Growth and the Fragmentation of Production*

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July 11, 2022

First draft, preliminary

Abstract

How much do changes in the fragmentation of production contribute to growth? Using detailed plant-level data on the manufacturing sector in India between 1990 and 2014, we study a version of Smithian growth, the link between greater fragmentation of supply chains and productivity. We propose a measure of a plant's vertical span, which corresponds roughly to the number of stages in a supply chain that the plant performs in-house; when plants have smaller vertical spans, production is more fragmented. We find that fragmentation increases with development in both the cross-section and time series. Further, within locations at a point in time, larger plants tend to have smaller vertical spans, and those that increase sales tend to decrease vertical span. Using changes in demand during the tariff liberalization in the 1990s, we provide evidence that increased demand causes specialization. We find evidence from economies of scale in specialization. We construct a general equilibrium model to rationalize these findings and estimate the sources and magnitude of scale economies. Goods are produced in a succession of steps, each combining labor and a set of intermediate inputs, giving rise to a tree-like structure. Firms exert effort to find suppliers for inputs, and choose the set of production stages (and thereby inputs) to produce the output at lowest cost. The structure implies that the returns to searching are more strongly diminishing for inputs that are further upstream. Firms with high productivity draws are therefore more likely to choose to be more vertically specialized.

KEYWORDS: Growth, Organization, Intermediate Inputs, Productivity, Production Networks, Specialization
JEL: O11, O14, L23, L25

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1 Introduction

Since Adam Smith, economists have been postulating that specialization is one of the primary drivers of improvements in standards of living.[1] Specialization can take many forms; perhaps the most salient is the fragmentation of supply chains, with different vertically related production stages often done by different firms in different locations.[2] Whether this increased specialization plays an important role in growth is ultimately an empirical question: What is the relationship between increased fragmentation and productivity growth?

In this paper we study the fragmentation of production in the Indian economy, from the time of reform at the onset of the 1990s, up until 2015. We use detailed data on plants' inputs and outputs from annual manufacturing surveys to construct a measure of plants' vertical spans of production, a measure intended to capture the range of sequential production steps performed within the plant. The first part of the paper explores the empirical relationship between plants' vertical specialization choices—as measured by their vertical spans of production—and productivity and market conditions. We first examine the macro-level relationship between specialization and development across time and space. We find that vertical span and income per capita are strongly related. At the macroeconomic level, plants in richer districts of India are more vertically specialized, and state-level income growth is associated with within-plant reductions in vertical span. This is consistent with Smith's famous statement that "the division of labour is limited by the extent of the market" (Stigler, 1951). Second, we assess the micro-level relationship between plant size and vertical specialization among plants that produce the same product in the same location at the same time. In contrast to the usual intuition that larger plants perform a wider range of activities, we find that those with higher sales tend to have shorter vertical spans, i.e., perform fewer sequential stages of production. This holds both across plants in the cross-section and also for within-plant changes over time.

These correlations suggest a link between vertical specialization and growth. To make inference about the direction of causality between size and vertical span, we exploit India's tariff liberalization at the start of the 1990's. We find that plants respond to negative demand shocks (induced by a reduction in the import tariff to the plant's output) by increasing their span, and to positive demand shocks by reducing their span, i.e., specializing in fewer sequential steps of production. In addition, increases in demand are associated with using fewer inputs.

Finally, inspired by Young (1928), we study the presence of network economies. When tariffs on a good ω are reduced, this negative demand shock propagates upstream to industries that supply ω. At the same time, we also find that industries that are downstream from these upstream industries also see a reduction in sales (even excluding ω itself). This suggests industry-level scale economies. These network economies could be the results of internal economies of scale in production in the

[1] As virtually all papers on Smithian growth have noted, Adam Smith begins the first chapter of *Wealth of Nations* with the statement: "The *greatest* improvement in the productive powers of labour [...] seem to have been the effects of the division of labour." (Wealth of Nations, Chapter 1, 1776, emphasis added). Smith goes on to present the famous example of the pin factory, and how an increased division of labor leads to higher labor productivity.

[2] See, for example, Johnson and Noguera (2017) and Antràs (2020).

upstream industries, or the result of external economies of scale operating along the value chain.

We rationalize and explain these findings using a simple model, which we later develop into a full quantitative model. In the simple model, firms produce an output ("shirts") using one of two (perfectly substitutable) production functions: either from cloth (in which case the firm is vertically specialized, and just tailors the cloth into shirts) or from yarn (in which case the firm performs both the weaving of the cloth from yarn, and the tailoring of the shirts). Both production steps follow a Cobb-Douglas production function in labor and the respective intermediate input. In the case of the firm being integrated, it produces using the nested production function. Firms are born with ex-ante heterogeneous Hicks-neutral productivity, and need to search for suppliers for each of the two inputs, paying a cost that is convex in search effort. More search effort leads to more matches with potential suppliers, and therefore on average a lower effective cost of that input. Following realization of these draws, firms choose one of the two production structures (as well as suppliers) to minimize cost.

The model implies that firms that are born with higher productivity have higher returns to searching, and will therefore search more and obtain even lower costs, a manifestation of scale economies. The sequential nature of production gives rise to a nonhomotheticity: when productivity increases, the firm searches more for both inputs, but disproportionally more so for the downstream input (cloth). This is because searching for cloth lowers the cost of the out-sourcing option by more than seraching for yarn lowers the cost of the in-house option, as the latter requires labor and th ecost of labor is invariant to search. Thus as the firm searches more, the expected expenditure share of cloth rises relative to the expected expenditure share of yarn. The firm responds to this by shifting it search effort toward suppliers of cloth, increasing the likelihood of a short vertical span of production. This mechanism, whereby more productive firms select into being specialized, is consistent with our empirical findings on the positive correlation between specialization and size.

The full quantitative model embeds this mechanism into a setup where value chains can be made up of an arbitrarily large number of sequential steps, where each step combines primary factors with an arbitrary number of inputs. Firms search for suppliers in all input markets in this tree, and face a make-or-buy decision for each step. Building on Kortum (1997) and Oberfield (2018), we choose functional form assumptions such that, conditional on search efforts, the cost of buying each input from the market follows a Weibull distribution. The make or buy decision at each node is tractable due to a functional form assumption that is new to the literature for the distribution of task-specific productivity draws, which yields a Weibull-distributed unit cost for in-house production with the same shape. Firms make profits from sales to households, and entry is elastic. The search process features network economies through a matching function that is increasing in the mass of potential suppliers. This elasticity, together with firm-level scale economies in the form of an elastic search effort, drives the propagation of shocks across industries and speaks to our empirical results.

In the final section, we describe an approach to solve and estimate the model. Estimation is still in progress.

1.1 Related Literature

Our paper builds on a large theoretical literature that studies the relationship between specialization, market size, and economic performance, including classic papers by Young (1928), Stigler (1951), Rosen (1978), and Becker and Murphy (1992), and more recent work by Yang and Borland (1991), Baumgardner (1988*a*), Rodriguez-Clare (1996), Kelly (1997), Chaney and Ossa (2013), Legros, Newman and Proto (2014), and Menzio (2020). In contrast to this work, our model is written in such a way that it can be taken to the data and used for quantitative work.

There is a good amount of evidence that the specialization among workers is limited by the extent of the market: Baumgardner (1988*b*) shows that the range of tasks performed by physicians shrinks as local labor market grows. Garicano and Hubbard (2009) show that the fraction of lawyers working in firms that specialize in particular fields grows with the size of the market. Duranton and Jayet (2011) provide evidence that scarce specialist occupations are over-represented in large cities. Tian (2018) shows that Brazilian manufacturing firms that are located in cities tend to hire workers in more occupations. Atalay, Sotelo and Tannenbaum (2021) show that workers are more specialized, relative to other workers in the same firm or the same occupation, when they work in larger markets. Hansman et al. (2020) studies the vertical specialization and its relationship to the demand for quality in the Peruvian fishmeal industry. Brown (1992), in a study of the German cotton textile industry at the beginning of the 20th century, also finds a positive correlation between scale and the degree of vertical specialization. He attributes the high degree of vertical integration (compared to British cotton textile) to the underdevelopment of input markets.

Our modeling approach builds on existing approaches. We allow firms to choose from a set of production functions, called "recipes", similarly to Boehm and Oberfield (2020), and also related to Acemoglu and Azar (2020), and Ciccone (2002). Our measure of the vertical span builds on work by Alfaro et al. (2019) and Boehm and Oberfield (2020). In terms of its application to the vertical scope of production, the paper relates to recent work by Chor, Manova and Yu (2021).

In terms of context, we build on work that studies the Indian trade liberalization in the early 1990s (Panagariya (2004), Sivadasan (2009), Topalova and Khandelwal (2011), and Goldberg et al. (2010), among others) and manufacturing productivity growth in India more generally (Hsieh and Klenow (2014), Bollard, Klenow and Sharma (2013)).

Finally, our work is related to the growing literature on endogenous production networks, e.g. Oberfield (2018), Eaton, Kortum and Kramarz (2022), Lim (2018), Chaney (2014), Dhyne et al. (2021), Startz (2021), Grant and Startz (2021), Taschereau-Dumouchel (2017), Huneeus (2018), Panigrahi (2021), and Miyauchi (2018), among others. Our work also builds on studies that focused on cross-country differences in input-output structure and its relation to development, Chenery et al. (1986), Jones (2013), Boehm (2020), Fadinger, Ghiglino and Teteryatnikova (2021) and Bartelme and Gorodnichenko (2015)

2 Reduced-form regressions

2.1 Data and Context

Our main dataset is the Indian Annual Survey of Industries (ASI), a panel of formal manufacturing plants that the Ministry of Statistics and Programme Implementation constructs and makes available to researchers. Each year the ASI surveys all plants that have more than 100 employees, and a fifth of all plants with more than 20 employees (or more than 10 employees, if the plant is using power). We use survey rounds from between the years 1989/90 and 2014/15 except 1990 to 1993 and 1995/96, for which no detailed data is available. Most relevant for our paper, the ASI contains detailed information about the plants' mix of outputs and intermediate inputs in addition to commonly available factor costs. We concord all product codes to the five-digit classification used in the 2007/08 round, which contains about 5,500 items that are similar in detail to six-digit HS codes. Some early ASI rounds have low quality for certain parts of the survey and we exclude these in our regressions; we discuss these choices along with more detail on concordances and summary statistics in Appendix A.

Our empirical strategy employs the large changes in India's barriers to importing from around India's vast trade liberalization at the beginning of the 1990's. Before this liberalization, India's economic strategy emphasized self-sufficiency (Panagariya, 2004). Imports were generally subject to licensing. While the list of items exempted from licensing grew towards the end of the 1980's, tariffs remained high. In 1991, India experienced a balance of payment crisis and depreciated the rupee. Following the crisis, in July 1991 abolished the import licensing regime on all but a few items, and started a series of large tariff cuts. Between 1991 and 1997, tariffs fell from an (unweighted) average of 113 percent to an average of about 35 percent. While there was substantial variation in tariffs before the reforms (the top tariff bracket was 355 percent), post-reform tariffs were much more even. Since the levels of pre-reform tariffs had been determined much earlier, Topalova and Khandelwal (2011) conclude that the "differential tariff changes across industries between 1991 and 1997 were as exogenous to the state of the industries as a researcher might hope for in a real-world setting."

We therefore combine the manufacturing survey data with information on import tariff levels. With the exception of a few years at the start of the 1990's, tariff data is available from UNCTAD; for the remaining years we digitize and transcribe commercial publications that contain effective import tariffs by six-digit HS code. We concord all tariffs to the common product classification used by the ASI in the 2000's. See Appendix A.1 for details.

2.2 Measuring the vertical span of production

We follow our earlier work (Boehm and Oberfield, 2020) to construct a measure of the vertical span of production, by which we mean the number of consecutive production steps performed in the plant. We give the precise definition in Appendix A.2, and convey the intuition here. In the

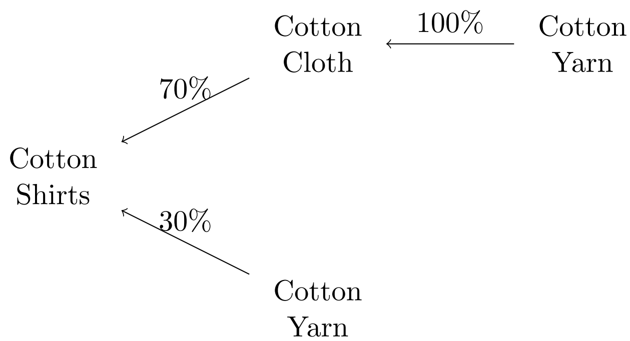


Figure 1 Example of Vertical Distance between Output (Shirts) and Inputs

first step, we construct a measure of vertical distance between each output-input pair (ω, ω'). If there is a single path in the IO matrix from input to the output in the IO matrix, then the vertical distance is the number of links in that path, or the number of plants along the value chain from the input to the output. If there are multiple such paths from the input to the output in the IO matrix, then vertical distance is simply the cost-weighted average of the number of links in each path. Suppose the (aggregate) materials expenditure mix of shirts is 70% cloth and 30% yarn, and that the materials mix of cloth is 100% yarn. The distance between the output shirts and the input cloth is one (because cloth only shows up directly as an input to shirts), and the distance between shirts and yarn is $0.3 \times 1 + 0.7 \times 1.0 \times 2 = 1.7$. In the second step, we construct the *vertical span* of a (single-product) plant, which measures whether a plant tends to use inputs that are distant or close. A plant's vertical span is the cost-weighted average distance between its output and each of its inputs. In this example, a plant that produces shirts and uses cloth as an input would have a vertical span of 1, whereas one that uses yarn as an input would have a longer vertical span of 1.7.

In our view, the measure of vertical span does not measure a primitive attribute of technology, as the vertical distance is partly determined by plant decisions in equilibrium. Further, the number of plants between an input and an output in a value chain is not the same as the number of steps between the input and the output, however one might define a "step." Nevertheless, when we compare plants that produce the same product, the plant with the higher vertical span arguably performs more steps in-house then the plant with the lower vertical span.

2.3 Motivating facts

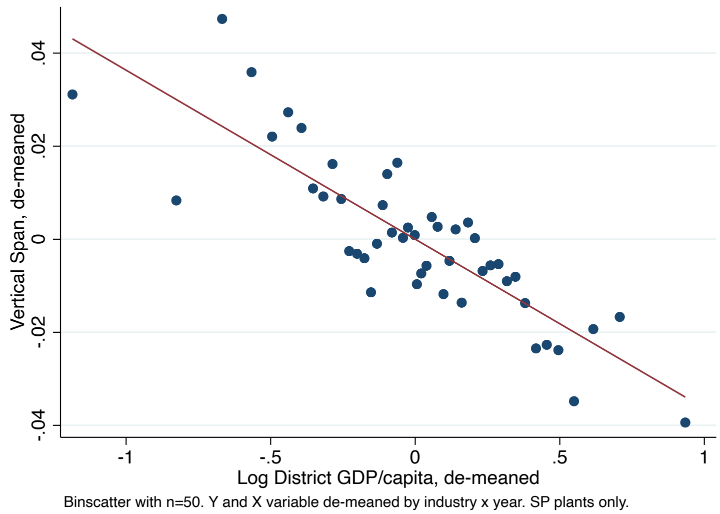
Our first fact is a robust relationship between the aggregate level of development and the degree of vertical fragmentation of production. This relationship holds both in the cross-section and within locations over time, and hence motivates our study of fragmentation in the context of economic growth:

Fact 1 (Macro-level correlation between development and fragmentation) *We have:*

(a) Plants in richer districts have shorter vertical spans of production (within their industry)

(b) *Plants in states that grew faster have, on average, reduced their vertical span of production more*

Figure 2 Plants in richer districts are more vertically specialized



Part (a) of Fact 1 is corroborated by Figure 2, which shows a binned scatter plot for the relationship between the (residualized) measure of the vertical span of production and the (residualized) log income per capita of the district where the plant is located in, after projecting on 5-digit industry dummies. Among plants in the same industry, those in richer districts have shorter vertical spans of production. Concerning part (b) of Fact 1, Table I shows that plants shortened their vertical spans of production more in states that grew faster.[3] In these (and subsequent) regressions we exclude multi-product plants that, while economically important, are not easy to measure the vertical span for.

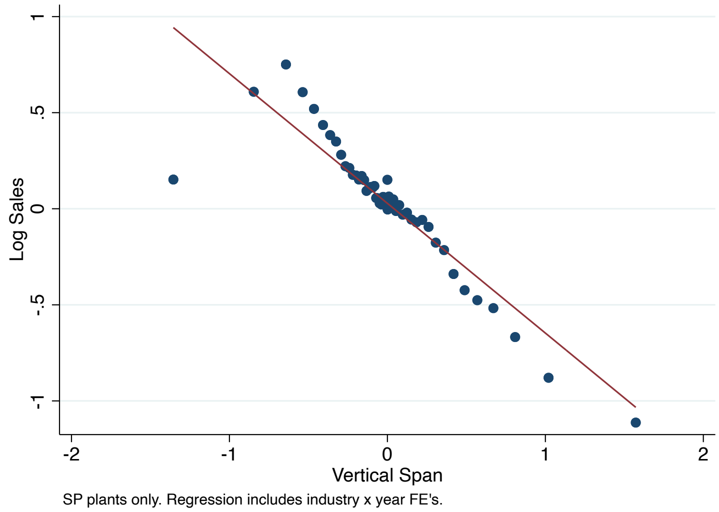
Table I Plants in states that grew faster have vertically specialized more

	Dependent variable: Vertical Span		
	(1)	(2)	(3)
Log GDP/capita$_{st}$	-0.0716*	-0.0601*	-0.0551*
	(0.028)	(0.026)	(0.026)
Year FE	Yes	Yes	
Plant FE	Yes	Yes	
5-digit Industry FE		Yes	
5-digit Industry $\times$ Year FE			Yes
Plant $\times$ 5-digit Industry FE			Yes
R^2	0.592	0.656	0.808
Observations	270003	269399	163668

Standard errors in parentheses, clustered at the state $\times$ 5-dgt industry level. SP plants only.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

[3] We use state-level growth rather than district-level growth because district-level income per capita is only available for a single cross-section in 2005.

Figure 3 Larger plants are more vertically specialized



We next study the relationship between plant-size and vertical span. In many settings, larger firms do more activities: it is well documented that they sell a wider variety of products and operate in a wider range of industries, operate more plants, and sell to a wider variety of destinations.[4] Nevertheless, we show here that larger and more successful plants tend to have shorter vertical spans, and hence do fewer steps.

Fact 2 (Micro-level correlations between size and vertical span) *We have:*

(a) Plants with higher sales within their industry on average have a shorter vertical span

(b) Plants that, within their industry, grow faster on average reduce their vertical span more.

Figure 2.3 shows (a) by plotting the relationship between vertical span and size within 5-digit industry $\times$ year cells. Table II shows that this correlation also holds within narrow industry $\times$ district $\times$ year cells, and also conditional on plant age and employment. Table III corroborates part (b) of Fact 2 and shows regressions of the change in sales on the change in vertical span, over all time horizons where firms produce the same single five-digit product. The point estimates are consistently negative across specifications.

Finally, we mention some other covariates of vertical span: firms with shorter vertical span have higher materials share of cost, are more likely to import, and have a higher share of relationship-specific inputs in their materials basket. These correlations hold in the cross-section (within industries) and over time (within plant-industries). Table XII in Appendix C shows these results.

2.4 Determinants of Vertical Specialization

A plant's vertical span and size are, of course, jointly determined. How a plant organizes itself may affect its marginal cost and therefore its size, and a plant's size might affect incentives to organize

[4]See, e.g., Bernard, Redding and Schott (2010), Cao et al. (2022), and Bernard and Jensen (1999).

Table II More specialized plants are larger in their industry

	Dependent variable: Log Sales				
	(1)	(2)	(3)	(4)	(5)
Vertical Span	-0.719**	-0.670**	-0.431**	-0.432**	-0.193**
	(0.024)	(0.023)	(0.034)	(0.034)	(0.015)
Age				0.00616**	-0.00314**
				(0.0012)	(0.00068)
Log Employment					1.067**
					(0.015)
Year FE	Yes	Yes	Yes	Yes	Yes
5-digit Industry FE	Yes	Yes			
District FE		Yes			
Industry × District × Year FE			Yes	Yes	Yes
R^2	0.394	0.440	0.700	0.701	0.859
Observations	353659	295094	140610	136831	136608

Standard errors in parentheses, clustered at the 5-dgt industry level.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

Table III Plant growth is correlated with increased vertical specialization

	Dependent variable: Δ log Sales			
	(1)	(2)	(3)	(4)
Δ Vertical Span	-0.0655**	-0.0445**	-0.0284*	-0.0259*
	(0.0082)	(0.0087)	(0.013)	(0.011)
Year FE	Yes			
Product × Year FE		Yes	Yes	Yes
Plant FE			Yes	
Plant × Product FE				Yes
R^2	0.00819	0.149	0.432	0.431
Observations	120436	111244	83026	74707

Changes within plant-products
Standard errors in parentheses, clustered at the state-industry level.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

its production in a particular way. In this section we attempt to identify a causal channel: if a plant grows in response to higher demand, how does it change its vertical span?

As discussed in Section 2.1, we use tariff changes as a source of exogenous variation. These tariff changes can act as a demand shock through import competition or as a supply shock by changing the cost of materials. Table IV shows that changes in import tariffs are positively correlated with changes in the sales of single-product plants that produce this good (the import competition channel). In column (2), we also include the industry's change in input tariffs (weighted by the shares in the aggregate industry's materials basket). The coefficient on this term is negative, meaning that plant sales increase as input tariffs decrease. These relationships mean tariff changes act as industry-level demand shifters and cost shifters.

We turn to exploring the direction of causality between scale and vertical span. Table V shows regressions of the plant's vertical span on sales (again, for single-product plants only), within plant

Table IV Impact of Import Competition on Plant Sales

	Dependent variable: Δ log Sales	
	(1)	(2)
Δ log Output Tariff	0.159^{+}	0.235^{*}
	(0.090)	(0.094)
$\Delta \log(1+\bar{\tau}^{\text{input}}_{\omega t})$		-0.222^{+}
		(0.12)
Year-Pair FE	Yes	Yes
R^2	0.0624	0.0626
Observations	104996	104985

Standard errors in parentheses, clustered at the state × industry level.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

× product pairs, and controlling for year effects. Columns (1)-(3) report OLS results, and find a negative correlation between in sales and in span. Columns (4)-(6) instrument sales by log import tariffs on the plants' output. The IV coefficient estimates are larger than the OLS ones, and point to a negative causal relationship: as plants shrink because of a reduced output tariff, they increase their vertical span of production. Because output tariff changes may be correlated with input tariff changes, columns (2) and (5) also control for plant-level input tariffs (where we weigh tariffs by the plants' expenditure shares at time of first observation), and whether those input tariff changes are larger for inputs that are more distant.

Taken together, the regressions in Table V suggest that Smith's famous adage about the division of labor being limited by the extent of the market applies to our context in the sense that vertical span captures the division of labor across production units.

Next, we turn to searching for network externalities. When firms receive a negative demand shock, they scale down or exit. This means that there may be in turn lower demand for the goods of producers further upstream, etc. In neoclassical models with constant returns, demand shocks propagate upstream. In the presence of scale economies or network externalities, they may also propagate downstream again.

In Table VI, we search for upstream propagation of demand shocks. We regress log sales of plants j onto the weighted average log number of producers of j's inputs in j's state, where the weights are again j's materials shares at the time of first observation:

$$p_j y_j = \beta \left(\sum_{\hat{\omega}\in\Omega} \frac{p_{j\hat{\omega}} x_{j\hat{\omega}}}{\sum_{\omega'} p_{j\omega'} x_{j\omega'}} \log(\#\text{producers of } \hat{\omega} \text{ in state } d)_t \right) + \alpha_{j\omega} + \alpha_{\omega t} + \varepsilon_{jt}$$

The inclusion of plant-product fixed effects means that we again look at changes in the dependent and independent variables. Columns (1) to (3) estimate this relationship using OLS. In columns (4) to (6) we instrument the number of producers in upstream industries $\hat{\omega}$ by the log tariff on goods $\tilde{\omega}$ weighted by the share of $\hat{\omega}$'s sales to $\tilde{\omega}$, but excluding j's industry ω itself:

Table V Firms vertically specialize when they receive a positive demand shock

	Dependent variable: Vertical Span					
	(1)	(2)	(3)	(4)	(5)	(6)
Log Sales	-0.0191**	-0.0190**	-0.0196**	-0.512^{+}	-0.382^{+}	-0.252^{+}
	(0.0020)	(0.0020)	(0.0024)	(0.28)	(0.22)	(0.13)
$\log(1+\bar{\tau}^{\text{input}}_{j\omega t})$		-0.0877**	-0.0209		-0.0830^{+}	0.0194
		(0.026)	(0.050)		(0.050)	(0.065)
$\sum_i \alpha_i \log(1+\bar{\tau}^{\text{input}}_{it})\overline{\text{span}}_j$			-0.0910			-0.218*
			(0.056)			(0.099)
$\sum_i \alpha_i \log(1+\bar{\tau}^{\text{input}}_{it})(\text{distance}_{\omega i} - \overline{\text{span}}_j)$			-0.122			-0.336*
			(0.095)			(0.15)
Year FE	Yes	Yes	Yes	Yes	Yes	Yes
Plant × Product FE	Yes	Yes	Yes	Yes	Yes	Yes
Estimator	OLS	OLS	OLS	IV	IV	IV
R^2	0.765	0.765	0.731	-1.049	-0.569	-0.229
Observations	186628	186628	145181	138204	138204	137060

Standard errors in parentheses, clustered at the state-industry level. $^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

Regression contains single-product plants only. The left-hand side is the plant's vertical span of production; the right-hand side is log sales. Columns (3) and (4) instrument log sales by the log tariff on the plant's output ω.

$$\sum_{\hat{\omega}\in\Omega} \frac{p_{j\hat{\omega}} x_{j\hat{\omega}}}{\sum_{\omega'} p_{j\omega'} x_{j\omega'}} \left(\sum_{\tilde{\omega}\in\Omega,\ \tilde{\omega}\neq\omega} \nu_{\hat{\omega}\tilde{\omega}} \log(1+\tau_{\tilde{\omega}t}) \right)$$

where $\nu_{\hat{\omega}\tilde{\omega}}$ denotes the share of sales of industry $\hat{\omega}$ to industry $\tilde{\omega}$ (but ignoring sales to ω). In other words, we instrument the number of producers in upstream industries by demand shocks to industries downstream from these industries, but excluding the industry of j itself. The regressions in Table XV of Appendix C confirm that tariff shocks in downstream industries affect entry.

The IV regressions in columns (4) to (6) find a positive and significant coefficient: the demand shocks in the form of tariff changes to industries that are not directly vertically related also change the cost of production in the industry. These changes to cost may be coming from internal economies of scale in upstream sectors, or from network externalities as postulated by Young (1928). The model in the next section will help to understand the role of both channels and serve as a basis for identifying them.

2.5 Spiders, Snakes, and Trees

In this section we provide evidence on the structure of a firm's production, studying how the number of inputs a plant uses relates to vertical specialization. Baldwin and Venables (2013) coined names of two commonly used models of the structure of production, *spiders* and *snakes*. In a spider, input use has only a horizontal dimension. There is only one production stage which combines

Table VI Demand shocks also propagate downstream

	Dependent variable: log Sales					
	(1)	(2)	(3)	(4)	(5)	(6)
Avg. log #Producers in Upstream Ind.	0.0466^{**}	0.0383^{**}	0.0375^{**}	0.0383^{*}	0.0613^{**}	0.0738^{**}
	(0.0041)	(0.0050)	(0.0060)	(0.017)	(0.017)	(0.018)
$\log(1+\bar{\tau}^{\text{input}}_{j\omega t})$			-0.0243			-0.0208
			(0.096)			(0.097)
$\sum_i \alpha_i \log(1+\bar{\tau}^{\text{input}}_{it})(\text{distance}_{\omega i} - \overline{\text{span}}_j)$			-0.330^{**}			-0.333^{**}
			(0.11)			(0.11)
Year FE	Yes			Yes		
Industry × Year FE		Yes	Yes		Yes	Yes
Plant × Industry FE	Yes	Yes	Yes	Yes	Yes	Yes
Estimator	OLS	OLS	OLS	IV	IV	IV
R^2	0.942	0.952	0.954	0.00183	0.000631	0.000277
Observations	215805	199039	142041	215805	199039	142041

Standard errors in parentheses, clustered at the industry-year level. $^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$
Regression contains single-product plants only. The left-hand side is the plant's sales; the right-hand side is the change in log sales. Changes are taken across all time horizons within the same plant-product pairs. Columns (3), (4), (7), and (8) instrument Δ log sales by the change in the log output tariff.

multiple intermediate inputs to produce output.[5] In a snake, input use is vertical. There are several sequential stages of production, and at each stage a single intermediate is combined with labor and the output of that stage is passed on to the next stage.[6] The length of a snake has a clear connection to a plant's vertical span, but no relation to the number of inputs. In contrast, the width of a spider has a clear connection to number of inputs, but an ambiguous connection to vertical span. A *production tree* combines the vertical dimension of a snake with the horizontal dimension of a spider. There are multiple sequentially related stages, and each stage may require more than one intermediate input.[7] Figure 4 depicts these three types of production structures.

The different production structures hard-wire different relationships between the number of tasks that are outsourced and the number of inputs. With a spider production structure, outsourcing more tasks would correspond to purchasing *more* intermediate inputs. With a snake production structure, the number of inputs used is invariant to outsourcing. With a tree production structure, outsourcing more tasks would correspond to purchasing *fewer* intermediate inputs. For example, the firm pictured in panel (c) can choose to purchase $\tilde{\omega}_1$ from a supplier or produce it in-house, in which case it would purchase $\tilde{\omega}_4$ and $\tilde{\omega}_5$ from suppliers. The former corresponds to a shorter vertical span and fewer inputs, whereas the latter corresponds to a longer vertical span and more inputs.

[5]Spider production functions are standard in the macro literature with input-output structures such as Long and Plosser (1983), Acemoglu et al. (2012), and Baqaee and Farhi (2019). Spider production structures with an endogenous number/measure of intermediate inputs include Acemoglu, Antràs and Helpman (2007), Eaton, Kortum and Kramarz (2022), Taschereau-Dumouchel (2017), Lim (2018), Huneeus (2018), Chan (2017), Blaum, Lelarge and Peters (2015), and Tintelnot et al. (2018).

[6]Snake production structures have been used Dixit and Grossman (1982), Yi (2003), Levine (2010), Costinot, Vogel and Wang (2013), Chaney and Ossa (2013), Fally and Hillberry (2018), and Antràs and de Gortari (2017).

[7]In graph theory, this is a rooted, directed tree.

We therefore study the relationship between the number of inputs, the vertical span, and the scale of production of plants. In particular, we present evidence that that is consistent with an important role for a vertical dimension of input use, consistent with a snake or a tree structure. Table VII shows correlations between the vertical span of plants and the number of five-digit materials inputs that the plant consumes in the production process (alternatively, the inverse Herfindahl of expenditure shares on 5-digit inputs[8]). In both the cross-section (within industries) and in the time dimension, plants that have a long vertical span on average use more inputs. We find that plants with larger share of intermediate inputs in total cost tend to use, if anything, fewer inputs (Appendix C.2), consistent with a tree but not with a spider.

Table VIII shows regressions of the plant's log number of inputs (columns (1) to (4)) or inverse HHI of cost shares of the plant's materials basket (columns (5) to (8)) on sales, within plant-product pairs. In the IV specifications, columns (2), (3), (7), and (8), we again instrument sales using the level of output tariffs. While the OLS estimates of the sales coefficient are perhaps positive, the IV estimates show a clear picture: decreases in scale coming from negative demand shocks are associated with an increase in the number of inputs, and vice versa.

Table VII Production Structures are Trees

	Dependent variable: Vertical Span			
	(1)	(2)	(3)	(4)
Log Number of Inputs	0.0169**	0.0533**		
	(0.0031)	(0.0045)		
Inverse HHI of Materials Cost Shares			0.0269**	0.0337**
			(0.0021)	(0.0042)
Year FE	Yes	Yes	Yes	Yes
Industry FE	Yes		Yes	
Plant × Industry FE		Yes		Yes
Estimator	OLS	OLS	OLS	OLS
R^2	0.302	0.766	0.304	0.767
Observations	353694	186641	353432	186486

Standard errors in parentheses, clustered at the state-industry level.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

Regression contains single-product plants only. The left-hand side is vertical span; the right-hand side variables are the plant's number of 5-digit materials inputs (columns (1) and (2)) and the inverse of the HHI of materials shares (columns (3) and (4)).

These patterns are consistent with a tree production structure in which reductions in vertical span and reductions in number of inputs go hand-in-hand. We found that reductions in vertical

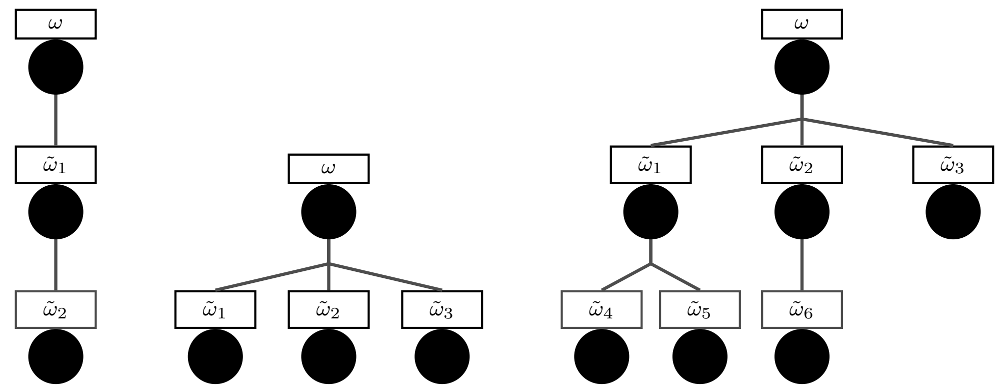
[8]The inverse of the Herfindahl index is a common, ad hoc measure of the diversity of inputs. Simply counting the number of inputs would not differentiate between a situation in which a plant splits its intermediate input spending equally on two inputs, and one in which the plant uses two intermediate inputs but 99.9% of its expenditure on intermediates is spent on one of them. In the latter case, we would argue that the plant "effectively" uses one input. Formally, the inverse Herfindahl can be interpreted as the number n of inputs such that, if cost shares were uniform across those n inputs, would deliver the same level of concentration of input spending as the actual input bundle. To see this, note that with firm used n inputs each with a share $1/n$, the inverse Herfindahl would be $\left(\sum_{i=1}^{n}\left(\frac{1}{n}\right)^2\right)^{-1} = n$. Thus we interpret the inverse Herfindahl index as capturing the "effective" number of inputs.

Table VIII Firms reduce the number of inputs when they receive a positive demand shock

	Dependent variable: log Number Of Inputs				Dependent variable: Inverse Input HHI			
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Log Sales	0.0477**	0.0479**	-1.321*	-0.674*	0.0101	0.0104	-1.888^{+}	-1.055^{+}
	(0.0033)	(0.0032)	(0.64)	(0.34)	(0.0072)	(0.0069)	(1.03)	(0.59)
$\log(1+\bar{\tau}^{\text{input}}_{j\omega t})$		-0.244**		-0.369**		-0.411^{+}		-0.498*
		(0.086)		(0.10)		(0.25)		(0.24)
Year FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Plant × Product FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Estimator	OLS	OLS	IV	IV	OLS	OLS	IV	IV
R^2	0.871	0.872	-6.543	-1.816	0.807	0.808	-3.437	-1.076
Observations	188868	188803	138938	138898	192809	192809	142270	142270

Standard errors in parentheses, clustered at the state-industry level.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$
Regression contains single-product plants only. The left-hand side is the plant's number of 5-digit materials inputs (columns (1) to (4)) and the inverse of the HHI of materials shares (columns (5) to (8)); the right-hand side is log sales. Columns (3), (4), (7), and (8) instrument log sales by the log output tariff.

span correspond to using *fewer* intermediate inputs. Further, the same increases in demand that cause plants to reduce vertical span also cause them to reduce the number of inputs. This finding will guide the theoretical framework described below.



(a) Snake (b) Spider (c) Tree

Figure 4 Production Structures

The figure shows three different ways inputs (denoted by $\tilde{\omega}_i$) can be combined to form an output ω. In a snake, each stage combines one input with primary factors to produce the stage output, and several of these stages are performed sequentially to produce the final output. In the spider, there is only one production stage which combines multiple intermediate inputs to produce output. In general the production structures form a tree, which combines multiple vertically related stages with the fact that more than one input may be required to produce the stage output.

3 A Simple Model of Vertical Span

In this section, we describe a simple model of a firm's vertical span of production. There are a large number of industries. Each industry is comprised of firms that produce imperfectly substitutable varieties. Firm j in industry ω produces a variety of ω, and is born with ex-ante productivity q_j. Each firm can, in principle, produce using different recipes that have different vertical spans. Each firm pays a convex cost to search for suppliers. Firms encounter potential suppliers according to a matching function.

The key mechanism operates through the decision to search for suppliers of different inputs. Firms born with higher productivity or that expect higher demand will tend to search more, which will be a source of increasing returns. As we will show, the vertical structure of the technology menu makes the returns to searching across inputs asymmetric, and search decisions respond asymmetrically to changes in productivity and demand. This non-homotheticity implies that ex-ante heterogeneous firms select differentially into different vertical spans, generating the size/specialization relationship emphasized in Section 2.1.

To build intuition, this section focuses on a simple version of the model. We focus mostly on the problem of a single firm that can produce performing either one task or two vertically related tasks arranged as a snake. In Section 5 we present a full quantitative model that allows for richer production structures including trees and that can be taken to the data.

3.1 Environment: Simple Model

There are a large number of industries arranged in a circle. Each industry is indexed by ω. In each industry there is a finite measure of firms that produce differentiated varieties. Each firm in industry ω can produce in two ways, either using a variety in industry $\omega - 1$ as an intermediate input or using a variety in industry $\omega - 2$ as an intermediate input. That is, a firm can perform one task in production, purchasing a variety of $\omega - 1$ and using labor to transform it into its variety of type ω; or perform two tasks, purchasing a variety of $\omega - 2$, using labor to transform it into $\omega - 1$, and then using labor again to transform the product into its variety of type ω.

Firm j is born with an idiosyncratic Hicks-neutral productivity draw q_j. Before producing, each firm j in industry ω exerts search effort h_{j1} and h_{j2} to find potential suppliers in industries $\omega - 1$ and $\omega - 2$ respectively. Search yields a set of matches with potential suppliers. For each match, firm j draws a match-specific, input-augmenting productivity z that is specific to using supplier s's variety as an input. After searching, firm j also gets an idiosyncratic productivity shock specific to the more upstream task, B_j. If firm j produces using supplier s in industry $\omega - 1$ as an input, its production function is

$$y_j = q_j a l_{j1}^{1-\alpha} \left[z_{j1s} x_{j1s}\right]^{\alpha},$$

whereas if it uses supplier s in industry $\omega - 2$ its production function is

$$y_j = q_j a l_{j1}^{1-\alpha} \left[B_j a l_{j2}^{1-\alpha} \left(z_{j2s} x_{j2s}\right)^{\alpha}\right]^{\alpha}$$

where $a \equiv (\alpha^{\alpha}(1-\alpha)^{1-\alpha})^{-1}$ is a normalizing constant. The nested structure of the production functions captures the sequential nature of the two tasks in the production process.

3.2 Market Structure

There is a representative household with nested CES preferences. It consumes an aggregate of industry bundles, $u \equiv \left(\sum_{\omega} \delta_{\omega}^{\frac{1}{\eta}} u_{\omega}^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}$, where the industry bundle for industry ω is $u_{\omega} = \left(\int_{J_{\omega}} u_{\omega j}^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}$ where J_{ω} is the set of firms producing varieties in industry ω and $\varepsilon > 1$. δ_{ω} is a preference parameter that shifts the household's demand for industry ω.

Firms sell their goods to the household and to other firms that are further downstream. We assume in this section that firms price at marginal cost when selling to other firms.[9] Firms engage in monopolistic competition when selling to the household.

The mass of firms in each industry is endogenous. Upon entry, each firm draws its ex-ante productivity q from a fixed distribution with CDF $Q(\cdot)$. We model entry by having a representative entrepreneur that takes as given the distribution of profit within each industry, and chooses a mass of entrants for each industry subject to a constant elasticity of transformation function

$$\max_{\{J_{\omega}\}} \sum_{\omega} J_{\omega}\bar{\pi}_{\omega} - w\frac{1}{1+1/\chi}\left(\sum_{\omega} J_{\omega}^{\frac{1+\beta}{\beta}}\right)^{\frac{\beta}{1+\beta}(1+1/\chi)}$$

where $\bar{\pi}_{\omega}$ is expected profit for an entrants into industry ω, or, equivalently, average profit among all entrants into the industry. Note that this nests free entry ($\beta, \chi \to \infty$) and inelastic entry ($\beta = \chi = 0$) as special cases. We assume $\beta < \infty$, so that an increase in demand is not fully absorbed by new entrants.

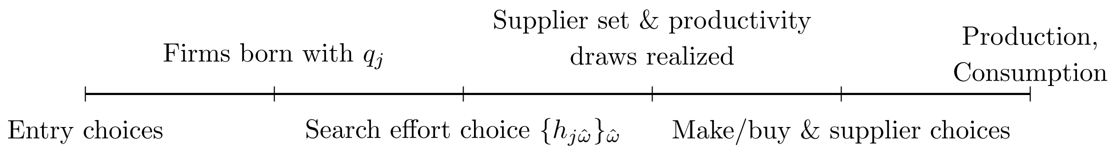
3.3 Search

We assume that the search happens according to a matching function. In particular, if a firm j in industry ω exerts search effort h_{j1} to find suppliers in industry $\omega-1$, the arrival of matches depends on $h_{j1}M(J_{\omega-1})$, where the matching function M is weakly increasing in the mass of firms $|J_{\omega-1}|$. Following the findings of Miyauchi (2018), we assume that the arrival rate increases with the mass of potential suppliers, but congestion from others buyers that are also searching for suppliers does not reduce the matching rate.[10] If M is strictly increasing, then the matching function exhibits increasing returns.

[9] With a notion of bargaining power as in Oberfield (2018), the assumption that each firm sets a price of marginal cost in firm-to-firm trade is equivalent to the equilibrium in which buyers have complete bargaining power.

[10] Miyauchi (2018) found that when a firm lost its supplier due to an unexpected supplier bankruptcy, the speed with which the firm found a new supplier increased with density of suppliers, but was uncorrelated with buyer density. In our view, this is the most credible evidence to date on matching functions in the context of firm-to-firm trade.

3.4 Timing



All entering firms draw their Hicks-neutral productivity shifter q_j. Then they simultaneously decide how much to search, choosing h_{j1} and h_{j2}. Nature then reveals each firm's set of potential suppliers along with match-specific productivities and each firm's task productivity B_j. Then all firms set prices and make production choices (i.e. choosing production function and suppliers to minimize cost, and choosing quantities to maximize profit) simultaneously. Finally, production and consumption occur.

For firm j in industry ω born with productivity q_j, let $\pi_{\omega j}^{gross}$ be the realized gross profit (gross of search costs). If that firm exerts search effort h_{j1} and h_{j2}, then $E\left[\pi_{\omega j}|q_j, h_{j1}, h_{j2}\right]$ is its expected profit taking as given the search decision of all other firms, and expectations are taken over realizations of all firms' matches and task productivities. We assume that the cost of search effort is isoelastic. Then the firm's choice of search effort maximizes expected profit net of search costs:

$$\max_{h_{j1},h_{j2}} E\left[\pi_{\omega j}^{gross}|q_j, h_{j1}, h_{j2}\right] - w\frac{k}{1+\gamma}h_{j1}^{1+\gamma} - w\frac{k}{1+\gamma}h_{j2}^{1+\gamma}.$$

3.5 Functional Form Assumptions

We make some functional form assumptions that will help in the characterization of the equilibrium.

First, we assume that if firm j chooses search effort (h_{j1}, h_{j2}), then the number of potential suppliers in industry $\omega - 1$ with match-specific productivity greater than z is Poisson with mean $h_{j1}M(J_{\omega-1})z^{-\zeta}$, and the number of potential suppliers in industry $\omega - 2$ with match-specific productivity greater than z is Poisson with mean $h_{j2}M(J_{\omega-2})z^{-\zeta}$. We will use the shorthand $m_1 = M(J_{\omega-1})$ and $m_2 = M(J_{\omega-2})$ when there is no risk of ambiguity. In our numerical implementations we parameterize M as $M(J) = J^{\mu}$.

Second, we assume that the task productivities are independent and identically distributed across firms and take the form $B_j = e^{b_j/\zeta}$, where b_j is a random variable with characteristic function $\frac{\Gamma(1-it)}{\Gamma(1-\alpha it)}$, or equivalently that $e^{b_j/\alpha}$ is an α-stable random variable. We discuss the role of this functional form assumption in footnote 11 below.

Third, we assume that the distribution of productivity with which firms are born has a sufficiently thin tail: $\lim_{q\to\infty} q^{\zeta\frac{1+\gamma}{\gamma}}\left[1 - Q(q)\right] = 0$. This assumption ensures that average profits remain finite.

3.6 Characterizing the Equilibrium

We characterize the equilibrium by working backward. Each firm's production cost exhibits constant returns to scale. Given its search effort and that of other firms, the realization of matches, and all firms' production choices, let c_j denote the realization of j's unit cost. Let $c^o_{j1} = \min_{s \in S_{j1}} \frac{p_s}{z_{js}}$ be the effective cost delivered by the best supplier of input $\omega - 1$, and $c^o_{j2} = \min_{s \in S_{j2}} \frac{p_s}{z_{js}}$ be the effective cost delivered by the best supplier of input $\omega - 2$. The cost of producing intermediate $\omega - 1$ in-house is $c^i_{j1} = \frac{1}{B_j} w^{1-\alpha} (c^o_{j2})^\alpha$. Thus the effective cost of intermediate $\omega - 1$ is $c_{j1} = \min\{c^o_{j1}, c^i_{j1}\}$. Finally, j's unit cost is $c_j = \frac{1}{q_j} w^{1-\alpha} c^\alpha_{j1}$. Let $F_\omega(c)$ denote the cumulative distribution of unit costs in industry ω.

We first characterize the possible realizations of firm j's unit cost, given its choices of search effort. We then use this to solve for the optimal search effort.

Proposition 1 *For a firm j in industry ω with productivity q_j that chooses search effort h_{j1}, h_{j2},*

(a) The effective cost of outsourcing intermediate input $\omega - k$ follows a Weibull distribution with counter-cumulative distribution

$$\Pr\left(c^o_{jk} > c | q_j, h_{j1}, h_{j2}\right) = e^{h_{jk} m_k \bar{c}^{-\zeta}_{\omega-k} c^\zeta}$$

with $\bar{c}_{\omega-k} \equiv \left(\int c^{-\zeta} dF_{\omega-k}(c)\right)^{-\frac{1}{\zeta}}$.

(b) The effective cost of producing $\omega - 1$ in-house follows a Weibull distribution with counter-cumulative distribution

$$\Pr\left(c^i_{j1} > c | q_j, h_{j1}, h_{j2}\right) = e^{h^\alpha_{j2} m^\alpha_2 \left(\bar{c}^\alpha_{\omega-2} w^{1-\alpha}\right)^{-\zeta} c^\zeta}.$$

(c) The conditional probability that the firm uses a supplier in recipe using $\omega - 1$ is chosen is

$$\frac{h_{j1} m_1 \bar{c}^{-\zeta}_{\omega-1}}{h_{j1} m_1 \bar{c}^{-\zeta}_{\omega-1} + h^\alpha_{j2} m^\alpha_2 \left(\bar{c}^\alpha_{\omega-2} w^{1-\alpha}\right)^{-\zeta}}$$

(d) The firm's expected profit is

$$E\left[\pi^{gross}_{\omega j} | q_j, h_{j1}, h_{j2}\right] = A_\omega \delta_\omega q^{\varepsilon-1}_j \left\{\left[h_{j1} m_1 \bar{c}^{-\zeta}_{\omega-1} + h^\alpha_{j2} m^\alpha_2 \left(\bar{c}^\alpha_{\omega-2} w^{1-\alpha}\right)^{-\zeta}\right]^{-\frac{\alpha}{\zeta}} w^{1-\alpha}\right\}^{1-\varepsilon}$$

where $A_\omega \equiv u p^\eta p^{\varepsilon-\eta}_\omega \frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^\varepsilon} \Gamma\left(1 - \alpha \frac{\varepsilon-1}{\zeta}\right)$

Parts (a) and (b) of Proposition 5 characterize the distribution of costs of different choices conditional on q_j and on search choices. Both costs, in-house production and outsourcing, follow a Weibull distribution with the same shape parameter, and the realizations are conditionally independent.[11] The probability that the firm chooses to outsource the input hence follows the well-known expression that depends on the relative scale parameters (part (c)). Part (d) gives the expected profit conditional on q_j and search choices, which allows us to restate the search effort choice problem as:

$$\max_{h_{j1},h_{j2}} A_\omega \delta_\omega q_j^{\varepsilon-1} \left\{ \left[h_{j1} m_1 \bar{c}_{\omega-1}^{-\zeta} + h_{j2}^\alpha m_2^\alpha \left(\bar{c}_{\omega-2}^\alpha w^{1-\alpha} \right)^{-\zeta} \right]^{-\frac{\alpha}{\zeta}} w^{1-\alpha} \right\}^{1-\varepsilon} - w \frac{k}{1+\gamma} h_{j1}^{1+\gamma} - w \frac{k}{1+\gamma} h_{j2}^{1+\gamma} \tag{1}$$

The optimal choices of search intensity imply a non-homotheticity in input use:

Proposition 2 *In industry ω, the probability of using $\omega-1$ is increasing in q_j. Sales are positively correlated with the decision to use $\omega-1$.*

The empirical counterpart to Proposition 2 is a negative correlation between vertical span and size within industries: firms are born with varying Hicks-neutral productivity q_j, and choose search efforts based on this draw. Firms born with higher productivity choose higher search intensities and but also increasingly tilt their search toward suppliers in industry $\omega-1$, resulting in a higher probability of choosing the production function in $\omega-1$. At the same time, those firms born with higher productivity are likely to have more customers and sell more to the household, and so will end up being larger on average.

What is the source of this nonhomotheticity? Mathematically, given the curvature in the expected profit as a function of the search intensities, the returns to search diminish more quickly for suppliers in $\omega-2$ than for suppliers in $\omega-1$ (see equation (1)). Hence, those firms that search more increasingly tilt their searches toward suppliers in $\omega-1$.

The intuition behind this result is subtle. $\omega-2$ is more complementary with labor than is $\omega-1$ (because labor used for the upstream task has a unitary elasticity with $\omega-2$ but is a substitute of $\omega-1$). Labor is special because, unlike for intermediate inputs, the firm cannot reduce the wage. A firm that increases search effort will gradually expand its expected expenditure share on $\omega-1$

[11] The functional form for the distribution of the task-specific productivity B_j is chosen to solve the following problem. The effective cost of purchasing input $\omega-1$ follows a Weibull distribution with shape ζ. The effective cost of purchasing input $\omega-2$ is also Weibull with shape ζ. Without B_j, the effective cost of producing $\omega-1$ in-house would be Weibull-distributed with shape $\frac{\zeta}{\alpha}$ because $\omega-2$ has an output elasticity of α in the production of $\omega-1$. This would be unfortunate because discrete choice problems with extreme-value random variables only work well when the random variables have the same shape parameter. To make the discrete choice problem tractable, the distribution of B_j is chosen so that the cost of producing $\omega-1$ in-house (incorporating B_j) follows a Weibull distribution with shape ζ.

Why does this particular functional form work? We build on the observation of Shanbhag and Sreehari (1977) that if Z is a standard exponential random variable and X is an α-stable random variable defined by the Laplace transform $E\left[e^{-uX}\right] = e^{-u^\alpha}$, then $\left(\frac{Z}{X}\right)^\alpha$ is also a standard exponential random variable. The proof is remarkably simple:

$$\Pr\left((Z/X)^\alpha > u\right) = \Pr\left(Z > u^{1/\alpha} X\right) = \int_0^\infty e^{-u^{1/\alpha} x} d\Pr(X \le x) = E\left[e^{-u^{1/\alpha} X}\right] = e^{-u} .$$

If we raise each side of the equation to the power $1/\zeta$, we obtain that $\frac{Z^{\alpha/\zeta}}{X^{\alpha/\zeta}}$ has the same distribution of $Z^{\frac{1}{\zeta}}$. In other words, the ratio of a Weibull-distributed random variable with shape ζ/α and the random variable $X^{\alpha/\zeta}$ is a Weibull-distributed random variable with shape ζ. B_j in the model plays the role of $X^{\alpha/\zeta}$.

relative to $\omega - 2$. This raises the return to searching for $\omega - 1$ relative to the return to searching for $\omega - 2$.[12]

An example might help to illustrate the mechanism. Consider a producer of polished diamonds who faces the choice of using cut diamonds or rough diamonds as inputs.[13] These two recipes are substitutable, but each production step requires her to combine labor and the intermediate input in an imperfectly substitutable manner. When faced with a higher q_j, the producer would exert a higher search effort for both cut and rough diamonds. She would, however, increase her search effort for rough diamonds (the upstream input) proportionally less than for cut diamonds, knowing that when she searches more for rough diamonds the cost of self-produced cut diamonds falls proportionally by less, because the rough diamonds would still need to be combined with labor, whose cost is not decreasing.

3.7 A Shift in Demand

We now consider a shift in household preferences that raises demand for the industry ω. The implications are summarized by Proposition 3

Proposition 3 *Suppose that entry is not completely elastic or inelastic ($\beta \in (0, \infty)$). An increase in δ_ω causes*

- *more entry in industry ω: $J_\omega \nearrow$*
- *the price levels in industry ω falls: $p_\omega, \bar{c}_\omega \searrow$*

[12] To see this, consider a related simpler problem: A firm produces with the production function $y = qf(l, h_1 x_1, h_2 x_2)$, faces isoelastic demand $y = \delta p^{-\varepsilon}$, and can choose h_1 and h_2 at costs $\frac{h_1^{1+\gamma}}{1+\gamma}$ and $\frac{h_2^{1+\gamma}}{1+\gamma}$. An increase in q raises incentives to search. Changing either h_1 or h_2 leads to a change in the marginal product of h_1 and of h_2. To determine whether a firm born with higher productivity would tilt their search toward h_1, one must compare (i) how raising h_1 would change the marginal product of h_1 relative to the marginal product of h_2, to (ii) how raising h_2 would change the marginal product of h_2 relative to the marginal product of h_1. These are encoded in the Morishima elasticities of substitution between h_1 and h_2: A firm born with higher productivity would tilt its search relatively more toward h_1 (i.e. $\frac{d\ln h_1}{d\ln q} > \frac{d\ln h_2}{d\ln q}$) if the direct Morishima elasticity of substitution of input 1 for input 2 is higher than the elasticity of input 2 for input 1.

More concretely, suppose that f takes the nested CES form,

$$y = q\left\{ (A_1 h_1 x_1)^{\frac{\eta-1}{\eta}} + \left[(A_0 l)^{\frac{\phi-1}{\phi}} + (A_2 h_2 x_2)^{\frac{\phi-1}{\phi}} \right]^{\frac{\phi}{\phi-1}\frac{\eta-1}{\eta}} \right\}^{\frac{\eta}{\eta-1}}.$$

Under parameter restrictions which guarantee the choices of search effort are interior, whether an increase in q leads to a larger proportional increase in h_1 or h_2 depend on whether $\eta \gtreqless \phi$. That is, $\eta > \phi$ implies $\frac{d\ln h_1}{d\ln q} > \frac{d\ln h_2}{d\ln q} > 0$ whereas $\eta < \phi$ implies $\frac{d\ln h_2}{d\ln q} > \frac{d\ln h_1}{d\ln q} > 0$. Importantly, the cost shares of each input play no role (i.e., the relative magnitudes of A_0, A_1, A_2 play no role).

In the model, expected profit takes a form similar to this nested CES production function, where the elasticity of substitution of the inner nest translates to one (the elasticity of substitution between labor and input $\omega - 2$ in the upstream production function), while the elasticity across outer nests translates to $1 + \zeta$ (the effective elasticity of substitution across production recipes induced by discrete choice across Weibull distributed random variables with shape parameter ζ). Since $1 + \zeta > 1$, $\frac{d\ln h_1}{d\ln q} > \frac{d\ln h_2}{d\ln q} > 0$.

[13] In the former case, the producer would just polish the cut diamonds, whereas in the latter case she would have to do both cutting and polishing.

- *the fraction of firms in industry ω using $\omega - 1$ increases.*

The proof shows that the positive demand shock sets in motion a sequence of events. First, the demand shock raises profit. This increases entry, but since entry is not fully elastic, expected demand for each firm rises. Firms increase their search intensity. Both the increased entry and increased search intensity lower the industry price index, due both to increased variety and to lower cost for each q. In addition, with increased search intensity, firms tilt their search toward suppliers in industry $\omega - 1$, for the same reason as above.[14]

We think of Proposition 3 as speaking to the empirical results in Table V, that demand shocks affect the vertical span.

3.8 A Shift in Demand for an Upstream Industry

We next consider the impact of a shift in household demand for industry $\omega - 1$.

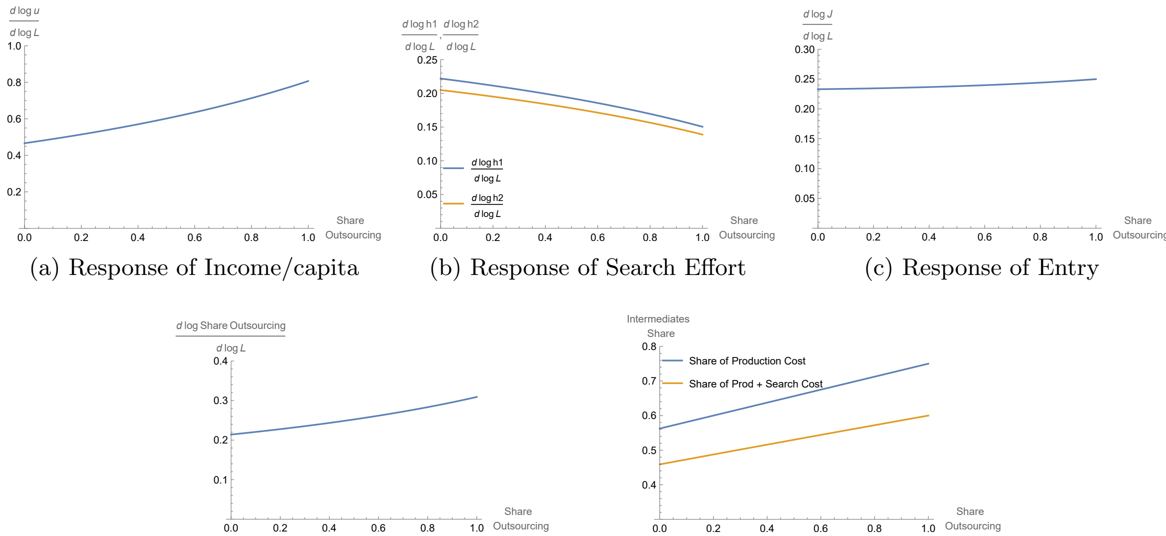
Proposition 4 *If $\delta_{\omega-1}$ increases, then if γ is sufficiently large (search effort not too elastic):*

- *more entry in industry $\omega - 1$: $J_{\omega-1} \nearrow$, $c_{\omega-1} \searrow$*
- *the fraction of firms in industry ω using $\omega - 1$ increases*
- *total sales in industry ω increase*

The first result—that the increase in $\delta_{\omega-1}$ raises entry in $\omega-1$ and increase the value of searching for suppliers in $\omega - 1$—is simply an application of Proposition 3. The increase in $v_{\omega-1}$ would lead to a reduction in the distribution of cost even if those in ω did not change their search effort. This implies that the price index p_ω declines. The fact that $v_{\omega-1}$ increases raises incentives to search, but the fact that p_ω declines reduces incentives to search. If γ is sufficiently large, we can show that the former dominates and all firms search more.[15] When firms search more, search effort increases more for suppliers in $\omega - 1$ than for suppliers in $\omega - 2$. This, along with the reduction in $\bar{c}_{\omega-1}$ increases the probability that firms use suppliers in $\omega - 1$. Finally, the fact that $\bar{c}_\omega$ falls and p_ω decreases means that total industry sales rise.

[14] If entry were completely elastic ($\beta \to \infty$), profit per firm would be independent of δ_ω, as the number of firms would increase until profit per firm matched the opportunity cost of entry. With the same profit per firm, incentives to search would not change, and the fraction of firms in ω using suppliers in $\omega - 1$ would not change. However, J_ω would increase, and gains from variety would imply that p_ω would fall. If $m(J_\omega)$ is strictly increasing in J_ω, then v_ω would rise as well. If entry were completely inelastic, ($\beta = 0$), the number of firms would be fixed by construction. However, the increased demand would increase incentives to search, leading to lower price index p_ω and higher v_ω.

[15] For any value of γ, the largest firms will become more likely to use $\omega - 1$ as a supplier. But if γ is low, the smallest firms may not. In particular, if the firms born with lowest productivity q are extremely unlikely to use a supplier in $\omega - 1$, then the decline in $\bar{c}_\omega$ will have less impact on profitability than the decline in p_ω. If search effort is sufficiently elastic (i.e., γ small enough), then the large firms would increase their search enough so that the decline in p_ω is large. Then the low-q firms may reduce search effort, tilting their search more toward suppliers in $\omega - 2$.



(a) Response of Income/capita (b) Response of Search Effort (c) Response of Entry

(d) Response of Share Outsourcing (e) Cost Share of Intermediates

Figure 5 Elasticity to an Increase in Scale

Note: This figure plots the respective elasticities of income per capita, search effort, entry, and the share of firms outsourcing to an increase in the labor force. The final plot shows the share of expenditures on intermediate inputs, both as a fraction of total production cost and as a fraction of total production and search cost. The values of parameters are $\alpha = 0.75$, $\varepsilon = 2.5$, $\zeta = 1.5$, $\chi = 0.3$, $\gamma = 3$, and $\mu = 0.6$.

3.9 An Economy-wide Increase in Scale

We now illustrate some key mechanisms in the model using a numerical example. We study the impact of an increase in the labor force, L. This could be interpreted as population growth or growth in efficiency units of labor per person. In particular, we study the elasticity of output per worker, entry, search effort, and vertical span to L. For this exercise, we assume that all industries are symmetric, and that the distribution of productivity within each industry is degenerate, so that all entrants have the same ex-ante productivity q. In this simplified environment, it turns out that these elasticities can be expressed in terms of parameters and a single endogenous statistic, O, the share of firms that choose a vertical span of 1 (or, alternatively,the total spending on intermediate inputs as a share of production costs which is equal to $O\alpha + (1-O)\alpha^2$).

First, in the special case in which there is a fixed set of firms ($\chi = 0$) and fixed search effort ($\gamma \to \infty$), income per capita is invariant to scale: $\frac{d \log u}{d \log L} = 0$. Second, suppose that search effort remains fixed ($\gamma \to \infty$) and allow for entry ($\chi > 0$), and allow gains from variety to consumption, except in returns to search ($\mu = 0$). In this case, there is only the usual returns to scale that would be present in a standard one-sector model with increasing varieties, $\frac{d \log u}{d \log L} = \frac{\chi}{\chi+1} \frac{1}{\varepsilon - 1}$.

Consider now the case of interest, with either endogenous search effort ($\gamma < \infty$) and/or entry and returns to scale in search ($\mu > 0$, $\chi > 0$). The economy exhibits increasing returns to scale: panel (a) of Figure 5 shows that the elasticity of income/capita to the size of the labor force is positive. When the labor force grows, there is more entry and firms search more, as shown in panels

(b) and (c). Both mechanisms raise income per capita, and both reduce firms' vertical spans, as in panel (d).

These effects are stronger when firms are more specialized, indicating the presence of an input-output multiplier effect. From the perspective of canonical input-output models (e.g. Long and Plosser (1983), Acemoglu et al. (2012) or Jones (2011, 2013)) this multiplier effect may appear puzzling, since input-output multiplier effects are usually only present with increases in neutral productivity, not labor-augmenting productivity.[16] In our model, there is a sense in which the expansion of labor supply is similar to an increase in labor-augmenting productivity, for which there is no multiplier effect. But, differently from canonical models, some of that labor gets channeled into search effort and into the creation of new firms. Both the search effort and the gains from additional potential suppliers act like neutral shifters of those suppliers' costs. That is, they both reduce the effective cost of suppliers in a neutral way, not just augmenting the labor of those suppliers.

More formally, recall that average effective cost of an intermediate input is $\bar{c} \equiv \left(\int c^{-\zeta} dF(c)\right)^{-\frac{1}{\zeta}}$. In this simple economy, this cost index satisfies $\bar{c} \propto w^{1-\alpha}\left\{h_1 M(J)\bar{c}^{-\zeta} + h_2^{\alpha} M(J)^{\alpha}(w^{1-\alpha}\bar{c}^{\alpha})^{-\zeta}\right\}^{-\alpha/\zeta}$. From this, we obtain that the expected effective cost of intermediate inputs falls with entry and search effort:

$$\frac{d\log \bar{c}/w}{d\log L} = -\frac{s}{1-s}\frac{1}{\zeta}\left\{\mu\frac{d\log J}{d\log L} + \frac{s_1}{s}\frac{d\log h_1}{d\log L} + \frac{s_2}{s}\frac{d\log h_2}{d\log L}\right\}$$

where s is the expected share of production costs spent on intermediate inputs, while s_1 and s_2 are the expected shares of production costs spent on inputs from $\omega - 1$ and $\omega - 2$ respectively, so that $s = s_1 + s_2$. The reduction in expected cost is coming fully from search efficiency and search effort.

Panel (d) shows that firms specialize more as the labor force expands. To see why, recall that O is the fraction of firms that choose to outsource input $\omega - 1$ rather than produce in-house, which can be expressed as

$$\frac{O}{1-O} = \frac{h_1 M(J)\bar{c}^{-\zeta}}{h_2^{\alpha} M(J)^{\alpha}(w^{1-\alpha}\bar{c}^{\alpha})^{-\zeta}}\ .$$

First, as firms exert more search effort, they tilt their search effort toward less distant inputs: panel (b) shows that $\frac{d\log h_1}{d\log L} > \frac{d\log h_2}{d\log L}$. Second, since $\alpha < 1$, firms tend to become more specialized as search effort rises, when the matching rate increases, or when the cost of intermediates falls relative to the wage. In fact, either of the first two forces would reduce the cost of intermediates relative to the wage, amplifying the direct impact.[17]

[16] Consider an economy with roundabout production so that output can either be consumed (U) or used as an intermediate input (X): the resource constraint is $Y \geq X + U$. Suppose that the production function is $Y = A(BL)^{1-\alpha}X^{\alpha}$, so that B is labor-augmenting productivity and A is neutral productivity. In a competitive equilibrium, $\frac{d\log U}{d\log B} = 1$ and $\frac{d\log U}{d\log A} = \frac{1}{1-\alpha}$.

[17] Interestingly, while endogenous search effort is needed for larger firms to be more specialized in the cross-section, it is not needed for specialization to rise with development; endogenous entry ($\chi > 0$) and a matching rate that rises with entry ($\mu > 0$) are sufficient.

4 Alternative Mechanisms

4.1 Cost of Coordination

Becker and Murphy (1992) emphasized the cost of coordinating tasks as an important delimiter of the range of activities done in a single team. An implication they draw is that higher demand (or productivity) larger firms find it in their interest to pay the coordination cost to do more activities.

In principle there are many types of coordinating costs. Here we discuss a few specific versions of a coordinating cost, and discuss why we think they do not explain our results about size and vertical specialization.

One such cost of coordination is a fixed cost that is increasing in the number of suppliers (a special case, of course, is a separate fixed cost for each supplier).[18] In such a setting, an increase in demand would make a firm more willing to pay the cost to engage with more suppliers. However, as discussed in Section 2.5, an increase in demand tends to cause firms to reduce the number of suppliers.

Alternatively, it is possible that there is a cost of coordinating many activities within a firm. For example, suppose a firm needed to do a range of horizontal tasks; it could perform each task in-house or outsource it, but there was a fixed cost that increased with the number of tasks performed in house (with, again, a special case of there being a fixed cost for each task).[19] The prediction of such a model is that an increase in demand would tend to make the firm do more tasks in house. This would reduce the share of intermediate inputs. However, as discussed in Table XII, an increase in demand tends to raise a firm's cost share of intermediate inputs.

Finally, if a firm faced a fixed cost that increased with the number of tasks performed in-house, but the tasks were arranged vertically as a snake, an increase in demand would cause the firm to do more tasks in house, increasing vertical span.

4.2 Limited Span of Control

Another factor that might affect vertical specialization is a limited span of control, as in Lucas (1978). For example, Bloom et al. (2013) found a correlation between the number of male family members and the number of plants operated by the firm.[20] While we are unaware of similar evidence indicating that span of control considerations limit plant size (or the size of single-plant firms), it is plausible that some plants are constrained in the quantity of labor that can be hired.

If a firm's employment is constrained, an increase in demand could induce a decline in vertical span. If demand rises, the firm might choose to meet that demand by shifting employment

[18] Such a production structure has been adopted by a large number of papers, including Acemoglu, Antràs and Helpman (2007), Lim (2018), Huneeus (2018), Tintelnot et al. (2018). In general, such models do not have clear implications about how vertical span would change with demand.

[19] For example, in Chan (2017), a firm faces a fixed cost of performing each task in house and a different fixed cost of outsourcing a task.

[20] Bloom et al. (2013) found that the number of "family members [that] could currently work as directors in the firm" accounts for 10% of the variation in the number of plants operated by a firm.

to a downstream task and outsourcing the upstream task. Indeed, the vertical structure of 3.1 augmented with a constraint on labor would lead to such a prediction.[21]

One difference between span of control limitations and endogenous search for suppliers is the predicted response of marginal cost to changes in demand. A limited span of control would suggest that marginal cost would rise with an increase in demand, whereas search for suppliers would imply that marginal cost would fall. The unit costs we can construct using our data are noisy– partly because the units in which quantities are denominated change over time—so at this point we are hesitant to make conclusive statements in either direction.

That said, our interpretation is consistent with the findings of Albornoz, Brambilla and Ornelas (2021), who study Argentine firms after the sudden removal of preferential tariffs by the US following an intellectual property dispute. They find that affected firms reduced sales to the US but also reduced sales to other export markets, consistent with marginal cost declining with size.

4.3 Other Mechanisms

Our empirical findings suggest that larger firms pay lower prices, on average, than smaller firms. We have posited that such a relationship arises naturally if firms can exert effort to such for suppliers. An alternative mechanism that yields similar cross-sectional results is that sellers practice second degree price discrimination and offer quantity discounts, as in Meleshchuk (2019).[22]

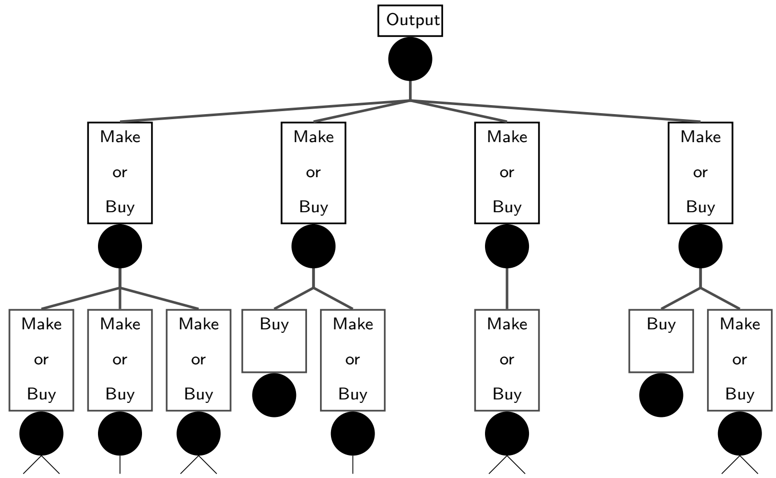
5 A Quantitative Model

In this section we extend the baseline model to so that production modules use multiple inputs. Household preferences, market structure, and timing are the same as in the simple model.

[21] Both a quantity constraint on labor and increase in search effort would lead to a decline in the cost of intermediates relative to the shadow cost of labor, leading to substitution toward the downstream intermediate which is a better substitute for labor (a higher Morishima elasticity of substitution).

[22] Monopsonistic behavior in intermediate input markets, on the other hand, is not consistent with marginal cost declining with size. Monopsonistic behavior requires an upward sloping supply curve for each intermediate input. While this could be consistent with larger firms attaining larger markdowns on intermediate input purchases, it also means that when the firm is larger and chooses to buy larger quantities of intermediate inputs, it must pay more for the marginal unit.

5.1 Recipes and Production



A firm's production possibilities can be described by a tree of production modules. A production module to make ω is a production function that uses a particular set of inputs $\hat{\Omega}_\omega$ and labor. The firm can either buy each input $\hat{\omega} \in \hat{\Omega}_\omega$ from a supplier or it can produce it in-house with another production module to produce $\hat{\omega}$. We say that $\hat{\Omega}_\omega^\infty$ is the set of all inputs that are nodes in the tree formed by the production modules.

The firm searches for suppliers. For each input $\hat{\omega} \in \hat{\Omega}_\omega^\infty$, its search effort $h_{\hat{\omega}}$ delivers a set of potential suppliers $S_{j\hat{\omega}}$ from whom it may purchase input $\hat{\omega}$. For each potential supplier $s \in S_{j\hat{\omega}}$, the firm draws a match-specific productivity z_{js}; with unit price p_s, the effective cost of purchasing that input from that supplier is $\frac{p_s}{z_{js}}$. Since the buyer will choose the supplier that delivers the lowest effective cost, the cost of outsourcing input $\hat{\omega}$ is $c^o_{j\hat{\omega}} = \min_{s \in S_{j\hat{\omega}}} \frac{p_s}{z_{js}}$.

The firm can also produce some inputs in-house using a production module. A production module to produce $\hat{\omega}$ in-house delivers an effective cost $c^i_{j\hat{\omega}}$ (which we characterize below). The firm's effective cost of input $\hat{\omega}$ is thus

$$c_{j\hat{\omega}} = \min\left\{c^i_{j\hat{\omega}}, c^o_{j\hat{\omega}}\right\}$$

We assume that each production module is Cobb Douglas. The production module to produce an input ω delivers an in-house unit cost of

$$c^i_{j\omega} = \frac{1}{B_{j\omega}} w^{\alpha^\omega_l} \prod_{\hat{\omega} \in \hat{\Omega}_\omega} c_{j\hat{\omega}}^{\alpha^\omega_{\hat{\omega}}}$$

$B_{j\tilde{\omega}}$ is a random firm-module specific productivity shifter and $\alpha^\omega_{\hat{\omega}}$ are the output elasticities of a module input $\hat{\omega}$.

A firm born in industry ω also has a core production module from which it produces ω. It cannot outsource the production module, and there is no productivity shifter B. This production module takes the form

$$c_{j\omega} = \frac{1}{q} w^{\alpha_l^\omega} \prod_{\hat{\omega} \in \hat{\Omega}_\omega} c_{j\hat{\omega}}^{\alpha_{\hat{\omega}}^\omega}$$

where q is a firm-specific productivity shifter. Every module has constant returns to scale, $\alpha_l^\omega + \sum_{\hat{\omega} \in \Omega_\omega} \alpha_{\hat{\omega}}^\omega = 1$, and labor is essential, $\alpha_l^\omega > 0$.

5.2 Entry

The mass of firms in each industry is endogenous. We model entry by having a representative entrepreneur that takes as given the distribution of profit within each industry, and chooses a mass of entrants for each industry subject to a constant elasticity of transformation function

$$\max_{\{J_\omega\}} \sum_\omega J_\omega \bar{\pi}_\omega - w \frac{k^E}{1 + 1/\chi} \left(\sum_\omega \left(\delta_\omega^E\right)^{-\frac{1}{\beta}} J_\omega^{\frac{1+\beta}{\beta}} \right)^{\frac{\beta}{1+\beta}(1+1/\chi)}$$

where $\bar{\pi}_\omega$ is average profit among all entrants into industry ω, and δ_ω^E is a shifter of the supply of entrants for industry ω. The solution is

$$J_\omega = \delta_\omega^E \bar{\pi}_\omega^\beta \bar{\pi}^{\chi - \beta} (wk^E)^{-\chi}$$

where $\bar{\pi} \equiv \left(\sum_\omega \delta_\omega^E \bar{\pi}_\omega^{1+\beta} \right)^{\frac{1}{1+\beta}}$ is an index of average profit. β indexes the ease of shifting entry between industries, and χ indexes the response of total entry to labor devoted to entry. Note that this specification nests free entry with $\beta, \chi \to \infty$, in which case equilibrium requires $\bar{\pi}_\omega = wk^E, \forall \omega$, and exogenous entry with $\beta, \chi \to 0$, in which case $J_\omega = \delta_\omega^E$. We assume that $\beta < \infty$, so that an increase in industry demand is not fully absorbed by new entrants.

Each new entrant draws the productivity of its core module, q, from an industry-specific distribution with distribution function $Q_\omega(q)$.

5.3 Functional Form Assumptions

In this section, we make several functional form assumptions that will be useful in facilitating a characterization of the equilibrium.

First, we specify the distribution of match-specific productivities. We assume that if firm j chooses search effort $h_{j\hat{\omega}}$ for some input $\hat{\omega}$ for use in a module to produce ω, then the number of potential suppliers of varieties of $\hat{\omega}$ with match-specific productivity greater than z is Poisson with mean $h_{j\hat{\omega}} m_{\omega\hat{\omega}} z^{-\zeta}$. $m_{\omega\hat{\omega}}$ is a measure of matches, which we will later endogenize.

Second, we specify the distribution of module-specific productivities, which are independent across firms and modules. For a module to produce ω with output elasticities α_l^ω, $\{\alpha_{\hat{\omega}}^\omega\}_{\hat{\omega} \in \hat{\Omega}_\omega}$, the productivity shifter $B_{j\omega}$ takes the form $e^{b/\zeta}$ with b drawn from a distribution with characteristic function[23]

$$\frac{\Gamma(1-it)}{\prod_{\hat{\omega}\in\Omega_\omega}\Gamma\left(1-\alpha_{\hat{\omega}}^{\omega}it\right)}$$

Third, we assume that the number of production modules in the tree is finite.

Finally, we assume that the distribution of productivity with which firms are born has a sufficiently thin tail: $\lim_{q\to\infty} q^{\zeta\frac{1+\gamma}{\gamma}}\left[1-Q_\omega(q)\right]=0$.

5.4 Characterizing the Equilibrium

We characterize the equilibrium by working backward. We normalize the wage to 1.

We first characterize possible realizations of firm j's unit cost, given its choices of search effort. We then use this to restate the firm's problem in order to solve for its choice of search effort.

Proposition 5 *For firm j in industry ω with productivity q_j that chooses search effort $\{h_{j\tilde{\omega}}\}_{\tilde{\omega}\in\hat{\Omega}_\omega^\infty}$,*

- *The effective unit cost of outsourcing input $\hat{\omega}\in\hat{\Omega}_{\tilde{\omega}}$ follows a Weibull distribution with counter-cumulative distribution*

$$\Pr(c_{j\hat{\omega}}^{o}>c)=e^{-h_{j\hat{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}c^{\zeta}}$$

  with $\bar{c}_{\hat{\omega}}\equiv\left(\int c^{-\zeta}dF_{\hat{\omega}}(c)\right)^{-\frac{1}{\zeta}}$ and $F_{\hat{\omega}}(\cdot)$ is the distribution function of unit cost among firms in industry $\hat{\omega}$.

- *For any input $\hat{\omega}\in\hat{\Omega}_{\tilde{\omega}}$ that can be produced in-house, the effective unit cost of producing it in-house follows a Weibull distribution with counter-cumulative distribution*

$$P\left(c_{j\hat{\omega}}^{i}>c\right)=e^{-T_{j\hat{\omega}}^{-\zeta}c^{\zeta}}$$

  where $\{T_{j\tilde{\omega}}\}_{\tilde{\omega}\in\Omega_\omega^\infty}$ are defined iteratively as

$$T_{j\tilde{\omega}}=w^{\alpha_l^{\tilde{\omega}}}\prod_{\hat{\omega}\in\hat{\Omega}_{\tilde{\omega}}}\left[h_{j\hat{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}+T_{j\hat{\omega}}^{-\zeta}\right]^{-\frac{\alpha_{\hat{\omega}}^{\tilde{\omega}}}{\zeta}}$$

  where we use the convention that $T_{j\hat{\omega}}=\infty$ if the input is a leaf of the production tree (i.e., if in-house production is infeasible).

- *The probability of outsourcing input $\hat{\omega}\in\hat{\Omega}_{\tilde{\omega}}$ conditional on using it in production is*

$$\frac{h_{j\hat{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}}{h_{j\hat{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}+T_{j\hat{\omega}}^{-\zeta}} \tag{2}$$

[23] There are two special cases in which such a random variable has a well-known name. If the recipe uses no intermediate inputs, b follows a Gumbel distribution. If the recipe uses exactly one intermediate input with output elasticity $\alpha_{\hat{\omega}}^{\omega}$, e^{-b} follows a one-sided Levy-stable distribution with characteristic exponent $\alpha_{\hat{\omega}}^{\omega}$, which has the Laplace transform $E\left[e^{-u\tilde{b}_{ji}}\right]=e^{-u^{\alpha_{\hat{\omega}}^{\rho}}}$. In Appendix E we show such a random variable exists, building on Shanbhag, Pestana and Sreehari (1977) who show the existence of a closely related random variable.

This is independent of the firm's cost or of the probability of outsourcing other inputs.

- *The firm's expected profit is*

$$A_\omega \delta_\omega q_j^{\varepsilon-1} T_{j\omega}^{1-\varepsilon} - \sum_{\hat{\omega}\in\hat{\Omega}_\omega^\infty} wk \frac{h_{j\hat{\omega}}^{1+\gamma}}{1+\gamma}$$

where $A_\omega \equiv up^\eta p_\omega^{\varepsilon-\eta} \frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^\varepsilon} \prod_{\hat{\omega}\in\hat{\Omega}_\omega} \Gamma\left(1-\alpha_{\hat{\omega}}^{\omega}\frac{\varepsilon-1}{\zeta}\right)$.

An implication of the functional form assumptions is that the effective unit cost of outsourcing each input $\hat{\omega}\in\hat{\Omega}_{\tilde{\omega}}$ follows a Weibull distribution with scale that depends on search effort and the availability of good suppliers, $h_{j\hat{\omega}} m_{\tilde{\omega}\hat{\omega}} \bar{c}_{\hat{\omega}}^{-\zeta}$ and shape ζ.[24] In addition, conditioning on search effort across all inputs, the unit cost for in-house production of each input follows a Weibull distribution with shape parameter ζ and scale $T_{j\hat{\omega}}^{\zeta}$.[25] Because the two Weibull distributions share the same shape parameter, the usual discrete-choice logic implies that a simple expression for the probability of outsourcing a particular input, (2).

Corollary 1 *Given search choices, the industry average cost index satisfies*

$$\bar{c}_\omega^{-\zeta} = \prod_{\hat{\omega}\in\hat{\Omega}_\omega} \Gamma\left(1-\alpha_{\hat{\omega}}^{\omega}\right) \int q^\zeta T_\omega(q)^{-\zeta} dQ_\omega(q)$$

and the industry price index satisfies

$$p_\omega^{1-\varepsilon} = \frac{\varepsilon}{\varepsilon-1} J_\omega \prod_{\hat{\omega}\in\hat{\Omega}_\omega} \Gamma\left(1-\alpha_{\hat{\omega}}^{\omega}\frac{\varepsilon-1}{\zeta}\right) \int q^{\varepsilon-1} T_\omega(q)^{1-\varepsilon} dQ_\omega(q)$$

5.5 Optimal Search Effort

With this, we can characterize the optimal choice of search effort. The firm's problem can be expressed as a recursive cost-minimization problem. In particular, for the module to produce ω, define the cumulative search cost function

$$K_\omega(T_\omega) = \min_{\{h_{\hat{\omega}}\}_{\hat{\omega}\in\hat{\Omega}_\omega^\infty}} \sum_{\hat{\omega}\in\hat{\Omega}_\omega^\infty} k \frac{h_{\hat{\omega}}^{1+\gamma}}{1+\gamma} .$$

[24] The logic closely follows Oberfield (2018).

[25] Why does the cost of in-house production follow a Weibull distribution? We show this by induction, starting upstream. For any input that is a leaf, the effective cost of the input follows a Weibull distribution with shape parameter ζ. Consider module to produce $\tilde{\omega}$, with inputs $\hat{\Omega}_{\tilde{\omega}}$. The cost of purchasing any input from a supplier follows a Weibull distribution with shape ζ. Suppose that unit cost of in-house production of input $\hat{\omega}$ follows a Weibull distribution with shape ζ. Since the firm will procure $\hat{\omega}$ in the way that delivers the lowest cost, and the family of Weibull distributions with the same shape is closed, the cost to the firm of procuring the input will be Weibull with shape ζ. The cost for the module is the product of the cost of each input, raised to the respective output elasticities $\{\alpha_{\hat{\omega}}^{\tilde{\omega}}\}_{\hat{\omega}\in\hat{\Omega}_{\tilde{\omega}}}$, multiplied by the random variable $B_{j\tilde{\omega}}$. While the product of independent Weibulls is not Weibull, the distribution of $B_{j\tilde{\omega}}$ is reverse engineered so that the resulting unit cost of $\tilde{\omega}$ follows a Weibull distribution with shape ζ.

This is the minimal search cost across all inputs upstream from the module required to deliver a distribution of unit cost for the module with scale parameter T_ω. With this notation, the firm's problem can be expressed as

$$\max_{T_{j\omega}} A_\omega \delta_\omega q_j^{\varepsilon-1} T_{j\omega}^{1-\varepsilon} - wK_\omega\left(T_{j\omega}\right) \ .$$

Further, the cumulative search cost function K_ω has an iterative representation

$$K_\omega\left(T_\omega\right) = \min_{\{h_{\hat{\omega}}, T_{\hat{\omega}}\}_{\hat{\omega}\in\hat{\Omega}_\omega}} \sum_{\hat{\omega}\in\hat{\Omega}_\omega} \left[\frac{k}{1+\gamma} h_{\hat{\omega}}^{1+\gamma} + K_{\hat{\omega}}\left(T_{\hat{\omega}}\right)\right]$$

subject to

$$T_\omega \geq w^{\alpha_l^\omega} \prod_{\hat{\omega}\in\hat{\Omega}_\omega} \left[h_{\hat{\omega}} m_{\omega\hat{\omega}} \bar{c}_{\hat{\omega}}^{-\zeta} + T_{\hat{\omega}}^{-\zeta}\right]^{-\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}}$$

The next lemma derives some useful properties of the cumulative search cost functions, K_ω.

Lemma 1 *Each $K_\omega(\cdot)$ is such that $K_\omega(0) = 0$, is strictly increasing, strictly convex, twice continuously differentiable, and the policy functions $\{h_{\hat{\omega}}(\cdot), T_{\hat{\omega}}(\cdot)\}_{\hat{\omega}\in\Omega_\omega}$ are differentiable and strictly increasing in T_ω.*

This Lemma is helpful in characterizing several features of firms' choices.

Proposition 6 *Among firms in ω, those born with higher productivity choose higher search effort for all nodes. Further, for any input for which it is feasible to produce in-house, the probability of outsourcing the input is strictly increasing in q.*

The proposition follows from Lemma 1 and the fact that a firm born with higher productivity would choose to invest more in lowering its cost by choosing higher T_ω. The proof of Lemma 1 notes first that, for any module, cost minimization implies that lowering its unit cost involves searching more for direct suppliers and for suppliers further upstream. That is, higher T_ω corresponds to both higher $h_{\hat{\omega}}$ and higher $T_{\hat{\omega}}$ for each of the module's inputs $\hat{\omega} \in \hat{\Omega}_\omega$. Second, for each of input for which in-house production is feasible, $h_{\hat{\omega}}$ rises proportionally more than $T_{\hat{\omega}}^\zeta$, $\frac{d\ln h_{\hat{\omega}}}{d\ln T_\omega} > \frac{d\ln T_{\hat{\omega}}^\zeta}{d\ln T_\omega}$. This happens because the cost of finding suppliers rises less steeply than the cost of reducing the in-house production cost by searching for suppliers further upstream. The critical step is that producing in-house requires labor, and searching for suppliers further upstream cannot reduce the cost of labor used in-house, whereas outsourcing an input replaces labor. The labor required to produce in-house ultimately limits the effectiveness of reducing cost by searching for upstream suppliers relative to doing so by searching for suppliers further downstream.[26]

With this, we can derive some cross sectional predictions.

[26] Another way to see the role of labor is to consider the following thought experiment. For a module to produce ω, no labor was required to produce in house, so that $\alpha_l^{\hat{\omega}} \to 0$ for all $\hat{\omega} \in \hat{\Omega}_\omega^\infty$. In that limit, $\frac{d\ln h_{\hat{\omega}}}{d\ln T_\omega} \to \frac{d\ln T_{\hat{\omega}}^{-\zeta}}{d\ln T_\omega}$, i.e., production becomes homothetic.

Proposition 7 *In industry ω, the probability of outsourcing any intermediate input (conditional on using it) is increasing in sales.*

The empirical counterpart to Proposition 7 is a negative correlation between vertical span and size within industries, as discussed in Section 2.3. The leap from the previous proposition is moving from q to sales. Firms with higher q search more and tend to have lower cost, enabling them to sell more, on average, to both the household and to other firms. To prove the result, we simply note that conditioning on q, a firm's sourcing decisions are independent of its sales.

6 Estimation and Quantitative Evaluation (In Progress)

6.1 Solving the Model

Before describing the algorithm, we collect the model's key equations in one place.

For a firm born with productivity q, the firm's problem gives its profit and the first order conditions give its choices:

$$\pi_\omega(q) = \max_{T_\omega} \delta_\omega A_\omega q^{\varepsilon-1} T_\omega^{1-\varepsilon} - wK_\omega\left(T_\omega\right)$$

where

$$A_\omega = up^\eta p_\omega^{\varepsilon-\eta} \frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^\varepsilon} \prod_{\hat{\omega}\in\hat{\Omega}_\omega} \Gamma\left(1-\alpha_{\hat{\omega}}^{\omega}\frac{\varepsilon-1}{\zeta}\right),$$

the industry price index satisfies

$$p_\omega^{1-\varepsilon} = \frac{\varepsilon}{\varepsilon-1} \prod_{\hat{\omega}\in\hat{\Omega}_\omega} \Gamma\left(1-\alpha_{\hat{\omega}}^{\omega}\frac{\varepsilon-1}{\zeta}\right) J_\omega \int q^{\varepsilon-1} T_\omega(q)^{1-\varepsilon} dQ_\omega(q),$$

and the cumulative search cost function K_ω for module to produce ω can be expressed iteratively as

$$K_\omega(T_\omega) = \min_{\{h_{\hat{\omega}}, T_{\hat{\omega}}\}_{\hat{\omega}\in\hat{\Omega}_\omega}} \sum_{\hat{\omega}\in\hat{\Omega}_\omega} \left[\frac{k}{1+\gamma} h_{\hat{\omega}}^{1+\gamma} + K_{\hat{\omega}}\left(T_{\hat{\omega}}\right)\right]$$

subject to

$$T_\omega \leq w^{\alpha_l^\omega} \prod_{\hat{\omega}\in\Omega_\omega} \left[h_{\hat{\omega}} m_{\omega\hat{\omega}} \bar{c}_{\hat{\omega}}^{-\zeta} + T_{\hat{\omega}}^{-\zeta}\right]^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}$$

where the average cost index satisfies

$$\bar{c}_\omega^{-\zeta} = \prod_{\hat{\omega}\in\hat{\Omega}_\omega} \Gamma\left(1-\alpha_{\hat{\omega}}^{\omega}\right) \int q^\zeta T_\omega(q)^{-\zeta} dQ_\omega(q).$$

Equilibrium entry decisions satisfy

$$J_\omega = \delta^E \bar{\pi}_\omega^\beta \bar{\pi}^{\chi-\beta} (wk^E)^{-\chi}$$

with $\bar{\pi} = \left(\sum_\omega \delta_\omega^E \bar{\pi}_\omega^{1+\beta}\right)^{\frac{1}{1+\beta}}$ and $\bar{\pi}_\omega = \int \left\{\delta_\omega A_\omega q^{\varepsilon-1} T_\omega(q)^{1-\varepsilon} - wK_\omega\left(T_\omega(q)\right) dQ_\omega(q)\right\} dQ_\omega(q)$.

Total spending up equals total revenue, which, in turn, equals the total wage bill for variable production times the markup, or $up = \frac{\varepsilon}{\varepsilon-1} wL^{Production}$.

The labor market clearing condition is

$$L^{production} + L^{Entry} + L^{Search} = L$$

Total labor used for search and for entry across all industries are

$$\begin{aligned} L^{Search} =& \sum_\omega J_\omega \bar{L}_\omega^{Search} = \sum_\omega J_\omega \int K_\omega\left(T_\omega(q)\right) dQ_\omega(q) \\ L^{Entry} =& \frac{k^E}{1+1/\chi}\left(\frac{\bar{\pi}}{wk^E}\right)^{1+\chi} . \end{aligned}$$

The Algorithm:

In the model, the structure of production is a directed acyclic graph. Our approach to solving the model will be to begin upstream to solve the decisions of firms and then iteratively work our way downstream. As we describe below, we can do this conditional on a single general equilibrium variable.

1. Guess $X \equiv \frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^\varepsilon} \frac{u(p/w)^\eta}{\left[\left(\frac{\bar{\pi}}{w}\right)^{\chi-\beta}(k^E)^{-\chi}\right]^{\frac{\varepsilon-\eta}{\varepsilon-1}}}$. This variable summarizes general equilibrium considerations.
2. Proceed iteratively, starting with the most upstream industry to solve for industry variables and cost functions and then proceed downstream. For industry ω:
   (a) First, solve for the cost function $K_\omega(\cdot)$. We first convert the cost to units of labor. Given the wage, w, let $\tilde{K}_\omega(\tilde{T}) \equiv K_\omega(w\tilde{T})$, so that $\tilde{T}$ is in units of labor. Then $\tilde{K}_\omega$ can be defined recursively as

   $$\tilde{K}_\omega(\tilde{T}_\omega) = \min_{\left\{h_{\hat{\omega}}, \tilde{T}_{\hat{\omega}}\right\}_{\hat{\omega}\in\hat{\Omega}_\omega}} \sum_{\hat{\omega}\in\hat{\Omega}_\omega} \left[\frac{k}{1+\gamma} h_{\hat{\omega}}^{1+\gamma} + \tilde{K}_{\hat{\omega}}\left(\tilde{T}_{\hat{\omega}}\right)\right]$$

   subject to

   $$\tilde{T}_\omega \leq \prod_{\hat{\omega}\in\Omega_\omega} \left[h_{\hat{\omega}} m_{\omega\hat{\omega}} \left(\frac{\bar{c}_{\hat{\omega}}}{w}\right)^{-\zeta} + \tilde{T}_{\hat{\omega}}^{-\zeta}\right]^{-\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}}$$

   The cost of any primary input is exogenous, and for any input $\hat{\omega}$ that is further upstream, we have already solved for $\tilde{K}_{\hat{\omega}}(\cdot)$ and $\frac{\bar{c}_{\hat{\omega}}}{w}$ from the previous step or, if a primary input, from technology.

   (b) We next solve for $\frac{A_\omega}{w}$. Given $\frac{A_\omega}{w}$ and $\tilde{K}_\omega$, we can solve for $\tilde{T}_\omega(q) = \frac{T_\omega(q)}{w}$ for each q. $\frac{A_\omega}{w}$ and $\frac{T_\omega(\cdot)}{w}$ are sufficient for expected profit for entrants, $\frac{\bar{\pi}_\omega}{w}$. Note that $\frac{A_\omega}{w}$ can be expressed as $\frac{A_\omega}{w} = X\left[\delta^E_\omega \left(\frac{\bar{\pi}_\omega}{w}\right)^\beta \int q^{\varepsilon-1}\left(\frac{T_\omega(q)}{w}\right)^{1-\varepsilon} dQ_\omega(q)\right]^{-\frac{\varepsilon-\eta}{\varepsilon-1}} \left\{\prod_{\hat{\omega}\in\hat{\Omega}_\omega} \Gamma\left(1-\alpha^\omega_{\hat{\omega}}\frac{\varepsilon-1}{\zeta}\right)\right\}^{\frac{\eta-1}{\varepsilon-1}}$.

(c) $\frac{T_\omega(\cdot)}{w}$ is also sufficient to recover the average cost index relative to labor, $\frac{\bar{c}_\omega}{w}$

3. Once we have these $\frac{\bar{\pi}_\omega}{w}$ for all industries, we can compute the average profit index $\frac{\bar{\pi}}{w} = \left(\sum_\omega \delta^E_\omega \left(\frac{\bar{\pi}_\omega}{w}\right)^{1+\beta}\right)^{\frac{1}{1+\beta}}$, which is sufficient to characterize the measure of entrants, J_ω, in each industry. With this, we can compute the industry price index $\frac{p_\omega}{w}$, and total labor used for search across all firms in the industry. With this, we can compute the aggregate price index $\frac{p}{w} = \left(\sum_\omega \delta_\omega \left(\frac{p_\omega}{w}\right)^{1-\eta}\right)^{\frac{1}{1-\eta}}$.

4. Compute $X \equiv \frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^\varepsilon}\frac{u(p/w)^\eta}{\left[\left(\frac{\bar{\pi}}{w}\right)^{\chi-\beta}(k^E)^{-\chi}\right]^{\frac{\varepsilon-\eta}{\varepsilon-1}}}$ and check whether it matches the initial guess. We already have $\frac{p}{w}$ and $\frac{\bar{\pi}}{w}$. We can recover u from $up = \frac{\varepsilon}{\varepsilon-1}w\left(L - L^{Search} - L^{Entry}\right)$.

6.2 Estimation of Model Parameters

6.2.1 Constructing the Production Tree

We take the model to the manufacturing data by assuming that each product ω corresponds to a five-digit product code. We proceed in the following steps. The details are in Appendix A.3.

1. We set some of the industries ω to be leaf industries, i.e. industries which produce only using labor. In the data, this corresponds to product codes that fall outside of manufacturing (i.e. agriculture, mining) as well as some goods where consumption vastly exceeds domestic production.

2. We then construct a directed acyclical graph of input-output relationships between products. Starting with leaf industries, we find all goods that produced only using leaf industry inputs, then all goods that are produced from those goods and leaf goods, etc. In order to allow the construction of such a graph without any cycles, we treat some less important observed inputs as primary inputs (cf. Tintelnot et al. (2018)).[27]

3. Starting again upstream, we recursively calibrate the production function parameters $\alpha^\omega_{\hat{\omega}}$ for each module ω and each direct input $\hat{\omega}$ to match the aggregate expenditure on direct inputs and inputs upstream from them. We calibrate α^ω_l to match the residual expenditure on primary inputs, netting out primary factor expenditure associated with intermediate input expenditures from non-direct inputs (which are determined by the α's of modules further upstream).

[27] Some large firms span many products, making it more likely that cycles arise. In contrast, the graph we are constructed is based on the production functions of plants that produce single products. As a result, fewer links need to be severed.

6.2.2 Calibration of Shifters

In this section, we take as given the production trees and the elasticities $\gamma, \beta, \chi, \zeta, \varepsilon, \eta, \phi$ and show how the we pin down all of the shifters in the economy.

We have data on shares of each industry in final consumption, HH_ω, the measure of firms in each industry, J_ω, and for each of an industry's direct inputs, the fraction of firms in an industry that purchase the input from a supplier, $\bar{O}_{\hat{\omega}}$. We also can compute each firm's value added and wage bill.[28]

1. Normalizations and choice of units
   (a) We choose units of labor so that $L = 1$.
   (b) We normalize each $\underline{q}_\omega$ to unity.[29]
   (c) We normalize the cost of search parameter k to unity.[30]
   (d) For any leaf input, normalize $\bar{c}_\omega$ and $m_{\omega\hat{\omega}}$ to equal the wage.[31]
   (e) We normalize the sum of the preference shifters and entry shifters to unity: $\sum_\omega \delta_\omega = \sum_\omega \delta^E_\omega = 1$.

2. Next, we infer $L^{Production}$, L^{Search}, and L^{Entry}. First, the sales to the household is equal to value added. Since firms charge the usual markup of $\frac{\varepsilon}{\varepsilon-1}$, the share of labor used in production satisfies $\frac{\text{Value Added}}{wL^{Production}} = \frac{\varepsilon}{\varepsilon-1}$. Second, we assume that labor used for entry does not appear in our data, so that the remaining labor in the wage bill represents search for suppliers: $wL^{Search} = \text{Wage Bill} - wL^{Production}$. Third, note also that the ratio of profit to expenditure on entry depends on the rate of diminishing returns to entry: $\frac{\sum_\omega J_\omega \bar{\pi}_\omega}{wL^{Entry}} = 1 + \frac{1}{\chi}$, and that total profit is the difference between aggregate Value Added and the Wage Bill. Since we have chosen units so that $L = 1$, these together imply

$$\begin{aligned}
L^{Entry} &= \frac{\frac{\chi}{\chi+1}(\text{Value Added} - \text{Wage Bill})}{\frac{\chi}{\chi+1}\text{Value Added} + \frac{1}{\chi+1}\text{Wage Bill}} \\
L^{Production} &= \frac{\frac{\varepsilon-1}{\varepsilon}\text{Value Added}}{\frac{\chi}{\chi+1}\text{Value Added} + \frac{1}{\chi+1}\text{Wage Bill}} \\
L^{Search} &= \frac{\text{Wage Bill} - \frac{\varepsilon-1}{\varepsilon}\text{Value Added}}{\frac{\chi}{\chi+1}\text{Value Added} + \frac{1}{\chi+1}\text{Wage Bill}}
\end{aligned}$$

3. Calibrate $\{m_{\omega\hat{\omega}}\}$ proceeding iteratively, starting with the most upstream industries.
   (a) Consider industry ω. For each direct input $\hat{\omega} \in \hat{\Omega}_\omega$, we already have the cumulative search cost function, $K_{\hat{\omega}}$.

[28] Note that here the wage bill corresponds to total expenditure on primary inputs.

[29] A model with $\{\underline{q}_\omega, \delta_\omega, \{m_{\omega\hat{\omega}}\}\}$ is equivalent to one with $\underline{q}'_\omega = 1$, $\delta'_\omega = \delta_\omega \underline{q}_\omega^{\varepsilon-1}$, and $m'_{\omega\hat{\omega}} = m_{\omega\hat{\omega}} \underline{q}^\zeta_{\hat{\omega}}$.

[30] A model with k and $\{\{m_{\omega\hat{\omega}}\}\}$ is equivalent to one with $k' = 1$ and $m'_{\omega\hat{\omega}} = \frac{m_{\omega\hat{\omega}}}{k^{\frac{1}{1+\gamma}}}$.

[31] Since production of any good is ultimately Cobb-Douglas across primary inputs, changing the cost of any primary input would simply be absorbed by changes in the demand shifters $\{\delta_\omega\}$.

(b) For each non-leaf, direct input, $\hat{\omega}$, guess $m_{\omega\hat{\omega}}$. We already have $\int q^{\zeta} T_{\hat{\omega}}^{\zeta} dQ_{\hat{\omega}}(q)$ from the previous step. Thus we have $v_{\omega\hat{\omega}}$.

(c) For each leaf direct input, $\hat{\omega}$, we have normalized $\bar{c}_{\hat{\omega}} = 1$ and $m_{\omega\hat{\omega}}$.

(d) Given $m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}$ for each direct input and $K_{\hat{\omega}}$ for each non-leaf direct input, we can solve for K_{ω}.

(e) We next solve for $\frac{p_\omega}{w}$. We guess a value for $\frac{p_\omega}{w}$. Then, noting that $HH_\omega = \frac{\delta_\omega p_\omega^{1-\eta}}{p^{1-\eta}}$ and $up = \frac{\varepsilon}{\varepsilon-1} w L^{production}$, imply that

$$\begin{aligned}\delta_\omega A_\omega =& \delta_\omega u p^\eta p_\omega^{\varepsilon-\eta} w^{-\varepsilon} = HH_\omega \frac{p^{1-\eta}}{p_\omega^{1-\eta}} u p^\eta p_\omega^{\varepsilon-\eta} w^{-\varepsilon} = HH_\omega \frac{1}{p_\omega^{1-\eta}} \frac{\varepsilon}{\varepsilon-1} w L^{production} p_\omega^{\varepsilon-\eta} w^{-\varepsilon} \\ =& HH_\omega \frac{\varepsilon}{\varepsilon-1} L^{production} (p_\omega/w)^{\varepsilon-1}\end{aligned}$$

With this, one can solve for $T_\omega(q)$ for each q. Finally, we check that $\left(\frac{p_\omega}{w}\right)^{1-\varepsilon} \propto J_\omega \int q T_\omega(q)^{\varepsilon-1} dQ_\omega(q)$.

(f) Compute the fraction of firms in industry ω that outsource each direct input. Check that these match the data. If not, adjust the guess of $\{m_{\omega\hat{\omega}}\}$.

4. After we have solved for each p_ω/w, we next solve for δ_ω. Using $HH_\omega = \frac{\delta_\omega p_\omega^{1-\eta}}{p^{1-\eta}}$ from household optimization and the normalization $\sum_\omega \delta_\omega = 1$ gives

$$\delta_\omega = \frac{HH_\omega p_\omega^{\eta-1}}{\sum_{\omega'} HH_{\omega'} p_{\omega'}^{\eta-1}}$$

5. Solve for δ_ω^E using $J_\omega = \delta_\omega^E \bar{\pi}_\omega^\beta \bar{\pi}^{\chi-\beta}(wk^E)^{-\chi}$ and the normalization $\sum_\omega \delta_\omega^E = 1$, which together yield

$$\delta_\omega^E = \frac{J_\omega \bar{\pi}_\omega^{-\beta}}{\sum_{\omega'} J_{\omega'} \bar{\pi}_{\omega'}^{-\beta}}$$

6. k^E is set to match labor used for entry

$$L^{Entry} = \frac{k^E}{1+1/\chi} \left(\frac{\bar{\pi}}{wk^E}\right)^{1+\chi} .$$

6.2.3 Calibration of Elasticities

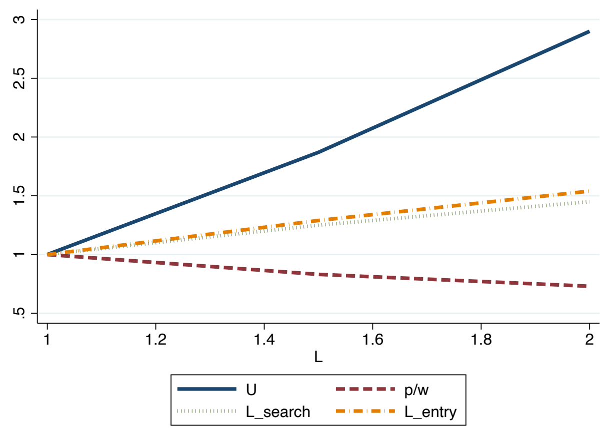
We have seven elasticities to pin down. We set $\varepsilon - 1 = \zeta = 1.5$. These parameters characterize the elasticity of substitution across varieties in the same industry of the household and of downstream firms. We assume that these elasticities are the same. We set the elasticity of substitution across baskets equal to the within-basket elasticity, $\eta = \varepsilon$. We set the overall entry elasticity $\chi = 0.3$, and $\beta = 1$. Together, these calibration choices imply that seven percent of labor is used on search, and 20 percent of labor is used on entry. For the distribution of firm productivities q we choose a

Pareto distribution with lower bound of unity and tail exponent of 8. For the matching function, we follow Miyauchi (2018) and set $M(J_\omega) = J_\omega^{0.6}$. In the baseline calibration, we set $\gamma = 3$.

6.3 A counterfactual increase in L

Similarly to Section 3.9, we explore numerically the economy's response in utility, the price level, and allocation of labor to a change in the labor force L. Figure 6 shows the proportional response of a simulated economy that contains all industries up to 10 production steps away the exogenous industries (which corresponds to 80% of all industries, and 84% of sales to households). Following a 100 percent increase in L, the price level in this simulated economy falls by 27 percent. The share of labor allocated to entry and to search increases by about one half.

Figure 6 Counterfactual increase in size of the economy L



7 Conclusion

How do firms organize the vertical extent of their activities, and what are the implications for the relationship between specialization and growth? We present data on the vertical span of production of Indian manufacturing plants. At the macro level, specialization—plants having short vertical spans—is positively correlated with the level of development. At the micro level, among plants that produce the same output, plants that produce at larger scales have shorter vertical spans. We present a theory where firms are born with ex-ante heterogeneous Hicks-neutral productivity and search for suppliers of inputs at different stages in their value chain, eventually choosing the cost-minimizing bundle of inputs and suppliers. When intermediate inputs are imperfectly substitutable with primary production factors, a non-homotheticity emerges: firms that are ex-ante more productive search disproportionately more for inputs further downstream, and end up more likely to have a short vertical span. A full quantitative model points to the presence of considerable economies of scale in production, creating a strong link between specialization and growth at the aggregate level.

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A Data Appendix

A.1 Data Sources and Variable Definitions

- *Plant-level data:* Our plant data is India's Annual Survey of Industries (ASI), published by the Central Statistics Office, Ministry of Statistics and Program Implementation (MOSPI). The data is at annual frequency, each reporting year starts on April 1st and ends on March 31st. Our data covers the years 1989/90, 1993/94, 1994/95, and 1996/97 to 2014/15. In the following, we denote ASI rounds by the year in which they end (e.g. '2004" for "2003/04"). Product codes for inputs and outputs vary across the years. We map all product codes to the ones used between 1998 and 2008 ("ASIC 2008"). We map the pre-1997 product codes to ASIC 2008 using a concordance that we create manually from the input and output descriptions published by MOSPI. We concord input and output product codes for 2008/09 and 2009/10 to ASIC 2008 using the concordance used in Boehm and Oberfield (2020). The years 2010/11 to 2014/15 use the NPCMS product classification, which we convert to ASIC 2008 codes using the concordance published by the Ministry. Codes for 1997 largely follow the 2008 classification, but unlike the other rounds, the 1997 ASI does not list packaging materials and auxiliary consumables (nuts, bolts, etc) as separate input categories, making the breakdown of the plants' materials basket inconsistent with the other years. We therefore exclude observations from that round in all regressions where variables that rely on the composition of plant inputs are used (most notably, regressions with the vertical span).

- *Total cost:* Sum of the user cost of capital, the total wage bill, energy, services, and materials inputs. Total cost is set to be missing if and only if the user cost of capital, the wage bill, or total materials are missing. The user cost of capital is constructed using the perpetual inventory method as in the Appendix of Greenstreet (2007), using depreciation rates of 0%, 5%, 10%, 20%, and 40% for land, buildings, machinery, transportation equipment, and computers & software, respectively. Capital deflators are from the Ministry's wholesale price index (except buildings, for which we use the CPWD building cost index up to 1994, and the Construction Cost Index CIDC Average after 1994), and the nominal interest rate is the India Bank Lending Rate, from the IMF's International Financial Statistics (on average about 11%). Services input expenditures include payments for "work done by others on materials provided by the factory", operating expenditures (inward freight and transportation charges, local taxes and licence fees), and non-operating expenditures (payments for communication, accounting services, financial and insurance services, legal services, contractor fees, and others).

- *Materials expenditure in total cost:* and subsequent tables) Total expenditure on 5-digit intermediate inputs (which excludes energy and services) divided by total cost (see above).

- *Rauch classification of goods*: From James Rauch's website, for 5-digit SITC codes. Concorded from SITC codes to ASIC via the SITC-CPC concordance from UNSTATS, and the NPCMS-ASIC concordance from the Indian Ministry of Statistics (NPCMS is based on CPC codes).

- *Dependence on relationship-specific inputs, by industry:* Total expenditure of single-product plants in an industry on relationship-specific inputs (according to the concorded Rauch classification), by 3-digit industry, divided by total expenditures on intermediate inputs that are associated with a 5-digit product code (which excludes services and most energy intermediate inputs).

- *Gross domestic product per capita, by district:* Nominal district domestic product was assembled from various state government reports, for the year 2005 (to maximize coverage). Missing for Goa and Gujarat and some union territories, and for some individual districts in the other states. Population data from the 2001 and 2011 Census of India, interpolated to 2005 assuming a constant population growth rate in each district. Whenever district domestic product per capita was unavailable, we used gross state domestic product per capita, as reported by the Ministry of Statistics and Program Implementation.

- *Gross state domestic product per capita:* Nominal gross state domestic product per capita, by state and year, from CMIE's "States of India" database.

- *Vertical Span:* See Appendix A.2.

- *Import Tariffs:* Effective applied tariffs, whenever available, otherwise MFN applied tariffs. Tariffs for the tariff years 1996 to 2014 are from TRAINS. For the tariff years 1989, 1993, and 1994, we digitized and coded the six-digit harmonized system codes from the respective edition of Arun Goyal's Customs Tariff books. This excludes tariffs for headings 84 ("nuclear reactors, boilers, machinery and mechanical applications, and parts thereof") and 85 ("electrical machinery and equipment, and parts thereof"), which cannot be reasonably mapped to the six-digit level because of long lists of exceptions (140 pages in Goyal's 1989 book). We concord all codes to the 2007 revision of the harmonized system, and then via CPC/NPCMS to ASIC 2008, as in Boehm and Oberfield (2020). We impute tariffs for years where they are missing by linearly interpolating within the ASIC code, and assign each tariff year to the ASI year that maximizes the overlap (e.g. tariff year 1989 to the ASI year 1989/90).

A.2 Definition of Vertical Span

See Appendix B of Boehm and Oberfield (2020) for definition and examples of vertical distance and vertical span. We follow these steps:

1. For a given product ω, construct the materials cost shares of industry ω on each input
2. Recursively construct the cost shares of the input industries (and inputs' inputs, etc...), excluding all products that are further downstream.

3. Vertical distance between ω and ω' is the average number of steps between ω and ω', weighted by the product of the cost shares.

Tables IX and X give examples of vertical distance.

Table IX Vertical distance examples for 63428: *Cotton Shirts*

	Mean Vertical Distance
Fabrics/Cloths	1.66
Yarns	2.58
Ginned & pressed cotton	3.44
Raw cotton	4.09

Table X Vertical distance examples for 73107: *Aluminium Ingots*

	Vertical Distance
Anodes, copper	1.00
Aluminium scrap	1.19
Aluminium oxide	1.25
Bauxite, calcined	2.18
Caustic soda (sodium hydroxide)	2.39
Bauxite, raw	3.03
Coal	3.43

The vertical span is the cost-weighted distance of a plant's inputs from its output. We define vertical span for single-product plants only.

A.3 Mapping the data to the model

A.3.1 Defining the sample for the quantitative exercise

Interpreting the data through the lens of our model imposes some requirements on the data. We therefore restrict the sample to observations that we can use for the estimation:

- We remove plant-year observations that have missing total cost (which arises mainly because of missing labor or capital expenditures).
- We remove plant-year observations that do not report at least one 5-digit materials input with positive nonmissing associated expenditures
- We remove observations from years 1996/97, 1998/99, and 1999/2000, because plants seem to underreport intermediate inputs in those rounds.

Furthermore, for the calibration and estimation we only use single-product plants.

A.3.2 Constructing the graph of input-output relationships

In general we assume that each five-digit ASIC code corresponds to an industry ω, and model their position in the production tree as determined by the inputs that plants use to produce them. That said, we model some industries ω as leaf nodes, i.e. we do not model their inputs explicitly:

- **Non-manufacturing industries:** Industries corresponding to 5-digit codes where total consumption exceeds total production by more than 30%, and some additional codes that are part of agriculture, mining, and services. Also industries where the aggregate materials share is below 2%.

- **MP-only industries:** 5-digit codes that are not produced by single-product plants with valid input codes

- **Residual industries:** Some industries defined as residuals ("not elsewhere classified", "others" etc), or govern broad types of inputs ("paint", "aromatic chemicals"). Plants in these industries usually have different input mixes because they produce different goods, not necessarily because the plants differ in organizational form.

The model assumes that the production tree (the input-output relationships) for each good takes the shape of a directed acyclical graph, i.e. no circular input structures. We clean the data manually to remove cycles:

(i) We merge a small number of goods that are chemically identical but differ in their state of matter (e.g. liquid and gaseous nitrogen), or where the product descriptions are virtually indistinguishable (e.g. "aluminium oxide" and "alumina (aluminium oxide)").

(ii) We remove a small number of input-output relationships that would otherwise give rise to cycles. Some of those cases seem to have arisen from survey respondents confusing inputs and outputs. Some of them arise because respondents seem to have included packing materials or capital goods in their reported intermediate inputs (e.g. tin plates in the production of mushrooms, hospital furniture in amoxycilin). These removed inputs account for less than TODO% of total intermediate input expenditure.

(iii) Finally, we remove all inputs that account for less than 5% of materials expenditures in an industry (unless they are reported by more than 15% of all plant-years, and at least 5), and all inputs that are identical to the outputs.

In the structural work we treat materials expenditures removed in steps (ii) and (iii) as primary input expenditures.

The resulting input-output relationships form a directed acyclical graph, which we construct recursively: we starting from the leaf goods and assign them a downstreamness of 0, then calculate the set of goods ω that only rely on inputs with a downstreamness of 0 and assign them a downstreamness of 1, then calculate the set of goods ω that only rely on inputs with a downstreamness of 0 or 1 and assign them a downstreamness of 2, etc. The resulting DAG has a maximum downstreamness of 40.

A.3.3 Calibrating the $\alpha_{\hat{\omega}}^{\rho(\hat{\omega})}$

We again start with goods that have a downstreamness of 0, and proceed by moving further downstream. For goods ω with a downstreamness of 0, we set $\alpha_l^{\rho(\omega)}$ equal to the share of expenditure on primary inputs in total cost, and we set each $\alpha_{\hat{\omega}}^{\rho(\omega)}$ to the share of expenditure on $\hat{\omega}$ in total cost. For goods ω with a downstreamness greater than 2:

1. We first set $\alpha_{\hat{\omega}}^{\rho(\omega)} = 0$ for all inputs $\hat{\omega}$ that have a downstreamness of more than one less than the downstreamness of ω. In other words, we set the input requirements for non-direct inputs to zero.
2. We calculate the $\alpha_{\hat{\omega}}^{\rho(\omega)}$ for direct inputs $\hat{\omega}$ as well as $\alpha_{l}^{\rho(\omega)}$ as expenditure shares of direct inputs and primary factors in the aggregate input basket. If a good $\tilde{\omega}$ is showing up multiple times upstream from ω, we split expenditures equally between all direct inputs. For each input observed in the data, we also account a share of primary input expenditures (as given by the α's to that input). The remainder of the observed primary factor expenditure is used to calculate the direct primary factor coefficient, $\alpha_{l}^{\rho(\omega)}$. Whenever this coefficient is below 5%, we assume that primary factors have been underreported and set the share to 5%.

B Robustness

C Further Results

C.1 Further correlates of vertical span

Table XI Cross-sectional correlates of Vertical Span

	Dependent variable: Vertical Span						
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Log Employment	-0.0589**						-0.0661**
	(0.0031)						(0.0028)
Sales/Cost Ratio		-0.0540**					0.00316
		(0.0051)					(0.0062)
Materials Share of Cost			-0.250**				-0.306**
			(0.018)				(0.023)
Importer Dummy				-0.163**			-0.0723**
				(0.0094)			(0.0084)
Exporter Dummy					-0.0903**		-0.0168*
					(0.0068)		(0.0075)
Share of R-Inputs in Materials Cost						-0.260**	-0.242**
						(0.021)	(0.022)
Year FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Industry FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes
R^2	0.317	0.304	0.310	0.309	0.322	0.322	0.371
Observations	353170	331685	332356	353694	150416	347548	147649

Standard errors in parentheses, clustered at the 5-dgt industry level.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

Table XII Other Plant-level Correlates of Vertical Span

	Dependent variable: Vertical Span					
	(1)	(2)	(3)	(4)	(5)	(6)
Materials Share of Cost	-0.250**			-0.119**		
	(0.018)			(0.015)		
Importer Dummy		-0.163**			-0.0143**	
		(0.0094)			(0.0055)	
Share of R-Inputs in Materials Cost			-0.260**			-0.181**
			(0.021)			(0.021)
Year FE	Yes	Yes	Yes	Yes	Yes	Yes
Industry FE	Yes	Yes	Yes			
Plant x Industry FE				Yes	Yes	Yes
R^2	0.310	0.309	0.322	0.774	0.765	0.773
Observations	332356	353694	347548	173141	186641	181958

Standard errors in parentheses, clustered at the 5-dgt industry level.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

C.2 Materials shares and number of inputs

Table XIII Materials Shares and Number of Inputs

	Dependent variable: Materials Share in Total Cost			
	(1)	(2)	(3)	(4)
Inverse HHI of Materials Cost Shares	-0.00727**	-0.000160		
	(0.00063)	(0.00097)		
Number of Inputs			-0.00575**	-0.00174*
			(0.00050)	(0.00076)
Year × 5-digit Industry × District FE	Yes	Yes	Yes	Yes
Plant × Product FE		Yes		Yes
R^2	0.722	0.928	0.722	0.928
Observations	137013	63325	137013	63325

SP plants only. Number of inputs exclude those with small cost shares (<5%).
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

Table XIV Input Adoption

	Dependent variable: Input Used Dummy $\mathbf{1}(X_{j\hat{\omega}t} > 0)$	
	(1)	(2)
$\log(1 + \tau_{it})$	-0.0506**	-0.0373**
	(0.0067)	(0.0071)
Year FE	Yes	Yes
Plant × Input FE	Yes	Yes
Plant × Product FE		Yes
R^2	0.337	0.361
Observations	2460831	2454899

Standard errors in parentheses.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

Regression at the level of plant × input × year, on single-product plants only. The left-hand side is a dummy for whether expenditure on input $\hat{\omega}$ is positive; the right-hand side is the log tariff on input $\hat{\omega}$ at time t.

Table XV Supply and demand shifters determine entry

	Dependent variable: log Producers $\|J\|_{d\omega t}$	
	(1)	(2)
$\log(1+\bar{\tau}_{it}^{\text{input}})$	-0.108**	-0.0496**
	(0.025)	(0.015)
$\log(1+\tau_{it}^{\text{output}})$	0.186**	0.251**
	(0.021)	(0.013)
Year FE	Yes	
State FE	Yes	
Industry FE	Yes	
State × Year FE		Yes
State × Industry FE		Yes
R^2	0.481	0.844
Observations	548180	537013

Standard errors in parentheses.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

The left-hand side is the log number of producers of a good ω at time t in state d. The right-hand side are log import tariffs on ω's input industries, weighted by I-O table cost shares ("log upstream tariff") and log import tariffs on downstream industries, weighted by ω's sales shares. Weights are invariant across time and space.

C.3 Pre- to post-liberalization

Table XVI Tariff changes act as demand and supply shocks: pre/post liberalization

	Dep. var.: Δ^t_{1990} log Sales	
	(1)	(2)
$\Delta^t_{1990}\log(1+\tau^{\text{output}}_{\omega t})$	1.302^{+}	1.533^{+}
	(0.75)	(0.79)
$\Delta^t_{1990}\log(1+\bar{\tau}^{\text{input}}_{\omega t})$		-1.188
		(0.77)
Year FE	Yes	Yes
R^2	0.0852	0.0903
Observations	2376	2376

Standard errors in parentheses, clustered at the state $\times$ industry level.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

The sample consists of all census plants that are observed in 1990 and post-1997 as single-product plants producing the same product. Changes are within plant-product.

Table XVII Vertical span and market size: pre/post liberalization

	Dependent variable: Δ^t_{1990} Vertical Span		
	(1)	(2)	(3)
Δ^t_{1990} log Sales	-0.147^{+}	-0.166^{+}	-0.237^{+}
	(0.084)	(0.086)	(0.12)
$\Delta^t_{1990}\log(1+\bar{\tau}^{\text{input}}_{it})$		0.194	1.421^{+}
		(0.24)	(0.77)
$\Delta\sum_i \alpha_i \log(1+\bar{\tau}^{\text{input}}_{it})(\text{distance}_{\omega i} - \overline{\text{span}}_j)$			-0.747
			(0.75)
$\Delta\sum_i \alpha_i \log(1+\bar{\tau}^{\text{input}}_{it})\overline{\text{span}}_j$			-1.031^{+}
			(0.62)
Year FE	Yes	Yes	Yes
R^2	-0.194	-0.255	-0.498
Observations	2179	2179	2128

Standard errors in parentheses, clustered at the state $\times$ industry level.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

The sample consists of all census plants that are observed in 1990 and post-1997 as single-product plants producing the same product. Changes are within plant-product. The regressions in both columns instrument Δ^t_{1990} log sales by the corresponding change in the log output tariff over the same horizon.

Table XVIII Vertical span and market size: pre/post liberalization

	Dependent variable: log Sales					
	(1)	(2)	(3)	(4)	(5)	(6)
Avg. log #Producers in Upstream Ind.	0.0655**	0.0560**	0.0551**	0.0201	0.119**	0.115**
	(0.013)	(0.018)	(0.018)	(0.043)	(0.044)	(0.044)
$\log(1+\bar{\tau}^{\text{input}}_{j\omega t})$			0.540*			0.519*
			(0.26)			(0.26)
Year FE	Yes			Yes		
Industry × Year FE		Yes	Yes		Yes	Yes
Plant × Industry FE	Yes	Yes	Yes	Yes	Yes	Yes
Estimator	OLS	OLS	OLS	IV	IV	IV
R^2	0.916	0.943	0.943	0.00262	-0.000638	0.000690
Observations	13683	9768	9757	13683	9768	9757

Standard errors in parentheses, clustered at the industry-year level.

$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

Analogously to Tables XVI and XVII, we limit the sample to plants that are observed as single-product plants in 1990 and post-1997. The dependent variable is log sales of plant j; the right-hand side variable is the average log number of producers in j's state in industries upstream from j (with weights taken at the time of j's first observation). Columns (4) to (6) instrument the average log number of producers in upstream industries by the tariff on industries downstream of these upstream industries, excluding the industry of the left-hand side plant.

D Proofs and Additional Theoretical Results

D.1 Proofs for Simple Model of Section 3

We first derive an expression for the effective cost of outsourcing each input. Let $F_\omega(c)$ be the distribution of unit cost among those in industry ω.

Lemma 2 *If firm j in industry ω exerts search effort h_{jk} to search for suppliers of $\omega - k$ then* $\Pr\left(c^o_{jk} > c | q_j, h_{j1}, h_{j2}\right) = e^{h_{jk} m_k \bar{c}_{\omega-k}^{-\zeta} c^\zeta}$ *with* $\bar{c}_{\omega-k} = \left(\int c^{-\zeta} dF_{\omega-k}(c)\right)^{-\frac{1}{\zeta}}$.

Proof. If firm j exerts search effort h_{jk}, the arrival rate of potential suppliers of $\omega - k$ with match-specific productivity better than z is $h_{jk} m_k z^{-\zeta}$. Thus the probability that the firm's best outside supplier delivers cost greater than c is

$$
\begin{aligned}
\Pr\left(c^o_{jk} > c\right) &= e^{-h_{jk} m_k \int\int 1\left\{\frac{c_s}{z} \le c\right\} dF_{\omega-k}(c_s) \zeta z^{-\zeta-1} dz} \\
&= e^{-h_{jk} m_k c^\zeta \int\int 1\left\{\frac{c_s}{u} \le 1\right\} dF_{\omega-k}(c_s) \zeta u^{-\zeta-1} du} \\
&= e^{-h_{jk} m_k \bar{c}_{\omega-k}^{-\zeta} c^\zeta}
\end{aligned}
$$

where $\bar{c}_{\omega-k}$ is defined to satisfy

$$\begin{aligned}
\bar{c}_{\omega-k}^{-\zeta} &\equiv \int\int 1\left\{\frac{c_s}{u} \leq 1\right\} dF_{\omega-k}(c_s)\zeta u^{-\zeta-1}du \\
&= \int\int 1\{t \leq 1\} c_s^{-\zeta} dF_{\omega-k}(c_s)\zeta t^{\zeta-1}dt \\
&= \int c_s^{-\zeta} dF_{\omega-k}(c_s)
\end{aligned}$$

■

Claim 1 *If firm j in industry ω exerts search effort h_2, its cost of producing $\omega-1$ in-house will follow a Weibull distribution*

$$\Pr\left(c_{j1}^i > c|h_{j2}\right) = e^{-h_{j2}^\alpha m_2^\alpha \left(w^{(1-\alpha)}\bar{c}_{\omega-2}^\alpha\right)^{-\zeta} c^\zeta}$$

Proof. If firm j in industry ω uses supplier s in industry $\omega-2$, its effective cost of $\omega-1$ will be $\frac{1}{B_j}w^{1-\alpha}\left(\frac{p_s}{z_s}\right)^\alpha$. Conditional on task productivity B_j and match-specific productivity z_s, the probability that the supplier's price p_s is low enough so that the the supplier delivers effective cost of $\omega-1$ weakly less than c–that is, $\frac{1}{B_j}w^{1-\alpha}\left(\frac{p_s}{z_s}\right)^\alpha < c$– is $F_{\omega-2}\left(B_j^{1/\alpha}z_s w^{-\frac{1-\alpha}{\alpha}}c^{1/\alpha}\right)$. Integrating across possible match-specific draws, the arrival rate of such a supplier is

$$\begin{aligned}
\int h_{j2}m_2F_{\omega-2}\left(B_j^{1/\alpha}zw^{-\frac{1-\alpha}{\alpha}}c^{1/\alpha}\right)\zeta z^{-\zeta-1}dz &= h_{j2}m_2\left(B_j^{1/\alpha}w^{-\frac{1-\alpha}{\alpha}}c^{1/\alpha}\right)^\zeta \int F_{\omega-2}(u)\,\zeta u^{-\zeta-1}dz \\
&= h_{j2}m_2w^{-\frac{1-\alpha}{\alpha}\zeta}c^{\frac{\zeta}{\alpha}}B_j^{\frac{\zeta}{\alpha}}\int u^{-\zeta}dF_{\omega-2}(u)\,du \\
&= h_{j2}m_2w^{-\frac{1-\alpha}{\alpha}\zeta}c^{\frac{\zeta}{\alpha}}B_j^{\frac{\zeta}{\alpha}}\bar{c}_{\omega-2}^{-\zeta}
\end{aligned}$$

Given B_j, the probability that no such supplier arrives is then

$$e^{-h_{j2}m_2w^{-\frac{1-\alpha}{\alpha}\zeta}(B_jc)^{\frac{\zeta}{\alpha}}\bar{c}_{\omega-2}^{-\zeta}}$$

Finally, integrating across realizations of B_j and using the functional form $E\left[e^{-uB^{\zeta/\alpha}}\right] = e^{-u^\alpha}$ gives

$$\Pr\left(c_{j1}^i > c|h_{j2}\right) = E\left[e^{-h_{j2}m_2w^{-\frac{1-\alpha}{\alpha}\zeta}c^{\frac{\zeta}{\alpha}}\bar{c}_{\omega-2}^{-\zeta}B_j^{\frac{\zeta}{\alpha}}}\right] = e^{-h_{j2}^\alpha m_2^\alpha\left(w^{(1-\alpha)}\bar{c}_{\omega-2}^\alpha\right)^{-\zeta}c^\zeta}$$

■

Claim 2 *If firm j chooses search effort h_{j1} and h_{j2}, the probability that the firm uses a supplier in $\omega-1$ is*

$$\frac{h_{j1}m_1\bar{c}_{\omega-1}^{-\zeta}}{h_{j1}m_1\bar{c}_{\omega-1}^{-\zeta} + h_{j2}^\alpha m_2^\alpha\left(w^{(1-\alpha)}\bar{c}_{\omega-2}^\alpha\right)^{-\zeta}}$$

Proof. Given h_1 and h_2, what is the probability that the firm chooses to use a supplier from $\omega - 1$?

$$\begin{aligned}
\Pr\left(c_{j1}^{i} \leq c_{j1}^{o} | h_{j1}, h_{j2}\right) &= \int \Pr\left(c_{j2}^{i} > c\right) d \Pr\left(c_{j1}^{o} < c\right) \\
&= \int e^{-h_{j2}^{\alpha} m_2^{\alpha} \left(w^{(1-\alpha)} \bar{c}_{\omega-2}^{\alpha}\right)^{-\zeta} c^{\zeta}} e^{-h_{j1} m_1 \bar{c}_{\omega-1}^{-\zeta} c^{\zeta}} h_{1j} m_1 \zeta c^{\zeta-1} dc \\
&= \frac{h_{j1} m_1 \bar{c}_{\omega-1}^{-\zeta}}{h_{j1} m_1 \bar{c}_{\omega-1}^{-\zeta} + h_{j2}^{\alpha} m_2^{\alpha} \left(w^{(1-\alpha)} \bar{c}_{\omega-2}^{\alpha}\right)^{-\zeta}}
\end{aligned}$$

■

Claim 3 *If firm j chooses search effort h_{j1} and h_{j2}, its expected gross profit is*

$$\mathbb{E}\left[\pi_j | q_j, h_{j1}, h_{j2}\right] = A_\omega \delta_\omega q_j^{\varepsilon-1} \left\{ \left[h_{j1} m_1 \bar{c}_{\omega-1}^{-\zeta} + h_{j2}^{\alpha} m_2^{\alpha} \left(w^{(1-\alpha)} \bar{c}_{\omega-2}^{\alpha}\right)^{-\zeta} \right]^{-\frac{\alpha}{\zeta}} w^{1-\alpha} \right\}^{1-\varepsilon}$$

where $A_\omega \equiv u p^{\eta} \delta_\omega p_\omega^{\varepsilon-\eta} \frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}} \Gamma\left(1 - \alpha \frac{\varepsilon-1}{\zeta}\right)$

Proof. Given h_1, and h_2, the probability that the firm's effective cost of $\omega - 1$ is greater than c is

$$\begin{aligned}
\Pr\left(\min\left\{c_1^{i}, c_1^{o}\right\} > c | h_1, h_2\right) &= \Pr\left(c_1^{o} > c | h_1\right) \Pr\left(c_1^{i} > c | h_2\right) \\
&= e^{-h_{j1} m_1 \bar{c}_{\omega-1}^{-\zeta} c^{\zeta}} e^{-h_{j2}^{\alpha} m_2^{\alpha} \left(w^{(1-\alpha)} \bar{c}_{\omega-2}^{\alpha}\right)^{-\zeta} c^{\zeta}} \\
&= e^{-T_1^{-\zeta} c^{\zeta}}
\end{aligned}$$

where $T_1 \equiv \left[h_{j1} m_1 \bar{c}_{\omega-1}^{-\zeta} + h_{j2}^{\alpha} m_2^{\alpha} \left(w^{(1-\alpha)} \bar{c}_{\omega-2}^{\alpha}\right)^{-\zeta}\right]^{-\frac{1}{\zeta}}$

Given q, h_1, and h_2, the probability that the firm's cost is greater than is c

$$\begin{aligned}
\Pr\left(\frac{1}{q} w^{1-\alpha} \left(\min\left\{c_1^{i}, c_1^{o}\right\}\right)^{\alpha} > c \middle| q, h_1, h_2\right) &= \Pr\left(\min\left\{c_1^{i}, c_1^{o}\right\} > \left(\frac{qc}{w^{1-\alpha}}\right)^{1/\alpha} \middle| q, h_1, h_2\right) \\
&= e^{-T_1^{-\zeta} \left(\frac{qc}{w^{1-\alpha}}\right)^{\zeta/\alpha}} \\
&= e^{-\left(\frac{qc}{T_1^{\alpha} w^{1-\alpha}}\right)^{\zeta/\alpha}}
\end{aligned}$$

The firms gross profit comes only from sales to the household. Given isoelastic demand, $u_j \leq u p^{\eta} \delta_\omega p_\omega^{\varepsilon-\eta} p_j^{-\varepsilon}$, the firm chooses price $p_j = \frac{\varepsilon}{\varepsilon-1} c_j$, so its expected gross profit is $\mathbb{E}\left[u p^{\eta} \delta_\omega p_\omega^{\varepsilon-\eta} \frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}} c_j^{1-\varepsilon}\right] =$

$up^{\eta}\delta_{\omega}p_{\omega}^{\varepsilon-\eta}\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}\mathbb{E}\left[c_j^{1-\varepsilon}\right]$. The final term is

$$\begin{aligned}
E\left[c^{1-\varepsilon}|q,h_1,h_2\right] &= \int_0^{\infty} c^{1-\varepsilon}d\left\{1-e^{-\left(\frac{qc}{T_1^{\alpha}w^{1-\alpha}}\right)^{\zeta/\alpha}}\right\} \\
&= \left(\frac{q}{T_1^{\alpha}w^{1-\alpha}}\right)^{\varepsilon-1}\int_0^{\infty} u^{1-\varepsilon}d\left\{1-e^{-u^{\zeta/\alpha}}\right\} \\
&= \left(\frac{q}{T_1^{\alpha}w^{1-\alpha}}\right)^{\varepsilon-1}\int_0^{\infty} v^{-\alpha\frac{\varepsilon-1}{\zeta}}d\left\{1-e^{-v}\right\} \\
&= \left(\frac{q}{T_1^{\alpha}w^{1-\alpha}}\right)^{\varepsilon-1}\Gamma\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)
\end{aligned}$$

■

D.1.1 Optimal Search Behavior

In this section, we normalize the wage to unity. Total expected profit is then for a firm with productivity q is

$$\max_{h_1,h_2} A_{\omega}\delta_{\omega}q^{\varepsilon-1}\left[h_1m_1\bar{c}_{\omega-1}^{-\zeta}+h_2^{\alpha}m_2^{\alpha}\bar{c}_{\omega-2}^{-\alpha\zeta}\right]^{\alpha\frac{\varepsilon-1}{\zeta}}-\frac{k}{1+\gamma}h_1^{1+\gamma}-\frac{k}{1+\gamma}h_2^{1+\gamma}$$

The first order conditions are

$$\begin{aligned}
kh_1^{\gamma} &= A_{\omega}\delta_{\omega}q^{\varepsilon-1}\left[h_1m_1\bar{c}_{\omega-1}^{-\zeta}+h_2^{\alpha}m_2^{\alpha}\bar{c}_{\omega-2}^{-\alpha\zeta}\right]^{\alpha\frac{\varepsilon-1}{\zeta}-1}\alpha\frac{\varepsilon-1}{\zeta}m_1\bar{c}_{\omega-1}^{-\zeta} \\
kh_2^{\gamma} &= A_{\omega}\delta_{\omega}q^{\varepsilon-1}\left[h_1m_1\bar{c}_{\omega-1}^{-\zeta}+h_2^{\alpha}m_2^{\alpha}\bar{c}_{\omega-2}^{-\alpha\zeta}\right]^{\alpha\frac{\varepsilon-1}{\zeta}-1}\alpha^2\frac{\varepsilon-1}{\zeta}h_2^{\alpha-1}m_2^{\alpha}\bar{c}_{\omega-2}^{-\alpha\zeta}
\end{aligned}$$

Note that $1+\gamma>\alpha\frac{\varepsilon-1}{\zeta}$ gurantees that the second order condition hold.

Let $T\equiv\left[h_1m_1\bar{c}_{\omega-1}^{-\zeta}+h_2^{\alpha}m_2^{\alpha}\bar{c}_{\omega-2}^{-\alpha\zeta}\right]^{-\frac{1}{\zeta}}$ amd $r=\frac{A_{\omega}\delta_{\omega}\alpha\frac{\varepsilon-1}{\zeta}}{k}$. Then these FOCs can be expressed as

$$\begin{aligned}
h_1^{\gamma} &= rq^{\varepsilon-1}\left(T^{-\zeta}\right)^{\alpha\frac{\varepsilon-1}{\zeta}-1}m_1\bar{c}_{\omega-1}^{-\zeta} \\
h_2^{\gamma} &= rq^{\varepsilon-1}\left(T^{-\zeta}\right)^{\alpha\frac{\varepsilon-1}{\zeta}-1}\alpha h_2^{\alpha-1}m_2^{\alpha}\bar{c}_{\omega-2}^{-\alpha\zeta}
\end{aligned}$$

Let $O=\frac{h_1m_1\bar{c}_{\omega-1}^{-\zeta}}{h_1m_1\bar{c}_{\omega-1}^{-\zeta}+h_2^{\alpha}m_2^{\alpha}\bar{c}_{\omega-2}^{-\alpha\zeta}}$ be the probability of outsourcing the more upstream task.

$$\frac{d\ln T^{-\zeta}}{d\ln q^{\varepsilon-1}}=O\frac{d\ln h_1}{d\ln q^{\varepsilon-1}}+(1-O)\,\alpha\frac{d\ln h_2}{d\ln q^{\varepsilon-1}}$$

$$\gamma \frac{d\ln h_1}{d\ln q^{\varepsilon-1}} = 1-\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{d\ln T^{-\zeta}}{d\ln q^{\varepsilon-1}}$$
$$\gamma \frac{d\ln h_2}{d\ln q^{\varepsilon-1}} = 1-\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{d\ln T^{-\zeta}}{d\ln q^{\varepsilon-1}} - (1-\alpha)\frac{d\ln h_2}{d\ln q^{\varepsilon-1}}$$

Together, these imply that

$$\begin{aligned}\frac{d\ln T^{-\zeta}}{d\ln q^{\varepsilon-1}} &= O\frac{1}{\gamma}\left[1-\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{d\ln T^{-\zeta}}{d\ln q^{\varepsilon-1}}\right] \\ &\quad + (1-O)\,\alpha\frac{1}{\gamma+(1-\alpha)}\left[1-\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{d\ln T^{-\zeta}}{d\ln q^{\varepsilon-1}}\right]\end{aligned}$$

or

$$\frac{d\ln T^{-\zeta}}{d\ln q^{\varepsilon-1}} = \frac{1}{\frac{1}{O\frac{1}{\gamma}+(1-O)\frac{\alpha}{\gamma+(1-\alpha)}}+1-\alpha\frac{\varepsilon-1}{\zeta}} > 0$$

Plugging this in gives

$$\gamma\frac{d\ln h_1}{d\ln q^{\varepsilon-1}} = \frac{1}{1+\left[O\frac{1}{\gamma}+(1-O)\frac{\alpha}{\gamma+(1-\alpha)}\right]\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)} > 0$$
$$(\gamma+1-\alpha)\frac{d\ln h_2}{d\ln q^{\varepsilon-1}} = \frac{1}{1+\left[O\frac{1}{\gamma}+(1-O)\frac{\alpha}{\gamma+(1-\alpha)}\right]\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)} > 0$$

Claim 4 $\frac{\Pr(use\ \omega-1)}{\Pr(use\ \omega-2)}$ *is strictly increasing in* q.

Proof. Let $O_\omega(q)$ be the probability that a firm in ω born with productivity q ends up using $\omega-1$ as a input. We have

$$\frac{O_\omega(q)}{1-O_\omega(q)} = \frac{h_1 m_1 \bar{c}_{\omega-1}^{-\zeta}}{h_2^{\alpha} m_2^{\alpha}\bar{c}_{\omega-2}^{-\alpha\zeta}}$$

Differentiating with respect to $q^{\varepsilon-1}$ and using the expressions for

$$\frac{d\ln\frac{O_\omega(q)}{1-O_\omega(q)}}{d\ln q^{\varepsilon-1}} = \frac{d\ln h_1}{d\ln q^{\varepsilon-1}} - \alpha\frac{d\ln h_2}{d\ln q^{\varepsilon-1}} = \frac{d\ln h_1}{d\ln q^{\varepsilon-1}} - \frac{\alpha\gamma}{\gamma+(1-\alpha)}\frac{d\ln h_1}{d\ln q^{\varepsilon-1}}$$

The conclusion that $\frac{d\ln\frac{O_\omega(q)}{1-O_\omega(q)}}{d\ln q^{\varepsilon-1}} > 0$ follows from $\left(1-\frac{\alpha\gamma}{\gamma+(1-\alpha)}\right) = \frac{(1-\alpha)(1+\gamma)}{\gamma+(1-\alpha)} > 0$ and $\frac{d\ln h_1}{d\ln q^{\varepsilon-1}}$. ■

Lemma 3 *Among those born with productivity* q*, sales is independent of input choice.*

Proof. Among those with productivity q, the realization of unit cost is independent of input choice:

$$\begin{aligned}\Pr\left(c_1^o < c_1^i | c_1 = c, q, h_1, h_2\right) &= \frac{\Pr\left(c_1^o = c, c_1^i > c | q, h_1, h_2\right)}{\Pr\left(c_1 = c | q, h_1, h_2\right)} \\ &= \frac{h_1 m_1 \bar{c}_{\omega-1}^{-\zeta} \zeta c^{\zeta-1} e^{-h_1 m_1 \bar{c}_{\omega-1}^{-\zeta} c^\zeta} e^{-h_2^\alpha m_2^\alpha \left(w^{(1-\alpha)} \bar{c}_{\omega-2}^\alpha\right)^{-\zeta} c^\zeta}}{\left[h_1 m_1 \bar{c}_{\omega-1}^{-\zeta} + h_2^\alpha m_2^\alpha \left(w^{(1-\alpha)} \bar{c}_{\omega-2}^\alpha\right)^{-\zeta}\right] \zeta c^{\zeta-1} e^{-\left[h_1 m_1 \bar{c}_{\omega-1}^{-\zeta} + h_2^\alpha m_2^\alpha \left(w^{(1-\alpha)} \bar{c}_{\omega-2}^\alpha\right)^{-\zeta}\right] c^\zeta}} \\ &= \frac{h_1 m_1 \bar{c}_{\omega-1}^{-\zeta}}{h_1 m_1 \bar{c}_{\omega-1}^{-\zeta} + h_2^\alpha m_2^\alpha \left(w^{(1-\alpha)} \bar{c}_{\omega-2}^\alpha\right)^{-\zeta}}\end{aligned}$$

Since this probability of producing in-house is the same for all values of c, it must be independent of sales, which is a random variable hat is measurable with respect to the firms unit cost but not with respect to its choice of input. ■

Claim 5 *Among those in industry ω, those with higher sales are more likely to use a supplier in industry $\omega - 1$*

Proof. Using the law of toal covariance, we can express this as

$$Cov\left(Sales, \text{use } \omega - 1\right) = E\left[Cov\left(Sales, \text{use } \omega - 1\right) | q\right] + cov\left[E\left[Sales | q\right], E\left[\text{use } \omega - 1 | q\right]\right]$$

The first term is zero and the second term is positive. ■

Since using a supplier in industry 1 maps directly into a shorter vertical span and higher intermediate input share, we immediately have the following corollary

Corollary 2 *Among those in industry ω, those with higher sales are more likely to have a shorter vertical span and higher intermediate input share*

D.1.2 Demand Shocks

We consider in this section the impact of an increase in the demand for industry ω, i.e., and increase in δ_ω.

Proposition 8 *If δ_ω increases, J_ω increases, p_ω and $\bar{c}_\omega$ fall, and fraction of firms using $\omega - 1$ rises.*

Proof. A shock to δ_ω does not change profit in $\omega - 1$ or $\omega - 2$, so it must be that $\bar{c}_{\omega-1}$ and $\bar{c}_{\omega-2}$ are unchanged. We first establish that $\frac{d \ln A_\omega \delta_\omega}{d \ln \delta_\omega} > 0$. Let $\bar{\pi}_\omega(q)$ be the expected profit of a firm born with productivity q.

$$\bar{\pi}_\omega\left(A_\omega \delta_\omega q^{\varepsilon-1}\right) = \max_{h_1, h_2} A_\omega \delta_\omega q^{\varepsilon-1} \left[h_1 m_1 \bar{c}_{\omega-1}^{-\zeta} + h_2^\alpha m_2^\alpha \bar{c}_{\omega-2}^{-\alpha\zeta}\right]^{\alpha \frac{\varepsilon-1}{\zeta}} - \frac{k}{1+\gamma} h_1^{1+\gamma} - \frac{k}{1+\gamma} h_2^{1+\gamma}$$

Using the envelope theroem, we have

$$\frac{d\bar{\pi}_\omega\left(A_\omega\delta_\omega q^{\varepsilon-1}\right)}{d\ln A_\omega\delta_\omega q^{\varepsilon-1}}=A_\omega\delta_\omega q^{\varepsilon-1}\left[h_1 m_1\bar{c}_{\omega-1}^{-\zeta}+h_2^\alpha m_2^\alpha\bar{c}_{\omega-2}^{-\alpha\zeta}\right]^{\alpha\frac{\varepsilon-1}{\zeta}}>0$$

The elasticity with respect to δ_ω is simply $\frac{d\bar{\pi}_\omega\left(A_\omega\delta_\omega q^{\varepsilon-1}\right)}{d\ln\delta_\omega}=\frac{d\bar{\pi}_\omega\left(A_\omega\delta_\omega q^{\varepsilon-1}\right)}{d\ln A_\omega\delta_\omega q^{\varepsilon-1}}\frac{d\ln A_\omega\delta_\omega}{d\ln\delta_\omega}$. With this, we can see how average profit $\bar{\pi}_\omega=\int\bar{\pi}_\omega(q)\,dQ(q)$ changes with δ_ω:

$$\frac{d\ln\bar{\pi}_\omega}{d\ln\delta_\omega}=\frac{\int\frac{d\bar{\pi}_\omega\left(A_\omega\delta_\omega q^{\varepsilon-1}\right)}{d\ln\delta_\omega}dQ(q)}{\int\bar{\pi}_\omega(q)\,dQ(q)}=\frac{\int\frac{d\bar{\pi}_\omega\left(A_\omega\delta_\omega q^{\varepsilon-1}\right)}{d\ln A_\omega\delta_\omega q^{\varepsilon-1}}dQ(q)}{\int\bar{\pi}_\omega(q)\,dQ(q)}\frac{d\ln A_\omega\delta_\omega}{d\ln\delta_\omega}$$

We next study the impact of δ_ω on $\int\mathbb{E}\left[c^{1-\varepsilon}|q\right]dQ(q)$. Note that $\mathbb{E}\left[c^{1-\varepsilon}|q\right]=\Gamma\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)q^{\varepsilon-1}\left[h_{j1}m_1\bar{c}_{\omega-1}^{-\zeta}+\right.$ Differentiating yields

$$\frac{d\ln\mathbb{E}\left[c^{1-\varepsilon}|q\right]}{d\ln\delta_\omega}=\alpha\frac{\varepsilon-1}{\zeta}\left[O_\omega(q)\frac{d\ln h_1}{d\ln\delta_\omega}+(1-O_\omega(q))\,\alpha\frac{d\ln h_2}{d\ln\delta_\omega}\right]$$

Similarly, the first order conditions for h_1 and h_2 imply that these are functions of $A_\omega\delta_\omega q^{\varepsilon-1}$. Following the logic about how serach effort changes with $q^{\varepsilon-1}$ in the introduction to the section, we have

$$\begin{aligned}\gamma\frac{d\ln h_1\left(A_\omega\delta_\omega q^{\varepsilon-1}\right)}{d\ln A_\omega\delta_\omega} &= \frac{1}{1+\left[O\frac{1}{\gamma}+(1-O)\frac{\alpha}{\gamma+(1-\alpha)}\right]\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)}\\ (\gamma+1-\alpha)\frac{d\ln h_2\left(A_\omega\delta_\omega q^{\varepsilon-1}\right)}{d\ln A_\omega\delta_\omega} &= \frac{1}{1+\left[O\frac{1}{\gamma}+(1-O)\frac{\alpha}{\gamma+(1-\alpha)}\right]\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)}\end{aligned}$$

Plugging these in yields

$$\frac{d\ln\mathbb{E}\left[c^{1-\varepsilon}|q\right]}{d\ln A_\omega\delta_\omega}=\frac{\alpha\frac{\varepsilon-1}{\zeta}\left[O_\omega(q)\frac{1}{\gamma}+(1-O_\omega(q))\frac{\alpha}{\gamma+1-\alpha}\right]}{1+\left[O_\omega(q)\frac{1}{\gamma}+(1-O_\omega(q))\frac{\alpha}{\gamma+1-\alpha}\right]\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)}>0$$

With these, we can derive an expression for the change in the household's price index for industry ω. This price index satisfies

$$\begin{aligned}p_\omega^{1-\varepsilon} &= \int_{J_\omega}p_{\omega j}^{1-\varepsilon}dj=J_\omega\int\mathbb{E}\left[\left(\frac{\varepsilon}{\varepsilon-1}c\right)^{1-\varepsilon}|q\right]dQ(q)\\ &= \left(\frac{\varepsilon}{\varepsilon-1}\right)^{1-\varepsilon}\bar{\pi}^{\chi-\frac{1}{\beta}}\bar{\pi}_\omega^\beta\int\mathbb{E}\left[c^{1-\varepsilon}|q\right]dQ(q)\end{aligned}$$

where we used $J_\omega = \bar{\pi}^{\chi - \frac{1}{\beta}} \bar{\pi}_\omega^\beta$. Differentiating yields

$$\begin{aligned}
(1-\varepsilon)\frac{d\ln p_\omega^{1-\varepsilon}}{d\ln\delta_\omega} &= \beta\frac{d\ln\bar{\pi}_\omega}{d\ln\delta_\omega} + \frac{d\ln\int\mathbb{E}\left[c^{1-\varepsilon}|q\right]dQ\left(q\right)}{d\ln\delta_\omega} \\
&= \left[\beta\frac{\int\frac{d\bar{\pi}_\omega\left(A_\omega\delta_\omega q^{\varepsilon-1}\right)}{d\ln A_\omega\delta_\omega}dQ\left(q\right)}{\int\bar{\pi}_\omega\left(q\right)dQ\left(q\right)} + \frac{\int\frac{d\mathbb{E}\left[c^{1-\varepsilon}|q\right]}{d\ln A_\omega\delta_\omega}dQ\left(q\right)}{\int\mathbb{E}\left[c^{1-\varepsilon}|q\right]dQ\left(q\right)}\right]\frac{d\ln A_\omega\delta_\omega}{d\ln\delta_\omega}
\end{aligned}$$

Let X denote the term in brackets, with $X > 0$. Since

$$\frac{d\ln A_\omega\delta_\omega}{d\ln\delta_\omega} = 1 + \left(\varepsilon-\eta\right)\frac{d\ln p_\omega}{d\ln\delta_\omega} = 1 - \frac{\varepsilon-\eta}{\varepsilon-1}\left[X\frac{d\ln A_\omega\delta_\omega}{d\ln\delta_\omega}\right]$$

Solving for $\frac{d\ln A_\omega\delta_\omega}{d\ln\delta_\omega}$ gives

$$\frac{d\ln A_\omega\delta_\omega}{d\ln\delta_\omega} = \frac{1}{1+\frac{\varepsilon-\eta}{\varepsilon-1}X} > 0$$

Some simple consequences are

$$\begin{aligned}
\frac{d\ln p_\omega}{d\ln\delta_\omega} &= -\frac{X}{\varepsilon-1}\frac{d\ln A_\omega\delta_\omega}{d\ln\delta_\omega} < 0 \\
\frac{d\ln J_\omega}{d\ln\delta_\omega} &= \beta\frac{d\ln\bar{\pi}_\omega}{d\ln A_\omega\delta_\omega}\frac{d\ln A_\omega\delta_\omega}{d\ln\delta_\omega} > 0 \\
\frac{d\ln\bar{c}_\omega}{d\ln\delta_\omega} &= \frac{\int\mathbb{E}\left[c^{-\zeta}|q\right]\frac{d\ln\mathbb{E}\left[c^{-\zeta}|q\right]}{d\ln A_\omega\delta_\omega}dQ\left(q\right)}{\int\mathbb{E}\left[c^{-\zeta}|q\right]dQ\left(q\right)}\frac{d\ln A_\omega}{d\ln\delta_\omega} > 0
\end{aligned}$$

where the last line follows the same logic as showing that $\frac{d\ln\int\mathbb{E}\left[c^{1-\varepsilon}|q\right]dQ(q)}{d\ln A_\omega\delta_\omega} > 0$. Finally, the fraction of firms in ω born with productivity q is that use a supplier in industry $\omega-1$ is $\frac{O_\omega(q)}{1-O_\omega(q)} = \frac{h_1 m_1 \bar{c}_{\omega-1}^{-\zeta}}{h_2^\alpha m_2^\alpha \bar{c}_{\omega-2}^{-\alpha\zeta}}$. Differentiating yields

$$\frac{d\ln\frac{O_\omega(q)}{1-O_\omega(q)}}{d\ln\delta_\omega} = \frac{d\ln h_1}{d\ln\delta_\omega} - \frac{d\ln h_2}{d\ln\delta_\omega} = \frac{\frac{1}{\gamma}-\frac{\alpha}{\gamma+1-\alpha}}{1+\left[O_\omega\left(q\right)\frac{1}{\gamma}+\left(1-O_\omega\left(q\right)\right)\frac{\alpha}{\gamma+1-\alpha}\right]\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)}\frac{d\ln A_\omega\delta_\omega}{d\ln\delta_\omega} > 0$$

Since this is true for each q, it must be true for the industry as a whole. ■

D.1.3 Upstream Demand Shock

In this section we study the impact on industry ω of a shock to $\delta_{\omega-1}$. We know from above that an increase in $\delta_{\omega-1}$ raises reduces $\bar{c}_{\omega-1}$. It has no impact on $\bar{c}_{\omega-2}$. It will, however have an impact on A_ω because firms in ω respond to the change in expected input cost.

Using the envelope theorem, we have

$$\bar{\pi}_\omega\left(q\right) = \max_{h_1,h_2} A_\omega\delta_\omega q^{\varepsilon-1}\left[h_1 m_1\bar{c}_{\omega-1}^{-\zeta} + h_2^\alpha m_2^\alpha\bar{c}_{\omega-2}^{-\alpha\zeta}\right]^{\alpha\frac{\varepsilon-1}{\zeta}} - \frac{k}{1+\gamma}h_1^{1+\gamma} - \frac{k}{1+\gamma}h_2^{1+\gamma}$$

$$\begin{aligned}\frac{d\bar{\pi}_{\omega}\left(q\right)}{d\ln\bar{c}_{\omega-1}^{-\zeta}} &= A_{\omega}\delta_{\omega}q^{\varepsilon-1}\left[h_1 m_1\bar{c}_{\omega-1}^{-\zeta}+h_2^{\alpha}m_2^{\alpha}\bar{c}_{\omega-2}^{-\alpha\zeta}\right]^{\alpha\frac{\varepsilon-1}{\zeta}}\frac{d\ln A_{\omega}}{d\ln\bar{c}_{\omega-1}^{-\zeta}}+A_{\omega}\delta_{\omega}q^{\varepsilon-1}\left[h_1 m_1\bar{c}_{\omega-1}^{-\zeta}+h_2^{\alpha}m_2^{\alpha}\bar{c}_{\omega-2}^{-\alpha\zeta}\right]^{\alpha\frac{\varepsilon-1}{\zeta}-1}\alpha\frac{\varepsilon-}{\zeta} \\ &= A_{\omega}\delta_{\omega}q^{\varepsilon-1}\left[h_1 m_1\bar{c}_{\omega-1}^{-\zeta}+h_2^{\alpha}m_2^{\alpha}\bar{c}_{\omega-2}^{-\alpha\zeta}\right]^{\alpha\frac{\varepsilon-1}{\zeta}}\left\{\frac{d\ln A_{\omega}}{d\ln\bar{c}_{\omega-1}^{-\zeta}}+\alpha\frac{\varepsilon-1}{\zeta}O_{\omega}\left(q\right)\right\}\end{aligned}$$

Let $\lambda=\frac{\int\pi_{\omega}^{gross}(q)dQ(q)}{\int\bar{\pi}_{\omega}(q)dQ(q)}>1$ be the ratio of gross profit (without subtracting search costs) to net profit (subtracting search costs), and for any quantity $x\left(q\right)$, let let $\bar{x}=\frac{\int\pi_{\omega}^{gross}(q)x(q)dQ(q)}{\int\pi_{\omega}^{gross}(q)dQ(q)}$ be the gross-profit-weighted average. With this we can derive an expression for how profit changes with upstream characteristics.

$$\frac{d\ln\bar{\pi}_{\omega}}{d\ln\bar{c}_{\omega-1}^{-\zeta}}=\frac{\int\frac{d\bar{\pi}_{\omega}(q)}{d\ln\bar{c}_{\omega-1}^{-\zeta}}dQ_{\omega}\left(q\right)}{\bar{\pi}_{\omega}}=\lambda\left(\frac{d\ln A_{\omega}}{d\ln\bar{c}_{\omega-1}^{-\zeta}}+\alpha\frac{\varepsilon-1}{\zeta}\bar{O}_{\omega}\right)$$

Next, we derive expressions for how $\frac{d\ln A_{\omega}}{d\ln\bar{c}_{\omega-1}^{-\zeta}}$ changes. Define

$$\begin{aligned}Y\left(q\right) &\equiv \left[h_1 m_1\bar{c}_{\omega-1}^{-\zeta}+h_2^{\alpha}m_2^{\alpha}\bar{c}_{\omega-2}^{-\alpha\zeta}\right]^{\alpha\frac{\varepsilon-1}{\zeta}} \\ \Sigma_0\left(q\right) &\equiv \frac{1}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left[O_{\omega}\left(q\right)\frac{1}{\gamma}+\left(1-O_{\omega}\left(q\right)\right)\frac{\alpha}{\gamma+1-\alpha}\right]} \\ \Sigma_1\left(q\right) &\equiv \frac{O_{\omega}\left(q\right)}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left[O_{\omega}\left(q\right)\frac{1}{\gamma}+\left(1-O_{\omega}\left(q\right)\right)\frac{\alpha}{\gamma+1-\alpha}\right]}.\end{aligned}$$

The FOCs for h_1 and h_2 can be expressed as

$$\begin{aligned}kh_1^{\gamma} &= A_{\omega}\delta_{\omega}q^{\varepsilon-1}Y^{-\frac{1-\alpha\frac{\varepsilon-1}{\zeta}}{\alpha\frac{\varepsilon-1}{\zeta}}}\alpha\frac{\varepsilon-1}{\zeta}m_1\bar{c}_{\omega-1}^{-\zeta} \\ kh_2^{\gamma} &= A_{\omega}\delta_{\omega}q^{\varepsilon-1}Y^{-\frac{1-\alpha\frac{\varepsilon-1}{\zeta}}{\alpha\frac{\varepsilon-1}{\zeta}}}\alpha^2\frac{\varepsilon-1}{\zeta}h_2^{\alpha-1}m_2^{\alpha}\bar{c}_{\omega-2}^{-\alpha\zeta}\end{aligned}$$

Differentiating yields

$$\begin{aligned}\gamma\frac{d\ln h_1\left(q\right)}{d\ln\bar{c}_{\omega-1}^{-\zeta}} &= \frac{d\ln A_{\omega}}{d\ln\bar{c}_{\omega-1}^{-\zeta}}-\frac{1-\alpha\frac{\varepsilon-1}{\zeta}}{\alpha\frac{\varepsilon-1}{\zeta}}\frac{d\ln Y}{d\ln\bar{c}_{\omega-1}^{-\zeta}}+1 \\ \left(\gamma+1-\alpha\right)\frac{d\ln h_2\left(q\right)}{d\ln\bar{c}_{\omega-1}^{-\zeta}} &= \frac{d\ln A_{\omega}}{d\ln\bar{c}_{\omega-1}^{-\zeta}}-\frac{1-\alpha\frac{\varepsilon-1}{\zeta}}{\alpha\frac{\varepsilon-1}{\zeta}}\frac{d\ln Y}{d\ln\bar{c}_{\omega-1}^{-\zeta}}\end{aligned}$$

and

$$\begin{aligned}\frac{d\ln Y(q)}{d\ln \bar{c}_{\omega-1}^{-\zeta}} &= \alpha\frac{\varepsilon-1}{\zeta}\left[O_\omega(q)\left(\frac{d\ln h_1(q)}{d\ln \bar{c}_{\omega-1}^{-\zeta}}+1\right)+(1-O_\omega(q))\frac{d\ln h_2(q)}{d\ln \bar{c}_{\omega-1}^{-\zeta}}\right]\\ &= \alpha\frac{\varepsilon-1}{\zeta}\left[\begin{array}{c} O_\omega(q)\left(\frac{1}{\gamma}\left(\frac{d\ln A_\omega}{d\ln \bar{c}_{\omega-1}^{-\zeta}}-\frac{1-\alpha\frac{\varepsilon-1}{\zeta}}{\alpha\frac{\varepsilon-1}{\zeta}}\frac{d\ln Y}{d\ln \bar{c}_{\omega-1}^{-\zeta}}+1\right)+1\right)\\ +(1-O_\omega(q))\frac{1}{\gamma+1-\alpha}\left(\frac{d\ln A_\omega}{d\ln \bar{c}_{\omega-1}^{-\zeta}}-\frac{1-\alpha\frac{\varepsilon-1}{\zeta}}{\alpha\frac{\varepsilon-1}{\zeta}}\frac{d\ln Y}{d\ln \bar{c}_{\omega-1}^{-\zeta}}\right)\end{array}\right]\end{aligned}$$

Solving for $\frac{d\ln Y}{d\ln \bar{c}_{\omega-1}^{-\zeta}}$ gives

$$\begin{aligned}\frac{d\ln Y(q)}{d\ln \bar{c}_{\omega-1}^{-\zeta}} &= \alpha\frac{\varepsilon-1}{\zeta}\frac{\left[O_\omega(q)\frac{1}{\gamma}+(1-O_\omega(q))\frac{1}{\gamma+1-\alpha}\right]\frac{d\ln A_\omega}{d\ln \bar{c}_{\omega-1}^{-\zeta}}+\left(\frac{1}{\gamma}+1\right)O_\omega(q)}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left[O_\omega(q)\frac{1}{\gamma}+(1-O_\omega(q))\frac{1}{\gamma+1-\alpha}\right]}\\ &= \alpha\frac{\varepsilon-1}{\zeta}\left[\frac{1-\Sigma_0}{1-\alpha\frac{\varepsilon-1}{\zeta}}\frac{d\ln A_\omega}{d\ln \bar{c}_{\omega-1}^{-\zeta}}+\left(\frac{1}{\gamma}+1\right)\Sigma_1\right]\end{aligned}$$

We next study the impact of $\bar{c}_{\omega-1}$ on $\int \mathbb{E}\left[c^{1-\varepsilon}|q\right]dQ(q)$. Note that $\mathbb{E}\left[c^{1-\varepsilon}|q\right]=\Gamma\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)q^{\varepsilon-1}Y(q)$. Differentiating yields

$$\frac{d\ln \mathbb{E}\left[c^{1-\varepsilon}|q\right]}{d\ln \bar{c}_{\omega-1}^{-\zeta}}=\frac{d\ln Y(q)}{d\ln \bar{c}_{\omega-1}^{-\zeta}}=\alpha\frac{\varepsilon-1}{\zeta}\left[\frac{1-\Sigma_0}{1-\alpha\frac{\varepsilon-1}{\zeta}}\frac{d\ln A_\omega}{d\ln \bar{c}_{\omega-1}^{-\zeta}}+\left(\frac{1}{\gamma}+1\right)\Sigma_1\right]$$

As a result, integrating yields

$$\frac{d\ln \int \mathbb{E}\left[c^{1-\varepsilon}|q\right]dQ(q)}{d\ln \bar{c}_{\omega-1}^{-\zeta}}=\alpha\frac{\varepsilon-1}{\zeta}\left[\frac{1-\bar{\Sigma}_0}{1-\alpha\frac{\varepsilon-1}{\zeta}}\frac{d\ln A_\omega}{d\ln \bar{c}_{\omega-1}^{-\zeta}}+\left(\frac{1}{\gamma}+1\right)\bar{\Sigma}_1\right]$$

With these, we can derive an expression for the change in the household's price index for industry ω. This price index satisfies

$$\begin{aligned}p_\omega^{1-\varepsilon} &= \int_{J_\omega}p_{\omega j}^{1-\varepsilon}dj=J_\omega\int\mathbb{E}\left[\left(\frac{\varepsilon}{\varepsilon-1}c\right)^{1-\varepsilon}|q\right]dQ(q)\\ &= \left(\frac{\varepsilon}{\varepsilon-1}\right)^{1-\varepsilon}\bar{\pi}^{\chi-\frac{1}{\beta}}\bar{\pi}_\omega^{\beta}\int\mathbb{E}\left[c^{1-\varepsilon}|q\right]dQ(q)\end{aligned}$$

where we used $J_\omega=\bar{\pi}^{\chi-\frac{1}{\beta}}\bar{\pi}_\omega^{\beta}$. Differentiating yields

$$\begin{aligned}(1-\varepsilon)\frac{d\ln p_\omega^{1-\varepsilon}}{d\ln \bar{c}_{\omega-1}^{-\zeta}} &= \beta\frac{d\ln\bar{\pi}_\omega}{d\ln \bar{c}_{\omega-1}^{-\zeta}}+\frac{d\ln\int\mathbb{E}\left[c^{1-\varepsilon}|q\right]dQ(q)}{d\ln \bar{c}_{\omega-1}^{-\zeta}}\\ &= \beta\lambda\left(\frac{d\ln A_\omega}{d\ln \bar{c}_{\omega-1}^{-\zeta}}+\alpha\frac{\varepsilon-1}{\zeta}\bar{O}_\omega\right)+\alpha\frac{\varepsilon-1}{\zeta}\left[\frac{1-\bar{\Sigma}_0}{1-\alpha\frac{\varepsilon-1}{\zeta}}\frac{d\ln A_\omega}{d\ln \bar{c}_{\omega-1}^{-\zeta}}+\left(\frac{1}{\gamma}+1\right)\bar{\Sigma}_1\right]\end{aligned}$$

Finally, since $\frac{d\ln A_\omega}{d\ln \bar{c}_\omega^{-\zeta}} = (\varepsilon-\eta)\frac{d\ln p_\omega}{d\ln \bar{c}_\omega^{-\zeta}} = -\frac{\varepsilon-\eta}{\varepsilon-1}(1-\varepsilon)\frac{d\ln p_\omega}{d\ln \bar{c}_\omega^{-\zeta}}$, we have

$$\frac{d\ln A_\omega}{d\ln \bar{c}_\omega^{-\zeta}} = -\frac{\varepsilon-\eta}{\varepsilon-1}\left\{\beta\lambda\left(\frac{d\ln A_\omega}{d\ln \bar{c}_{\omega-1}^{-\zeta}}+\alpha\frac{\varepsilon-1}{\zeta}\bar{O}_\omega\right)+\alpha\frac{\varepsilon-1}{\zeta}\left[\frac{1-\bar{\Sigma}_0}{1-\alpha\frac{\varepsilon-1}{\zeta}}\frac{d\ln A_\omega}{d\ln \bar{c}_{\omega-1}^{-\zeta}}+\left(\frac{1}{\gamma}+1\right)\bar{\Sigma}_1\right]\right\}$$

Solving for $\frac{d\ln A_\omega}{d\ln \bar{c}_\omega^{-\zeta}}$ gives

$$\frac{d\ln A_\omega}{d\ln \bar{c}_\omega^{-\zeta}} = -\frac{\alpha\frac{\varepsilon-1}{\zeta}\left(\beta\lambda\bar{O}_\omega+\left(\frac{1}{\gamma}+1\right)\bar{\Sigma}_1\right)}{\frac{\varepsilon-1}{\varepsilon-\eta}+\beta\lambda+\frac{\alpha\frac{\varepsilon-1}{\zeta}}{1-\alpha\frac{\varepsilon-1}{\zeta}}\left(1-\bar{\Sigma}_0\right)} < 0$$

Claim 6 *If* $\frac{\eta-1}{\varepsilon-\eta} \geq \frac{1}{\gamma}$ *then* $\frac{d\ln A_\omega}{d\ln v_{\omega-1}} \geq -\bar{O}_\omega\alpha\frac{\varepsilon-1}{\zeta}$ *and* $\frac{d\ln \bar{\pi}_\omega}{d\ln v_{\omega-1}} > 0$ *and* $\frac{d\ln J_\omega}{d\ln v_{\omega-1}} > 0$

Proof. First, we have that $\frac{d\ln J_\omega}{d\ln v_{\omega-1}} = \beta\lambda\left(\frac{d\ln A_\omega}{d\ln \bar{c}_{\omega-1}^{-\zeta}}+\alpha\frac{\varepsilon-1}{\zeta}\bar{O}_\omega\right)$. Under what conditions is the term in $\frac{d\ln A_\omega}{d\ln \bar{c}_{\omega-1}^{-\zeta}}+\alpha\frac{\varepsilon-1}{\zeta}\bar{O}_\omega$ positive? We have

$$\begin{aligned}
\frac{\frac{d\ln A_\omega}{d\ln \bar{c}_{\omega-1}^{-\zeta}}}{\alpha\frac{\varepsilon-1}{\zeta}\bar{O}_\omega}+1 &= -\frac{\beta\lambda\alpha\frac{\varepsilon-1}{\zeta}\bar{O}_\omega+\alpha\frac{\varepsilon-1}{\zeta}\frac{\gamma+1}{\gamma}\bar{\Sigma}_1}{\frac{\varepsilon-1}{\varepsilon-\eta}+\beta\lambda+\frac{\alpha\frac{\varepsilon-1}{\zeta}}{1-\alpha\frac{\varepsilon-1}{\zeta}}\left(1-\bar{\Sigma}_0\right)}\frac{1}{\alpha\frac{\varepsilon-1}{\zeta}\bar{O}_\omega}+1 \\
&= -\frac{\beta\lambda+\frac{\gamma+1}{\gamma}\frac{\bar{\Sigma}_1}{\bar{O}_\omega}}{\frac{\varepsilon-1}{\varepsilon-\eta}+\beta\lambda+\frac{\alpha\frac{\varepsilon-1}{\zeta}}{1-\alpha\frac{\varepsilon-1}{\zeta}}\left(1-\bar{\Sigma}_0\right)}+1 \\
&= \frac{\frac{\varepsilon-1}{\varepsilon-\eta}+\frac{\alpha\frac{\varepsilon-1}{\zeta}}{1-\alpha\frac{\varepsilon-1}{\zeta}}\left(1-\bar{\Sigma}_0\right)-\frac{\gamma+1}{\gamma}\frac{\bar{\Sigma}_1}{\bar{O}_\omega}}{\frac{\varepsilon-1}{\varepsilon-\eta}+\beta\lambda+\frac{\alpha\frac{\varepsilon-1}{\zeta}}{1-\alpha\frac{\varepsilon-1}{\zeta}}\left(1-\bar{\Sigma}_0\right)}
\end{aligned}$$

We are looking for conditions under which the numerator is positive. Since $O_\omega(q)$ is increasing in q, and $\Sigma_0(q) = \frac{1}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left[O_\omega(q)\frac{1}{\gamma}+(1-O_\omega(q))\frac{\alpha}{\gamma+1-\alpha}\right]}$ is decreasing in q, we have that $\bar{\Sigma}_0\bar{O}_\omega > \bar{\Sigma}_1$, so that the numerator is bounded below by

$$\frac{\varepsilon-1}{\varepsilon-\eta}+\frac{\alpha\frac{\varepsilon-1}{\zeta}}{1-\alpha\frac{\varepsilon-1}{\zeta}}\left(1-\bar{\Sigma}_0\right)-\frac{\gamma+1}{\gamma}\bar{\Sigma}_0$$

The RHS is decreasing in $\bar{\Sigma}_0$, so since since $\bar{\Sigma}_0 < \frac{1}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{\alpha}{\gamma+1-\alpha}}$, the following condition guarantees the numerator is positive

$$\begin{aligned}
0 &< \frac{\varepsilon-1}{\varepsilon-\eta}+\frac{\alpha\frac{\varepsilon-1}{\zeta}}{1-\alpha\frac{\varepsilon-1}{\zeta}}\left(1-\frac{1}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{\alpha}{\gamma+1-\alpha}}\right)-\frac{\gamma+1}{\gamma}\frac{1}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{\alpha}{\gamma+1-\alpha}}\\
&= \frac{\varepsilon-1}{\varepsilon-\eta}+\frac{\alpha\frac{\varepsilon-1}{\zeta}\frac{\alpha}{\gamma+1-\alpha}-\frac{\gamma+1}{\gamma}}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{\alpha}{\gamma+1-\alpha}}\\
&= \frac{\eta-1}{\varepsilon-\eta}+1+\frac{\alpha\frac{\varepsilon-1}{\zeta}\frac{\alpha}{\gamma+1-\alpha}-\frac{\gamma+1}{\gamma}}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{\alpha}{\gamma+1-\alpha}}\\
&= \frac{\eta-1}{\varepsilon-\eta}+\frac{\frac{\alpha}{\gamma+1-\alpha}-\frac{1}{\gamma}}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{\alpha}{\gamma+1-\alpha}}\\
&\leq \frac{\eta-1}{\varepsilon-\eta}-\frac{\frac{1}{\gamma}-\frac{\alpha}{\gamma+1-\alpha}}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{\alpha}{\gamma+1-\alpha}}
\end{aligned}$$

or

$$\frac{\eta-1}{\varepsilon-\eta}\geq\frac{\frac{1}{\gamma}-\frac{\alpha}{\gamma+1-\alpha}}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{\alpha}{\gamma+1-\alpha}}$$

A simpler sufficient condition that guarantees this condition holds is

$$\frac{\eta-1}{\varepsilon-\eta}\geq\frac{1}{\gamma}$$

■

Claim 7 $\frac{d\ln\frac{O_\omega(q)}{1-O_\omega(q)}}{d\ln\bar{c}_{\omega-1}^{-\zeta}}>0$.

Proof. Differentiating each side of $\frac{O_\omega(q)}{1-O_\omega(q)}=\frac{h_1m_1\bar{c}_{\omega-1}^{-\zeta}}{h_2^\alpha m_2^\alpha\bar{c}_{\omega-2}^{-\alpha\zeta}}$ gives

$$\begin{aligned}
\frac{d\ln\frac{O_\omega(q)}{1-O_\omega(q)}}{d\ln\bar{c}_{\omega-1}^{-\zeta}} &= \frac{d\ln h_1(q)}{d\ln\bar{c}_{\omega-1}^{-\zeta}}+1-\alpha\frac{d\ln h_2(q)}{d\ln\bar{c}_{\omega-1}^{-\zeta}}\\
&= \frac{1}{\gamma}\left(\frac{d\ln A_\omega}{d\ln\bar{c}_{\omega-1}^{-\zeta}}-\frac{1-\alpha\frac{\varepsilon-1}{\zeta}}{\alpha\frac{\varepsilon-1}{\zeta}}\frac{d\ln Y}{d\ln\bar{c}_{\omega-1}^{-\zeta}}+1\right)+1-\frac{\alpha}{\gamma+1-\alpha}\left(\frac{d\ln A_\omega}{d\ln\bar{c}_{\omega-1}^{-\zeta}}-\frac{1-\alpha\frac{\varepsilon-1}{\zeta}}{\alpha\frac{\varepsilon-1}{\zeta}}\frac{d\ln Y}{d\ln\bar{c}_{\omega-1}^{-\zeta}}\right)\\
&= \left(\frac{1}{\gamma}-\frac{\alpha}{\gamma+1-\alpha}\right)\left(\frac{d\ln A_\omega}{d\ln\bar{c}_{\omega-1}^{-\zeta}}-\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left[\frac{1-\Sigma_0}{1-\alpha\frac{\varepsilon-1}{\zeta}}\frac{d\ln A_\omega}{d\ln\bar{c}_{\omega-1}^{-\zeta}}+\left(\frac{1}{\gamma}+1\right)\Sigma_1\right]\right)+\left(\frac{1}{\gamma}+1\right.\\
&= \left(\frac{1}{\gamma}-\frac{\alpha}{\gamma+1-\alpha}\right)\left(\Sigma_0\frac{d\ln A_\omega}{d\ln\bar{c}_{\omega-1}^{-\zeta}}-\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left(\frac{1}{\gamma}+1\right)\Sigma_1\right)+\left(\frac{1}{\gamma}+1\right)
\end{aligned}$$

Since $\frac{d\ln A_\omega}{d\ln\bar{c}_{\omega-1}^{-\zeta}}<0$ and bounded below by $-\alpha\frac{\varepsilon-1}{\zeta}$, $\Sigma_0(q)$ is decreasing in q and bounded above by $\frac{1}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{\alpha}{\gamma+1-\alpha}}$, and $\Sigma_1(q)$ is increasing in q and bounded above by $\frac{1}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{1}{\gamma}}$. Together, these give a lower bound of

$$
\begin{aligned}
\Sigma_0\frac{d\ln A_\omega}{d\ln\bar{c}_{\omega-1}^{-\zeta}}-\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left(\frac{1}{\gamma}+1\right)\Sigma_1 \quad &\geq \quad \Sigma_0\left(-\alpha\frac{\varepsilon-1}{\zeta}\right)-\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left(\frac{1}{\gamma}+1\right)\Sigma_1 \\
&= \quad -\frac{\alpha\frac{\varepsilon-1}{\zeta}+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left(\frac{1}{\gamma}+1\right)O_\omega(q)}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left[O_\omega(q)\frac{1}{\gamma}+\left(1-O_\omega(q)\right)\frac{\alpha}{\gamma+1-\alpha}\right]} \\
&= \quad -\frac{\alpha\frac{\varepsilon-1}{\zeta}+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left(\frac{1}{\gamma}+1\right)O_\omega(q)}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{\alpha}{\gamma+1-\alpha}+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left(\frac{1}{\gamma}-\frac{\alpha}{\gamma+1-\alpha}\right)O_\omega(q)} \\
&= \quad -\frac{\alpha\frac{\varepsilon-1}{\zeta}+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left(\frac{1}{\gamma}+1\right)O_\omega(q)}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{\alpha}{\gamma+1-\alpha}+\frac{1-\alpha}{\gamma+1-\alpha}\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{\gamma+1}{\gamma}O_\omega(q)} \\
&= \quad -\frac{\alpha\frac{\varepsilon-1}{\zeta}+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left(\frac{1}{\gamma}+1\right)O_\omega(q)}{\left[1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{\alpha}{\gamma+1-\alpha}-\alpha\frac{\varepsilon-1}{\zeta}\frac{1-\alpha}{\gamma+1-\alpha}\right]+\frac{1-\alpha}{\gamma+1-\alpha}\left[\alpha\frac{\varepsilon-1}{\zeta}+\left(1-\alpha\right.\right.}
\end{aligned}
$$

Since the first term of the denominator is positive, this is minimized when the fraction is maximized, or $O_\omega(q)\to 1$. This gives the bound

$$
\begin{aligned}
\Sigma_0\frac{d\ln A_\omega}{d\ln\bar{c}_{\omega-1}^{-\zeta}}-\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left(\frac{1}{\gamma}+1\right)\Sigma_1 \quad &\geq \quad -\frac{\alpha\frac{\varepsilon-1}{\zeta}+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\left(\frac{1}{\gamma}+1\right)}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{1}{\gamma}} \\
&= \quad -\frac{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{1}{\gamma}}{1+\left(1-\alpha\frac{\varepsilon-1}{\zeta}\right)\frac{1}{\gamma}} \\
&= \quad -1
\end{aligned}
$$

Plugging this back in yields

$$
\frac{d\ln\frac{O_\omega(q)}{1-O_\omega(q)}}{d\ln\bar{c}_{\omega-1}^{-\zeta}}\geq\left(\frac{1}{\gamma}-\frac{\alpha}{\gamma+1-\alpha}\right)(-1)+\left(\frac{1}{\gamma}+1\right)=\frac{\gamma+1}{\gamma+1-\alpha}>0
$$

■

Claim 8 *If γ is large enough,* $\frac{d\ln\bar{c}_\omega}{d\ln\delta_{\omega-1}}<0$.

Proof. Following the same logic as for the expression for $\frac{d\ln\int\mathbb{E}\left[c^{1-\varepsilon}|q\right]dQ(q)}{d\ln\bar{c}_{\omega-1}^{-\zeta}}$, we have $\mathbb{E}\left[c^{-\zeta}|q\right]=\Gamma(1-\alpha)\,q^{\zeta}\left[h_1m_1\bar{c}_{\omega-1}^{-\zeta}+h_2^{\alpha}m_2^{\alpha}\bar{c}_{\omega-2}^{-\alpha\zeta}\right]^{\alpha}$, so that differentiating yields

$$
\frac{d\ln\mathbb{E}\left[c^{-\zeta}|q\right]}{d\ln\bar{c}_{\omega-1}^{-\zeta}}=\alpha\left[\frac{1-\Sigma_0}{1-\alpha\frac{\varepsilon-1}{\zeta}}\frac{d\ln A_\omega}{d\ln\bar{c}_{\omega-1}^{-\zeta}}+\left(\frac{1}{\gamma}+1\right)\Sigma_1\right]
$$

Integrating with across firms gives

$$\frac{d\ln \bar{c}_\omega^{-\zeta}}{d\ln \bar{c}_{\omega-1}^{-\zeta}} = \frac{d\ln\int \mathbb{E}\left[c^{-\zeta}|q\right]dQ\left(q\right)}{d\ln \bar{c}_{\omega-1}^{-\zeta}} = \alpha\left[\frac{1-\bar{\Sigma}_0^\zeta}{1-\alpha\frac{\varepsilon-1}{\zeta}}\frac{d\ln A_\omega}{d\ln \bar{c}_{\omega-1}^{-\zeta}} + \left(\frac{1}{\gamma}+1\right)\bar{\Sigma}_1^\zeta\right]$$

where we define the weighted average $\bar{x}^\zeta \equiv \frac{\mathbb{E}\left[c^{-\zeta}x\right]}{\mathbb{E}\left[c^{-\zeta}\right]}$. Since $\lim_{\gamma\to\infty}\bar{\Sigma}_1^\zeta = \bar{O}_\omega^\zeta$ and $\lim_{\gamma\to\infty}\bar{\Sigma}_0^\zeta = 1$, taking the limit as $\gamma\to\infty$ gives

$$\lim_{\gamma\to\infty}\frac{1}{\alpha}\frac{d\ln\int C_\omega\left(q\right)^{-\zeta}dQ\left(q\right)}{d\ln v_\omega} \geq \lim_{\gamma\to\infty}\bar{O}_\omega^\zeta$$

Finally, we show that $\lim_{\gamma\to\infty}\bar{O}_\omega^\zeta > 0$. The FOCs for h_1 and h_2 can be rearranged as

$$\begin{aligned}
h_1 m_1 \bar{c}_{\omega-1}^{-\zeta} &= \left\{\frac{1}{k}A_\omega\delta_\omega q^{\varepsilon-1}O_\omega^{1-\alpha\frac{\varepsilon-1}{\zeta}}\alpha\frac{\varepsilon-1}{\zeta}\right\}^{\frac{1}{1+\gamma-\alpha\frac{\varepsilon-1}{\zeta}}}\left(m_1\bar{c}_{\omega-1}^{-\zeta}\right)^{\frac{1+\gamma}{1+\gamma-\alpha\frac{\varepsilon-1}{\zeta}}} \\
h_2^\alpha m_2^\alpha \bar{c}_{\omega-2}^{-\alpha\zeta} &= \left\{\frac{1}{k}A_\omega\delta_\omega q^{\varepsilon-1}\left(1-O_\omega\right)^{1-\alpha\frac{\varepsilon-1}{\zeta}}\alpha^2\frac{\varepsilon-1}{\zeta}\right\}^{\frac{1}{\frac{1+\gamma}{\alpha}-\alpha\frac{\varepsilon-1}{\zeta}}}\left(m_2^\alpha\bar{c}_{\omega-2}^{-\alpha\zeta}\right)^{\frac{\frac{1+\gamma}{\alpha}}{\frac{1+\gamma}{\alpha}-\alpha\frac{\varepsilon-1}{\zeta}}}
\end{aligned}$$

Taking the limit of the ratio gives

$$\lim_{\gamma\to\infty}\frac{O_\omega\left(q\right)}{1-O_\omega\left(q\right)} = \lim_{\gamma\to\infty}\frac{h_1 m_1\bar{c}_{\omega-1}^{-\zeta}}{h_2^\alpha m_2^\alpha\bar{c}_{\omega-2}^{-\alpha\zeta}} = \lim_{\gamma\to\infty}\frac{\left\{\frac{1}{k}A_\omega\delta_\omega q^{\varepsilon-1}O^{1-\alpha\frac{\varepsilon-1}{\zeta}}\alpha\frac{\varepsilon-1}{\zeta}\right\}^{\frac{1}{1+\gamma-\alpha\frac{\varepsilon-1}{\zeta}}}\left(m_1\bar{c}_{\omega-1}^{-\zeta}\right)^{\frac{1+\gamma}{1+\gamma-\alpha\frac{\varepsilon-1}{\zeta}}}}{\left\{\frac{1}{k}A_\omega\delta_\omega q^{\varepsilon-1}\left(1-O\right)^{1-\alpha\frac{\varepsilon-1}{\zeta}}\alpha^2\frac{\varepsilon-1}{\zeta}\right\}^{\frac{1}{\frac{1+\gamma}{\alpha}-\alpha\frac{\varepsilon-1}{\zeta}}}\left(m_2^\alpha\bar{c}_{\omega-2}^{-\alpha\zeta}\right)^{\frac{\frac{1+\gamma}{\alpha}}{\frac{1+\gamma}{\alpha}-\alpha\frac{\varepsilon-1}{\zeta}}}} = \frac{m}{m}$$

Since this is the same for all q, we have $\lim_{\gamma\to\infty}\bar{O}_\omega^\zeta = \frac{m_1\bar{c}_{\omega-1}^{-\zeta}}{m_1\bar{c}_{\omega-1}^{-\zeta}+m_2^\alpha\bar{c}_{\omega-2}^{-\alpha\zeta}} > 0$. ■

Claim 9 *If $\eta > 1$ and γ is large enough, an increase in $\delta_{\omega-1}$ leads to an increase in industry ω sales*

Proof. Since $\eta > 1$ and p_ω declines, so sales to household increases. Further, since $\bar{c}_\omega$ declines, sales for intermediate use rise as well. ■

D.2 Proofs for Quantitative Model of Section 5

In this section, we prove the propositions in Section 5. Several of te results parallel proofs for the simple model of Section 3. Nevertheless, we include the results here both for completeness and because the notation is not always the same.

We first derive an expression for the effective cost of outsourcing each input. Let $F_\omega(c)$ be the distribution of unit cost among those in industry ω.

Lemma 4 *If firm j exerts search effort $h_{j\hat{\omega}}$ to search for suppliers of input $\hat{\omega}\in\hat{\Omega}_{\tilde{\omega}}$, then* $\Pr\left(c_{j\hat{\omega}}^o > c\right) = e^{-h_{j\hat{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}c^\zeta}$ *with* $\bar{c}_{\hat{\omega}} = \left(\int c^{-\zeta}dF_{\hat{\omega}}\left(c\right)\right)^{-\frac{1}{\zeta}}$.

Proof. If firm j exerts search effort $h_{j\hat{\omega}}$, the arrival rate of potential suppliers of $\hat{\omega}$ with match-specific productivity better than z is $h_{\hat{\omega}} m_{\tilde{\omega}\hat{\omega}} z^{-\zeta}$. Thus the probability that the firm's best outside supplier delivers cost greater than c is

$$\begin{aligned}
\Pr\left(c^o_{\hat{\omega}} > c\right) &= e^{-h_{\hat{\omega}} m_{\tilde{\omega}\hat{\omega}} \int\int 1\left\{\frac{c_s}{z} \le c\right\} dF_{\hat{\omega}}(c_s)\zeta z^{-\zeta-1}dz} \\
&= e^{-h_{\hat{\omega}} m_{\tilde{\omega}\hat{\omega}} c^{\zeta} \int\int 1\left\{\frac{c_s}{u} \le 1\right\} dF_{\hat{\omega}}(c_s)\zeta u^{-\zeta-1}du} \\
&= e^{-h_{\hat{\omega}} m_{\tilde{\omega}\hat{\omega}} \bar{c}_{\hat{\omega}}^{-\zeta} c^{\zeta}}
\end{aligned}$$

where $\bar{c}_{\hat{\omega}}$ is defined to satisfy

$$\begin{aligned}
\bar{c}_{\hat{\omega}}^{-\zeta} &\equiv \int\int 1\left\{\frac{c_s}{u} \le 1\right\} dF_{\hat{\omega}}\left(c_s\right)\zeta u^{-\zeta-1}du \\
&= \int\int 1\left\{t \le 1\right\} c_s^{-\zeta} dF_{\hat{\omega}}\left(c_s\right)\zeta t^{\zeta-1}dt \\
&= \int c_s^{-\zeta} dF_{\hat{\omega}}\left(c_s\right)
\end{aligned}$$

■

Lemma 5 *Suppose that there are independent exponential random variables X_1,...,X_n that with $\Pr\left(X_i > x\right) = e^{-\kappa_i x}$, and that b is a random variable with characteristic function $E\left[e^{bit}\right] = \frac{\Gamma(1-it)}{\Gamma(1-\alpha_1 it)...\Gamma(1-\alpha_n it)}$. Suppose that $\alpha_1, ..., \alpha_n$ are non-negative numbers such that $\alpha_1 + ... + \alpha_n \le 1$. Then $Y = e^{-b} X_1^{\alpha_1}...X_n^{\alpha_n}$ is an exponential random variable with countercumalutive distribution*

$$\Pr\left(Y > y\right) = e^{-\kappa_1^{\alpha_1}...\kappa_n^{\alpha_n} y}$$

Proof. Note that $E\left[X_j^{\alpha_j it}\right] = \int_0^\infty x^{\alpha_j it}\kappa_j e^{-\kappa_j x}dx = \kappa_j^{-\alpha_j it}\int_0^\infty u^{\alpha_j it}e^{-u}du = \kappa_j^{-\alpha_j it}\Gamma\left(1+\alpha_j it\right)$. With this, we have that

$$\begin{aligned}
E\left[Y^{it}\right] &= E\left[\left(e^{-b}X_1^{\alpha_1}...X_n^{\alpha_n}\right)^{it}\right] = E\left[e^{-bit}\right]E\left[X_1^{\alpha_1 it}\right]...E\left[X_n^{\alpha_n it}\right] \\
&= \frac{\Gamma\left(1+it\right)}{\Gamma\left(1+\alpha_1 it\right)...\Gamma\left(1+\alpha_n it\right)}\kappa_1^{-\alpha_1 it}\Gamma\left(1+\alpha_1 it\right)...\kappa_1^{-\alpha_n it}\Gamma\left(1+\alpha_n it\right) \\
&= \frac{1}{\left(\kappa_1^{\alpha_1}...\kappa_n^{\alpha_n}\right)^{it}}\Gamma\left(1+it\right)
\end{aligned}$$

Suppose that Z is an exponential random variable with $\Pr\left(Z > z\right) = e^{-\kappa_1^{\alpha_1}...\kappa_n^{\alpha_n}z}$. Then the characteristic function of $\log Z = E\left[Z^{it}\right] = \frac{1}{\left(\kappa_1^{\alpha_1}...\kappa_n^{\alpha_n}\right)^{it}}\Gamma\left(1+it\right)$. Since there is a one-to one mapping between distribution functions and characteristic functions, Y and Z have the same distribution. ■

Lemma 6 *Suppose that there are independent Weibull variables $M_1, ..., M_n$ so that $\Pr\left(M_i > m\right) = e^{-\kappa_i m^{\zeta}}$, and that b is a random variable with characteristic function $E\left[b^{it}\right] = \frac{\Gamma(1-it)}{\Gamma(1-\alpha_1 it)...\Gamma(1-\alpha_n it)}$.*

Suppose that $\alpha_1, ..., \alpha_n$ are non-negative numbers such that $\alpha_1 + ... + \alpha_n \leq 1$. Then $Y = e^{-b/\zeta} M_1^{\alpha_1} ... M_n^{\alpha_n}$ is an Weibull random variable with .

$$\Pr(Y > y) = e^{-\kappa_1^{\alpha_1} ... \kappa_n^{\alpha_n} y^\zeta}$$

Proof. If M is Weibull, $X = M^\zeta$ is exponential, as $\Pr(X > x) = \Pr\left(M^\zeta > x\right) = \Pr\left(M > x^{1/\zeta}\right) = e^{-\kappa_1\left(x^{1/\zeta}\right)^\zeta} = e^{-\kappa_1 x}$. Since $Y^\zeta = e^{-b}\left(M_1^\zeta\right)^{\alpha_1} ... \left(M_n^\zeta\right)^{\alpha_n} = e^{-b} X_1^{\alpha_1} ... X_n^{\alpha_n}$, it follows that from the previous lemma that Y^ζ follows an exponential distribution with $\Pr\left(Y^\zeta > t\right) = e^{-\kappa_1^{\alpha_1} ... \kappa_n^{\alpha_n} t}$, which implies that

$$\Pr(Y > y) = \Pr\left(Y^\zeta > y^\zeta\right) = e^{-\kappa_1^{\alpha_1} ... \kappa_n^{\alpha_n} y^\zeta}$$

■

Lemma 7 *Given search effort $\{h_{j\tilde{\omega}}\}_{\tilde{\omega} \in \hat{\Omega}_{\tilde{\omega}}^\infty}$, for any input $\tilde{\omega}$, $\Pr\left(c_{j\tilde{\omega}}^i > c\right) = e^{-T_{j\tilde{\omega}}^{-\zeta} c^\zeta}$ where $\{T_{j\hat{\omega}}\}_{\hat{\omega} \in \Omega_{\tilde{\omega}}^\infty}$ satisfy the relationships*

$$T_{j\tilde{\omega}} = w^{-\alpha_l^{\tilde{\omega}}\zeta} \prod_{\hat{\omega} \in \hat{\Omega}_{\tilde{\omega}}} \left(h_{j\hat{\omega}} m_{\tilde{\omega}\hat{\omega}} \bar{c}_{\hat{\omega}}^{-\zeta} + T_{j\hat{\omega}}^{-\zeta}\right)^{-\frac{\alpha_{\hat{\omega}}^{\tilde{\omega}}}{\zeta}}$$

where we use the convention that $T_{j\hat{\omega}} = \infty$ if in-house production of $\hat{\omega}$ is infeasible.

Proof. We proceed by induction. Consider first a terminal module. The cost of producing in house is

$$c_{\tilde{\omega}}^i = \frac{1}{B_{\tilde{\omega}}} w^{\alpha_l^{\tilde{\omega}}} \prod_{\hat{\omega} \in \hat{\Omega}_{\tilde{\omega}}} c_{\hat{\omega}}^{\alpha_{\hat{\omega}}^{\tilde{\omega}}}$$

or

$$\frac{c_{\tilde{\omega}}^i}{w} = \frac{1}{B_{\tilde{\omega}}} \prod_{\hat{\omega} \in \hat{\Omega}_{\tilde{\omega}}} \left(\frac{c_{\hat{\omega}}}{w}\right)^{\alpha_{\hat{\omega}}^{\tilde{\omega}}}$$

Since the distribution of each $\frac{c_{\tilde{\omega}}}{w}$ is Weibull with $\Pr\left(\frac{c_{\hat{\omega}}}{w} > x\right) = e^{-w^\zeta h_{\hat{\omega}} m_{\tilde{\omega}\hat{\omega}} c_{\hat{\omega}}^{-\zeta} x^\zeta}$, the previous lemma implies that $\frac{c_{\tilde{\omega}}^i}{w}$ is Weibull with

$$\Pr\left(\frac{c_{\tilde{\omega}}^i}{w} > x\right) = e^{-\prod_{\hat{\omega} \in \hat{\Omega}_{\tilde{\omega}}}\left(w^\zeta m_{\tilde{\omega}\hat{\omega}} c_{\hat{\omega}}^{-\zeta}\right)^{\alpha_{\hat{\omega}}^{\tilde{\omega}}} x^\zeta}$$

or

$$\Pr\left(c_{\tilde{\omega}}^i > c\right) = \Pr\left(\frac{c_{\tilde{\omega}}^i}{w} > \frac{c}{w}\right) = e^{-\prod_{\hat{\omega} \in \hat{\Omega}_{\tilde{\omega}}}\left(w^\zeta h_{\hat{\omega}} m_{\tilde{\omega}\hat{\omega}} \bar{c}_{\hat{\omega}}^{-\zeta}\right)^{\alpha_{\hat{\omega}}^{\tilde{\omega}}} \left(\frac{c}{w}\right)^\zeta} = e^{-w^{-\alpha_l^{\tilde{\omega}}\zeta} \prod_{\hat{\omega} \in \hat{\Omega}_{\tilde{\omega}}}\left(h_{\hat{\omega}} m_{\tilde{\omega}\hat{\omega}} \bar{c}_{\hat{\omega}}^{-\zeta}\right)^{\alpha_{\hat{\omega}}^{\tilde{\omega}}} c^\zeta} = e^{-T_{\tilde{\omega}}^{-\zeta} c^\zeta}$$

Suppose that for all modules with depth of $D-1$, the $\Pr\left(c_{\hat{\omega}}^i > c\right) = e^{-T_{\hat{\omega}}^{-\zeta} c^\zeta}$. Then since the cost of the outsourcing the input is also Weibull, $\Pr\left(c_{\hat{\omega}}^o > c\right) = e^{-h_{\hat{\omega}} m_{\tilde{\omega}\hat{\omega}} \bar{c}_{\hat{\omega}}^{-\zeta} c^\zeta}$, we have that the effective cost of input $\hat{\omega}$ follows a Weibull distribution

$$\Pr\left(c_{\hat{\omega}} > c\right) = \Pr\left(c^{i}_{\hat{\omega}} > c\right)\Pr\left(c^{o}_{\hat{\omega}} > c\right) = e^{-T_{\hat{\omega}}^{-\zeta}c^{\zeta}}e^{-h_{\hat{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}c^{\zeta}} = e^{-\left[T_{\hat{\omega}}^{-\zeta}+h_{\hat{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right]c^{\zeta}}$$

Again, since the distribution of each $\frac{c_{\hat{\omega}}}{w}$ is Weibull with $\Pr\left(\frac{c_{\hat{\omega}}}{w} > x\right) = e^{-w^{\zeta}\left[T_{\hat{\omega}}^{-\zeta}+h_{\hat{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right]x^{\zeta}}$, the previous lemma implies that $\frac{c^{i}_{\tilde{\omega}}}{w}$ is Weibull with

$$\Pr\left(\frac{c^{i}_{\tilde{\omega}}}{w} > x\right) = e^{-\prod_{\hat{\omega}\in\hat{\Omega}_{\tilde{\omega}}}\left(w^{\zeta}\left[T_{\hat{\omega}}^{-\zeta}+h_{\hat{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right]\right)^{\alpha^{\tilde{\omega}}_{\hat{\omega}}}x^{\zeta}}$$

or

$$\Pr\left(c^{i}_{\tilde{\omega}} > c\right) = \Pr\left(\frac{c^{i}_{\tilde{\omega}}}{w} > \frac{c}{w}\right) = e^{-\prod_{\hat{\omega}\in\hat{\Omega}_{\tilde{\omega}}}\left(w^{\zeta}\left[T_{\hat{\omega}}^{-\zeta}+h_{\hat{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right]\right)^{\alpha^{\tilde{\omega}}_{\hat{\omega}}}\left(\frac{c}{w}\right)^{\zeta}} = e^{-w^{-\alpha^{\tilde{\omega}}_{l}\zeta}\prod^{\alpha^{\tilde{\omega}}_{\hat{\omega}}}_{\hat{\omega}\in\hat{\Omega}_{\tilde{\omega}}}\left[T_{\hat{\omega}}^{-\zeta}+h_{\hat{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right]^{\alpha^{\tilde{\omega}}_{\hat{\omega}}}c^{\zeta}} = e^{-T_{\tilde{\omega}}^{-\zeta}c^{\zeta}}$$

■

Claim 10 *Given search effort $\{h_{\omega}\}$, the probability of outsourcing input i conditional on using it in production is*

$$\frac{h_{\tilde{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}}{h_{\tilde{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}+T_{\tilde{\omega}}^{-\zeta}}$$

This is independent of the firm's cost or of the probability of outsourcing other inputs.

Proof. This follows from usual math with Weibulls

$$\begin{aligned}
\Pr\left(c^{o}_{\tilde{\omega}} \leq c^{i}_{\tilde{\omega}} | c_{\tilde{\omega}} = c\right) &= \frac{\Pr\left(c^{o}_{\tilde{\omega}} \leq c^{i}_{\tilde{\omega}}, c_{\tilde{\omega}} = c\right)}{\Pr\left(c^{o}_{\tilde{\omega}} = c\right)} \\
&= \frac{\Pr\left(c^{o}_{\tilde{\omega}} \leq c^{i}_{\tilde{\omega}}, c^{o}_{\tilde{\omega}} = c\right)}{\Pr\left(c^{o}_{\tilde{\omega}} = c\right)} \\
&= \frac{\Pr\left(c^{i}_{\tilde{\omega}} \geq c, c^{o}_{\tilde{\omega}} = c\right)}{\Pr\left(c^{i}_{\tilde{\omega}} \geq c, c^{o}_{\tilde{\omega}} = c\right)+\Pr\left(c^{o}_{\tilde{\omega}} \geq c, c^{i}_{\tilde{\omega}} = c\right)} \\
&= \frac{h_{\tilde{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\zeta c^{\zeta-1}e^{-h_{\tilde{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}c^{\zeta}}e^{-T_{\tilde{\omega}}^{-\zeta}c^{\zeta}}}{\Pr\left(c^{o}_{\tilde{\omega}} = c\right)} \\
&= \frac{h_{\tilde{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}}{h_{\tilde{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}+T_{\tilde{\omega}}^{-\zeta}}\frac{\left[h_{\tilde{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}+T_{\tilde{\omega}}^{-\zeta}\right]\zeta c^{\zeta-1}e^{-h_{\tilde{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}c^{\zeta}}e^{-T_{\tilde{\omega}}^{-\zeta}c^{\zeta}}}{\Pr\left(c^{o}_{\tilde{\omega}} = c\right)} \\
&= \frac{h_{\tilde{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}}{h_{\tilde{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}+T_{\tilde{\omega}}^{-\zeta}}
\end{aligned}$$

Since this is true for all c, it follows that the choice of in-house or outsourcing is independent of decision downstream from the input $\tilde{\omega}$, and the same argument used for inputs upstream from $\tilde{\omega}$ implies that outsourcing decisions are independent. Finally, integrating givesPr $\left(c^{o}_{\tilde{\omega}} \leq c^{i}_{\tilde{\omega}}\right) = E\left[\Pr\left(c^{o}_{\tilde{\omega}} \leq c^{i}_{\tilde{\omega}} | c_{\tilde{\omega}} = c\right)\right] = \frac{h_{\tilde{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}}{h_{\tilde{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}+T_{\tilde{\omega}}^{-\zeta}}$. ■

We next turn to the firm's expected profit

Claim 11 *Given its productivity q and its search effort $\{h_{\tilde{\omega}}\}_{\tilde{\omega}\in\Omega^o_\omega}$, a firm's expected profit is*

$$A_\omega q^{\varepsilon-1}T_\omega^{1-\varepsilon}-\sum_{\hat{\omega}\in\hat{\Omega}^o_\omega} wk\frac{h_{\hat{\omega}}^{1+\gamma}}{1+\gamma}$$

where

$$T_{\tilde{\omega}}=\prod_{\hat{\omega}\in\hat{\Omega}_{\tilde{\omega}}}\left(h_{\hat{\omega}}m_{\tilde{\omega}\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}+T_{\hat{\omega}}\right)^{\alpha^{\omega}_{\hat{\omega}}},\qquad \forall\tilde{\omega}\in\Omega^i_\omega$$

with the convention that $T_{\hat{\omega}}=\infty$ for inputs that for which in-house production is infeasible, i.e., $\hat{\omega}\notin\Omega^i_\omega$.

Proof. For a firm in industry ω with cost c, profit gross of search costs is $\frac{1}{\varepsilon}\left(\frac{\varepsilon}{\varepsilon-1}\right)^{1-\varepsilon}up^\eta p_\omega^{\varepsilon-\eta}c^{1-\varepsilon}$. It is thus sufficient to show that $E\left[c^{1-\varepsilon}|q,\{h_{\tilde{\omega}}\}_{\tilde{\omega}\in\hat{\Omega}^o_\omega}\right]=\kappa q^{\varepsilon-1}T_\omega^{1-\varepsilon}$. Since $c=\frac{1}{q}w^{\alpha^\omega_l}\prod_{\hat{\omega}\in\hat{\Omega}_\omega}c_{\hat{\omega}}^{\alpha^\omega_{\hat{\omega}}}$ and the cost of each input is independent conditional on search effort, we have

$$\begin{aligned}
E\left[\left(\frac{c}{w}\right)^{1-\varepsilon}|q,\{h_{\tilde{\omega}}\}_{\tilde{\omega}\in\hat{\Omega}^o_\omega}\right] &= E\left[\left(\frac{1}{q}\prod_{\hat{\omega}\in\hat{\Omega}_\omega}\left(\frac{c_{\hat{\omega}}}{w}\right)^{\alpha^\omega_{\hat{\omega}}}\right)^{1-\varepsilon}|q,\{h_{\tilde{\omega}}\}_{\tilde{\omega}\in\hat{\Omega}^o_\omega}\right]\\
&= q^{\varepsilon-1}\prod_{\hat{\omega}\in\hat{\Omega}_\omega}E\left[\left(\frac{c_{\hat{\omega}}}{w}\right)^{\alpha^\omega_{\hat{\omega}}(1-\varepsilon)}|q,\{h_{\tilde{\omega}}\}_{\tilde{\omega}\in\hat{\Omega}^o_\omega}\right]
\end{aligned}$$

Note that if X follows a Weibull distribution with $\Pr(X>x)=e^{-\kappa x^\zeta}$, Then $E\left[\left(\frac{x}{w}\right)^{-r}\right]=w^r\int x^{-r}\kappa\zeta x^{\zeta-1}e^{-\kappa x^\zeta}dx=w^r\kappa^{\frac{r}{\zeta}}\int u^{-\frac{r}{\zeta}}e^{-u}du=w^r\Gamma\left(1-\frac{r}{\zeta}\right)\kappa^{\frac{r}{\zeta}}$. Thus we can express this as

$$\begin{aligned}
E\left[\left(\frac{c}{w}\right)^{1-\varepsilon}|q,\{h_{\tilde{\omega}}\}_{\tilde{\omega}\in\hat{\Omega}^o_\omega}\right] &= q^{\varepsilon-1}w^{\left(1-\alpha^\omega_l\right)(\varepsilon-1)}\prod_{\hat{\omega}\in\hat{\Omega}_\omega}\Gamma\left(1-\alpha^\omega_{\hat{\omega}}\frac{\varepsilon-1}{\zeta}\right)\left(h_{\hat{\omega}}m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}+T_{\hat{\omega}}^{-\zeta}\right)^{\alpha^\omega_{\hat{\omega}}\frac{\varepsilon-1}{\zeta}}\\
&= \left\{w^{\left(1-\alpha^\omega_l\right)(\varepsilon-1)}\prod_{\hat{\omega}\in\hat{\Omega}_\omega}\Gamma\left(1-\alpha^\omega_{\hat{\omega}}\frac{\varepsilon-1}{\zeta}\right)\right\}q^{\varepsilon-1}T_\omega^{1-\varepsilon}
\end{aligned}$$

Therefore $A_\omega=\frac{1}{\varepsilon}\left(\frac{\varepsilon}{\varepsilon-1}\right)^{1-\varepsilon}up^\eta p_\omega^{\varepsilon-\eta}\left\{w^{\left(1-\alpha^\omega_l\right)(\varepsilon-1)}\prod_{\hat{\omega}\in\hat{\Omega}_\omega}\Gamma\left(1-\alpha^\omega_{\hat{\omega}}\frac{\varepsilon-1}{\zeta}\right)\right\}$. ■

D.2.1 Optimal Search Effort

The firms problem can be expressed as a cost-minimization problem. let $K_\omega(T_\omega)$ be the minimum cost of delivering a cost distribution indexed by T_ω. Then the firm's problem can be expressed as

$$E\left[\Pi|q,\{h_{\hat{\omega}}\}_{\hat{\omega}\in\hat{\Omega}^o_\omega}\right]=\max_{T_\omega}A_\omega q^{\varepsilon-1}T_\omega^{\frac{1-\varepsilon}{\zeta}}-wK_\omega(T_\omega)$$

where each cost function K_ω can be expressed recursively

$$K_\omega\left(T_\omega\right)=\min_{\{h_{\hat\omega},T_{\hat\omega}\}_{\hat\omega\in\hat\Omega_\omega}}\sum_{\hat\omega\in\hat\Omega_\omega}\left[\frac{k}{1+\gamma}h_{\hat\omega}^{1+\gamma}+K_{\hat\omega}\left(T_{\hat\omega}\right)\right]$$

subject to

$$T_\omega\leq w^{1-\alpha_l^\omega}\prod_{\hat\omega\in\Omega_\omega}\left[h_{\hat\omega}m_{\omega\hat\omega}\bar c_{\hat\omega}^{-\zeta}+T_{\hat\omega}^{-\zeta}\right]^{\alpha_{\hat\omega}^\omega}$$

We now derive some properties of K_ω. In doing so, it will be useful to define $\{\kappa_\omega,\beta_\omega\}$ iteratively as follows.

$$\beta_\omega=\sum_{\hat\omega\in\hat\Omega_\omega^{leaf}}\alpha_{\hat\omega}^\omega+\sum_{\hat\omega\in\hat\Omega_\omega^{non-leaf}}\beta_{\hat\omega}\alpha_{\hat\omega}^\omega$$

$$\kappa_\omega=\prod_{\hat\omega\in\hat\Omega_\omega^{leaf}}\left(\left(m_{\omega\hat\omega}\left(\frac{\bar c_{\hat\omega}}{w}\right)^{-\zeta}\right)^{1+\gamma}\frac{\alpha_{\hat\omega}^\omega}{\beta_\omega}\right)^{\frac{\alpha_{\hat\omega}^\omega}{\beta_\omega}}\prod_{\hat\omega\in\hat\Omega_\omega^{non-leaf}}\left(\kappa_{\hat\omega}\frac{\beta_{\hat\omega}\alpha_{\hat\omega}^\omega}{\beta_\omega}\right)^{\frac{\beta_{\hat\omega}\alpha_{\hat\omega}^\omega}{\beta_\omega}}$$

Claim 12 *Each $K_\omega(\cdot)$ is such that $K_\omega(0)=0$, strictly decreasing, strictly convex, twice continuously differentiable, and the policy functions $\{h_{\hat\omega}(\cdot),T_{\hat\omega}(\cdot)\}_{\hat\omega\in\Omega_\omega}$ are differentiable and strictly monotonic, with $\frac{d\ln h_{\hat\omega}}{d\ln T_\omega}>\frac{d\ln T_{\hat\omega}^{-\zeta}}{d\ln T_\omega}>0$, and $\frac{d\ln O_{\hat\omega}}{d\ln T_\omega}$. The elasticity of $K_\omega'(\cdot)$ is bounded by $\frac{\zeta(1+\gamma)}{\beta_\omega}\leq\frac{K_\omega''(T_\omega)T_\omega}{-K_\omega'(T_\omega)}-1\leq\frac{\zeta(1+\gamma)}{1-\alpha_l^\omega}$. Finally*

$$\begin{aligned}\lim_{T_\omega\to0}K_\omega\left(T_\omega\right)T_\omega^{\frac{\zeta(1+\gamma)}{1-\alpha_l^\omega}}&=\frac{k}{1+\gamma}\frac{1}{\prod_{\hat\omega\in\hat\Omega_\omega}\left(\left(m_{\omega\hat\omega}\bar c_{\hat\omega}^{-\zeta}\right)^{1+\gamma}\frac{\alpha_{\hat\omega}^\omega}{1-\alpha_l^\rho}\right)^{\frac{\alpha_{\hat\omega}^\omega}{1-\alpha_l^\omega}}}\\\lim_{T_\omega\to\infty}K_\omega\left(T_\omega\right)T_\omega^{\frac{\zeta(1+\gamma)}{\beta_\omega}}&=\frac{k}{1+\gamma}\frac{1}{\kappa_\omega}\end{aligned}$$

Proof. By induction. If there is no in-house option (i.e., a module of depth 1) then we have

$$K_\omega\left(T_\omega\right)=\min_{\{h_{\hat\omega}\}}\sum_{\hat\omega\in\hat\Omega_\omega}\frac{k}{1+\gamma}h_{\hat\omega}^{1+\gamma}\quad\text{subject to}\quad T_\omega\geq w^{\alpha_l^\omega}\prod_{\hat\omega\in\hat\Omega_\omega}\left(h_{\hat\omega}m_{\omega\hat\omega}\bar c_{\hat\omega}^{-\zeta}\right)^{-\frac{\alpha_{\hat\omega}^\omega}{\zeta}}$$

Letting λ be the multiplier, we have, the FOCs are

$$kh_{\hat\omega}^\gamma=w^{\alpha_l^\omega}\lambda\frac{\alpha_{\hat\omega}^\omega}{\zeta}\frac{1}{h_{\hat\omega}}T_\omega$$

or

$$h_{\hat\omega}^{1+\gamma}=\frac{\alpha_{\hat\omega}^\omega}{\zeta}\frac{w^{\alpha_l^\omega}\lambda T_\omega}{k}$$

Multiplying each side by $\left(m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right)^{1+\gamma}$ and combining across $\hat{\omega}$ gives

$$\prod_{\hat{\omega}\in\Omega_{\omega}}\left(\left(h_{\hat{\omega}}m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right)^{1+\gamma}\right)^{\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}=\left(\frac{w^{\alpha_{l}^{\omega}}\lambda}{k}T_{\omega}\right)^{\frac{1-\alpha_{l}^{\omega}}{\zeta}}\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}}\left(\left(m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right)^{1+\gamma}\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}\right)^{\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}$$

Using the constraint and rearranging, we have

$$\frac{\left(T_{\omega}/w^{\alpha_{l}^{\omega}}\right)^{-(1+\gamma)}}{\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}}\left(\left(m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right)^{1+\gamma}\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}\right)^{\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}}=\left(\frac{w^{\alpha_{l}^{\omega}}\lambda}{k}T_{\omega}\right)^{\frac{1-\alpha_{l}^{\omega}}{\zeta}}$$

We can then use these to derive an explicit expression for the cost function:

$$\begin{aligned}
K_{\omega}\left(T_{\omega}\right) &= \sum_{\hat{\omega}\in\hat{\Omega}_{\omega}}\frac{k}{1+\gamma}h_{\hat{\omega}}^{1+\gamma}=\sum_{\hat{\omega}\in\hat{\Omega}_{\omega}}\frac{k}{1+\gamma}\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}\frac{w^{\alpha_{l}^{\omega}}\lambda T_{\omega}}{k}=\frac{\left(1-\alpha_{l}^{\omega}\right)k}{\zeta\left(1+\gamma\right)}\frac{w^{\alpha_{l}^{\omega}}\lambda T_{\omega}}{k}\\
&= \frac{\frac{\left(1-\alpha_{l}^{\omega}\right)k}{\zeta(1+\gamma)}}{\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}}\left(\left(m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right)^{1+\gamma}\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}\right)^{\frac{\alpha_{\hat{\omega}}^{\omega}}{1-\alpha_{l}^{\omega}}}}\left(\frac{T_{\omega}}{w^{\alpha_{l}^{\omega}}}\right)^{-\frac{\zeta(1+\gamma)}{1-\alpha_{l}^{\omega}}}\\
&= \frac{k}{1+\gamma}\frac{1}{\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}}\left(\left(m_{\omega\hat{\omega}}\left(\bar{c}_{\hat{\omega}}/w\right)^{-\zeta}\right)^{1+\gamma}\frac{\alpha_{\hat{\omega}}^{\omega}}{1-\alpha_{l}^{\omega}}\right)^{\frac{\alpha_{\hat{\omega}}^{\omega}}{1-\alpha_{l}^{\omega}}}}\left(\frac{T_{\omega}}{w}\right)^{-\frac{\zeta(1+\gamma)}{1-\alpha_{l}^{\omega}}}
\end{aligned}$$

Suppose that the hypothesis is true for all depths less than D. The problem for a module at depth D is

$$K_{\omega}\left(T_{\omega}\right)=\min_{\{h_{\hat{\omega}},T_{\hat{\omega}}\}}\sum_{\hat{\omega}\in\Omega_{\omega}}\left\{\frac{k}{1+\gamma}h_{\hat{\omega}}^{1+\gamma}+K_{\hat{\omega}}\left(T_{\hat{\omega}}\right)\right\}$$

subject to

$$T_{\omega}\geq w^{\alpha_{l}^{\omega}}\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}}\left(h_{\hat{\omega}}m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}+T_{\hat{\omega}}^{-\zeta}\right)^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}$$

Let λ be the multplier, and define $O_{\hat{\omega}}=\frac{h_{\hat{\omega}}m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}}{h_{\hat{\omega}}m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}+T_{\hat{\omega}}^{-\zeta}}$. If input $\hat{\omega}$ is a leaf so that it cannot be produced in-house, then FOC is

$$kh_{\hat{\omega}}^{\gamma}=w^{\alpha_{l}^{\omega}}\lambda T_{\omega}\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta h_{\hat{\omega}}}$$

Otherwise, the FOCs are

$$\begin{aligned}
h_{\hat{\omega}} &: kh_{\hat{\omega}}^{\gamma}=w^{\alpha_{l}^{\omega}}\lambda O_{\hat{\omega}}T_{\omega}\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta h_{\hat{\omega}}}\\
T_{\hat{\omega}} &: K_{\hat{\omega}}'\left(T_{\hat{\omega}}\right)=-w^{\alpha_{l}^{\omega}}\lambda\left(1-O_{\hat{\omega}}\right)T_{\omega}\frac{\alpha_{\hat{\omega}}^{\omega}}{T_{\hat{\omega}}}
\end{aligned}$$

These along with the constraint define the policy functions and the Lagrange multiplier. The implicit function theorem implies that the Lagrange multiplier is differentiable with repect to T_ω. Differentiating the FOCs with respect to $T_\omega^{-\zeta}$ and noting that $d\ln O_{\hat\omega} = (1-O_{\hat\omega})\, d\ln h_{\hat\omega} - (1-O_{\hat\omega})\, d\ln T_{\hat\omega}^{-\zeta}$ gives

$$
\begin{aligned}
(1+\gamma)\frac{d\ln h_{\hat\omega}}{d\ln T_\omega^{-\zeta}} &= \frac{d\ln\lambda T_\omega}{d\ln T_\omega^{-\zeta}} + (1-O_{\hat\omega})\frac{d\ln h_{\hat\omega}}{d\ln T_\omega^{-\zeta}} - (1-O_{\hat\omega})\frac{d\ln T_{\hat\omega}^{-\zeta}}{d\ln T_\omega^{-\zeta}} &\quad (3)\\
-\frac{1}{\zeta}\left(\frac{K_{\hat\omega}''(T_{\hat\omega})\,T_{\hat\omega}}{K_{\hat\omega}'(T_{\hat\omega})}+1\right)\frac{d\ln T_{\hat\omega}^{-\zeta}}{d\ln T_\omega^{-\zeta}} &= \frac{d\ln\lambda T_\omega}{d\ln T_\omega^{-\zeta}} - O_{\hat\omega}\frac{d\ln h_{\hat\omega}}{d\ln T_\omega^{-\zeta}} + O_{\hat\omega}\frac{d\ln T_{\hat\omega}^{-\zeta}}{d\ln T_\omega^{-\zeta}} &\quad (4)
\end{aligned}
$$

We can use these two expressions to solve for $\frac{d\ln h_{\hat\omega}}{d\ln T_\omega}$ and $\frac{d\ln T_{\hat\omega}^{-\zeta}}{d\ln T_\omega^{-\zeta}}$ in terms of $\frac{d\ln\lambda T_\omega}{d\ln T_\omega^{-\zeta}}$ for the nonleaf nodes:

$$
\begin{aligned}
\frac{d\ln h_{\hat\omega}}{d\ln T_\omega^{-\zeta}} &= \frac{1}{\gamma+O_{\hat\omega}+r_{\hat\omega}(1-O_{\hat\omega})}\frac{d\ln\lambda T_\omega}{d\ln T_\omega^{-\zeta}}\\
\frac{d\ln T_{\hat\omega}^{-\zeta}}{d\ln T_\omega^{-\zeta}} &= \frac{r_{\hat\omega}}{\gamma+O_{\hat\omega}+r_{\hat\omega}(1-O_{\hat\omega})}\frac{d\ln\lambda T_\omega}{d\ln T_\omega^{-\zeta}}
\end{aligned}
$$

where we defined $r_{\hat\omega} \equiv -\frac{\gamma}{\frac{1}{\zeta}\left(\frac{K_{\hat\omega}''(T_{\hat\omega})T_{\hat\omega}}{K_{\hat\omega}'(T_{\hat\omega})}+1\right)+1}$.

To solve for $\frac{d\ln\lambda T_\omega}{d\ln T_\omega^{-\zeta}}$, we differentiate the constraint to get

$$
1 = \sum_{\hat\omega\in\hat\Omega_\omega^{leaf}} \alpha_{\hat\omega}^{\omega}\frac{d\ln h_{\hat\omega}}{d\ln T_\omega^{-\zeta}} + \sum_{\hat\omega\in\hat\Omega_\omega^{non-leaf}} \alpha_{\hat\omega}^{\omega}\left\{O_{\hat\omega}\frac{d\ln h_{\hat\omega}}{d\ln T_\omega^{-\zeta}} + (1-O_{\hat\omega})\frac{d\ln T_{\hat\omega}^{-\zeta}}{d\ln T_\omega^{-\zeta}}\right\}
$$

Using the expressions for $\frac{d\ln h_{\hat\omega}}{d\ln T_\omega^{-\zeta}}$ and $\frac{d\ln T_{\hat\omega}^{-\zeta}}{d\ln T_\omega^{-\zeta}}$ and solving for $\frac{d\ln\lambda T_\omega}{d\ln T_\omega^{-\zeta}}$ gives

$$
\frac{d\ln\lambda T_\omega}{d\ln T_\omega^{-\zeta}} = \left\{\sum_{\hat\omega\in\hat\Omega_\omega^{leaf}} \alpha_{\hat\omega}^{\omega}\frac{1}{\gamma+1} + \sum_{\hat\omega\in\hat\Omega_\omega^{non-leaf}} \alpha_{\hat\omega}^{\omega}\frac{O_{\hat\omega}+r_{\hat\omega}(1-O_{\hat\omega})}{\gamma+O_{\hat\omega}+r_{\hat\omega}(1-O_{\hat\omega})}\right\}^{-1}
$$

The bounds on $\frac{K_{\hat\omega}''(T_{\hat\omega})T_{\hat\omega}}{-K_{\hat\omega}'(T_{\hat\omega})}$ imply that r_ω is bounded inside the $[0,1]$ interval

$$
0 < \frac{\gamma}{\frac{1+\gamma}{\beta_{\hat\omega}}-1} \le r_{\hat\omega} \le \frac{\gamma}{\frac{1+\gamma}{1-\alpha_l^{\hat\omega}}-1} < 1 \qquad (5)
$$

In particular, Since $r_{\hat\omega}$ is strictly smaller than 1 and larger than $\frac{\gamma}{\frac{1+\gamma}{\beta_{\hat\omega}}-1}$, we can bound $\frac{d\ln\lambda T_\omega}{d\ln T_\omega^{-\zeta}}$ above and below:

$$\begin{aligned}
\frac{d\ln\lambda T_\omega}{d\ln T_\omega^{-\zeta}} &\leq \left\{\sum_{\hat\omega\in\hat\Omega_\omega}\alpha_{\hat\omega}^{\omega}\frac{1}{\gamma+1}\right\}^{-1}=\frac{1+\gamma}{1-\alpha_l^\omega}\\
\frac{d\ln\lambda T_\omega}{d\ln T_\omega^{-\zeta}} &\geq \left\{\sum_{\hat\omega\in\hat\Omega_\omega^{leaf}}\alpha_{\hat\omega}^{\omega}\frac{1}{\gamma+1}+\sum_{\hat\omega\in\hat\Omega_\omega^{non-leaf}}\alpha_{\hat\omega}^{\omega}\frac{\underline{r}_{\hat\omega}}{\gamma+\underline{r}_{\hat\omega}}\right\}^{-1}=\frac{1+\gamma}{\beta_\omega}
\end{aligned}$$

where the last line used $\frac{\underline{r}_{\hat\omega}}{\gamma+\underline{r}_{\hat\omega}}=\frac{\beta_{\hat\omega}}{1+\gamma}$ and the definition of β_ω.

Three consquences are that the changes in the policy functions are bounded away from 0:

$$\begin{aligned}
\frac{d\ln h_{\hat\omega}}{d\ln T_\omega^{-\zeta}} &= \frac{1}{\gamma+O_{\hat\omega}+r_{\hat\omega}\left(1-O_{\hat\omega}\right)}\frac{d\ln\lambda T_\omega}{d\ln T_\omega^{-\zeta}}\geq\frac{1}{\gamma+\underline{r}_{\hat\omega}}\left\{\sum_{\hat\omega\in\hat\Omega_\omega}\alpha_{\hat\omega}^{\omega}\frac{\underline{r}_{\hat\omega}}{\gamma+\underline{r}_{\hat\omega}}\right\}^{-1}>0\\
\frac{d\ln T_{\hat\omega}^{-\zeta}}{d\ln T_\omega^{-\zeta}} &= \frac{r_{\hat\omega}}{\gamma+O_{\hat\omega}+r_{\hat\omega}\left(1-O_{\hat\omega}\right)}\frac{d\ln\lambda T_\omega}{d\ln T_\omega^{-\zeta}}\geq\frac{\underline{r}_{\hat\omega}}{\gamma+\underline{r}_{\hat\omega}}\left\{\sum_{\hat\omega\in\hat\Omega_\omega}\alpha_{\hat\omega}^{\omega}\frac{\underline{r}_{\hat\omega}}{\gamma+\underline{r}_{\hat\omega}}\right\}^{-1}>0\\
\frac{d\ln\frac{O_{\hat\omega}}{1-O_{\hat\omega}}}{d\ln T_\omega^{-\zeta}} &= \frac{d\ln h_{\hat\omega}}{d\ln T_\omega^{-\zeta}}-\frac{d\ln T_{\hat\omega}^{-\zeta}}{d\ln T_\omega^{-\zeta}}=\frac{1-r_{\hat\omega}}{\gamma+O_{\hat\omega}+r_{\hat\omega}\left(1-O_{\hat\omega}\right)}\frac{d\ln\lambda T_\omega}{d\ln T_\omega^{-\zeta}}\\
&\geq \frac{1-\bar r_{\hat\omega}}{\gamma+\underline{r}_{\hat\omega}}\left\{\sum_{\hat\omega\in\hat\Omega_\omega}\alpha_{\hat\omega}^{\omega}\frac{\underline{r}_{\hat\omega}}{\gamma+\underline{r}_{\hat\omega}}\right\}^{-1}>0
\end{aligned}$$

In addition, since all of these derivatives are positive and bounded away from 0, we have the following asymptotic properties.

$$\begin{aligned}
\lim_{T_\omega\to0}h_{\hat\omega}\left(T_\omega\right) &= \infty\\
\lim_{T_\omega\to0}T_{\hat\omega}\left(T_\omega\right) &= 0\\
\lim_{T_\omega\to0}O_{\hat\omega}\left(T_\omega\right) &= 1
\end{aligned}$$

$$\begin{aligned}
\lim_{T_\omega\to\infty}h_{\hat\omega}\left(T_\omega\right) &= 0\\
\lim_{T_\omega\to\infty}T_{\hat\omega}\left(T_\omega\right) &= \infty\\
\lim_{T_\omega\to\infty}O_{\hat\omega}\left(T_\omega\right) &= 0
\end{aligned}$$

We next derive bounds on $\frac{K_\omega''(T_\omega)T_\omega}{K_\omega'(T_\omega)}$. The envelope theorem implies that $K_\omega'\left(T_\omega\right)=-\lambda_\omega<0$, so taking logs and differentiating gives

$$\frac{K_\omega''\left(T_\omega\right)T_\omega}{K_\omega'\left(T_\omega\right)}=\frac{d\ln\lambda}{d\ln T_\omega}=\frac{d\ln\lambda T_\omega}{d\ln T_\omega}-1=-\zeta\frac{d\ln\lambda T_\omega}{d\ln T_\omega^{-\zeta}}-1$$

This can be rearranged as

$$\frac{K_{\omega}''(T_{\omega})T_{\omega}}{-K_{\omega}'(T_{\omega})}-1=\zeta\frac{d\ln\lambda T_{\omega}}{d\ln T_{\omega}^{-\zeta}}$$

The bounds on $\frac{d\ln\lambda T_{\omega}}{d\ln T_{\omega}^{-\zeta}}$ give

$$\frac{\zeta(1+\gamma)}{\beta_{\omega}}\leq\frac{K_{\omega}''(T_{\omega})T_{\omega}}{-K_{\omega}'(T_{\omega})}-1\leq\frac{\zeta(1+\gamma)}{1-\alpha_{l}^{\omega}}$$

Finally, we turn to the asymptotic behavior of K_{ω}. We first solve for limiting behavior of λT_{ω} by rearranging the constraint

$$\begin{aligned}
T_{\omega} &= w^{\alpha_{l}^{\omega}}\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}}\left(h_{\hat{\omega}}m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}+T_{\hat{\omega}}^{-\zeta}\right)^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}\\
&= w^{\alpha_{l}^{\omega}}\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}^{leaf}}\left(h_{\hat{\omega}}m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right)^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}^{non-leaf}}\left(h_{\hat{\omega}}m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}+T_{\hat{\omega}}^{-\zeta}\right)^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}\\
&= w^{\alpha_{l}^{\omega}}\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}}\left(h_{\hat{\omega}}m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right)^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}^{non-leaf}}\left(1+\frac{T_{\hat{\omega}}^{-\zeta}}{h_{\hat{\omega}}m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}}\right)^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}\\
&= w^{\alpha_{l}^{\omega}}\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}}\left(h_{\hat{\omega}}m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right)^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}^{non-leaf}}\left(\frac{1}{O_{\hat{\omega}}}\right)^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}
\end{aligned}$$

The FOCs for $h_{\hat{\omega}}$ imply

$$\begin{aligned}
\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}}\left(h_{\hat{\omega}}m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right)^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}} &= \prod_{\hat{\omega}\in\hat{\Omega}_{\omega}}\left(\left(\frac{w^{\alpha_{l}^{\omega}}\lambda T_{\omega}}{k}\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}\right)^{\frac{1}{1+\gamma}}m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right)^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}^{non-leaf}}\left((O_{\hat{\omega}})^{\frac{1}{1+\gamma}}\right)^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}\\
&= \left(\frac{w^{\alpha_{l}^{\omega}}\lambda T_{\omega}}{k}\right)^{-\frac{1-\alpha_{l}^{\omega}}{(1+\gamma)\zeta}}\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}}\left(\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}\left(m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right)^{1+\gamma}\right)^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{(1+\gamma)\zeta}}\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}^{non-leaf}}O_{\hat{\omega}}^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta(1+\gamma)}}
\end{aligned}$$

Plugging this in gives

$$T_{\omega}=\left(\frac{w^{\alpha_{l}^{\omega}}\lambda T_{\omega}}{k}\right)^{-\frac{1-\alpha_{l}^{\omega}}{(1+\gamma)\zeta}}w^{\alpha_{l}^{\omega}}\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}}\left(\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}\left(m_{\omega\hat{\omega}}\bar{c}_{\hat{\omega}}^{-\zeta}\right)^{1+\gamma}\right)^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{(1+\gamma)\zeta}}\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}^{non-leaf}}O_{\hat{\omega}}^{\frac{\gamma}{1+\gamma}\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}$$

$$T_{\omega}=\left(\frac{w^{\alpha_{l}^{\omega}}\lambda T_{\omega}}{k}\right)^{-\frac{1-\alpha_{l}^{\omega}}{(1+\gamma)\zeta}}w\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}}\left(\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}\left(m_{\omega\hat{\omega}}\left(\frac{\bar{c}_{\hat{\omega}}}{w}\right)^{-\zeta}\right)^{1+\gamma}\right)^{-\frac{\alpha_{\hat{\omega}}^{\omega}}{(1+\gamma)\zeta}}\prod_{\hat{\omega}\in\hat{\Omega}_{\omega}^{non-leaf}}O_{\hat{\omega}}^{\frac{\gamma}{1+\gamma}\frac{\alpha_{\hat{\omega}}^{\omega}}{\zeta}}$$

Rearranging and taking the limit and using $\lim_{T_\omega\to 0} O_{\hat{\omega}} = 1$ gives

$$
\begin{aligned}
\lim_{T_\omega\to 0}\frac{w^{\alpha_l^\omega}\lambda T_\omega}{k}\left(\frac{T_\omega}{w}\right)^{\frac{\zeta(1+\gamma)}{1-\alpha_l^\omega}} &= \lim_{T_\omega\to 0}\frac{1}{w^{\alpha_l^\omega}}\left\{\prod_{\hat{\omega}\in\hat{\Omega}_\omega}\left(\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}\left(m_{\omega\hat{\omega}}\left(\frac{\bar{c}_{\hat{\omega}}}{w}\right)^{-\zeta}\right)^{1+\gamma}\right)^{-\frac{\alpha_{\hat{\omega}}^\omega}{\zeta(1+\gamma)}}\prod_{\hat{\omega}\in\hat{\Omega}_\omega} O_{\hat{\omega}}^{\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}\frac{\gamma}{1+\gamma}}\right\}^{\frac{\zeta(1+\gamma)}{1-\alpha_l^\omega}} \\
&= \left\{\prod_{\hat{\omega}\in\hat{\Omega}_\omega}\left(\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}\left(m_{\omega\hat{\omega}}\left(\frac{\bar{c}_{\hat{\omega}}}{w}\right)^{-\zeta}\right)^{1+\gamma}\right)^{\frac{\alpha_{\hat{\omega}}^\omega}{1-\alpha_l^\omega}}\right\}^{-1}
\end{aligned}
$$

We next use the objective function

$$
\begin{aligned}
K_\omega(T_\omega) &= \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}}\frac{k}{1+\gamma}h_{\hat{\omega}}^{1+\gamma} + \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}}\left\{\frac{k}{1+\gamma}h_{\hat{\omega}}^{1+\gamma} + K_{\hat{\omega}}(T_{\hat{\omega}})\right\} \\
&= \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}}\frac{k}{1+\gamma}h_{\hat{\omega}}^{1+\gamma} + \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}}\frac{k}{1+\gamma}h_{\hat{\omega}}^{1+\gamma}\left(1+\frac{K_{\hat{\omega}}(T_{\hat{\omega}})}{\frac{k}{1+\gamma}h_{\hat{\omega}}^{1+\gamma}}\right)
\end{aligned}
$$

Using the FOCs for h_ω and rearranging gives

$$
\begin{aligned}
K_\omega(T_\omega) &= \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}}\frac{w^{\alpha_l^\omega}\lambda T_\omega\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}}{1+\gamma} + \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}}\frac{w^{\alpha_l^\omega}\lambda O_{\hat{\omega}}T_\omega\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}}{1+\gamma}\left(1+\frac{K_{\hat{\omega}}(T_{\hat{\omega}})}{\frac{k}{1+\gamma}h_{\hat{\omega}}^{1+\gamma}}\right) \\
&= \frac{k}{1+\gamma}\frac{w^{\alpha_l^\omega}\lambda T_\omega}{k}\left\{\sum_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}}\frac{\alpha_{\hat{\omega}}^\omega}{\zeta} + \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}}\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}O_{\hat{\omega}}\left(1+\frac{K_{\hat{\omega}}(T_{\hat{\omega}})}{\frac{k}{1+\gamma}h_{\hat{\omega}}^{1+\gamma}}\right)\right\}
\end{aligned}
$$

Note that since $\frac{K_{\hat{\omega}}(T_{\hat{\omega}})}{h_{\hat{\omega}}^{1+\gamma}} = K_{\hat{\omega}}(T_{\hat{\omega}})\,T_\omega^{\frac{\zeta(1+\gamma)}{1-\alpha_l^\omega}}\left(\frac{T_\omega^{-\zeta}}{h_{\hat{\omega}}}\right)^{1+\gamma}T_\omega^{\zeta(1+\gamma)\left(1-\frac{1}{1-\alpha_l^\omega}\right)}$, taking the limit gives $\lim_{T_\omega\to 0}\frac{K_{\hat{\omega}}(T_{\hat{\omega}})}{h_{\hat{\omega}}^{1+\gamma}} = 0$. As result, multiplying through by $\left(\frac{T_\omega}{w}\right)^{\frac{\zeta(1+\gamma)}{1-\alpha_l^\omega}}$ and taking the limit yields

$$
\begin{aligned}
\lim_{T_\omega\to 0}K_\omega(T_\omega)\left(\frac{T_\omega}{w}\right)^{\frac{\zeta(1+\gamma)}{1-\alpha_l^\omega}} &= \frac{k}{1+\gamma}\lim_{T_\omega\to 0}\left(\left(\frac{T_\omega}{w}\right)^{\frac{\zeta(1+\gamma)}{1-\alpha_l^\omega}}\frac{w^{\alpha_l^\omega}\lambda T_\omega}{k}\right)\left\{\sum_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}}\frac{\alpha_{\hat{\omega}}^\omega}{\zeta} + \sum_{\hat{\omega}\in\hat{\Omega}_\omega}\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}\lim_{T_\omega\to 0}O_{\hat{\omega}}\left(1+\frac{K_{\hat{\omega}}(T_{\hat{\omega}})}{\frac{k}{1+\gamma}h_{\hat{\omega}}^{1+\gamma}}\right)\right\} \\
&= \frac{k}{1+\gamma}\frac{\sum_{\hat{\omega}\in\hat{\Omega}_\omega}\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}}{\left\{\prod_{\hat{\omega}\in\hat{\Omega}_\omega}\left(\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}\left(m_{\omega\hat{\omega}}\left(\frac{\bar{c}_{\hat{\omega}}}{w}\right)^{-\zeta}\right)^{1+\gamma}\right)^{\frac{\alpha_{\hat{\omega}}^\omega}{1-\alpha_l^\omega}}\right\}} \\
&= \frac{k}{1+\gamma}\frac{1}{\prod_{\hat{\omega}\in\hat{\Omega}_\omega}\left(\frac{\alpha_{\hat{\omega}}^\omega}{1-\alpha_l^\omega}\left(m_{\omega\hat{\omega}}\left(\frac{\bar{c}_{\hat{\omega}}}{w}\right)^{-\zeta}\right)^{1+\gamma}\right)^{\frac{\alpha_{\hat{\omega}}^\omega}{1-\alpha_l^\omega}}}
\end{aligned}
$$

We next turn to the limit as $T_\omega \to 0$ and take a similar approach

Suppose that $\lim_{T_{\hat{\omega}}\to\infty} K_{\hat{\omega}}\left(T_{\hat{\omega}}\right) T_{\hat{\omega}}^{\frac{\zeta(1+\gamma)}{\beta_{\hat{\omega}}}} \to \frac{k}{1+\gamma}\frac{1}{\kappa_{\hat{\omega}}}$. Then $\lim_{T_{\hat{\omega}}\to 0} \frac{K'_{\hat{\omega}}(T_{\hat{\omega}})T_{\hat{\omega}}}{K_{\hat{\omega}}(T_{\hat{\omega}})} = -\frac{\zeta(1+\gamma)}{\beta_{\hat{\omega}}}$ and

$$\lim_{T_{\hat{\omega}}\to\infty} K'_{\hat{\omega}}\left(T_{\hat{\omega}}\right) T_{\hat{\omega}}^{\frac{\zeta(1+\gamma)}{\beta_{\hat{\omega}}}+1} = \lim_{T_{\hat{\omega}}\to\infty} \frac{K'_{\hat{\omega}}\left(T_{\hat{\omega}}\right) T_{\hat{\omega}}}{K_{\hat{\omega}}\left(T_{\hat{\omega}}\right)} K_{\hat{\omega}}\left(T_{\hat{\omega}}\right) T_{\hat{\omega}}^{\frac{\zeta(1+\gamma)}{\beta_{\hat{\omega}}}} = \left(-\frac{\zeta\left(1+\gamma\right)}{\beta_{\hat{\omega}}}\right)\left(\frac{k}{1+\gamma}\frac{1}{\kappa_{\hat{\omega}}}\right) = -\frac{\zeta}{\beta_{\hat{\omega}}} k \frac{1}{\kappa_{\hat{\omega}}}$$

The FOCs give

$$\begin{aligned}
T_\omega &= w^{\alpha_l^\omega} \prod_{\hat{\omega}\in\hat{\Omega}_\omega} \left(h_{\hat{\omega}} m_{\omega\hat{\omega}} \bar{c}_{\hat{\omega}}^{-\zeta} + T_{\hat{\omega}}^{-\zeta}\right)^{-\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}} \\
&= w^{\alpha_l^\omega} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}} \left(h_{\hat{\omega}} m_{\omega\hat{\omega}} \bar{c}_{\hat{\omega}}^{-\zeta}\right)^{-\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \left(h_{\hat{\omega}} m_{\omega\hat{\omega}} \bar{c}_{\hat{\omega}}^{-\zeta} + T_{\hat{\omega}}^{-\zeta}\right)^{-\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}} \\
&= w^{\alpha_l^\omega} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}} \left(h_{\hat{\omega}} m_{\omega\hat{\omega}} \bar{c}_{\hat{\omega}}^{-\zeta}\right)^{-\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} T_{\hat{\omega}}^{\alpha_{\hat{\omega}}^\omega} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \left(\frac{h_{\hat{\omega}} m_{\omega\hat{\omega}} \bar{c}_{\hat{\omega}}^{-\zeta}}{T_{\hat{\omega}}^{-\zeta}} + 1\right)^{-\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}} \\
&= w^{\alpha_l^\omega} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}} \left(h_{\hat{\omega}} m_{\omega\hat{\omega}} \bar{c}_{\hat{\omega}}^{-\zeta}\right)^{-\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} T_{\hat{\omega}}^{\alpha_{\hat{\omega}}^\omega} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \left(1 - O_{\hat{\omega}}\right)^{\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}}
\end{aligned}$$

Plug in the FOCs to get

$$\begin{aligned}
T_\omega &= w^{\alpha_l^\omega} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}} \left(\left(\frac{w^{\alpha_l^\omega}\lambda T_\omega}{k}\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}\right)^{\frac{1}{1+\gamma}} m_{\omega\hat{\omega}} \bar{c}_{\hat{\omega}}^{-\zeta}\right)^{-\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \left(\frac{w^{\alpha_l^\omega}\lambda\left(1-O_{\hat{\omega}}\right) T_\omega \alpha_{\hat{\omega}}^\omega}{-K'_{\hat{\omega}}\left(T_{\hat{\omega}}\right) T_{\hat{\omega}}^{\frac{\zeta(1+\gamma)}{\beta_{\hat{\omega}}}+1}}\right)^{-\frac{\beta_{\hat{\omega}}\alpha_{\hat{\omega}}^\omega}{\zeta(1+\gamma)}} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} (1 - \\
\frac{T_\omega}{w} &= \left(\frac{w^{\alpha_l^\omega}\lambda T_\omega}{k}\right)^{-\frac{\beta_\omega}{\zeta(1+\gamma)}} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}} \left(\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}\left(m_{\omega\hat{\omega}}\left(\frac{\bar{c}_{\hat{\omega}}}{w}\right)^{-\zeta}\right)^{1+\gamma}\right)^{-\frac{\alpha_{\hat{\omega}}^\omega}{\zeta(1+\gamma)}} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \left(\frac{-K'_{\hat{\omega}}\left(T_{\hat{\omega}}\right) T_{\hat{\omega}}^{\zeta\beta_{\hat{\omega}}+1}}{k\alpha_{\hat{\omega}}^\omega}\right)^{\frac{\beta_{\hat{\omega}}\alpha_{\hat{\omega}}^\omega}{\zeta(1+\gamma)}} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{non}}
\end{aligned}$$

Rearranging and taking the limit gives

$$\begin{aligned}
\lim_{T_\omega\to\infty} \frac{w^{\alpha_l^\omega}\lambda T_\omega}{k}\left(\frac{T_\omega}{w}\right)^{\frac{\zeta(1+\gamma)}{\beta_\omega}} &= \lim_{T_\omega\to\infty} \frac{\left\{\prod_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \left(1-O_{\hat{\omega}}\right)^{\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}\left(1-\frac{\beta_{\hat{\omega}}}{1+\gamma}\right)}\right\}^{\frac{\zeta(1+\gamma)}{\beta_\omega}}}{\left\{\prod_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}} \left(\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}\left(m_{\omega\hat{\omega}}\left(\frac{\bar{c}_{\hat{\omega}}}{w}\right)^{-\zeta}\right)^{1+\gamma}\right)^{\alpha_{\hat{\omega}}^\omega} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \left(\frac{k\alpha_{\hat{\omega}}^\omega}{-K'_{\hat{\omega}}(T_{\hat{\omega}})T_{\hat{\omega}}^{\zeta\beta_{\hat{\omega}}+1}}\right)^{\beta_{\hat{\omega}}\alpha_{\hat{\omega}}^\omega}\right.} \\
&= \left\{\prod_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}} \left(\frac{\alpha_{\hat{\omega}}^\omega}{\zeta}\left(m_{\omega\hat{\omega}}\left(\frac{\bar{c}_{\hat{\omega}}}{w}\right)^{-\zeta}\right)^{1+\gamma}\right)^{\frac{\alpha_{\hat{\omega}}^\omega}{\beta_\omega}} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \left(\frac{\alpha_{\hat{\omega}}^\omega}{\frac{\zeta}{\beta_{\hat{\omega}}}\frac{1}{\kappa_{\hat{\omega}}}}\right)^{\frac{\beta_{\hat{\omega}}\alpha_{\hat{\omega}}^\omega}{\beta_\omega}}\right\}^{-1} \\
&= \zeta\left\{\prod_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}} \left(\alpha_{\hat{\omega}}^\omega\left(m_{\omega\hat{\omega}}\left(\frac{\bar{c}_{\hat{\omega}}}{w}\right)^{-\zeta}\right)^{1+\gamma}\right)^{\frac{\alpha_{\hat{\omega}}^\omega}{\beta_\omega}} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \left(\kappa_{\hat{\omega}}\beta_{\hat{\omega}}\alpha_{\hat{\omega}}^\omega\right)^{\frac{\beta_{\hat{\omega}}\alpha_{\hat{\omega}}^\omega}{\beta_\omega}}\right\}^{-1}
\end{aligned}$$

We rearrange the value of the objective function K_ω

$$
\begin{aligned}
K_\omega\left(T_\omega\right) &= \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}} \frac{k}{1+\gamma} h_{\hat{\omega}}^{1+\gamma} + \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \left\{ \frac{k}{1+\gamma} h_{\hat{\omega}}^{1+\gamma} + K_{\hat{\omega}}\left(T_{\hat{\omega}}\right) \right\} \\
&= \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}} \frac{k}{1+\gamma} h_{\hat{\omega}}^{1+\gamma} + \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \left\{ \frac{k}{1+\gamma} h_{\hat{\omega}}^{1+\gamma} + \frac{K_{\hat{\omega}}}{-K'_{\hat{\omega}}\left(T_{\hat{\omega}}\right) T_{\hat{\omega}}} \left(-K'_{\hat{\omega}}\left(T_{\hat{\omega}}\right) T_{\hat{\omega}}\right) \right\}
\end{aligned}
$$

Usign the FOCs, this is

$$
\begin{aligned}
K_\omega\left(T_\omega\right) &= \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}} \frac{k}{1+\gamma} \frac{w^{\alpha_l^\omega} \lambda T_\omega}{k} \frac{\alpha_{\hat{\omega}}^\omega}{\zeta} + \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \left\{ \frac{k}{1+\gamma} \frac{w^{\alpha_l^\omega} \lambda T_\omega}{k} O_{\hat{\omega}} \frac{\alpha_{\hat{\omega}}^\omega}{\zeta} + \frac{K_{\hat{\omega}}}{-K'_{\hat{\omega}}\left(T_{\hat{\omega}}\right) T_{\hat{\omega}}} \left(w^{\alpha_l^\omega} \lambda \left(1-O_{\hat{\omega}}\right) T_\omega \alpha_{\hat{\omega}}^\omega\right) \right. \\
&= \frac{k}{1+\gamma} \frac{w^{\alpha_l^\omega} \lambda T_\omega}{k} \left\{ \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}} \frac{\alpha_{\hat{\omega}}^\omega}{\zeta} + \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \frac{\alpha_{\hat{\omega}}^\omega}{\zeta} \left\{ O_{\hat{\omega}} + \zeta\left(1+\gamma\right) \frac{K_{\hat{\omega}}\left(T_{\hat{\omega}}\right)}{-K'_{\hat{\omega}}\left(T_{\hat{\omega}}\right) T_{\hat{\omega}}} \left(1-O_{\hat{\omega}}\right) \right\} \right\}
\end{aligned}
$$

Multiplying through by $\left(T_\omega/w\right)^{\frac{\zeta(1+\gamma)}{\beta_\omega}}$ and taking the limit gives

$$
\begin{aligned}
\lim_{T_\omega\to\infty} K_\omega\left(T_\omega\right) \left(\frac{T_\omega}{w}\right)^{\frac{\zeta(1+\gamma)}{\beta_\omega}} &= \frac{k}{1+\gamma} \lim_{T_\omega\to\infty} \frac{w^{\alpha_l^\omega} \lambda T_\omega}{k} \left(\frac{T_\omega}{w}\right)^{\frac{\zeta(1+\gamma)}{\beta_\omega}} \lim_{T_\omega\to\infty} \left\{ \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}} \frac{\alpha_{\hat{\omega}}^\omega}{\zeta} + \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \frac{\alpha_{\hat{\omega}}^\omega}{\zeta} \left\{ O_{\hat{\omega}} + \zeta\,(1 \right.\right. \\
&= \frac{k}{1+\gamma} \frac{\sum_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}} \alpha_{\hat{\omega}}^\omega + \sum_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \alpha_{\hat{\omega}}^\omega \beta_{\hat{\omega}}}{\prod_{\hat{\omega}\in\hat{\Omega}_\omega^{leaf}} \left(\alpha_{\hat{\omega}}^\omega \left(m_{\omega\hat{\omega}} \left(\frac{\bar{c}_{\hat{\omega}}}{w}\right)^{-\zeta}\right)^{1+\gamma}\right)^{\frac{\alpha_{\hat{\omega}}^\omega}{\beta_\omega}} \prod_{\hat{\omega}\in\hat{\Omega}_\omega^{non-leaf}} \left(\kappa_{\hat{\omega}} \beta_{\hat{\omega}} \alpha_{\hat{\omega}}^\omega\right)^{\frac{\beta_{\hat{\omega}} \alpha_{\hat{\omega}}^\omega}{\beta_\omega}}} \\
&= \frac{k}{1+\gamma} \frac{1}{\kappa_\omega}
\end{aligned}
$$

■

E Notes about Characteristic Function

In this section we prove the existence of an infinitely divisible random with characteristic function $\frac{\Gamma(1-it)}{\prod_j \Gamma(1-\alpha_j it)}$ for any $\alpha_1, ..., \alpha_J$ such that $\alpha_j > 0$ and $\sum_j \alpha_j \leq 1$. The proof builds on Shanbhag, Pestana and Sreehari (1977) who showed the existence of a closely related random variable.[32]

[32] Shanbhag, Pestana and Sreehari (1977) show the existence of a random variable with characteristic function V such that $E\left[e^{-\theta V}\right] = \frac{\Gamma\left(1+\theta\frac{\beta}{\alpha}\right)}{\Gamma(1+\theta)\Gamma(1+\theta\beta)}$ for $\beta \geq \frac{\alpha}{1-\alpha}$ and $\mathrm{Re}\,\theta > -\alpha/\beta$. One can use this result to show the existence of our random variable. To see this, their result implies the existence of a random variable with characteristic function $\frac{\Gamma(1-it)}{\Gamma(1-pit)\Gamma(1-(1-p)it)}$ for any $p \in (0,1)$ by using $\theta = -i(1-p)$, $\alpha = p$, and $\beta = \frac{p}{1-p}$. Next, letting $\bar{\alpha}_j = \sum_{\tilde{j}=1}^{j} \alpha_{\tilde{j}}$, we can decompose the characteristic function into a product.

$$
\frac{\Gamma\left(1-it\right)}{\Gamma\left(1-\alpha_1 it\right)...\Gamma\left(1-\alpha_n it\right)} = \frac{\Gamma\left(1-\bar{\alpha}_2 it\right)}{\Gamma\left(1-\alpha_1 it\right)\Gamma\left(1-\alpha_2 it\right)} \frac{\Gamma\left(1-\bar{\alpha}_3 it\right)}{\Gamma\left(1-\bar{\alpha}_2 it\right)\Gamma\left(1-\alpha_3 it\right)} \cdots \frac{\Gamma\left(1-it\right)}{\Gamma\left(1-\bar{\alpha}_n it\right)}
$$

Lemma 8 *(Malmsten's formula): If* $\mathrm{Re}(z) > 0$*, then*

$$\log\Gamma(z) = \int_0^\infty \left\{ e^{-t}(z-1) + \frac{e^{-tz} - e^{-t}}{1-e^{-t}} \right\} \frac{dt}{t}$$

Proof. Start with Gauss's expression for the digamma function $\psi(z) \equiv \frac{\Gamma'(z)}{\Gamma(z)}$ when $\mathrm{Re}(z) > 0$: $\psi(z) = \int_0^\infty \left\{ \frac{e^{-t}}{t} - \frac{e^{-tz}}{1-e^{-t}} \right\} dt$ and integrate from 1 to z

$$\begin{aligned}
\log\Gamma(z) &= \int_1^z \psi(x)\, dx \\
&= \int_1^z \int_0^\infty \left\{ \frac{e^{-t}}{t} - \frac{e^{-tx}}{1-e^{-t}} \right\} dtdx \\
&= \int_0^\infty \int_1^z \left\{ \frac{e^{-t}}{t} - \frac{e^{-tx}}{1-e^{-t}} \right\} dxdt \\
&= \int_0^\infty \left\{ \frac{e^{-t}}{t}(z-1) + \left[\frac{1}{t}\frac{e^{-tz}}{1-e^{-t}} - \frac{1}{t}\frac{e^{-t}}{1-e^{-t}} \right] \right\} dt \\
&= \int_0^\infty \left\{ e^{-t}(z-1) + \frac{e^{-tz} - e^{-t}}{1-e^{-t}} \right\} \frac{dt}{t}
\end{aligned}$$

■

Lemma 9 *If* α *positive and real and* $\mathrm{Re}(\theta) > -\alpha$*, then the following two identities hold:*

$$\log\frac{\Gamma(\alpha+\theta)}{\Gamma(\alpha)} = \psi(\alpha)\theta + \int_{-\infty}^0 \left(e^{\theta t} - 1 - \theta t \right) \frac{e^{\alpha t}}{|t|(1-e^t)} dt \quad (6)$$

$$\log\frac{\Gamma(\alpha+\theta)}{\Gamma(\alpha)} = [\psi(\alpha) + b_\alpha]\theta + \int_{-\infty}^0 \left[e^{\theta t} - 1 - \frac{\theta t}{1+t^2} \right] \frac{e^{\alpha t} dt}{(1-e^t)|t|} \quad (7)$$

where $\psi(\cdot)$ *is the digamma function and* $b_\alpha \equiv \int_0^\infty \frac{e^{-\alpha s}}{1-e^{-s}} \frac{s^2}{(1+s^2)} ds$ *is positive and bounded.*

Proof. Malmsten's formula can be rearranged, changing variables, to deliver

$$\begin{aligned}
\log\Gamma(z) &= \int_0^\infty \left\{ \frac{e^{-zu} - e^{-u}}{1-e^{-u}} + (z-1)e^{-u} \right\} \frac{du}{u} \\
&= \int_0^{-\infty} \left\{ \frac{e^{zt} - e^t}{1-e^t} + (z-1)e^t \right\} \frac{dt}{t} \\
&= \int_{-\infty}^0 \left\{ \frac{e^{zt} - e^t}{1-e^t} + (z-1)e^t \right\} \frac{dt}{|t|}
\end{aligned}$$

Since $\mathrm{Re}(\alpha+\theta) > 0$ and $\mathrm{Re}(\alpha) > 0$, we can apply Malmsten's formula to both $\Gamma(\alpha+\theta)$ and $\Gamma(\alpha)$

Let X_j be a random variable with characteristic function $\frac{\Gamma(1-it)}{\Gamma\left(1-\frac{\bar{\alpha}_{j-1}}{\bar{\alpha}_j}it\right)\Gamma\left(1-\frac{\alpha_j}{\bar{\alpha}_j}it\right)}$. Similarly, let S be the random variable with characteristic function $\frac{\Gamma(1-it)}{\Gamma(1-\bar{\alpha}_n it)}$ (this is the log of a stable random variable). Then $S + \sum_{j=2}^n \bar{\alpha}_j X_j$ has the characteristic function $\frac{\Gamma(1-it)}{\prod_j \Gamma(1-\alpha_j it)}$.

and simplify to get

$$\begin{aligned}
\log \frac{\Gamma\alpha+\theta)}{\Gamma(\alpha)} &= \int_{-\infty}^{0}\left\{\frac{e^{(\alpha+\theta)t}-e^{t}}{1-e^{t}}+(\alpha+\theta-1)\,e^{t}\right\}\frac{dt}{|t|}-\int_{-\infty}^{0}\left\{\frac{e^{\alpha t}-e^{t}}{1-e^{t}}+(\alpha-1)\,e^{t}\right\}\frac{dt}{|t|}\\
&= \int_{-\infty}^{0}\left\{\frac{e^{(\alpha+\theta)t}}{1-e^{t}}-\frac{e^{\alpha t}}{1-e^{t}}+\theta e^{t}\right\}\frac{dt}{|t|}\\
&= \int_{-\infty}^{0}\left[e^{\theta t}-1\right]\frac{e^{\alpha t}dt}{(1-e^{t})\,|t|}+\int_{-\infty}^{0}\theta e^{t}\frac{dt}{|t|} \qquad (8)
\end{aligned}$$

To get (6) we subtract and add θt from the inner brackets of the first term and simplify to get

$$\begin{aligned}
\log \frac{\Gamma(\alpha+\theta)}{\Gamma(\alpha)} &= \int_{-\infty}^{0}\left[e^{\theta t}-1-\theta t\right]\frac{e^{\alpha t}dt}{(1-e^{t})\,|t|}+\int_{-\infty}^{0}\theta t\frac{e^{\alpha t}dt}{(1-e^{t})\,|t|}+\int_{-\infty}^{0}\theta e^{t}\frac{dt}{|t|}\\
&= \int_{-\infty}^{0}\left[e^{\theta t}-1-\theta t\right]\frac{e^{\alpha t}dt}{(1-e^{t})\,|t|}+\theta\int_{-\infty}^{0}\left(\frac{t}{|t|}\frac{e^{\alpha t}}{(1-e^{t})}+\frac{e^{t}}{|t|}\right)dt\\
&= \int_{-\infty}^{0}\left[e^{\theta t}-1-\theta t\right]\frac{e^{\alpha t}dt}{(1-e^{t})\,|t|}+\theta\int_{0}^{\infty}\left(\frac{e^{-s}}{s}-\frac{e^{-\alpha s}}{1-e^{-s}}\right)ds\\
&\int_{-\infty}^{0}\left[e^{\theta t}-1-\theta t\right]\frac{e^{\alpha t}dt}{(1-e^{t})\,|t|}+\theta\psi(\alpha)
\end{aligned}$$

where the last line uses Gauss's expression for the Digamma function.

To get (7),we subtract and add $\frac{\theta t}{1+t^2}$ from the inner brackets of the first term in 8 and simplify to get

$$\begin{aligned}
\log \frac{\Gamma(\alpha+\theta)}{\Gamma(\alpha)} &= \int_{-\infty}^{0}\left[e^{\theta t}-1-\frac{\theta t}{1+t^{2}}\right]\frac{e^{\alpha t}dt}{(1-e^{t})\,|t|}+\int_{-\infty}^{0}\frac{\theta t}{1+t^{2}}\frac{e^{\alpha t}dt}{(1-e^{t})\,|t|}+\int_{-\infty}^{0}\theta e^{t}\frac{dt}{|t|}\\
&= \int_{-\infty}^{0}\left[e^{\theta t}-1-\frac{\theta t}{1+t^{2}}\right]\frac{e^{\alpha t}dt}{(1-e^{t})\,|t|}+\theta\int_{-\infty}^{0}\left(\frac{t}{1+t^{2}}\frac{e^{\alpha t}}{(1-e^{t})\,|t|}+\frac{e^{t}}{|t|}\right)dt
\end{aligned}$$

The second integral can be expressed as

$$\begin{aligned}
\int_{-\infty}^{0}\left(\frac{t}{1+t^{2}}\frac{e^{\alpha t}}{(1-e^{t})\,|t|}+\frac{e^{t}}{|t|}\right)dt &= \int_{0}^{\infty}\left(\frac{e^{-s}}{s}-\frac{1}{1+s^{2}}\frac{e^{-\alpha s}}{1-e^{-s}}\right)ds\\
&= \int_{0}^{\infty}\left\{\frac{e^{-s}}{s}-\frac{e^{-\alpha s}}{1-e^{-s}}\right\}ds+\int_{0}^{\infty}\left\{\frac{e^{-\alpha s}}{1-e^{-s}}-\frac{e^{-\alpha s}}{(1-e^{-s})\,(1+s^{2})}\right\}ds\\
&= \psi(\alpha)+b_{\alpha}
\end{aligned}$$

b_α is positive. To show that b_α is bounded, we use $\frac{1}{1+s^2} \leq 1$ and $\frac{s}{1-e^{-s}} \leq 1+s$ (the latter follows from adding se^s to each side of $0 \leq e^s-(1+s)$ to get $se^s \leq (e^s-1)+s(e^s-1)$, or $s \leq (1+s)(1-e^{-s})$). Together, these give

$$b_{\alpha}=\int_{0}^{\infty}\frac{e^{-\alpha s}}{1-e^{-s}}\frac{s^{2}}{1+s^{2}}ds=\int_{0}^{\infty}e^{-\alpha s}s\frac{s}{1-e^{-s}}\frac{1}{1+s^{2}}ds\leq\int_{0}^{\infty}e^{-\alpha s}s\,(1+s)\,sds<\infty$$

■

Lemma 10 *Let $\phi_r(x) = \frac{1-x^r}{1-x}$, $0 < x < 1$, $r > 0$. Then for fixed $r \leq 1$, $\phi_r(x)$ is decreasing in x; for $r \geq 1$, $\phi_r(x)$ is increasing in x.*

Proof. Consider first the case where $r \leq 1$. We will use a Taylor expansion of $f(t) = 1 - (1-t)^r$ around $t = 0$ for $t \in (0,1)$. Noting that $f'(t) = r(1-t)^{r-1}$ and $f^{(k)}(t) = (1-t)^{r-k} r \prod_{j=1}^{k-1}(j-r)$, $k \geq 2$, the Taylor expansion is thus

$$f(t) = f(0) + f'(0)t + \sum_{k=2}^{\infty} \frac{f^{(k)}(0)t^k}{k!} = 0 + rt + \sum_{k=2}^{\infty} \frac{\left[r \prod_{j=1}^{k-1}(j-r)\right] t^k}{k!}$$

Then, we have, for $x \in (0,1)$,

$$\phi_r(x) = \frac{1-x^r}{1-x} = \frac{f(1-x)}{1-x} = r + \sum_{k=2}^{\infty} \frac{\left[r \prod_{j=1}^{k-1}(j-r)\right](1-x)^{k-1}}{k!} > 0$$

Second, for the case of $r \geq 1$, we have $\phi_r(x) = \frac{1-x^r}{1-x} = \frac{1}{\frac{1-(x^r)^{1/r}}{1-x^r}} = \frac{1}{\phi_{1/r}(x^r)}$. $\phi_{1/r}(x^r)$ is decreasing because $\frac{1}{r} \leq 1$. Therefore $\phi_r(x) = \phi_{1/r}(x^r)^{-1}$ is increasing. ■

Lemma 11 *For every fixed α_1,...,α_J such that $\alpha_j > 0$ and $\sum_j \alpha_j \leq 1$ and $x \in (0,1)$. Then*

$$\frac{1}{x}\left(\frac{x}{1-x} - \sum_j \frac{x^{\frac{1}{\alpha_i}}}{1-x^{\frac{1}{\alpha_i}}}\right) \geq 0$$

Proof. Consider the case in which $\sum_j \alpha_j = 1$. The previous lemma implies that $\log\left(\frac{1-x^{1/\alpha_j}}{1-x}\right)$ is increasing, and hence $\sum_j \alpha_j \log\left(\frac{1-x^{1/\alpha_j}}{1-x}\right)$ is increasing, for $x \in (0,1)$.

$$\sum_j \alpha_j \log\left(\frac{1-x^{1/\alpha_j}}{1-x}\right) = [-\log(1-x)] - \sum_j \left[-\alpha_j \log\left(1-x^{1/\alpha_j}\right)\right] = \int_0^x \frac{1}{u}\left(\frac{u}{1-u} - \sum_j \frac{u^{1/\alpha_j}}{1-u^{1/\alpha_j}}\right) du$$

(because $\left[-\frac{\log[1-x^w]}{w}\right] = \int_0^x \frac{u^{w-1}}{1-u^w} dx$). Taking the derivative of each side gives

$$0 \leq \frac{d}{dx}\left\{\sum_j \alpha_j \log\left(\frac{1-x^{1/\alpha_j}}{1-x}\right)\right\} = \frac{1}{x}\left(\frac{x}{1-x} - \sum_i \frac{x^{1/\alpha_j}}{1-x^{1/\alpha_j}}\right)$$

Next, if $\sum_j \alpha_j < 1$, then let $\tilde{\alpha}_i = \alpha_i$ for $i = 1, ..., J$ and $\tilde{\alpha}_{J+1} = 1 - \sum_j \alpha_j$, and apply the result from the previous case to $\tilde{\alpha}_1, ..., \tilde{\alpha}_{J+1}$. ■

Lemma 12 *For any real and positive number* α*,*

$$\log\Gamma\left(1-\alpha it\right)=-itc_{\alpha}+\int_{0}^{\infty}\left(e^{itu}-1-\frac{uit}{1+u^{2}}\right)\frac{e^{-u/\alpha}}{u\left(1-e^{-u}\right)}du$$

with c_α *finite.*

Proof. We begin by rearranging $\log\Gamma\left(\frac{\theta}{x}+1\right)$ in the case where x is real and positive $\mathrm{Re}\left(\theta\right)>-x$. Using (6) and the change of variables $s=\frac{t}{x}$, and then using (6) again, we have

$$\begin{aligned}
\log\Gamma\left(\frac{\theta}{x}+1\right) &= \log\frac{\Gamma\left(\frac{\theta}{x}+1\right)}{\Gamma\left(1\right)}=\psi\left(1\right)\theta+\int_{-\infty}^{0}\left(e^{\frac{\theta}{x}t}-1-\frac{\theta}{x}t\right)\frac{e^{t}}{\left|t\right|\left(1-e^{t}\right)}dt \\
&= \psi\left(1\right)\theta+\int_{-\infty}^{0}\left(e^{\theta s}-1-\theta s\right)\frac{e^{xs}}{\left|s\right|\left(1-e^{s}\right)}ds \\
&= \psi\left(1\right)\theta-\psi\left(x\right)\theta+\psi\left(x\right)\theta+\int_{-\infty}^{0}\left(e^{\theta s}-1-\theta s\right)\frac{e^{xs}}{\left|s\right|\left(1-e^{s}\right)}ds \\
&= \left[\psi\left(1\right)-\psi\left(x\right)\right]\theta+\frac{\Gamma\left(\theta+x\right)}{\Gamma\left(x\right)}
\end{aligned}$$

Then using (7) and then a change of variables $u=-s$, we have

$$\begin{aligned}
\log\Gamma\left(\frac{\theta}{x}+1\right) &= \left[\psi\left(1\right)-\psi\left(x\right)\right]\theta+\left[\psi\left(x\right)+b_{x}\right]\theta+\int_{-\infty}^{0}\left(e^{\theta s}-1-\theta\frac{s}{1+s^{2}}\right)\frac{e^{xs}}{\left|s\right|\left(1-e^{s}\right)}ds \\
&= \left[\psi\left(1\right)+b_{x}\right]\theta+\int_{0}^{\infty}\left(e^{-\theta u}-1+\theta\frac{u}{1+u^{2}}\right)\frac{e^{-xu}}{u\left(1-e^{-u}\right)}du
\end{aligned}$$

The result follows from evaluating this expression for $\theta=-i$ and $x=\frac{1}{\alpha}$, and defining $c_{\alpha}\equiv\psi(1)+b_{1/\alpha}$. ■

Proposition 9 *Fix* $\alpha_1,...,\alpha_J$ *such that* $\alpha_j>0$ *and* $\sum_j\alpha_j\leq1$*. There exists an infinitely divisible random variable with characteristic function* $\frac{\Gamma(1-it)}{\prod_j\Gamma(1-\alpha_j it)}$*.*

Proof. We show this by showing that such a characteristic function has a Levy-Canonical Representation (see, for example, Lukacs (1960), Thm 5.5.2). Specifically, we can rearrange the function as

$$\begin{aligned}
\log\frac{\Gamma\left(1-it\right)}{\prod_{j}\Gamma\left(1-\alpha_{j}it\right)} &= \log\Gamma\left(1-it\right)-\sum_{j}\log\Gamma\left(1-\frac{it}{1/\alpha_{j}}\right) \\
&= -it\left(c_{1}-\sum_{j}c_{1/\alpha_{j}}\right)+\int_{0}^{\infty}\left(e^{itu}-1-it\frac{u}{1+u^{2}}\right)\frac{1}{u}\left[\frac{e^{-u}}{1-e^{-u}}-\sum_{j}\frac{e^{-\frac{1}{\alpha_{j}}u}}{1-e^{-u}}\right]du \\
&= -it\left(c_{1}-\sum_{j}c_{1/\alpha_{j}}\right)+\int_{0}^{\infty}\left(e^{itu}-1-it\frac{u}{1+u^{2}}\right)dN(u)
\end{aligned}$$

where $N(x)$ is defined for $x \geq 0$ as

$$N(x) \equiv -\int_x^\infty \frac{1}{u}\left[\frac{e^{-u}}{1-e^{-u}} - \sum_j \frac{e^{-\frac{1}{\alpha_j}u}}{1-e^{-u}}\right] du \; .$$

To verify that this conforms to a Levy Canonical representation, we must verify that N satisfies several conditions. First, $N(x)$ is non-decreasing on $(0, \infty)$ because Lemma 11 guarantees that $N'(x) \geq 0$. Second, $\lim_{x\to\infty} N(x) = 0$. Finally, $\int_0^\varepsilon u^2 dN(u) < \infty$ for all $\varepsilon > 0$:

$$\begin{aligned}
\int_0^\varepsilon u^2 dN(u) &\leq \int_0^\infty u^2 dN(u) \\
&= \int_0^\infty u \left[\frac{e^{-u}}{1-e^{-u}} - \sum_j \frac{e^{-\frac{1}{\alpha_j}u}}{1-e^{-u}}\right] du \\
&\leq \int_0^\infty u \frac{e^{-u}}{1-e^{-u}} du \\
&\leq \int_0^\infty (1+u)e^{-u} du \\
&= 2
\end{aligned}$$

where the last inequality uses $\frac{u}{1-e^{-u}} \leq 1 + u$ as discussed in the proof of Lemma 9 ■