Appendix. Hilbert's stroke-numerals and Freudenthal

What is Hilbert's stroke-numeral supposed to be? A notation for the pure whole number 3? Then it could as well be "3." Does Hilbert's stroke-numeral *picture* three?
Is the stroke-numeral III identical to three? Is the pure whole number three a display of 3 vertical lines?
Hilbert: '3' has the numeral III as meaning; III is a concrete object. 3 —> III.

H explanation: Hilbert wants III to be designator, notation, and semantics: on the level of metamathematics. In metamathematics, numerals are the meanings of numbers.
["3" would designate the stroke-numeral III, which *is* three.]
'1,' '2,' '3' are numerals also: they are names of Platonic integers. H is not using 'numeral' as a word for designator.

Back to Freudenthal: Even if an alien culture guessed that the writing IIIII means the abstraction 5 by showing 5, it might not accept that IIIII means five blips (sound sequence). **How can you identify co-present entities with events that come and go?**
Freudenthal: 3 can be pictured. So he says; then what about the pure-mathematics explanation of 3?
Should 5 bleeps (not to say dots) communicate the idea of the number 5? Well! To Hilbert, '5' should mean five dots. The idea of five has been lost.

HF. Assembling three scratches doesn't make the abstract number "three." Numbers must be abstractions.
Hilbert's theory doesn't deal with the tenet that three is an abstraction. Hilbert says he will evade that, he doesn't need to use it. But in that case, there would be a different word for three for each species counted.
Need equinumerousness [to be] a property possessed by qualitatively different assemblies.
You can't use stroke-numerals unless you have mastered abstract counting.

Consistency of Excluded Middle and mathematical induction
Hilbert's purpose: It is OK for Arabic numerals to have meaning in the sky: because by using stroke-numerals, Hilbert will prove the formalism to be consistent. A stroke-numeral is a concrete model for <an Arabic numeral which metaphysically designates an integer>. A concrete model proves the consistency of objects in the sky.
2 + 3 = 3 + 2 is mathematics and presumably has a (metaphysical) meaning. It is replaced by II + III = III + II. This has no meaning. This is a game played with signs.
Let's factor meaning out. Even though you know IIII = IV for the Romans, you don't use that ancient equivalence in a proof.

In Foundations, in the 1920s, the most problematic principle of inference was Mathematical Induction. If there is any inconsistency in arithmetic, would be because of induction. The purpose of stroke-notation was to avoid it.
Excluded Middle also. If there is any inconsistency in arithmetic, Excluded Middle could be at fault.

A concrete proxy which is a canonical notation is important to Hilbert: don't need mathematical induction, replaced by direct observation. That will be why you will accept mathematial induction later (when it goes beyond observation).
Hilbert uses this for a whole consistency proof for arbitrary finite-length proofs—beyond observation.

$$\underbrace{\mathbf{I} \ldots \mathbf{I}}_{100} \overset{?}{=} \underbrace{\mathbf{I} \ldots \mathbf{I}}_{101}$$

is it important to distinguish two such large numbers?

$$\underbrace{\mathbf{I} \ldots \mathbf{I}}_{10^{100}} \overset{?}{=} \underbrace{\mathbf{I} \ldots \mathbf{I}}_{10^{100}+1}$$

a computer can actually check this, so it can "know" something a human cannot directly know. then you forego comprehension as a test of truth

HF. Hilbert's semantics (or idiographic/ideographic ontology) for the natural numbers overlooks that the numbers are abstractions. Hilbert can't allow for that.
1. Without it, there are different numbers for different kinds of things.

2. The following can be added to Hilbert's metamathematics as an axiom, and will be consistent because it will be inaccessible to any ***derivation***.

$$\underbrace{\mathbf{II} \ldots \mathbf{I}}_{10^{12}} = \underbrace{\mathbf{II} \ldots \mathbf{I}}_{10^{12}+1}$$

But a machine could check this. Yet we cannot, and cannot directly check whether the machine makes a mistake. The machine is now like a God relative to us. [free association: Turing's Oracle] We have abandoned personal comprehension. **The theory of transverifiable knowledge.**