mathfrak{D}[X]$ the category of $R[X]$-modules whose underlying R-module is in $\mathfrak{D}$, a restriction of the isomorphism given in Theorem A.3 gives $\mathfrak{D}[X] \cong \mathrm{Flow}(\mathfrak{D})$.

A.1. Invariants and length functions

Given a ring R, an *invariant* of $\mathrm{Mod}(R)$ is a map $i : \mathrm{Mod}(R) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ such that $i(0) = 0$ and $i(M) = i(M')$ whenever $M \cong M'$.

To be able to deal with the invariants of $\mathrm{Mod}(R)$, without any assumption on the structure of the ring R, we impose three strong hypotheses considered in [36,48,42]. More precisely, an invariant i of $\mathrm{Mod}(R)$ is said to be:

(i) *additive* if for every exact sequence $0 \to A \to B \to C \to 0$, we have $i(B) = i(A) + i(C)$;
(ii) *upper continuous* if for every $M \in \mathrm{Mod}(R)$, $i(M) = \sup_{F \in \mathscr{F}(M)} i(F)$;
(iii) *discrete* if the set of finite values of i is order-isomorphic to $\mathbb{N}$.
(iv) *faithful*, if $i(M)$ for some $M \in \mathrm{Mod}(R)$ yields $M = 0$.

Now we come to the main definition of this subsection:

Definition A.4. ([36]) An additive upper continuous invariant is said to be a *length function*.

The first appearance of the concept of length function on a category of modules was in the paper [36] due to Northcott and Reufel. The length functions were deeply investigated also by Vámos in [48,46] and recently by Zanardo in [57].

The most investigated examples of discrete length functions are the composition length $l(M)$ of an R-module M, for R an arbitrary ring; the rank $\mathrm{rk}(M)$ of an R-module M over an integral domain R; and $\log|M|$ when M is an Abelian group. It is understood that the value of these length functions is ∞ whenever it is not finite. It is clear that, given a class $\mathfrak{D} \subseteq \mathrm{Mod}(R)$, it makes sense to speak about length functions on $\mathfrak{D}$, provided $\mathfrak{D}$ is closed under isomorphisms, submodules and quotients.

A.2. Algebraic L-entropy

In this subsection we introduce a notion of algebraic entropy associated with a given length function. Our references for this subsection are [42,41]. It turns out that the theory of length functions and the related notion of algebraic entropy can be described more clearly in the setting of modules over commutative Noetherian rings [45,49].

Given a length function L on $\mathrm{Mod}(R)$, we say that a module F is L-finite if $L(F) < \infty$. We denote by Fin_L the class of L-finite modules. Furthermore, for every module M we let $\mathrm{Fin}_L(M) = \{F \leqslant M \mid L(F) < \infty\}$ be the family of all the L-finite submodules of M.

Let now $\phi : M \to M$ be an endomorphism. For every submodule F of M and every $n \in \mathbb{N}_+$, one can consider the n-th ϕ-trajectory $T_n(\phi, F)$ of F and the ϕ-trajectory $T(\phi, F)$ of F, defined as for Abelian groups.

It is an easy consequence of the additivity of L that $T_n(\phi, F)$ is L-finite for every $n \in \mathbb{N}_+$, provided F is L-finite. Thus it makes sense to define the *L-entropy of ϕ with respect to $F \in \mathrm{Fin}_L(M)$* as

$$H_L(\phi, F) = \lim_{n\to\infty} \frac{L(T_n(\phi, F))}{n}$$

and the *L-entropy of ϕ* as $\mathrm{ent}_L(\phi) = \sup\{H_L(\phi, F) \mid F \in \mathrm{Fin}_L(M)\}$.

It is easily seen that, given an Abelian group G, an endomorphism $\phi : G \to G$ and letting $L = \log|-|$, the definition of $\mathrm{ent}_L(\phi)$ is just the same of $\mathrm{ent}(\phi)$.

We conclude this subsection showing that $\mathrm{ent}_L(-)$ takes the values one could expect in some basic cases.

Example A.5. Let F be an L-finite module.

(i) $\mathrm{ent}_L(\phi) = 0$ for every $\phi : F \to F$ (see [42, Proposition 1.8]).
(ii) Denoting as usual the right Bernoulli shift $\rho_F : F^{\mathbb{N}} \to F^{\mathbb{N}}$, we have $\mathrm{ent}_L(\rho_F \restriction_{F^{(\mathbb{N})}}) = L(F)$ (see [41, Example 2.14]).

A.3. Algebraic L-entropy as a length function

Consider for a while an Abelian group G and an endomorphism $\phi : G \to G$. Clearly every finite subgroup of G is contained in its torsion part $t(G)$ which is ϕ-invariant. Using this argument one could show that $\mathrm{ent}(\phi) = \mathrm{ent}(\phi \restriction_{t(G)})$. This is the reason for which the function $\mathrm{ent}(-)$ is only well behaved on torsion groups.

Returning now to our general situation, that is, given a ring R, a discrete length function L of $\mathrm{Mod}(R)$ and an endomorphism $\phi : M \to M$, we want to find a suitable generalization of the above observation. Indeed, we say that M is *locally L-finite* if every cyclic submodule of M is L-finite, i.e.

$$L(xR) < \infty, \quad \forall x \in M.$$

Equivalently, $\mathscr{F}_R(M) \subseteq \mathrm{Fin}_L$. We denote by lFin_L the subclass of $\mathrm{Mod}(R)$ consisting of the locally L-finite modules. Obviously Fin_L is contained in lFin_L. Furthermore, when $R = \mathbb{Z}$ and $L = \log|-|$ then lFin_L is the class of all torsion Abelian groups. In the class of locally L-finite modules one can prove very strong results for the algebraic L-entropy:

Theorem A.6. *([41,50]) Let R be a ring, $L : \mathrm{Mod}(R) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ a discrete length function, M a locally L-finite module and $\phi : M \to M$ an endomorphism. Then*

(i) $\mathrm{ent}_L(\phi) = \sup\{H_L(\phi, F)\colon F \in \mathscr{F}(M)\}$;
(ii) *if N is a ϕ-invariant submodule of M, then $\mathrm{ent}_L(\phi) = \mathrm{ent}_L(\phi \restriction_N) + \mathrm{ent}_L(\bar\phi)$, where $\bar\phi : M/N \to M/N$ is the induced map.*

The above theorem is very deep as part (i) implies continuity of $\mathrm{ent}_L(-)$ on direct limits and part (ii) is an Addition Theorem for $\mathrm{ent}_L(-)$. To better understand these results it is convenient to pass to the category of flows and so, as we explained at the beginning of this appendix, to $\mathrm{Mod}(R[X])$. More precisely, denote by $\mathrm{lFin}_L[X]$ the class of the $R[X]$-modules that are *locally L-finite* when considered as R-modules.

Now we can change a little bit our point of view about algebraic L-entropy. In fact, we can see $\mathrm{ent}_L(-)$ as a function

$$\mathrm{ent}_L : \mathrm{lFin}_L[X] \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}, \quad M_\phi \mapsto \mathrm{ent}_L(\phi).$$

Viewed in this way, $\mathrm{ent}_L(-)$ is an invariant of $\mathrm{lFin}_L[X]$. Furthermore, the above Theorem A.6 can be rephrased saying that $\mathrm{ent}_L(-)$ is upper continuous (part (i)) and additive (part (ii)), that is, $\mathrm{ent}_L(-)$ is a length function. In this setting we can state the following

Theorem A.7 *(Uniqueness Theorem). ([41,50]) Let R be a ring and* $L : \mathrm{Mod}(R) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *a discrete length function. Then* $\mathrm{ent}_L(-)$ *is the unique length function of* $\mathrm{lFin}_L[X]$ *that satisfies the following two conditions for every* $F \in \mathrm{Fin}_L$*.*

(i) $\mathrm{ent}_L(\phi) = 0$ *for every endomorphism* $\phi \in \mathrm{End}_R(F)$;
(ii) $\mathrm{ent}_L(\rho_F \restriction_{F^{(\mathbb{N})}}) = L(F)$*, where* $\rho_F \restriction_{F^{(\mathbb{N})}} : F^{(\mathbb{N})} \to F^{(\mathbb{N})}$ *is the restriction to* $F^{(\mathbb{N})}$ *of the right Bernoulli shift.*

This theorem was improved recently in [40, 2.1] as follows: if either L is a faithful invariant, or R is an integral domain, then item (i) follows from item (ii), so the theorem can be formulated with item (ii) only.

A.4. Algebraic L-entropy vs multiplicity

In the previous subsection we saw that any discrete length function L on $\mathrm{Mod}(R)$ induces in an essentially unique way a length function on (a suitable subclass of) the category of modules over $R[X]$, namely the algebraic L-entropy $\mathrm{ent}_L(-)$. When the ring R is right Noetherian, there is another way to induce a length function of $\mathrm{Mod}(R[X])$ starting with a length function L of $\mathrm{Mod}(R)$. This new function is called the *multiplicity* of L.

Multiplicity was firstly defined in terms of the composition length of modules in [35, Chapter 7]. The ideas of Northcott where then extended by Vámos in his PhD thesis [46], where he defined the multiplicity of an arbitrary length function. In this subsection we recall such definition, and we show the strong relation existing between the algebraic L-entropy and the multiplicity of L.

Let R be a right Noetherian ring and let L be a length function on $\mathrm{Mod}(R)$. The following remark is very useful when dealing with finitely generated $R[X]$-modules.

Remark A.8. We already noticed that we can consider an $R[X]$-module as an R-module with an endomorphism acting on it. Then is not difficult to see that an $R[X]$-module N_ϕ is finitely generated if and only if $N = T(\phi, F)$ for some $F \in \mathscr{F}_R(N)$.

The first step is to define multiplicity for finitely generated $R[X]$-modules. Indeed, given $N_\phi \in \mathscr{F}_{R[X]}$, we set

$$\mathrm{Mult}_L(N_\phi) = \begin{cases} L(N/\phi(N)) - L(\ker(\phi)) & \text{if } L(N/\phi(N)) < \infty; \\ \infty & \text{otherwise.} \end{cases} \tag{A.2}$$

Now, for an arbitrary $R[X]$-module M_ϕ define

$$\mathrm{Mult}_L(\phi) = \sup\{\mathrm{Mult}_L(N_\phi) \colon N_\phi \in \mathscr{F}_{R[X]}(M_\phi)\}.$$

The first thing to check is that $\mathrm{Mult}_L(-)$ is an invariant. Surprisingly, this is not an easy task. The difficult part is to prove that $\mathrm{Mult}_L(-)$ takes values in $\mathbb{R}_{\geqslant 0} \cup \{\infty\}$; we refer to [46, Chapter 5, Corollary 1] for this verification.

Theorem A.9. *([46]) Let R be a right Noetherian ring and let L be a length function on* $\mathrm{Mod}(R)$*. Then* $\mathrm{Mult}_L : \mathrm{Mod}(R[X]) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *is a length function.*

Proof. We already mentioned that $\mathrm{Mult}_L(-)$ is an invariant. Furthermore, the proof that $\mathrm{Mult}_L(-)$ is upper continuous follows directly from the definition. It remains to verify that $\mathrm{Mult}_L(-)$ is additive.

As we assumed R to be right Noetherian, also $R[X]$ is right Noetherian by Hilbert's Basis Theorem. Then $\mathscr{F}_{R[X]}$ coincides the class of Noetherian $R[X]$-modules. In [46, Chapter 5, Proposition 3] it is proved that the restriction of $\mathrm{Mult}_L(-)$ to $\mathscr{F}_{R[X]}$ is an additive invariant. Now the additivity on the whole $\mathrm{Mod}(R[X])$ follows by [45, Theorem 1] or the following lemma. □

Lemma A.10. *If* $L : \mathscr{F}_R \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *is an additive invariant for a right Noetherian ring R, then the function* $\widehat{L} : \mathrm{Mod}(R) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *defined by*

$$\widehat{L}(M) = \sup\{L(F) \colon F \in \mathscr{F}(M)\},$$

for every $M \in \mathrm{Mod}(R)$*, is a length function.*

Proof. $\widehat{L}$ is an invariant since L is. Furthermore, upper continuity follows by definition. It remains to verify additivity. Indeed, let $N \leqslant M$ and choose $F \leqslant \mathscr{F}(M)$. Since the ring R is right Noetherian, the class $\mathscr{F}_R$ is closed under taking submodules, so

$F \cap N \in \mathscr{F}(N)$ and $(F+N)/N \in \mathscr{F}(M/N)$. By the additivity of L on $\mathscr{F}_R$, we obtain $L(F) = L(F \cap N) + L((F+N)/N)$ which proves that $\widehat{L}(M) \leqslant \widehat{L}(N) + \widehat{L}(M/N)$ by the arbitrariness of F.

On the other hand, let $F_1 \in \mathscr{F}(N)$ and $\overline{F}_2 \in \mathscr{F}(M/N)$. Chose a module $F_2 \in \mathscr{F}(M)$ such that $(F_2+N)/N = \overline{F}_2$ and set $F = F_1 + F_2$. Then

$$L(F) = L(F \cap N) + L\big((F+N)/N\big) \geqslant L(F_1) + L(\overline{F}_2),$$

since $\overline{F}_2 = (F+N)/N$ (being F_1 contained in N) and $F_1 \subseteq F \cap N$. This gives $\widehat{L}(M) \geqslant \widehat{L}(N) + \widehat{L}(M/N)$, concluding the proof. □

As the definition of multiplicity is quite formal, it is useful to give here some example of computation. In the next example we give two general observations, while in the following one we collect some computations for Abelian groups.

Example A.11. Let L be a length function on $\mathrm{Mod}(R)$, M an R-module and $\phi : M \to M$ an endomorphism.

(i) If $M_\phi \in \mathscr{F}_{R[X]}$ and if ϕ is an automorphism, then $\mathrm{Mult}_L(M_\phi) = 0$. Indeed, if ϕ is an automorphism then $M/\phi M = 0$ and $\ker(\phi) = 0$, so $\mathrm{Mult}_L(M_\phi) = 0$.
(ii) If $L(M) < \infty$, then $\mathrm{Mult}_L(\phi) = 0$ (see [41, Lemma 4.2]).

Part (i) of the above example is no longer true if M_ϕ is not finitely generated. That is, there exist R-modules M and automorphisms $\phi : M \to M$ such that $\mathrm{Mult}_L(\phi) > 0$. An example of such situation is part (ii) of the following example.

Example A.12. In this example we let $R = \mathbb{Z}$ and $L = \log|-|$. We also fix a non-zero integer n.

(i) Let $\phi : \mathbb{Z} \to \mathbb{Z}$ be the multiplication by n. Clearly $\mathbb{Z}_\phi \in \mathscr{F}_{\mathbb{Z}[X]}$ and so we can apply directly the definition in (A.2). Hence $\mathrm{Mult}_L(\phi) = \log|\mathbb{Z}/n\mathbb{Z}| - \log|\ker(\phi)| = \log|n|$.
(ii) Part (i) can be generalized as follows. Let G be a non-trivial subgroup of $\mathbb{Q}$ and let $\phi : G \to G$ be the multiplication by n. As we mentioned in Remark A.8, the finitely generated $\mathbb{Z}[X]$-modules of G_ϕ are of the form $T(\phi, F)$ with F a finitely generated subgroup of G. By the choice of the morphism, in our particular case one has $T(\phi, F) = F$, for every $F \leqslant G$. Hence the finitely generated $\mathbb{Z}[X]$-submodules of G_ϕ are all isomorphic to $\mathbb{Z}_\phi$. Applying part (i), we have again $\mathrm{Mult}_L(\phi) = \log|n|$. Note that if the group G is n-divisible (e.g., $G = \mathbb{Q}$), then $\phi : G \to G$ is an automorphism with non-zero multiplicity, provided $n > 1$.
(iii) Consider the endomorphism $\phi : \mathbb{Q} \to \mathbb{Q}$ given by the multiplication by $1/n$, and let F be a subgroup of $\mathbb{Q}$. Then F is ϕ-invariant if and only if it is n-divisible, but then $\phi F = F$ that is, the restriction of ϕ to every ϕ-invariant subgroup of $\mathbb{Q}$ is an automorphism. An application of Example A.11(i) shows that $\mathrm{Mult}_L(F_\phi) = 0$ for every $F_\phi \in \mathscr{F}_{R[X]}(\mathbb{Q}_\phi)$ and so $\mathrm{Mult}_L(\phi) = 0$.
(iv) Let $\phi = \rho_{\mathbb{Z}(n)} \restriction_{\mathbb{Z}(n)^{(\mathbb{N})}} : \mathbb{Z}(n)^{(\mathbb{N})} \to \mathbb{Z}(n)^{(\mathbb{N})}$ be the restriction of the right Bernoulli shift to the direct sum $\mathbb{Z}(n)^{(\mathbb{N})}$. Then $(\mathbb{Z}(n)^{(\mathbb{N})})_\phi$ is finitely generated, in fact we can choose a set of generators of the form $(x, 0, 0, \ldots)$, where x ranges in $\mathbb{Z}(n)$. Now, $\mathbb{Z}(n)^{(\mathbb{N})}/\phi(\mathbb{Z}(n)^{(\mathbb{N})}) \cong \mathbb{Z}(n)$ and $\ker(\phi) = 0$, hence $\mathrm{Mult}_L(\phi) = \log|n|$.

The following result connects multiplicity and algebraic L-entropy.

Theorem A.13. *([41]) Let R be a right Noetherian ring and $L : \mathrm{Mod}(R) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ a discrete length function. Then* $\mathrm{ent}_L(\phi) = \mathrm{Mult}_L(\phi)$ *for every* $M_\phi \in \mathrm{lFin}_L[X]$.

The above theorem is a direct application of the Uniqueness Theorem for the algebraic L-entropy. Indeed, Theorem A.9 asserts that $\mathrm{Mult}_L(-)$ is a length function and so, for the proof of Theorem A.13 one has only to verify that $\mathrm{Mult}_L(-)$ satisfies conditions (i)–(ii) in Theorem A.7.

As a consequence of Theorem A.13 we get the following

Corollary A.14. *Let R be a right Noetherian ring and L a discrete length function of* $\mathrm{Mod}(R)$. *For every locally L-finite right R-module M and every injective endomorphism $\phi : M \to M$ we have that*

$$\mathrm{ent}_L(\phi) = \sup\left\{L\left(\frac{T(\phi, F)}{\phi(T(\phi, F))}\right):\ F \in \mathscr{F}_R(M)\right\}.$$

Proof. Theorem A.13 implies that the algebraic L-entropy $\mathrm{ent}_L(\phi)$ equals the multiplicity $\mathrm{Mult}_L(\phi)$. Hence we have to check that

$$\mathrm{Mult}_L(\phi) = \sup\big\{L\big(T(\phi, F)/\phi\big(T(\phi, F)\big)\big):\ F \in \mathscr{F}_R(M)\big\}.$$

This comes directly from the definition of multiplicity recalling that by Remark A.8 every $N_\phi \in \mathscr{F}_{R[X]}(M_\phi)$ has the form $T(\phi, F)$ for some $F \in \mathscr{F}_R(M)$, and that $L(\ker(\phi \restriction_{T(\phi, F)})) = L(0) = 0$, by the injectivity of ϕ. □

We conclude with two observations

Remark A.15.

(i) Let R be a right Noetherian ring, $L: \mathrm{Mod}(R) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ a length function, M an R-module and $\phi: M \to M$ an endomorphism. Then, by [46, Theorem 6, Ch. 5], the multiplicity $\mathrm{Mult}_L(\phi)$ can be computed via *Lech's Limit Formula*, that is

$$\mathrm{Mult}_L(\phi) = \sup\left\{\lim_{n\to\infty} \frac{L(N/\phi^n(N))}{n}:\ N_\phi \in \mathscr{F}_{R[X]}(M_\phi)\right\}.$$

Rephrased in the context of torsion Abelian groups this gives

$$\mathrm{ent}(\phi) = \sup\left\{\lim_{n\to\infty} \frac{\log|T(\phi,F)/\phi^n(T(\phi,F))|}{n}:\ F \text{ is a finite subgroup of } M\right\}.$$

This formula is somehow surprising since the limit $\lim_{n\to\infty} \frac{\log|T(\phi,F)/\phi^n(T(\phi,F))|}{n}$ seems to be genuinely smaller than $H_A(\phi, F)$ in general.

(ii) We saw in Theorem A.13 that there are many similarities between algebraic L-entropy and the multiplicity of L. Nevertheless, these two concepts are intrinsically different. In fact, we can think at algebraic entropy as an "operator" that transforms a discrete length function on $\mathrm{Mod}(R)$ in a length function defined on (a suitable subclass of) $\mathrm{Mod}(R[X])$, while multiplicity in its original formulation is an "operator" that transforms a length function on a given Noetherian ring to another length function on the same ring. For some deeper insight on these aspects of multiplicity we refer to [35,46].

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