Tilings of $2 \times n$ boards with dominos and L-shaped trominos

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Abstract

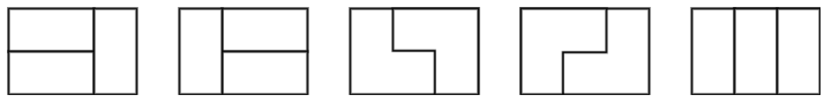
We count the tilings of a $2 \times n$ rectangle with dominos and L-shaped trominos, and we discover some new identities for old sequences. Some of our results are listed on the Online Encyclopedia of Integer Sequences at A052980, A080204, and A332647.

1 Introduction

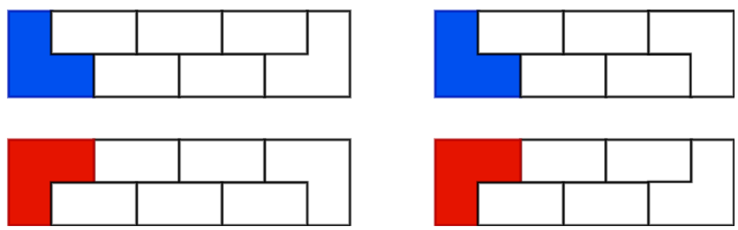
How many ways are there to tile a $2 \times n$ board? Naturally, it all depends on which tiles are allowed. Using just dominos we obtain the familiar Fibonacci sequence. For dominos and squares, we look to McQuistan and Lichtman [8] and the sequence A030186, and for colored dominos and squares we have papers by Katz and Stenson [7], Kahkeshani [5], and Kahkeshani and Arab [6]. Squares, dominos, and (straight) trominos were covered by Haymaker and Robertson [4] and are counted by A278815. For L-shaped trominos and squares, we can turn to to a recent paper by Chinn, Grimaldi, and Heubach [3] as seen in A077917. An interesting variation on the $2 \times n$ board was given by Bodeen, Butler, Kim, Sun, and Wang [2] who looked at tiling a $2 \times n$ lozenge with triangles, giving nice combinatorial interpretations to the sequences A000129, A000133, A097075, and A097076.

In this paper, we use a slightly different collection of tiles for the $2 \times n$ board: dominos and L-shaped trominos. We show some surprising results and identities. In particular, we are able to relate this question to the problem of tiling a $1 \times n$ board with squares, dominos, and colored k-minos, and this allows us to establish new identities for the sequences A052980, A080204, and A332647. Our work follows closely the tiling techniques studied by Benjamin and Quinn in their book, "Proofs That Really Count" [1].

To begin with, let's define a_n to be the number of different ways to tile a $2 \times n$ board with dominos and "bent trominos" in the shape of the letter L (henceforth, we will simply call these "trominos"). As an example, the following image gives all five possible tilings for $n = 3$, thus demonstrating that $a_3 = 5$.



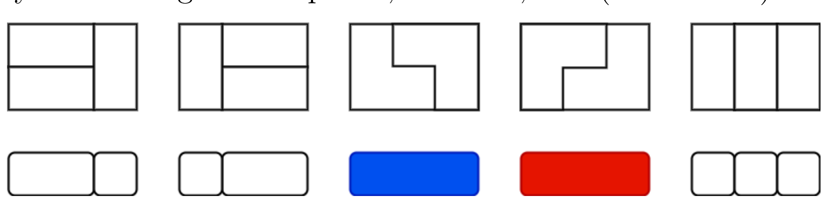
A few minutes of work with pencil and paper give the sequence $1, 1, 2, 5, 11, 24, 53, \ldots$ (starting with $a(0) = 1$). In a few pages, we will show that this sequence equals A052980. But first, inspired by a similar coloring trick in [3], we will show that we can reduce this $2 \times n$ tiling problem to a $1 \times n$ problem: our sequence a_n is exactly the number of ways to tile a $1 \times n$ strip with (white) squares, (white) dominos, and colored (blue or red) k-minos of arbitrary length $k \geq 3$. To see this equivalence, first note that in any tiling the L-shaped trominos must appear in pairs that face each other, as shown in this picture.



In the two pictures on the top, the left-side tromino is oriented like the letter "b", and for the two pictures on the bottom, the left-side tromino is oriented like the letter "r". Next, note that if we know the orientation of that left-side tromino, and if we know the length of the paired group (either even or odd) then we can reconstruct the tiling. Finally, note that the only times that two horizontal dominos appear unaligned on top of each other is when they are inside one of these paired trominos; everywhere else, the dominos must be either vertical, or in aligned horizontal pairs.

This is everything we need to establish the equivalence. We take any $2 \times n$ tiling (with dominos and L-shaped trominos) and we map each vertical domino to a (white) square, each aligned pair of horizontal dominos to a (white) domino, each tromino pairing where the left tromino is oriented like the letter "b" to a blue k-mino ("b" for blue), and each tromino pairing where the left tromino is oriented like the letter "r" to a red k-mino ("r" for red). Henceforth, when we talk about a k-mino we assume that $k \geq 3$. This mapping is reversible, and so we have our equivalence.

As an example, we show here how the five tilings of the 2×3 board shown earlier can be represented by 1×3 tilings with squares, dominos, and (red or blue) 3-minos.



As an aside, we note that similar types of tilings (of a board of length n using some single-color tiles and some multi-colored tiles) have interesting connections to other sequences. For example, Milan Janjic pointed out that the numerators of the continued fraction expansion for $\sqrt{2}$ also count the number of tilings using single-color squares and two-colored k-minos for $k \geq 2$ (see A001333 for details), and if we instead use three-colored k-minos for $k \geq 2$ we obtain A026150. Thus, we shouldn't be surprised that our tilings (with single-color squares and dominos, and two-color k-minos for $k \geq 3$) also turns up in the OEIS, and that is the subject of the next section.

2 Establishing that our tiling sequence equals A052980.

The sequence A052980, defined as the sequence with generating function $(1-x)/(1-2x-x^3)$, has initial values 1, 1, 2, 5, 11 and recurrence formula $x_n = 2x_{n-1} + x_{n-3}$. We now show (as a corollary to the following theorem) that our sequence has the same recurrence, and since it also has the same initial values then it must equal A052980.

Theorem 1. *For $n \geq 3$, we have $a_n = a_{n-1} + a_{n-2} + 2\left(a_{n-3} + a_{n-4} + \cdots + a_1 + a_0\right)$.*

Proof. In the spirit of [1], we ask: how many ways can we tile a board of length n using squares, dominos, and red or blue k-minos with $k \geq 3$? On one hand, by our equivalence discussed in the introduction there are a_n ways to tile it. On the other hand, we can condition based on the first tile. If it's a square, there are a_{n-1} ways to tile the remaining length of $n-1$. Likewise, if the first tile is a domino, we have a_{n-2} ways. Finally, if the tiling begins with a k-mino for $k \geq 3$, we recall that each k-mino has 2 possible colorings, and can also be any length greater than or equal to three. Therefore, any tiling starting with a k-mino has $2a_{n-k}$ ways to tile the rest of the strip. Summing up all the different ways, and comparing it to a_n, we have our desired formula. □

Corollary 2. *For $n \geq 3$, we have $a_n = 2a_{n-1} + a_{n-3}$.*

Proof. From our previous theorem, we have that

$$a_n = a_{n-1} + a_{n-2} + 2\sum_{i=0}^{n-3} a_i$$

which also implies

$$a_{n-1} = a_{n-2} + a_{n-3} + 2\sum_{i=0}^{n-4} a_i.$$

Subtracting the second equation from the first, and noticing that just about everything cancels, we get

$$a_n - a_{n-1} = a_{n-1} + a_{n-3}.$$

Therefore, $a_n = 2a_{n-1} + a_{n-3}$, as desired. □

Thanks to Corollary 2 above, we can now conclude that our sequence $a(n)$ is indeed the sequence A052980, as both have the same initial values and the same recurrence formula.

It's worth noting that Corollary 2 can be proved directly using the following clever tiling argument: on the one hand, there are a_n ways to tile a strip of length n with squares, dominos, and red or blue k-minos. On the other hand, we can look at the first tile. If a square, there are a_{n-1} tilings for the rest of the strip. If a red 3-mino, there are a_{n-3} tilings for the strip. For the other cases: if a domino, replace it with a square to get an $n-1$ tiling that begins with a square; if a blue 3-mino, replace it with a domino to get an $n-1$ tiling beginning with a domino, and if a k-mino for $k \geq 4$, reduce the k-mino by one to get an $n-1$ tiling beginning with a k-mino for $k \geq 3$, so in total we have a_{n-1} tilings. Adding up all the cases, we have $a_{n-1} + a_{n-3} + a_{n-1}$ tilings, giving us the desired formula.

(While this proof is clever, we actually prefer the original proof of Corollary 2 as it derives from the very natural and well-motivated proof of Theorem 1. Furthermore, we will need Theorem 1 in our proof of Theorem 4, below.)

3 Additional Identities.

The following lemma will become relevant when we look at a "bracelet" sequence in a few more pages.

Lemma 3. *For $n \geq 4$ we have $a_n = 3a_{n-1} - 2a_{n-2} + a_{n-3} - a_{n-4}$.*

Proof. From Theorem 2, we have that $a_n = 2a_{n-1} + a_{n-3}$ and hence $a_{n-1} = 2a_{n-2} + a_{n-4}$. We subtract and simplify to obtain the desired result. □

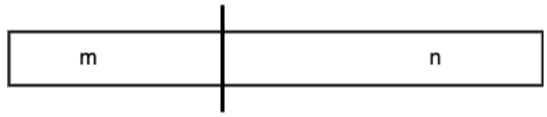
Let's look at some more identities. Benjamin and Quinn's book is full of ingenious proofs of various Fibonacci identities using careful countings of tilings. Since our sequence a_n is also a tiling sequence, we can use the same methods to come up with new identities for our sequence, just as we did in Theorem 1. For example, this next formula comes from looking at where the tiling breaks into two. Although it's fairly easy to prove this theorem (and the subsequent ones) by induction, it's more enjoyable to do so by counting tilings, in the spirit of Benjamin and Quinn's book.

Theorem 4. *For $n, m \geq 2$, we have*

$$a_{m+n} = a_m a_n - a_{m-1} a_{n-1} + \frac{1}{2}(a_{m+2} - a_{m+1} - a_m)(a_{n+2} - a_{n+1} - a_n).$$

Proof. Just like in Theorem 1, we ask: how many ways can we tile a board of length $m+n$ with squares, dominos, and red or blue k-minos with $k \geq 3$? On the one hand, this is simply a_{m+n}. On the other hand, we can count the tilings by conditioning on whether or not the tiling breaks between m and $m+1$. There are three options.

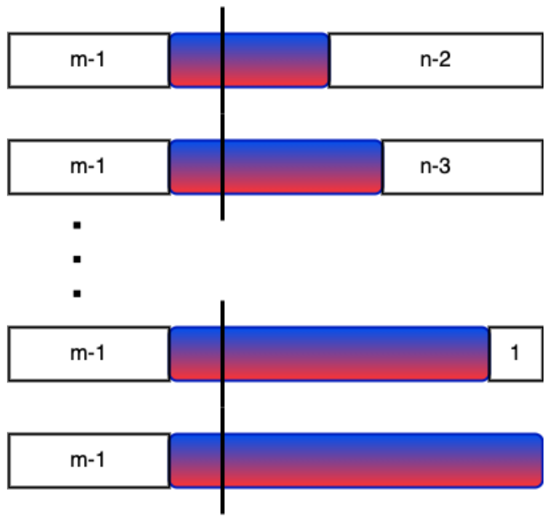
1. The tiling breaks between position m and $m+1$.



In this case, it is easy to see that there are a_m tilings on the left of the break, and a_n tilings on the right, giving $a_m a_n$ ways to tile this board.

2. There is a domino covering cells m and $m+1$. Using the same argument as above, we can see that the total number of ways to tile this board is $a_{m-1}a_{n-1}$.

3. There is a red or blue k-mino covering the break.



We imagine all possible red or blue k-minos that start at position m, as shown above. These could have length 3 (covering just positions $m, m+1$, and $m+2$) or they could stretch an arbitrary distance further to the right. The number of such tilings can be written as

$$2 \cdot a_{m-1} \sum_{j=2}^{n} a_{n-j} \;=\; 2 \cdot a_{m-1} \sum_{j=1}^{n} a_{n-j} - 2a_{m-1}a_{n-1}.$$

(We need that 2 out front, as every k-mino has two possible colorings of red or blue). Applying this same logic, for k-minos starting at position $m-2$ and crossing over the "break" there are $2a_{m-2}\sum_{j=1}^{n} a_{n-j}$ ways (note that here j starts at 1 and not 2), and for position $m-3$ there are $2a_{m-3}\sum_{j=1}^{n} a_{n-j}$, and so on, all the way down to $2a_0\sum_{j=1}^{n} a_{n-j}$. Adding these together, we have

$$2\sum_{i=2}^{m} a_{m-i} \sum_{j=1}^{n} a_{n-j}.$$

We have now covered all possible cases. Adding together the expressions from option 1, option 2, and the two from option 3 gives us, after a bit of re-arranging,

$$\begin{aligned} a_{m+n} &= a_m a_n - a_{m-1}a_{n-1} + 2a_{m-1}\sum_{j=1}^{n} a_{n-j} + 2\sum_{i=2}^{m} a_{m-i}\sum_{j=1}^{n} a_{n-j} \\ &= a_m a_n - a_{m-1}a_{n-1} + 2\sum_{i=1}^{m} a_{m-i}\sum_{j=1}^{n} a_{n-j}. \end{aligned}$$

We can now apply Theorem 1 (which says that $\sum_{i=1}^{n} a_{m-i} = \frac{1}{2}(a_{m+2} - a_{m+1} - a_m)$ and likewise for the sum with a_{n-j}) to the two summations in the above equation to obtain our desired formula. □

If we replace m with n in the above theorem and simplify, we immediately obtain the following.

Corollary 5. *For $n \geq 2$, we have $a_{2n} = a_n^2 - a_{n-1}^2 + \frac{1}{2}\Big(a_n + a_{n-1} + a_{n-2}\Big)^2$.*

For the next theorem, we must remind our reader (once again) of the Fibonacci numbers, traditionally defined as $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. If we define $f_n = F_{n+1}$ then the sequence f_n is exactly the number of ways to tile a board of length n with squares and dominos [1]. With this in mind, we present the following theorem.

Theorem 6. *For $n \geq 3$, we have*

$$a_n = F_{n+1} + 2\sum_{i=0}^{n-3} a_i\left(F_{n-i} - 1\right),$$

and also

$$a_{n+2} = F_{n+1} + a_{n+1} + 2\sum_{i=0}^{n} a_i F_{n-i}.$$

We will prove this theorem in just a moment by conditioning on the location of the last k-mino. But first, we note that if we were to instead condition on the location of the last blue k-mino, we obtain the following delightful weighted-sum theorem, where the Fibonaccis from above are replaced by powers of two.

Theorem 7. *For $n \geq 3$, then $\sum_{i=0}^{n} a_i 2^{n-i} = a_{n+3} - 2^{n+2}$.*

Proof of Theorem 6. As before, we count up the total number of tilings of a strip of length n and set it equal to a_n to obtain our formula.

If a particular tiling has no k-minos, it must be entirely made up of squares and dominos. By the statement made in the introduction, this corresponds to $f_n = F_{n+1}$ unique tilings. (Recall that when we talk about k-minos we always assume that $k \geq 3$).

Suppose, instead, that the tiling has at least one k-mino. If we look at the last (rightmost) k-mino, we see that this splits the length-n tiling into a tiling of length i (to the left) with squares, dominos and k-minos, and a tiling of length $n-i-k$ (to the right) with just squares and dominos. An illustration is shown below.

There are exactly a_i ways to tile everything to the left of the last k-mino, and there are f_{n-i-k} ways to tile everything to the right of the last k-mino, as there can only be squares or dominos beyond this point.

We now imagine fixing i at some permissible value, $0 \leq i \leq n-3$. Since $k \geq 3$, then $n-i-k$ ranges from 0 to $n-i-3$, and so for this fixed value of i we have $a_i \cdot 2 \cdot (f_0 + f_1 + f_2 + \cdots + f_{n-i-3})$ ways to tile the board. (The a_i, of course, represents the number of tilings to the left of the last k-mino, and the 2 represents that the last k-mino can be either red or blue, and the sum of Fibonacci numbers covers all possible tilings to the right).

We now sum up these values over i as i ranges from 0 to $n-3$, and we have

$$2\sum_{i=0}^{n-3} a_i(f_0 + f_1 + f_2 + \cdots + f_{n-i-3}).$$

Thanks to a well-known identity, that sum of consecutive Fibonacci numbers in the above expression can be replaced with $f_{n-i-1} - 1$. If we make this substitution, and also add in f_n (coming from the tilings with no k-minos) we obtain

$$a_n = f_n + 2\sum_{i=0}^{n-3} a_i(f_{n-i-1} - 1)$$

and if we use the identity $f_n = F_{n+1}$ we have the first formula of the theorem,

$$a_n = F_{n+1} + 2\sum_{i=0}^{n-3} a_i(F_{n-i} - 1).$$

It's now a simple matter to arrive at the second formula of the theorem. We substitute

Theorem 1 in the above expression to obtain

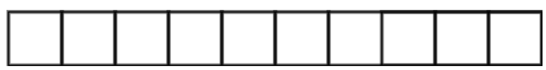
$$\begin{aligned} a_n &= F_{n+1} + 2\sum_{i=0}^{n-3} a_i F_{n-i} \;-\; (a_n - a_{n-1} - a_{n-2}) \\ &= F_{n+1} + 2\sum_{i=0}^{n} a_i F_{n-i} \;-\; 2(a_{n-2}F_2 + a_{n-1}F_1 + a_n F_0) \;-\; (a_n - a_{n-1} - a_{n-2}) \\ &= F_{n+1} + 2\sum_{i=0}^{n} a_i F_{n-i} \;-\; (a_n + a_{n-1} + a_{n-2}). \end{aligned}$$

The last expression gives us $2a_n + a_{n-1} + a_{n-2} = F_{n+1} + 2\sum_{i=0}^{n} a_i F_{n-i}$, and by applying Corollary 2 twice we can replace the left-hand side with $a_{n+2} - a_{n+1}$, which gives us the desired second formula

$$a_{n+2} = F_{n+1} + a_{n+1} + 2\sum_{i=0}^{n} a_i F_{n-i}.$$

□

Proof of Theorem 7. Once again we consider a strip of length n, but this time we condition on the location of the last blue k-mino. If there are no blue k-minos, this means that the tiling consists of single-color tiles of arbitrary length. In order to find the number of ways to tile such a board, consider the following image of a length-n board.



Note that there are exactly $n-1$ interior "dividing lines" which define the lengths of various tiles. We can create unique tiling patterns by simply removing (some of the) dividing lines. For each dividing line, there are only two options: keep it or remove it. Doing this for every dividing line gives 2^{n-1} possible tiling patterns.

Next, we suppose there is at least one blue k-mino, and we consider the location of the last such tile.

There are exactly a_i ways to tile everything to the left of the last blue k-mino. For the rest of the tiling, imagine cutting the blue k-mino into a tromino and a $(k-3)$-mino. This means that from position $i+3$ to n we have an arbitrary number of tiles of arbitrary length on a strip of length $n-i-2$, and by our previous argument there are 2^{n-i-3} such ways to tile this.

We now sum up these values over i as i ranges from 0 to $n-3$, and we add in our first value of 2^{n-1}, and we obtain

$$a_n = 2^{n-1} + \sum_{i=0}^{n-3} a_i 2^{n-i-3},$$

which then gives us our theorem. □

4 Bracelet numbers

Recall that the Fibonacci numbers f_n count the number of ways to tile a $1 \times n$ strip with squares and dominos. It's well known (see [1]) that the Lucas numbers L_n, defined as $L_0 = 2, L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, count the number of ways to tile a $1 \times n$ *bracelet* with squares and dominos. Inspired by this idea, we can ask: what are the corresponding "bracelet" numbers for our sequence a_n?

Surprisingly, there are two answers to this question! While a_n counts both the tilings of a $1 \times n$ strip (with squares, dominos, and red/blue k-minos) and a $2 \times n$ strip (with dominos and trominos), this equivalence breaks down when talking about bracelets. The key issue lies in this quote from the introduction to this article:

> Finally, note that the only times that two horizontal dominos appear unaligned on top of each other [in a $2 \times n$ *strip*] is when they are inside one of these paired trominos; everywhere else, the dominos must be either vertical, or in aligned horizontal pairs.

For $2 \times n$ *bracelets*, this isn't quite correct; when n is even we can have two configurations of unaligned horizontal dominos on a $2 \times n$ bracelet which are *not* flanked by paired trominos but instead go "all the way around the bracelet". (Imagine one such configuration; if we rotate the bracelet by one cell we will have the other configuration).

With this in mind, we define b_n to be the number of ways to tile a $1 \times n$ bracelet with squares, dominos, and red or blue k-minos, and b'_n the number of ways to tile a $2 \times n$ bracelet with dominos and trominos. By our discussion above, $b'_n = b_n + 2$ for n even and $b'_n = b_n$ for n odd. In what follows, we will focus on b_n.

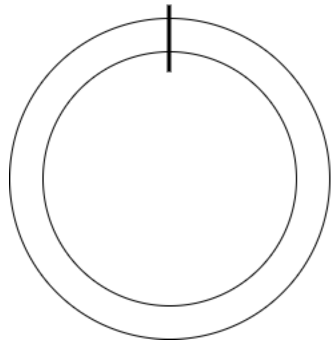
Once again, a few minutes with pencil and paper give the initial values $1, 3, 10, 23, \ldots$ for b_n (starting with $b_1 = 1$), and we will show in a moment that this is the sequence A080204 (which, interestingly enough, comes from a Kolakoski sequence which has nothing to do with tilings). To show this equality, we need to begin with the following theorem.

Theorem 8. *For a_n and b_n defined as above, we have*

$$b_n = a_n + a_{n-2} + 2\sum_{k=3}^{n}(k-1)a_{n-k}$$

Proof. Inspired by the proofs of Theorems 1 and 4, we count the number of ways to tile a $(1 \times n)$ bracelet using squares, dominos, and red or blue k-minos, and we set that equal to b_n. We condition on the tile covering the "break" at the top of the bracelet, as shown in the following images. There are three options.

1. The tiling is breakable at the top, meaning that no single tile crosses over from the n^{th} position (immediately to the left of the break) to the first position (immediately to the right).



As the bracelet is breakable here, there are simply a_n ways to tile the $1 \times n$ strip we obtain when we unfold the bracelet.

2. The bracelet is not breakable at the top, and the break is being covered by a domino. For this, we simply remove the domino and unfold the bracelet to give us a strip of length $n-2$, and hence there are a_{n-2} ways to tile it.

3. There is a red or blue k-mino covering the break. In this case, for each k-mino covering the break, there are exactly $k-1$ ways to shift it such that the bracelet remains unbreakable at the top. There are two colors for the k-mino (red or blue), and the rest of the bracelet has length $n-k$. Hence, there are $2(k-1)a_{n-k}$ ways to tile this particular bracelet. Since k can range from 3 to n, we get that the total number of ways to tile in this situation is $2\sum_{k=3}^{n}(k-1)a_{n-k}$.

We have now covered all three possible cases. Adding the results will give the desired formula. □

To conclude, we present the following results for our bracelet sequence b_n. These can all be proved by induction or by tilings; we leave the details to the reader.

Theorem 9. *For a_n and b_n defined as above, we have*

$$b_n = 3b_{n-1} - 2b_{n-2} + b_{n-3} - b_{n-4},$$

$$b_n = 2b_{n-1} + b_{n-3} + 2,$$

$$b_n = \frac{1}{2}\Big(5a_n - a_{n-1} - a_{n-2}\Big) - 1,$$

$$b_n = \theta_1^n + \theta_2^n + \theta_3^n - 1, \quad \textit{for } \theta_1, \theta_2, \theta_3 \textit{ the roots of } x^3 - 2x^2 - 1 = 0.$$

Note that the first equation gives exactly the same recurrence for b_n as that for a_n in Lemma 3; only the initial values are different. With this recurrence relation, and with our initial values for b_n of $1, 3, 10$, and 23, we can conclude that we do indeed have the sequence A080204.

We can rewrite our second equation in Theorem 9 as $(b_n + 1) = 2(b_{n-1} + 1) + (b_{n-3} + 1)$, which tells us that the sequence $b_n + 1$ has a particularly nice recurrence formula (in fact, the same recurrence formula as for a_n in Theorem 2). This sequence $b_n + 1$ appears in the OEIS as A332647.

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2010 *Mathematics Subject Classification*: Primary 05B45; Secondary 52C20.
Keywords: domino, tromino, tiling, Fibonacci, rectangle.

(Concerned with sequences A052980, A080204, and A332647.)
