Whirlpool Permutations

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(dedicated to my first calculus student, Richard Peter Stanley)
(with special thanks for *Enumerative Combinatorics*)

I spent the past few days playing with a new(?) toy called "whirlpool permutations," and I'm writing this just in case you too find them amusing. I plan to mention them in an exercise for Section 7.2.2.3 of *The Art of Computer Programming*.

An $m \times n$ matrix has $(m-1)(n-1)$ submatrices of size 2×2. An $m \times n$ *whirlpool permutation* is an $m \times n$ matrix containing mn distinct numbers, in which the relative order of the elements in each of those submatrices is a "vortex" — that is, it travels a cyclic path from smallest to largest, either clockwise or counterclockwise.

Thus there are eight 2×2 whirlpool permutations of $\{1, 2, 3, 4\}$:

$$\begin{pmatrix} 1\,2 \\ 4\,3 \end{pmatrix} \quad \begin{pmatrix} 1\,4 \\ 2\,3 \end{pmatrix} \quad \begin{pmatrix} 2\,1 \\ 3\,4 \end{pmatrix} \quad \begin{pmatrix} 2\,3 \\ 1\,4 \end{pmatrix} \quad \begin{pmatrix} 3\,2 \\ 4\,1 \end{pmatrix} \quad \begin{pmatrix} 3\,4 \\ 2\,1 \end{pmatrix} \quad \begin{pmatrix} 4\,1 \\ 3\,2 \end{pmatrix} \quad \begin{pmatrix} 4\,3 \\ 1\,2 \end{pmatrix}.$$

One simple test for a vortex in $\begin{smallmatrix} a\,b \\ c\,d \end{smallmatrix}$ is to compare $a:b$, $b:d$, $d:c$, and $c:a$; the number of '<' cases must be odd. (Consequently the number of '>' cases must also be odd.)

Here's a not-quite-random example of a 4×4 whirlpool permutation:

$$\begin{pmatrix} 16 & 3 & 2 & 13 \\ 9 & 7 & 8 & 10 \\ 5 & 6 & 12 & 11 \\ 4 & 15 & 14 & 1 \end{pmatrix}.$$

It becomes non-whirlpool if we interchange any two adjacent elements.

One simple exercise, which I leave to the reader, is to find the lexicographically smallest $m \times n$ whirlpool permutation of $\{1, 2, \ldots, mn\}$.

Another, which I didn't think of until after whirlpooling for more than a day, is a cute proof that the number of $m \times n$ whirlpool permutations is a multiple of mn: We can add 1 to all entries of the matrix, modulo mn(!). This preserves vortices and also their orientation.

Vortices and their orientation are also preserved by adding any constant, modulo any number M that exceeds the difference between the largest and smallest element.

Among the problems I thought of but cannot solve are:

(a) Enumerate $m \times n$ whirlpool permutations for which every pair of consecutive elements is horizontally or vertically adjacent. (This is a subset of the $m \times n$ Hamiltonian rook paths.)

(b) If consecutive elements of a whirlpool permutation are not adjacent, we can interchange them and get another whirlpool permutation. (For example, 1 and 2 can be exchanged in the 4×4 example above; also 3 and 4, 8 and 9, 9 and 10, 12 and 13, 13 and 14, 15 and 16.) What do the equivalence classes look like? (The equivalence classes of size 1 are, of course, the solutions to (a).)

(c) Given any $m \times n$ matrix filled with a non-whirlpool permutation, is there always a pair of adjacent elements whose interchange would reduce the number of non-vortices?

Empirical enumeration. A problem that I did solve was to enumerate whirlpool permutations for small m and n. Here are the results for $1 \le m \le 6$ and $1 \le n \le 5$:

$m = 1$	1	2	6	24	120
$m = 2$	2	8	84	1632	51040
$m = 3$	6	84	5904	1064304	402671760
$m = 4$	24	1632	1064304	2456909824	15584878111040
$m = 5$	120	51040	402671760	15584878111040	21798753441871296000
$m = 6$	720	2340480	273315542400	217353588326290944	823110394280028294344640

My program WHIRLPOOL-COUNT, which is available online [1], does not exploit the "mod mn" property, because I wanted that program to enumerate $m \times n$ matrices of permutations that satisfy much more general constraints, with *arbitrary* relations applied to each 2×2 submatrix — provided that the relations only depend on the relative order of the elements (not on the actual values). Each of those constraints might be different, and there are $2^{4!} = 16777216$ possibilities for each of them.

I don't have space here to describe the enumeration techniques introduced in that program, except to mention that it's a somewhat mind-boggling variant of dynamic programming that I've not seen before. It needs to represent $n + 1$ elements of a permutation of t elements, where t is at most mn, and there are up to $(mn)^{n+1}$ such partial permutations; so I can't expect to solve the problem unless m and n are fairly small. Even so, when I can solve the problem it's kind of thrilling, because the program makes use of a really interesting way to represent t^{n+1} counts in computer memory.

Connection to up-up-or-down-down permutations. Of course I looked in the OEIS [2] after computing the results above — and found that the $2 \times n$ permutations have indeed been studied before: Case $2 \times n$ is sequence A261683, and it is the "running example" of a seminal paper by Nicolas Basset [3].

Let W_n be the number of $2 \times n$ whirlpool permutations. If we assume that there's exactly one such permutation for $n = 0$, the sequence therefore begins

$$(W_0, W_1, W_2, W_3, W_4, W_5, \dots) = (1, 2, 8, 84, 1632, 51040, 2340480, \dots).$$

Basset found the exponential generating function

$$W_0 + W_1 \frac{z^2}{2!} + W_2 \frac{z^4}{4!} + \cdots = \frac{2}{1 - (z/\sqrt{2}) \tanh(z/\sqrt{2})} - 1.$$

Since we know that W_n is a multiple of $2n$, let's define

$$U_n = \frac{W_n}{2n},$$

so that

$$(U_1, U_2, U_3, U_4, U_5, \dots) = (1, 2, 14, 204, 5104, 195040, \dots).$$

Sure enough, this sequence too is in the OEIS, number A122647; and we learn that it is the number of permutations of $2n - 1$ elements for which every "peak" or "valley" occurs at an even position. I shall call such a permutation an *up-up-or-down-down permutation*, because we take either two upward steps or two downward steps at a time when we read it.

Up-up-or-down-down permutations have been studied by Michael La Croix [7], who found a nice combinatorial proof that $U_n = V_n / 2^{n-1}$, where

$$(V_1, V_2, V_3, V_4, V_5, \dots) = (1, 4, 56, 1632, 81664, 6241280, \dots)$$

is the number of permutations whose peak positions (but not necessarily valley positions) are required to be even. From this paper I learned that Ira Gessel had connected V_n to greedy algorithms in [5], and that the exponential generating function

$$V_1 \frac{z}{1!} + V_2 \frac{z^3}{3!} + V_3 \frac{z^5}{5!} + \cdots = \frac{\tanh z}{1 - z \tanh z}$$

had already been well known to a number of researchers.

None of the authors mentioned so far connected their work with anything like a whirlpool permutation; the constraints they considered were strictly one-dimensional. Thus the fact that W_n has a completely one-dimensional interpretation was a complete surprise to me. I suspect that it will also be surprising to Basset and Gessel and others, when I tell them about the present note!

Furthermore, the OEIS led me to an intriguing unpublished paper by Guo-Niu Han [6], who introduced a large variety of comparison-constrained permutations that can naturally be called *standard puzzles*, and he

showed that dozens of special cases are equivalent to familiar sequences such as the Catalan and Genocchi numbers. According to Han's terminology, $m \times n$ whirlpool permutations can be described as "the standard puzzles of support AFHKQSUZ whose shape is an $m \times n$ matrix." (Each "support" in his terminology is one of the 24 possible constraints of a 2×2 submatrix, and has its own identifying letter. For example, the Catalan numbers correspond to $2 \times n$ matrices of support BC. Han did not, however, specifically mention the eight-piece set of supports AFHKQSUZ that defines a whirlpool vortex.)

A bijection. Finding that W_n agrees with OEIS sequence A261683 for small n does not, of course, prove that $2 \times n$ whirlpool permutations have the same cardinality as up-up-or-down-down permutations for *all* n — although I was 100% convinced after noticing the agreement for $n \le 6$ with computer-obtained counts. That agreement simply could not be mere coincidence, so it begged the question of finding a formal proof.

The gold standard for such a proof would be to find a one-to-one correspondence between the objects known to be counted by U_n (namely the number of $(2n-1)$-element up-up-or-down-down permutations) and the $2 \times n$ whirlpool permutations of $\{0, 1, \ldots, 2n-1\}$ for which the element in the lower left corner is 0. Fortunately I stumbled on such a bijection, and it turns out to be surprisingly simple, although it's quite different from any other bijection I've ever seen before. (During a couple days of struggle I also came up with lots of *non*-bijections, which failed to work except in small cases. Indeed I was very near to abandoning this quest entirely, several times; but Lady Luck has been helping me along during this coronavirus lockdown.)

The bijection below has nice corollaries, because it also shows unexpectedly strong connections between these two families of permutations. For example,

$$\text{the whirlpool} \quad \begin{pmatrix} 5 & 7 & 6 & 9 & 3 \\ 0 & 8 & 4 & 1 & 2 \end{pmatrix} \quad \text{corresponds to} \quad 5\,7\,9\,8\,1\,2\,6\,4\,3;$$

and you can see that the clockwise vortices on the left appear in the same two places as the up-up subsequences appear on the right. In general, we can assign a "signature" of $\pm$ signs, of length $n-1$, assigning $\pm$ to the orientation of each $2 \times n$ whirlpool and to each $(2n-1)$-element up-up-or-down-down; the bijection preserves this signature. The two entries '5 7' at the top left of this two-dimensional whirlpool also agree with the first two entries of its one-dimensional mate. That is not just a coincidence, it's true in general.

Rather than presenting a formal description, I'll simply sketch the idea here by working out the example above. (Full details appear in the short computer programs WHIRLPOOL2N-ENCODE and WHIRLPOOL2N-DECODE, found on my website [1].) Given a whirlpool permutation P_n with n columns, let the two leftmost elements of its first row be x and y. Detach the first column from the other $n-1$ columns, and map the latter columns into $\{0, 1, \ldots, 2n-3\}$, preserving order, calling the result $\widehat{P}_{n-1}$. (In our example, $n = 5$, $x = 5$, $y = 7$, $P_5 = \begin{pmatrix} 5 & 7 & 6 & 9 & 3 \\ 0 & 8 & 4 & 1 & 2 \end{pmatrix}$, and $\widehat{P}_4 = \begin{pmatrix} 5 & 4 & 7 & 2 \\ 6 & 3 & 0 & 1 \end{pmatrix}$. Notice that the mapping reduces elements $< x$ by 1 but elements $> x$ by 2. In particular, y has been reduced by 2 in this example, because it exceeds x.) Now add an appropriate number to each element of $\widehat{P}_{n-1}$, modulo $2(n-1)$, so that its lower-left entry is 0, and call the result P_{n-1}. (In the example, we add 2, hence $P_4 = \begin{pmatrix} 7 & 6 & 1 & 4 \\ 0 & 5 & 2 & 3 \end{pmatrix}$.) If $n > 1$, apply the method recursively to P_{n-1}.

Continuing the example, in P_4 we have $(x, y) = (7, 6)$ and we get $\widehat{P}_3 = \begin{pmatrix} 5 & 0 & 3 \\ 4 & 1 & 2 \end{pmatrix}$, then $P_3 = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 3 & 4 \end{pmatrix}$. In P_3 we have $(x, y) = (1, 2)$ and we get $\widehat{P}_2 = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$, then $P_2 = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$. Almost done! Finally $(x, y) = (3, 2)$, $\widehat{P}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $P_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The value of P_1 will *always* be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Finally, let the respective pairs (x, y) be (x_n, y_n), (x_{n-1}, y_{n-1}), $\ldots$, (x_2, y_2), and let $x_1 = 1$. In general we'll have $1 \le x_k, y_k < 2k$, because x_k and y_k were elements of a "compressed" whirlpool with k columns. We now "uncompress" their values, increasing them to (x'_k, y'_k), where x'_k and y'_k are the x_kth- and y_kth-smallest positive integers not used in x_j or y_j for $j > k$. Then $x'_n y'_n x'_{n-1} y'_{n-1} \ldots x'_2 y'_2 x' 1$ is an up-up-or-down-down permutation of $\{1, 2, \ldots, 2n-1\}$. (In our example, $(x_5, y_5) = (5, 7)$, $(x_4, y_4) = (7, 6)$, $(x_3, y_3) = (1, 2)$, $(x_2, y_2) = (3, 2)$, and $x_1 = 1$; hence the output permutation $x'_5 y'_5 \ldots x'_1$ is $5\,7\,9\,8\,1\,2\,6\,4\,3$, as stated earlier.)

This construction is easily seen to be invertible. The key observation that we need, in order to prove that the resulting permutation is indeed up-up-or-down-down, is of course that we'll have $y'_k > x'_{k-1}$ when $x_k > y_k$, $y_k < x'_{k-1}$ if $x_k < y_k$. That property follows readily by taking a close look at x'_{n-1}, using the fact that $y > z$ if and only if $x > y$, where z is the element directly below y in P_n.

Asymptotic behavior. Since the number of vortices divided by the total number of permutations of four elements is $8/24 = 1/3$, we might expect that the number of $m \times n$ whirlpool permutations grows roughly

as $(mn)!/3^n$. But the table of values above for small m and n suggests that the true growth is significantly larger. Individual eddies seem to reinforce each other, instead of being mutually repellent.

When $m = 2$ we have an explicit generating function, so we can evaluate the limiting behavior explicitly in this case. There's a singularity of the generating function for V_n when $z \tanh z = 1$; this constant μ is known to be ≈ 1.19967864. (See §4.8 in [4]. In fact, $\mu = \sqrt{\lambda^2 + 1}$, where $\lambda \approx 0.6627434$ is the "Laplace limit constant.") Thus the factor 3^n in our heuristic estimate should be replaced in the case $m = 2$ by $((2\mu)^2)^n$, where $2\mu^2 \approx 2.878457679781290301551959273309705896777283552987821$.

[Incidentally, Cauchy computed the constants μ, μ^2, and λ in his historic memoir of 1831 to the Turin Academy. See page 101 in [9]!]

This asymptotic question suggests a natural problem, to be added to the three open problems already stated in the introduction:

(d) Is there a "simple" e.g.f. for the number of $3 \times n$ whirlpool permutations—or perhaps even for $m \times n$ whirlpool permutations when m is any fixed number?

The fact that enumeration is possible by dynamic programming, which is basically a fancy transfer-matrix scheme, suggests that each $m \times n$ e.g.f. will be differentiably finite [8], with a fairly decent chance of having singularities that aren't incredibly difficult to determine. But I haven't time to investigate this myself.

Concluding remarks. If the topic of whirlpool permutations stimulates you to come up with any interesting thoughts, please let me know, so that I can include them with the exercise in *TAOCP*. Don't get sucked in; but let's pool our ideas!

References.

[1] http://cs.stanford.edu/~knuth/programs.html, the source files of many downloadable programs.

[2] http://oeis.org/, The On-Line Encyclopedia of Integer Sequences®, founded in 1964 by N. J. A. Sloane. The table on page 1 of this note became sequences A334516–A334519 on 06 May 2020.

[3] Nicolas Basset, "Counting and generating permutations in regular classes," *Algorithmica* **76** (2016), 989–1034.

[4] Steven R. Finch, *Mathematical Constants*, Encyclopedia of Mathematics and Its Applications **94** (Cambridge University Press, 2003).

[5] Ira M. Gessel, "A coloring problem," *American Mathematical Monthly* **98** (1991), 530–533.

[6] Guo-Niu Han, "Enumeration of standard puzzles," (Université de Strasbourg, 27 May 2011), 25 pages.

[7] Michael A. La Croix, "A combinatorial proof of a result of Gessel and Greene," *Discrete Mathematics* **306** (2006), 2251–2256.

[8] R. P. Stanley, "Differentiably finite power series," *European Journal of Combinatorics* **1** (1980), 175–188.

[9] *Œuvres complètes d'Augustin Cauchy* (2) **12** (Paris, Gauthier-Villars, 1916).

News flash (10 May 2020)! Filip Stappers has found the following 5×5 non-whirlpool whose two non-vortices cannot be reduced by adjacent swaps:

$$\begin{pmatrix} 2 & 8 & 1 & 5 & 9 \\ 6 & 7 & 3 & 4 & 10 \\ 16 & 12 & 18 & 14 & 15 \\ 17 & 11 & 19 & 23 & 20 \\ 21 & 22 & 13 & 24 & 25 \end{pmatrix}$$

So the answer to problem (c) is "No."