Foundations for Mathematical Structuralism*

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Abstract

We investigate the form of mathematical structuralism that acknowledges the existence of structures and their distinctive structural elements. This form of structuralism has been subject to criticisms recently, and our view is that the problems raised are resolved by proper, mathematics-free theoretical foundations. Starting with an axiomatic theory of abstract objects, we identify a mathematical structure as an abstract object encoding the truths of a mathematical theory. From such foundations, we derive consequences that address the main questions and issues that have arisen. Namely, elements of different structures are different. A structure and its elements ontologically depend on each other. There are no haecceities and each element of a structure must be discernible within the theory. These consequences are not developed piecemeal but rather follow from our definitions of basic structuralist concepts.

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1 Introduction

Mathematical structuralism is the view that pure mathematics is about abstract structure or structures (see, e.g., Hellman 1989, Shapiro 1997).[1] This philosophical view comes in a variety of forms. In this paper, we investigate, and restrict our use of the term 'structuralism' to the form that acknowledges that abstract structures exist, that the pure objects of mathematics are in some sense elements of, or places in, those structures, and that there is nothing more to the pure objects of mathematics than can be described by the basic relations of their corresponding structure (e.g., Dedekind 1888 [1963], Resnik 1981, 1997, Parsons 1990, and Shapiro 1997).[2] We shall not suppose that structures are sets nor assume any set theory; our goal is to give an analysis of mathematics that doesn't presuppose any mathematics. Our work is motivated by two insights. First, as we discuss in Section 2.1, abstract objects are connected to the properties that define them in a different way than ordinary objects are connected to the properties they bear. Second, as we discuss in Section 3.1, theorems and truths about abstract relations are more important in defining mathematical structures than mathematical entities.

Recently, the literature on structuralism has centered on a variety of questions and problems. These issues arise, in part, because the philosophical view has not been given proper, mathematics-free theoretical foundations.[3] We shall show how to view mathematical structures as

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[1]In Hellman 1989, we find that 'mathematics is concerned principally with the investigation of structures ... in complete abstraction from the nature of individual objects making up those structures' (vii), and in Shapiro 1997, we find that '[P]ure mathematics is the study of structures, independently of whether they are exemplified in the physical realm, or in any realm for that matter' (75).

[2]Parsons (1990), for example, says 'By the 'structuralist view' of mathematical objects, I mean the view that reference to mathematical objects is always in the context of some background structure, and that the objects have no more to them than can be expressed in terms of the basic relations of the structure' (303). By contrast, eliminative forms of structuralism, such as modal structuralism (e.g., Putnam 1967, Hellman 1989), avoid commitments to the existence of mathematical structures and their special structural elements. See the discussion in Reck & Price 2000.

[3]Shapiro (1997) offers axioms (93–5), but these are not mathematics-free. Nor is it clear exactly which primitive notions are required for his list of axioms. For example, the Powerstructure axiom (94) asserts: Let S be a structure and s its collection of places. Then there is a structure T and a binary relation R such that for each subset $s' \subseteq s$, there is a place x of T such that $\forall z(z \in s' \equiv Rxz)$. Here it is clear that the notions of set theory have been used in the theory of structures, and so the resulting theory is not mathematics-free. Moreover, it is not clear just which primitives are

abstract objects and how to analyze elements and relations of such structures. We shall show why the elements of structures are incomplete and prove that the essential properties of an element of a structure are just those mathematical properties by which it is conceived.

We then examine the consequences of our view of structuralism on the following issues:

1. Can there be identities between elements of different structures?

2. Do the elements of a mathematical structure ontologically depend on the structure?

3. Do the elements of a structure have haecceities?

4. Is indiscernibility a problem for structuralism?

We begin with a brief review of an analysis of mathematics that can be given in terms of an axiomatic theory of abstract objects. We then interpret this theory as a foundation for structuralism and show how it yields theorems that decide the above issues. Our answers to these issues are compared with those from other recent defenses of structuralism. We hope to show that one can give principled, rather than piecemeal, answers to the issues that have been the subject of much debate.

2   Background

2.1   Axiomatic Theory of Abstract Objects

Our background theory is based on an insight into the nature of abstract objects and predication, namely, that abstract objects are constituted by the properties through which we conceive or theoretically define them and therefore are connected to those properties in a way that is very different from the way ordinary objects bear their properties. We shall say that mathematical and other abstract objects *encode* these constitutive properties, though they may *exemplify* (i.e., instantiate in the traditional sense) or even *necessarily exemplify*, other properties independently of their encoded properties. By contrast, ordinary objects only exemplify their properties.

So, for example, ordinary triangular objects (e.g., the faces of some physical pyramid, musical triangles, etc.) exemplify properties like having sides with a particular length, having interior angles of particular magnitudes, being made of a particular substance, etc. By contrast, the mathematical object, The Euclidean Triangle, doesn't exemplify any of these properties—indeed, it exemplifies their negations. Instead, it encodes only the theoretical properties implied by being triangular, such as being trilateral, having interior angles summing to 180 degrees, etc. Whereas every object x whatsoever (including ordinary triangular objects and The Euclidean Triangle) is complete with respect to the properties it exemplifies (i.e., for every property F, either x exemplifies F or the negation of F, given classical negation), The Euclidean Triangle encodes no other properties than those implied by being triangular. Thus, although classical logic requires that the exemplification mode of predication exclude objects that are incomplete, the encoding mode of predication allows us to assert the existence of abstract objects that are incomplete with respect to the properties they encode. Thus, we might use the encoding mode of predication to assert the existence of abstract objects whose only encoded properties are those they are theoretically-defined to have according to some mathematical theory. This will become important later as it calls to mind Benacerraf's view that the elements of an abstract structure 'have no properties other than those relating them to other 'elements' of the same structure' (1965, 70).

To say that an abstract object x encodes property F is, roughly, to say that property F is one of the defining or constitutive properties of x.[4] This idea leads directly to the two main theoretical innovations underlying the axiomatic theory of abstract objects ('object theory'). First, the theory includes a special atomic formula, xF, to express primitive encoding predications of the form object x encodes property F. (The theory will retain the traditional mode of predication (Fx) that is used to assert that object x—whether ordinary or abstract—*exemplifies* property F.) Second, the theory includes a comprehension principle that asserts, for any given formula φ that places a condition on properties, the existence

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needed for this axiom—such notions seem to include *structure*, *place*, *set*, *membership*, and *relation*. By contrast, in what follows, we shall not appeal to sets or membership; structures and places will be identified with objects axiomatized antecedently.

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[4]The distinction between encoding and exemplifying a property has appeared in the work of other philosophers, though under different names. Meinwald (1992, 378) argues that it appears in Plato; Boolos (1987, 3) argues that it appears in Frege; Kripke appeals to it in his Locke Lectures (1973 [2013], Lecture III, 74), and the philosophers Castañeda (1974), Rapaport (1978), and van Inwagen (1983) have invoked something like this distinction in their own work.

of an abstract object that encodes exactly the properties satisfying φ.

These theoretical innovations may be developed in a syntactically second-order, quantified S5 modal logic (with both first- and second-order Barcan formulas) that has two kinds of atomic formulas $F^n x_1 \ldots x_n$ ($n \geq 0$) ('$x_1, \ldots, x_n$ *exemplify* F^n') and xF^1 ('x *encodes* F^1'), a distinguished predicate '$E!$' ('being concrete'), λ-expressions $[\lambda y_1 \ldots y_n \, \varphi]$ ('being $y_1, \ldots, y_n$ such that φ', where φ has no encoding subformulas), and rigid definite descriptions of the form $\imath x \varphi$ ('the x such that φ', interpreted rigidly). Without loss of clarity, we drop the superscript on F^n, since the number of argument places make it clear what the superscript must be. Thus, we shall write Fx, xF, Rxy, etc.[5]

In terms of this language and logic, an *ordinary* object ('$O!x$') is defined as $\Diamond E!x$ (possibly concrete) and an *abstract* object ('$A!x$') as $\neg\Diamond E!x$ (not possibly concrete). Given this partition of the domain of individuals, a general notion of identity for individuals is defined disjunctively:

$$x = y \;\overset{\text{def}}{=}\; [O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)] \;\lor\; [A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF)]$$

Intuitively, this means that ordinary objects are identical if they exemplify the same properties and abstract objects are identical if they encode the same properties. Furthermore, identity is defined for properties (1-place relations) and propositions (0-place relations) as follows:

$$F = G \;\overset{\text{def}}{=}\; \Box\forall x(xF \equiv xG)$$
$$p = q \;\overset{\text{def}}{=}\; [\lambda y \, p] = [\lambda y \, q]$$

Identity can be defined generally for n-place relations ($n \geq 2$), but we omit the definition here.

Using these definitions, $\alpha = \alpha$ (α any variable) can be derived and the rule of identity substitution is taken as axiomatic. Moreover, β-, η-, and α-conversion are assumed as part of the logic of λ-expressions, and the Russell axiom is formulated as a contingent logical axiom schema for definite descriptions.[6] From this logical basis, one can derive the second-order

comprehension principle for relations from β-conversion, generalization and the rule of necessitation:

Comprehension for Relations
$\exists F^n \Box\forall x_1 \ldots \forall x_n(F^n x_1 \ldots x_n \equiv \varphi)$, where $n \geq 0$ and φ has no encoding subformulas

Finally, the logic of encoding is captured by the principle that $\Diamond xF \to \Box xF$. That is, if x possibly encodes F, then x necessarily encodes F.

The theory of abstract objects may then be stated in terms of the following two non-logical axioms:

No Encoding for Ordinary Objects
$O!x \to \Box\neg\exists F(xF)$

Comprehension for Abstract Objects
$\exists x(A!x \,\&\, \forall F(xF \equiv \varphi))$, for any φ with no free xs

The following theorem schema follows immediately, given the identity conditions for abstract objects and the abbreviation $\exists!x\varphi$ for asserting that there is a unique x such that φ:

Comprehension! for Abstract Objects
$\exists!x(A!x \,\&\, \forall F(xF \equiv \varphi))$, for any φ with no free xs

This theorem ensures that definite descriptions of the form $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$ are always well-defined, in the following sense:

Canonical Descriptions for Abstract Objects
$\exists y(y = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi)))$, for any φ with no free xs

Such canonical definite descriptions are therefore governed by the following theorem schema:

Abstraction for Abstract Objects
$\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \;\equiv\; \varphi^G_F$

As a simple example, consider the following instance of the above, where o denotes Obama and P denotes the property of being president:

$$\imath x(A!x \,\&\, \forall F(xF \equiv Fo))P \;\equiv\; Po$$

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[5]The rest of this section may be skipped by those familiar with object theory and its applications to mathematics. It is included here so that the present paper is self-contained. For those encountering object theory for the first time and interested in a gentler introduction, see the presentation in Zalta 2006. For a more complete treatment, see Zalta 1983, 1988a.

[6]See Zalta 1988b for a full discussion. When the Russell axiom governs definite descriptions that are interpreted rigidly, its instances become contingent logical truths.

The presence of contingent logical axioms requires that we place a restriction on the Rule of Necessitation; it may not be applied to any line of a proof that depends for its justification on a contingent logical truth.

This asserts: the abstract object x, which encodes exactly the properties F that Obama exemplifies, encodes the property of being president if and only if Obama exemplifies being president. From the simple exemplification on the right-hand side of the biconditional (Obama exemplifies being president), the left-hand side abstracts out an object (the complete individual concept of Obama) and asserts that it encodes a property (being president).

2.2   Analysis of Mathematics

This formal system has been used for the analysis of mathematical statements in Zalta 1983, 1999, 2000, 2006, and Linsky & Zalta 1995, 2006. Following Zalta 2006, we distinguish natural mathematics and theoretical mathematics. Natural mathematics consists of the ordinary, pretheoretic claims we make about mathematical objects, such as that the triangle has 3 sides, the number of planets is eight, there are more individuals in the class of insects than in the class of humans, lines a and b have the same direction, figures a and b have the same shape, etc. By contrast, theoretical mathematics is constituted by claims that occur in the context of some explicit or implicit (formal or informal) mathematical theory, for example, theorems. In what follows, we take the data from theoretical mathematics to consist of claims including a prefixed 'theory operator' that identify the principles from which the claim can be proved, such as:

> In ZF, the null set is an element of the unit set of the null set.
> In Real Number Theory, 2 is less than or equal to π.

Object theory uses one technique for the analysis of natural mathematics and a different technique for the analysis of theoretical mathematics. The objects of the former are analyzed using the theory we have presented thus far, while the objects of the latter require us to extend the theory with the data mentioned above. Since the thesis of interest to us in the present paper, that pure mathematics is about structures, can be validated by the analysis of theoretical mathematics, we omit further discussion of (the analysis of) natural mathematics and focus only on the analysis of mathematical theories and structures. (Readers interested in natural mathematical objects may look to Zalta 1999, Pelletier & Zalta 2000, and Anderson & Zalta 2004.)

The analysis of theoretical mathematical language in Zalta 2006 starts with the idea that there is a subdomain of abstract objects that encode

only properties F such that F is a propositional property of the form $[\lambda y\, p]$. We called such abstract objects *situations*:

$$Situation(x) \;\overset{\text{def}}{=}\; A!x \,\&\, \forall F(xF \to \exists p(F = [\lambda y\, p]))$$

In what follows, when F is the property $[\lambda y\, p]$, for some proposition p, we shall say that F is constructed out of p. We may therefore define a sense in which a proposition may be true in an abstract object x, where x is a situation:

$$p \text{ is true in } x \;(`x \models p`) \;\overset{\text{def}}{=}\; x[\lambda y\, p]$$

In other words, p is true in x just in case x encodes the property F constructed out of p. In what follows, we shall often read '$x \models p$' as: in (situation) x, p (is true).

If we now conceive of theories as situations, then we may use the following principle to identify the theory T:

> **Identity of Theory T**
> (The theory) $T = \imath x(A!x \,\&\, \forall F(xF \equiv \exists p(T \models p \,\&\, F = [\lambda y\, p])))$

This principle asserts that the theory T is the abstract object that encodes exactly the propositional properties F constructed out of propositions p that are true in T. Note that although T appears on both sides of the identity sign, this is a principle governing theories T and not a definition of T. But it should be clear what we are doing here: we are abstracting the theory T from a body of data of the form 'In theory T, p' in which T isn't yet parsed as such.

Next we use the above definitions to ground our analysis of mathematical statements by extending object theory through the importation of new claims. Let κ be a primitive term of theory T, and let the expression 'κ_T' represent the term κ *indexed* to T, where we read κ_T as: the κ of theory T. Then we adopt the following rule for importing new claims into object theory:

> **Importation Rule**
> For each formula φ that is an axiom or theorem of T, add to object theory the truth $T \models \varphi^*$, where φ^* is the result of replacing every well-defined singular term κ in φ by the indexed term κ_T.

So, for example, if *zero* ('0') is the primitive constant of Peano Number Theory (PNT) and *number* ('N') is a 1-place primitive predicate of PNT,

then the axiom 'Zero is a number' of PNT would be imported into object theory as: $\text{PNT} \models N0_{\text{PNT}}$. This new sentence of object theory asserts: in PNT, the 0 of PNT exemplifies being a number.

We are taking the view that a mathematical theory is constituted by its theorems (where the theorems include the axioms). It is common nowadays for logicians to identify a theory T with the set of its theorems. The theorems of PNT, for example, define what that theory is. Thus, the axioms and theorems are jointly *constitutive* of PNT, and from this, we believe we are justified in thinking that the claim $\text{PNT} \models N0_{\text{PNT}}$, and other similarly imported truths, might be considered analytic truths, though nothing much of any consequence for the present paper hangs on whether or not the imported truths are in fact analytic. In what follows, we shall say that when we apply the Importation Rule, the axioms and theorems of a theory T become imported into object theory *under the scope of the theory operator.*

Since all of the theorems of a theory T are imported, our procedure validates the following Rule of Closure:

Rule of Closure
    If $p_1, \ldots, p_n \vdash q$, and $T \models p_1, \ldots, T \models p_n$, then $T \models q$

In other words, if q is derivable as a logical consequence from $p_1, \ldots, p_n$, and $p_1, \ldots, p_n$ are all true in T, then q is true in T. Such a rule confirms that reasoning within the scope of the theory operator is classical.

Now that object theory has been extended with these new (analytic) truths, we may use the following axiom to theoretically identify the denotation of any well-defined singular term κ_T of mathematical theory T:

Reduction Axiom for Individuals
    $\kappa_T = \imath x(A!x \,\&\, \forall F(xF \equiv \ T \models F\kappa_T))$

This asserts that the κ of theory T (κ_T) is the abstract object that encodes exactly the properties F satisfying the condition: in theory T, κ_T exemplifies F. It is important to note that this is *not* a definition of κ_T, but rather a principle that identifies κ_T in terms of data in which κ_T is used. Here are two examples, where 0_{PNT} denotes the zero of Peano Number Theory (PNT) and $\emptyset_{\text{ZF}}$ denotes the null set of Zermelo-Fraenkel set theory (ZF):

$0_{\text{PNT}} = \imath x(A!x \,\&\, \forall F(xF \equiv \ \text{PNT} \models F0_{\text{PNT}}))$
$\emptyset_{\text{ZF}} = \imath x(A!x \,\&\, \forall F(xF \equiv \ \text{ZF} \models F\emptyset_{\text{ZF}}))$

An **Equivalence Theorem for Individuals** is an immediate consequence of the Reduction Axiom for Individuals, namely, that $\kappa_T F \equiv T \models F\kappa_T$.

To give a simple example of this analysis, let $\Re$ denote real number theory. By mathematical practice, we know $\vdash_\Re 2 < \pi$, and assuming λ-conversion is available in $\Re$, it also follows that $\vdash_\Re [\lambda x\ x < \pi]2$. Thus, by the Importation Rule, object theory is extended to include the following truths: $\Re \models 2_\Re < \pi_\Re$ and $\Re \models [\lambda x\ x < \pi_\Re]2_\Re$. By the Reduction Axiom for Individuals and the Equivalence Theorem for Individuals, the following are both theorems of object theory: $2_\Re F \equiv \Re \models F2_\Re$ and $2_\Re[\lambda x\ x < \pi_\Re]$. Finally, since encoding is a mode of predication and can be used to disambiguate ordinary claims of the form 'x is F', the latter theorem serves as the true reading of the unprefixed mathematical claim '2 *is* less than π', when uttered in the context of $\Re$.

This analysis of the ordinary statements of mathematics can be extended so that the terms denoting relations in mathematical statements become analyzed as well. To accomplish this, we have, in previous work, formulated the theory of abstract objects in a type-theoretic environment and then appealed to the third-order case (e.g., Zalta 2000). Thus, where i is the type for individuals, and $\langle t_1, \ldots, t_n \rangle$ is the type of relations among entities with types $t_1, \ldots, t_n$, respectively, we may reformulate the language, definitions, and axioms of object theory in a typed language, and then deploy the following typed principle:

Comprehension for Abstract Objects of Type t
    $\exists x^t(A^{\langle t \rangle}!x \,\&\, \forall F^{\langle t \rangle}(xF \equiv \varphi))$, where φ has no free x^ts

Intuitively, the theory has now been redeveloped so that at each logical type t, the domain has been divided into ordinary and abstract objects of type t, and the latter encode properties that can be predicated of objects of type t. So, for example, let R be a variable of type $\langle i, i \rangle$, $\boldsymbol{F}$ be a variable of type $\langle \langle i, i \rangle \rangle$, and $\boldsymbol{A}!$ denote the property of being abstract (type: $\langle \langle i, i \rangle \rangle$). Then we have the following third-order instance of typed comprehension:

$\exists R(\boldsymbol{A}!R \,\&\, \forall \boldsymbol{F}(R\boldsymbol{F} \equiv \varphi))$, where φ has no free Rs

This asserts the existence of an abstract relation among individuals that encodes exactly the properties of relations among individuals that satisfy φ.

Now let Π be a primitive predicate of theory T, and let the expression 'Π_T' represent the term Π *indexed* to T, where we read Π_T as: the Π (relation) of theory T. Then, we extend our rule for importing new claims into object theory as follows:

Importation Rule
For each axiom/theorem φ of T, add the truths of the form $T \models \varphi^*$, where φ^* is the result of replacing every well-defined singular term κ and well-defined predicate Π in φ by κ_T and Π_T.

Again, this validates a **Rule of Closure**.

With object theory extended by new truths, we may theoretically identify any relation Π of mathematical theory T by using the following:

Reduction Axiom for Relations
$\Pi_T = \iota R(\boldsymbol{A!}R \ \& \ \forall \boldsymbol{F}(R\boldsymbol{F} \equiv \ T \models \boldsymbol{F}\Pi_T))$

As an example, if we let $<_{\Re}$ denote the *less than* ordering relation of $\Re$, we may theoretically identify this relation as follows:

$$<_{\Re} \ = \ \iota R(\boldsymbol{A!}R \ \& \ \forall \boldsymbol{F}(R\boldsymbol{F} \equiv \Re \models \boldsymbol{F}<_{\Re}))$$

Again, an **Equivalence Theorem for Relations**, to the effect that $\Pi_T\boldsymbol{F} \equiv \ T \models \boldsymbol{F}\Pi_T$, is an immediate consequence of the Reduction Axiom for Relations.

To continue and complete our example from before, from $\vdash_{\Re} 2 < \pi$ and λ-conversion, it follows that $\vdash_{\Re} [\lambda R \ 2R\pi]<$ (the λ-expression maintains the infix notation). Thus, by the Importation Rule, object theory is extended to include the following truths: $\Re \models 2_{\Re} <_{\Re} \pi_{\Re}$ and $\Re \models [\lambda R \ 2_{\Re}R\pi_{\Re}]<_{\Re}$. By the Reduction Axiom for Relations and the Equivalence Theorem for Relations, the following are both theorems of object theory: $<_{\Re}\boldsymbol{F} \equiv \Re \models \boldsymbol{F}<_{\Re}$ and $<_{\Re}[\lambda R \ 2_{\Re}R\pi_{\Re}]$. The latter identifies one of the properties of relations encoded by the *less-than* relation of $\Re$. Indeed, if we define xyS ('x and y encode the relation S') to mean that $x[\lambda z \ zSy] \ \& \ y[\lambda z \ xSz] \ \& \ S[\lambda R \ xRy]$, then a simple ordinary (relational) statement of $\Re$, such as '2 *is* less than or equal to π', may become analyzed as $2_{\Re}\pi_{\Re} <_{\Re}$. Thus, an ordinary relational statement of $\Re$ tells us not only encoding facts about the objects denoted by the singular terms but also an encoding fact about the relation denoted by the predicate. Such an analysis is easily extended to complex statements for arbitrary mathematical theories (Zalta 2006, 2000). In addition, though function

terms in mathematical theories could be eliminated before importation, a more elegant treatment can be given.[7]

3   A Structure and Its Parts

We now show how to define a mathematical structure as a kind of abstract object given the axiomatic theory of abstract objects outlined in the previous section. We then investigate elements and relations of structures, show a precise sense in which elements are incomplete and characterize the essential properties of elements.

3.1   The Concept of a Structure

Different authors have different intuitions about what a structure is. Dedekind suggests that a structure is something which neglects the special character of its elements (1888 [1963], 68), and Benacerraf (1965, 70) similarly suggests that the elements of a structure have only mathematical properties and no others. By contrast, some structuralists (Shapiro 1997, 89) see a structure and the physical systems having that structure as being either in a type-to-token or a universal-to-instance relationship. Still other structuralists (Resnik 1981) see structures essentially as patterns. These intuitions don't offer a definition of the notion of structure, but rather constitute desiderata or features that a theory of structure must capture. We propose to identify, for each mathematical theory, an

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[7]For example, the successor function $s(\ )$ of Peano Number Theory may be analyzed as follows (where κ now ranges over indexed singular terms and S is the primitive successor relation of PNT):

$s_{\mathrm{PNT}}(\kappa) = \iota y(\mathrm{PNT} \models S_{\mathrm{PNT}}y\kappa)$

In other words, the PNT-successor of κ is the object y which, according to Peano Number Theory, succeeds$_{\mathrm{PNT}}$ κ.

In what follows, we'll make use of the following analyses of functional notation in set theory:

The unit set of κ in ZF:
$\{\kappa\}_{\mathrm{ZF}} = \iota y(\mathrm{ZF} \models \forall z(z \in_{\mathrm{ZF}} y \equiv z =_{\mathrm{ZF}} \kappa))$

The union of κ and κ' in ZF:
$\kappa \cup_{\mathrm{ZF}} \kappa' = \iota y(\mathrm{ZF} \models \forall z(z \in_{\mathrm{ZF}} y \equiv z \in_{\mathrm{ZF}} \kappa \ \lor \ z \in_{\mathrm{ZF}} \kappa'))$

These analyses go back to the Whitehead and Russell's elimination of function terms using relations and definite descriptions in *Principia Mathematica*, *30 · 01: where f is the functional relation R, $f(x)$ is defined as $\iota y(Rxy)$.

abstract object that has these features and that can rightly be called *the structure* for that theory.

We unfold our proposal by first showing how to unite the above intuitions about structures. At the beginning of Section 2.1, we saw that object theory is well-suited to defining structural elements whose only properties are their mathematical properties. These elements will play a role in our definition of a structure. Similarly, it is reasonable to analyze the type-token distinction as follows: a type is an abstract object that encodes just the properties that all of the tokens of that type exemplify in common (thus, the type abstracts away from the properties that distinguish the tokens from each other). So, for example, The Euclidean Triangle is a type that encodes only the properties that all triangular objects exemplify. While not itself a universal, The Euclidean Triangle *encodes* all the properties implied by the universal *being triangular*— that is, the properties that are exemplified by all triangular objects.[8] It thereby constitutes an imprint or pattern which is *preserved in* (by being exemplified by) all those objects that are triangular.

This way of uniting the various intuitions about what structures are leads us to the following suggestion. Structures are abstract objects that are incomplete with respect to the properties they encode; in particular, they encode only those properties that *make true* all of the theorems of the theory that define the structure. The theorems of a mathematical theory are sentences that have the semantic property of being true in virtue of the properties of a structure. But the theorems of a theory are propositions, and as we've seen, a theory can itself be identified with its objectified content. We've identified the theory T with the abstract object that encodes the truths in T (as per the **Identity of Theory T** principle). This leads us to collapse the distinction between theories and structures and identify the structure with the theory! In this way, a structure becomes precisely that abstract object that encodes only the properties needed to make true all of the theorems of its defining theory.

Therefore, assuming that we've imported the theorems of the theory T into object theory, we may define:

The structure T $\stackrel{\text{def}}{=}$ The theory T

As we shall see in Section 3.3, this definition identifies structures as incomplete objects and this helps to distinguish them from the objects found in traditional Platonism. So, on our view, the objectified content of mathematical theory T just is the structure T and that is the sense in which mathematics is about abstract structures.[9] It is a theorem that the structure T is an abstract object.

While traditional understandings of structuralism focus on mathematical entities, our view is that a structure is composed of the *truths* that organize its elements and relations. Mathematical entities are simply the means to truths concerning abstract relations in the form of theorems. They have a role to play to the extent they contribute to these truths. So the heart of a mathematical *structure* lies not in the entities, but in the properties and theorems it yields about abstract relations. Awodey captures this understanding (2004, 59):

> Rather it is characteristic of mathematical statements that the particular nature of the entities involved plays no role, but rather their relations, operations, etc.—the 'structures' that they bear—are related, connected, and described in the statements and proofs of theorems. ... [M]athematical statements (theorems, proofs, etc., even definitions) are about connections, operations, relations, properties of connections, operations on relations, connections between relations on properties, and so on.

When a mathematician establishes an isomorphism or an embedding between a known structure and one under study, it is important because it helps us to understand the properties that hold—that is, the truths—in the new structure.

Although a structure T is constituted by the truths in T, we should note that the elements are not the truths themselves. In the next subsection we shall define the *elements* and *relations* of a structure and show that they are abstracted from a body of mathematical truths simultaneously with the structure itself.

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[8]We note, but do not prove here, that these kinds of facts position the theory to avoid the reconstructed third-man objection for structuralism found in Hand 1993. As we shall see, a structure will not exemplify itself, since it will not itself be a universal. For a full discussion of how object theory solves the third-man problem, see Pelletier & Zalta 2000.

[9]Note that distinct axiomatizations of a mathematical theory will lead to the same structure as long as they yield the same set of theorems. Moreover the various constructions or models used to interpret a theory do not result in different *theories*. For example there is one theory of the reals with various possible set-theoretic constructions, such as Cauchy sequences, Dedekind cuts, etc. These don't therefore yield different structures!

3.2   Elements and Relations of Structures

We suggest that the elements of a structure are not to be defined model-theoretically. Model-theory typically assumes a mathematical theory of sets, and if we rely on concepts from set theory to help us identify the elements of a structure, we would have to add primitive mathematical notions (e.g., set membership) to our ontology. Such a move would leave set theory as an 'ontological dangler', since we wouldn't have a mathematics-free account of structures and their elements. So, using $=_T$ to denote the identity relation that is explicit or implicit in every mathematical theory, we define[10]

> x is an element of (structure) T $\overset{\text{def}}{=}$ $T \models \forall y(y\neq_T x \rightarrow \exists F(Fx \;\&\; \neg Fy))$

> R is a relation of (structure) T $\overset{\text{def}}{=}$ $T \models \forall S(S\neq_T R \rightarrow \exists \boldsymbol{F}(\boldsymbol{F}R \;\&\; \neg\boldsymbol{F}S))$

In other words, the elements of the structure T are just those objects which are distinguishable within T from every other object in T. Similarly, the relations of the structure T are those relations that are discernible within T from every other relation in T.[11] This defines the elements and relations of a structure to be entities that encode only their mathematical properties (where these include any relational properties they may bear to other elements and relations of their structure). This, we claim, captures Dedekind's idea when he says 'we entirely neglect the special character of the elements, merely retaining their distinguishability and taking into account only the relations to one another' (Dedekind 1888 [1963], 68). Object theory presents us with a clearly defined method of abstraction, and given our application of it, the elements and relations

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[10]Note that we are in agreement with Ketland 2006 (311) and Leitgeb & Ladyman 2008 (390, 393–4) who allow for a primitive notion of identity. They take identity to be an 'integral' component of a structure and accept primitive identity facts. We use the identity symbol indexed to theory T to express these facts.

[11]This corrects an error in Zalta 2000 (232), where an object x of theory T was defined more simply as: $\exists F(T \models Fx)$. It is clear from the discussion in Zalta 2000 (233, 251–2) that object theory was being applied to the analysis of the *well-defined* singular terms and predicates of a mathematical theory. But the definition of 'object of theory T' used in that paper wasn't restrictive enough to ensure this. It will become apparent, in Section 4.4 below on indiscernibles, how our understanding of the elements and relations of a structure, as defined in the current text, plays a role in our response to the issue of whether indiscernibility poses a problem for structuralism.

of a structure are abstract objects.[12] Unlike Dedekind, however, we've defined the elements and relations of a structure without appealing to any mathematical notions, and in particular, without appealing to any model-theoretic or set-theoretic notions.

3.3   Elements Are Incomplete

A significant issue for any structuralist view is to explain the intuition that elements of a structure are incomplete—in the sense that they have only those properties they are required to have by their governing mathematical definitions or theories. From the point of view of object theory, the elements and relations of a structure are incomplete with respect to the properties they encode, but complete with respect to the properties they exemplify. Let x^t range over abstract entities of type t, and $F^{\langle t \rangle}$ range over properties of entities of type t, and $\bar{F}^{\langle t \rangle}$ denote the negation of the property $F^{\langle t \rangle}$, that is, $[\lambda y^t \; \neg F^{\langle t \rangle} y]$. Then we may define

> x^t is incomplete $\overset{\text{def}}{=}$ $\exists F^{\langle t \rangle}(\neg xF \;\&\; \neg x\bar{F})$

So if there is a property such that neither it nor its negation are encoded by an object, we say that object is incomplete. Clearly, the examples of mathematical objects and relations that we've discussed—such as $2_\Re$, $\emptyset_{\text{ZF}}$, $<_\Re$, etc.—are incomplete in this sense. Consider the more general case of an arbitrary element κ_T of the structure T. Given the Equivalence Theorem for Individuals, that for any property F, $\kappa_T F \equiv T \models F\kappa_T$, it should be clear that there is a property (e.g., $E!$) making κ_T incomplete.[13]

Notice that the structuralist slogan 'Mathematical objects possess only structural (relational) properties' is ambiguous with respect to the theory of abstract objects. The notion 'possess' can be represented by two different forms of predication, exemplification and encoding. The slogan is false when 'possess' is read as 'exemplifies', yet true when read as 'encodes'.

This immediately undermines two classical objections that have been put forward against structuralist accounts of mathematical objects. Recall that Dedekind understood numbers as having no special character other than their relational properties. This is one of the most important

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[12]See Reck 2003, for a thorough discussion of what Dedekind had in mind in developing his notion of a simply infinite system.

[13]To see this, note that real number theory $\Re$ does not include the concept of being concrete ($E!$). So, it is not the case that $\Re \models E!2$ nor is it the case that $\Re \models \bar{E}!2$. Thus, in object theory we have $\neg 2_\Re E! \;\&\; \neg 2_\Re \bar{E}!$, showing that $2_\Re$ is not complete.

ways in which Dedekind's structuralism is distinguishable from traditional Platonism, for the latter doesn't explicitly endorse abstract objects that are in any sense incomplete. Russell (1903) objected to Dedekind's understanding of the numbers by saying:

> [I]t is impossible that the ordinals should be, as Dedekind suggests, nothing but the terms of such relations as constitute a progression. If they are to be anything at all, they must be intrinsically something; they must differ from other entities as points from instants, or colours from sounds. What Dedekind intended to indicate was probably a definition by means of the principle of abstraction ... But a definition so made always indicates some class of entities having ... a genuine nature of their own. (249)

He goes on:

> What Dedekind presents to us is not the numbers, but any progression: what he says is true of all progressions alike, and his demonstrations nowhere—not even where he comes to cardinals—involve any property distinguishing numbers from other progressions. ... Dedekind's ordinals are not essentially either ordinals or cardinals, but the members of any progression. (Russell 1903, 249–51)

From the point of view of object theory, Russell's conclusion—that it is impossible that the ordinals should be nothing but the terms of such relations as constitute a progression—doesn't follow, if understood as an objection to the idea of special incomplete structural elements. Nor does Benacerraf's (1965) conclusion follow:

> Therefore, numbers are not objects at all, because in giving the properties (that is, necessary and sufficient) of numbers you merely characterize an *abstract structure*—and the distinction lies in the fact that the 'elements' of the structure have no properties other than those relating them to other 'elements' of the same structure. (70)

Clearly, $2_\Re$, as identified above, is an object despite its encoding only the relational properties to other elements of the structure $\Re$. Benacerraf's argument fails for structural elements that both encode and exemplify properties.

Our notion of incompleteness undermines other alleged counterexamples to the structuralist view that mathematical objects are incomplete. For example, Shapiro (2006, 114) and Linnebo (2008, 64) both note that a mathematical object can't have only the mathematical properties defined by its structure—since, for example, 3 has the property of being the number of Shapiro's children, being one's favorite number, being abstract, etc. From the point of view of object theory, however, these are not counterexamples but rather *exemplified* properties. The examples are consistent with there being objects that are incomplete with respect to their encoded properties.[14] Thus, full retreat from incompleteness isn't justified.

Finally, note that the notion of incompleteness defined above applies also to mathematical relations. We take this to be a way to understand *indeterminacy* in mathematics. Philosophers have remarked on the fact that many of our mathematical concepts are simply indeterminate. For example, given that the Continuum Hypothesis is independent of ZFC, it is claimed that the notion of set and set membership is indeterminate (see, e.g., Field 1994). On our view, this indeterminacy has a natural explanation and analysis, since it is captured by the idea that $\in_{\text{ZFC}}$ is incomplete with respect to the mathematical properties of relations that it encodes.

3.4   Essential Properties of Elements

The question of the essential properties of the elements of a structure has recently been the subject of some attention. After considering whether the natural numbers have any non-structural necessary properties or whether they have any non-structural mathematical properties, Shapiro notes (2006,

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[14]Actually, in object theory one has to be somewhat careful in asserting what properties there are. It is straightforward to represent, in object theory, the claims that the natural number 3 and the Peano Number 3 both exemplify being abstract, and exemplify being one's favorite number, but the theory suggests that the way in which the natural number 3 is the number of Shapiro's children is different from the way in which the Peano Number 3 is the number of Shapiro's children. In the former case, the natural number 3 encodes the property of being a child of Shapiro (because the natural number 3 is defined, following Frege, as the object that encodes all and only those properties that have exactly three exemplification instances), whereas in the latter case, one has to define a 1-1 correspondence between the first three elements of PNT and the objects exemplifying the property of being a child of Shapiro. See Zalta 1999 for a full discussion of how object theory reconstructs the natural numbers by following Frege's definitions in the *Grundgesetze*.

115) that the official view of his 1997 book is that individual natural numbers do not have any non-structural *essential* properties. He then says:

> Kastin and Hellman point out that, even so, numbers seem to have some non-structural essential properties. For example, the number 2 has the property of being an abstract object, the property of being non-spatio-temporal, and the property of not entering into causal relations with physical objects. ... Abstractness is certainly not an *accidental* property of a number—or is it? (2006, 116)

He goes on to note that abstractness is not a mathematical property, and then wonders whether the abstractness of 2 follows from its characterization as a place in the natural number structure plus some conceptual and metaphysical truths (116–7). At one point, there is an extended discussion (117–20) on whether abstractness is a contingent property of structures and their objects.

On the present analysis, it is clear that Shapiro is driven to such considerations about what are the essential and necessary properties of the numbers, in part, because he doesn't have the means of distinguishing the essential properties of the numbers from their necessary properties. This is precisely the treatment developed in detail in Zalta 2006. There it is noted that in the context of object theory, one can easily distinguish the necessary properties of abstract objects from their essential properties. Zalta first notes, as have others, that the traditional definition of 'F is essential to x' has an otiose clause, namely, the antecedent in the modalized conditional: $\Box(E!x \rightarrow Fx)$. Such a definition, on a traditional view of mathematical objects as necessary existents, simply reduces the essential properties of mathematical objects to their necessary properties.[15] If we simply get rid of the otiose clause so as to retain a definition of the necessary properties of an abstract object x, we may distinguish the necessary from the essential properties of abstract objects (using x here as a restricted variable), defining

$$x \text{ exemplifies } F \text{ necessarily} \quad \overset{\text{def}}{=} \quad \Box Fx$$
$$F \text{ is essential to } x \quad \overset{\text{def}}{=} \quad xF$$

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[15]Zalta (2006) also notes that there are several notions of essential property that apply to ordinary objects, and that these are to be defined differently from those that apply to abstract objects. See his definitions of weakly essential, strongly essential, and necessary properties of ordinary objects in 2006 (Section 3, 678–85).

These definitions make it clear that encoded properties are more important to the identity of the elements of a structure: such encoded properties are the ones by which they are identified via the Reduction Axiom for Individuals and by which they are individuated by the definition of identity for abstract objects. These essential (encoded) properties are the ones by which they are conceived within our mathematical theories. It is not the necessarily exemplified properties but the *encoded* properties that make abstract objects the objects that they are.

Thus, the examples of Kastin and Hellman that concerned Shapiro in the passage quoted above are not compelling. The following properties are all necessary without being essential: being abstract, being non-spatiotemporal, not entering into causal relations with physical objects, not being a building, not being extended, etc. We need not carry on an extended discussion of whether abstractness is an essential or contingent property of the elements of structures. It is a false dichotomy. Abstractness is *provably* non-essential (yet remains necessary), and this *validates* one of Shapiro's preferred solutions, when he says 'To summarize, on the resolution in question, we ... claim that the purported non-mathematical counterexamples, like abstractness, are not essential to the natural numbers' (2006, 120). Moreover, we may retain and give precision to Shapiro's original thesis in 1997, namely, that by defining the properties essential to the elements of a structure as those they encode, they have only their mathematical, structural properties essentially. For a full and complete discussion, including an explanation of why it is essential to {Socrates} that it have Socrates as an element, without the property of being an element of {Socrates} being essential to Socrates, see Zalta 2006.

Additional concerns from Hellman are also readily dispatched. Consider the following passage in Hellman 2001 (193):

> We say that Dedekind described the natural numbers, that Shapiro referred to the number 7, that 9 enumerates the planets, etc. The idea of objects with 'only these properties', specified in some axioms, seems incoherent. Perhaps what is intended is that the positions have only those essential properties required by the defining axioms. But even this cannot be quite right: surely places are non-concrete, for example, and necessarily so. But mathematical axioms say nothing about such matters.

There is no incoherency of the kind Hellman suggests, given objects that

have a dimension in which they are incomplete, and moreover, it is *exactly* right to say numbers 'have only those essential properties required by the defining axioms'.

From our definition of the elements of a structure T (Section 3.2), it is clear that the identity conditions for such elements are given by the right disjunct of the definition of $x = y$ in Section 2.1: the elements of structures may be identified whenever they encode the same properties. We may legitimately reject Keränen's (2001, 2006) stipulation that a structuralist can't appeal to properties denoted by λ-expressions containing singular terms that themselves denote elements of systems or 'places' in structures. He suggests (2001) that such constraints are needed so as to allow the realist structuralist 'to say both that (a) one can individuate places without appealing to elements in any particular system, and yet (b) in some sense places are individuated by the very relations their occupants have to one another' (317). But if we take the 'places' in structures to be elements of a structure (as this is defined above), we see no reason to accept those constraints. For clearly the realist structuralist, if she is to be able to say, with Dedekind, Benacerraf, etc., that the elements of a structure have no properties other than those relational properties that they bear to other elements of the structure, should also be allowed to say that the element 1 of Peano Number Theory 'has' such properties as being the successor$_\mathrm{PNT}$ of the element 0 of Peano Number Theory, that is, that 1_PNT encodes $[\lambda x\ S_\mathrm{PNT} x 0_\mathrm{PNT}]$. Thus 1_PNT's identity comprises such relational properties.

Nevertheless, we are in full compliance with Keränen's requirement (2001, 312 and 2006, 147) that realist structuralism offer an account of identity that fills in the blank of his 'identity schema' (IS). Since Keränen admits that the axiom of Extensionality (2006, 152) constitutes a non-trivial account of the identity of the objects in Zermelo-Fraenkel set theory, he must admit that our definition of identity, which has as a theorem that $\forall x, y[(A!x\ \&\ A!y\ \&\ \forall F(xF \equiv yF)) \to x = y]$, counts as a non-trivial account of the identity of abstract objects, and hence, of the elements of structures.[16] And, as noted above, we aren't smuggling in any illicit model-theoretic facts about the places in structures (or elements of systems) when identifying them as we have done.

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[16]This generalizes to the relations of structures as well. It is a theorem that $\forall R, S(\boldsymbol{A}!R\ \&\ \boldsymbol{A}!S \to (\forall \boldsymbol{F}(R\boldsymbol{F} \equiv S\boldsymbol{F}) \to R = S))$.

4   Consequences of the View

We now show how to resolve the issues (1)–(4) noted in Section 1 as consequences of the definitions from the previous section. We show, in turn, that the elements of different structures are different objects, that a structure and its elements ontologically depend on each other, that haecceities of abstract and mathematical objects don't exist, and that the problem of indiscernibles has a natural solution.

4.1   Elements of Different Structures Are Different

In Resnik 1981, it is claimed that when we fix the occurrence of a pattern, for the purposes of giving a semantics to the terms of a theory, 'there is, in general, no fact of the matter concerning whether the occurrence supposedly fixed is or is not the same as some other occurrence' (545). Resnik's 'referential relativity' here, and elsewhere (e.g., Resnik 1997, 90, 214), leads MacBride (2005) to say 'According to Resnik, there is no fact of the matter concerning whether the natural number 2 is identical to or distinct from the real number 2 (since these numbers are introduced by distinct theories)' (570).

By contrast, Parsons (1990) believes 'one should be cautious in making such assertions as that identity statements involving objects of different structures are meaningless or indeterminate' (334). And Shapiro's view has evolved on this issue, from one of cautious indeterminacy (1997) to that of claiming 'places from different structures are distinct' (2006, 128). Historically, from the remarks Frege makes in 1893/1903 (Volume II, §159ff), it is clear that he took the real number 2 to be distinct from the natural number 2. He introduced different numerals for the natural numbers and the reals, and defined both in terms of their applicability: the natural numbers are defined so as to answer questions like 'How many Fs are there?', while the real numbers are defined for use in measurement.

Our approach is based on the idea that different conceptions of objects (relations) yield different objects (relations).[17] If different properties are

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[17]We caution against a misunderstanding of our principle that if the conceptions differ the objects differ. One might think this view implies that there must be distinct sets of real numbers, for example, one arising from Dedekind's conception of them as cuts and one arising from Cantor's conception of them as equivalence classes of Cauchy sequences. But, in fact, no such conclusion is warranted. Any construction of the real numbers must satisfy the usual set of axioms which uniquely determine their

required to characterize the abstract objects x and y, or to characterize the abstract relations F and G, then the abstract objects or relations so characterized encode different properties and thus are different. To think otherwise is to suppose that abstract objects and relations are somehow out there, independent of our theories of them, waiting to be discovered. This is another way in which our view differs from traditional Platonism.

With this in mind, object theory sustains the view Shapiro adopted in 2006 (128–9) by way of the following, *principled* resolution of the issue before us: if the structures are different, the properties encoded by the elements and relations of the structure will be different, and thus the elements and relations themselves will be different. The natural number 2 is not the same as the number 2 of Real Number Theory, and indeed, the natural number 2 is distinct from the number 2 of Peano Number Theory (for the reason, see footnote 14).

Moreover, our theory yields the conclusion that neither the natural number 2 nor the number 2 of PNT are identical to the third element of any ω-sequence of sets that can be defined in any sufficiently strong set theory. The elements of ω-sequences have a *set-theoretic* structure, and thus set-theoretic relational properties, not shared by the natural numbers or the Peano numbers.

Notice how this offers a response to MacBride's (2005) objection concerning Shapiro's (then forthcoming) 2006 position. MacBride notes that Shapiro's position has 'two key assumptions: (i) the same object cannot belong to different structures; (ii) mathematical objects of different kinds belong to different structures' (2005, 579). MacBride says in each case that 'nothing has been established to preclude' either the possibility that an object belongs to more than one structure or the possibility that 2_{natural} is $2_{\Re}$ (579). One gets the sense that MacBride believes that from the perspective of mathematical practice, 2_{natural} ought to be the same as $2_{\Re}$. But MacBride's reflections don't get a purchase in the present theory. Our view doesn't assume (i) and (ii); rather, they are principled consequences of the analysis. We have grounded reasons for denying that 2_{natural} is $2_{\Re}$. One might think that these versions of 2 should be collapsed because there ought to be one 'true' universe of mathematics. But we think our view—which keeps the objects of the various theories separate—

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properties and relations. Thus, there is only one domain of real numbers: Dedekind cuts and equivalence classes of Cauchy sequences are different conceptions of how to *represent* real numbers in set theory, not different conceptions of the real numbers.

is closer to mathematical practice. Similarly, Awodey (2004, 56) holds:

> As opposed to this one-universe, 'global foundational' view, the 'categorical-structural' one we advocate is based instead on the idea of specifying, for a given theorem or theory only the required or relevant degree of information or structure, the essential features of a given situation, for the purpose at hand, without assuming some ultimate knowledge, specification, or determination of the 'objects' involved. The laws, rules, and axioms involved in a particular piece of reasoning, or a field of mathematics, may vary from one to the next, or even from one mathematician or epoch to another. The statement of the inferential machinery involved thus becomes a (tacit) part of the mathematics ... Thus according to our view, there is neither a once-and-for-all universe of all mathematical objects, nor a once-and-for-all system of all mathematical inferences.

Consequently, we see no reason to accept MacBride's suggestion (in the same passage) that we have more motivation to multiply types of relations (i.e., postulating natural successor, real successor, etc.) than to multiply different kinds of objects. As far as we know, there is no mathematical practice of defining a successor relation on the reals. Moreover, we would argue that there is, in any case, a fluidity to this practice of embedding and identifying structures that is part and parcel of mathematical practice. The fact that we can define, in object theory, a mapping that embeds the natural numbers in the reals doesn't imply that we have to identify structures at the ontological level.

Finally, we should return to an earlier example in light of the fact our theory is also based on the idea that different conceptions of relations yield different relations. Object theory yields the theorem that $\in_{\text{ZFC}}$ and $\in_{\text{ZFC+CH}}$ are distinct relations. As soon as the axioms for ZFC+CH are formulated so as to decide the question of the Continuum Hypothesis, the axioms capture a different conception of a membership relation. Since ZFC and ZFC+CH embody different conceptions of membership, the relations conceived, $\in_{\text{ZFC}}$ and $\in_{\text{ZFC+CH}}$, are different. The latter underwrites theorems and properties of sets derived from the truth of CH while the former does not. Our work in Section 2 above leaves us with a way of understanding the distinctness of these, and similar, relations

in terms of the different properties of relations they encode.[18] Such relations are not 'out there waiting to be discovered', but are the way that our various theories of them describe them to be.

4.2   Ontological Interdependence

The issue of ontological dependence is sometimes thought to distinguish structuralism from Platonism. Platonic mathematical objects allegedly exist independently of each other and independently of any mathematical structures, while on some structuralist conceptions, mathematical objects are thought to depend ontologically on their structure or on each other. An early version of the view might be found in Resnik (1982, 95), when he claims 'Mathematical objects ... have their identities determined by their relationships to other positions in the structure to which they belong'. But this is not as explicit as Shapiro, who says 'The structure is prior to the mathematical objects it contains' (1997, 78) and elsewhere says

> The number 2 is no more and no less than the second position in the natural number structure; and 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as positions in this structure, neither number is independent of the other. (2000, 258)

And more recently, Shapiro, having quoted the passage from 1997 just cited, writes, 'With these passages, I said (or meant to say) that a given *structure* is ontologically or metaphysically prior to its places' (2006, 142). Linnebo (2008, 67–8) is even more explicit about the issue, since he distinguishes the thesis, that each mathematical object in a mathematical structure depends on every other object in that structure, from the thesis that each mathematical object depends on the structure to which it belongs. Linnebo goes on to argue that there are some cases where both

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[18]Note that the infamous Julius Caesar problem (Frege 1884, §66, §68) doesn't affect the present theory: no context of the form '$\#F = x$' is left undefined for arbitrary x, unlike in Frege's theory. In object theory, $x = y$ is generally defined, and the substitution instances can be terms denoting any objects whatsoever. So the theory provides clear truth conditions for the claim 'The number of Fs is Julius Caesar' (i.e., $\#F = c$), namely, either $\#F$ and c are both ordinary objects that necessarily exemplify the same properties or they are both abstract objects that necessarily encode the same properties. We claim, therefore, that the various dimensions of the Caesar problem discussed in MacBride 2006b (metaphysical, epistemological, semantical) get no traction on the above account.

dependence claims hold, for example, in some group structures (2008, 74ff). However, he suggests that in the case of set theory, both dependence claims fail in the upward direction, since the elements of a set S do not ontologically depend either on S or on the entire structure of sets (2008, 72ff). He agrees, however, that in the downward direction, a set S does ontologically depend on its elements (2008, 72). Linnebo arrives at his set-theoretic counterexamples to the dependence claims by way of considerations from the standard conception of sets.

But the present theory yields a different conclusion: a structure and its relations and elements ontologically depend upon one another, and the elements of a given structure depend on each other in the sense that they are all abstractions governed by the same principles. One can see why the present theory yields these conclusions: the structure and its relations/elements all exist as abstractions grounded in facts of the form $T \models p$ (or even more precisely, in facts of the form *In theory* T, $\varphi(\Pi_1, \ldots, \Pi_n, \kappa_1, \ldots, \kappa_m)$, where the Π_i are relation terms and κ_j are singular terms occurring in φ). The very definition of *the structure* T given above grounds the identity of T in such facts, and the same holds of the Reduction Axioms for the denoting terms. If one wants a metaphor that captures the idea, then think of how office-buildings and their offices ontologically depend upon one another: the office-building doesn't exist as an office-building without the offices in it, and the offices don't exist without the office-building.

Our conclusions differ from Linnebo (2008, 72ff) for several reasons. One is that Linnebo relies on Fine's 1994 notions of essence and ontological dependence; by contrast, our theory of essence derives from Zalta 2006. A second is that the asymmetry in pure set theory that Linnebo argues for (a set depends on its members but the members don't depend on the set) may get its purchase from his focus on what might be called 'unit set theory': Null Set, Extensionality, and Unit Set Axiom, that is, $\forall x \exists y \forall z (z \in y \equiv z = x)$. If you restrict your attention to unit set theory, no reference to large cardinals, for example, is made when individuating unit sets. This might lead one to believe that no set depends on the universe of sets. But, of course, it is worth pointing out here that the individuation of any set, whether in unit set theory or full ZF, goes by way of Extensionality, which quantifies over every set. Thus, there is some reason to think that the individuation of both $\emptyset_{ZF}$, $\{\emptyset\}_{ZF}$, or indeed, of any object that encodes the property Set_{ZF}, is defined by the structure

ZF.[19]

Note how the present position avoids the circularity objections concerning ontological dependence raised in Hellman 2001 and considered in MacBride 2006a. Hellman considers the (modal) structuralist view that endorses (the very possibility of) an abstract structure in which the positions are 'entirely determined by the successor function φ, and derivative from it ... ' (194). He goes on to note that 'if the relata are not already given but depend for their very identity upon a given ordering, what content is there to talk of 'the ordering'?' (194). He concludes there is a 'vicious circularity: in a nutshell, to understand the relata, we must be given the relation, but to understand the relation, we must already have access to the relata' (194). But no such vicious circularity exists for the present position, for the identities of the structure itself and its relational and objectual elements are abstracted from, and thus grounded in, real invariances that exist in the use of singular terms T (for theories) and in the use of relational terms $\Pi_1, \ldots, \Pi_n$ and singular terms $\kappa_1, \ldots, \kappa_m$ in facts of the form *In theory* T, $\varphi(\Pi_1, \ldots, \Pi_n, \kappa_1, \ldots, \kappa_m)$. There is no problem with circularity here.

4.3   No Haecceities

The issue of whether the elements of a structure have haecceities comes up frequently in discussions of structuralism. For example, Shapiro writes (2006, 137):

> What reason is there to think that the realm of properties and propositional functions is up to the task of individuating each and every object? Unless, of course, there are haecceities, in which case the identity of indiscernibles is trivially true, and not very interesting.

Keränen replies (2006, 156) that Shapiro *must* accept haecceities.

Again, we shall not rehearse the arguments here. Instead, we develop theoretical reasons for thinking that abstract objects (and thus, the elements of a structure) do not have haecceities. First, note the weaker fact

that the present theory does not guarantee that haecceities for abstract objects exist. Identity (simpliciter) was defined (Section 2.1) in terms of encoding formulas. Neither '$[\lambda xy\ x=y]$' nor '$[\lambda x\ x=a]$' are well-defined; when the defined notation '$=$' is eliminated, encoding formulas appear in the definiens. So we can't use these formulas inside λ-expressions to form haecceities. Could one extend the theory by adding such expressions and asserting that they denote relations? Actually, no, and metatheoretical considerations show why this is so. For the remainder of the present paragraph only, let us appeal to ZF set theory for the purposes of modeling the theory of abstract objects. Models of the theory developed by Dana Scott and by Peter Aczel have been reported in previous work (see Zalta 1983 and 1999, respectively). Intuitively, in these models, abstract objects can be modeled by sets of properties, and Comprehension for Abstract Objects is made true by the fact that the domain of abstract objects is identified with the power set of the domain of properties. But then one can't, for each distinct set b of properties, formulate a distinct property $[\lambda x\ x=b]$; that is, there would be a violation of Cantor's theorem if distinct sets of properties b and c could always be correlated with distinct properties $[\lambda x\ x=b]$ and $[\lambda x\ x=c]$, for that would constitute a 1-1 correlation from the power set of the set of properties with a subset of the set of properties. This explains why the present theory disallows haecceities for abstract objects.

It would serve well to describe a few deeper theoretical facts about object theory as we prepare for our discussion of indiscernibles in Section 4.4. Recall that the left disjunct of the definition of $=$ is free of encoding formulas. Indeed, the following λ-expression, $[\lambda xy\ \forall F(Fx \equiv Fy)]$, is perfectly well-defined and, by Comprehension, denotes a relation. It is an interesting fact about $[\lambda xy\ \forall F(Fx \equiv Fy)]$ that, as an equivalence relation, it is well-behaved only with respect to ordinary objects. Not only is it provably reflexive, symmetrical, and transitive on the ordinary objects, but it is also provable that whenever there are ordinary objects d, e such that $d \neq e$, then $[\lambda x\ x =_E d] \neq [\lambda x\ x =_E e]$.

This becomes significant when we consider how $[\lambda xy\ \forall F(Fx \equiv Fy)]$ behaves on abstract objects. Some interesting results trace back to the following theorem of object theory (where a, b are abstract objects and R any relation):[20]

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[19]In recent, as yet unpublished work, Linnebo and Horsten show that, at least in the case of natural set theory (i.e., set theory specified by putting restrictions either on Basic Law V or on Comprehension for Properties), it is possible to individuate sets *without* quantifying over the whole domain, by building a model of identity statements in stages.

[20]Consider an arbitrary R. By Comprehension for Abstract Objects,

$$\exists x(A!x\ \&\ \forall F(xF \equiv \exists y(A!y\ \&\ F=[\lambda z\ Rzy]\ \&\ \neg yF)))$$

$$\forall R \exists a, b(a \neq b \ \& \ [\lambda x \ Rxa] = [\lambda x \ Rxb])$$

In other words, for every relation R, there are distinct abstract objects a, b such that the property of bearing R to a is the *same* property as that of bearing R to b. This interesting result has a rather natural explanation in terms of the models temporarily assumed in the paragraph above. If a, b are modeled as distinct sets of properties, then clearly we may *not* form distinct new properties $[\lambda x \ Rxa]$ and $[\lambda x \ Rxb]$, by the constraints of Cantor's Theorem explained previously.

If we apply the above theorem to the relation $[\lambda xy \forall F(Fx \equiv Fy)]$, we get the result (where a, b are abstract objects) that

$$\exists a, b(a \neq b \ \& \ \forall F(Fa \equiv Fb))$$

In other words, there are distinct abstract objects (i.e., they encode different properties) that are indiscernible from the point of view of exemplification.

What is happening here is that there are too many abstract objects for the traditional notion of exemplification to distinguish. Abstract objects don't have haecceities, and some are indiscernible with respect to the properties they exemplify despite being discernible with respect to the properties they encode. Although these indiscernible abstract objects don't actually play a role in our analysis of indiscernibles in mathematics, it is nevertheless important to have an understanding of the above facts, as will be seen in the next section.

Call an arbitrary such object k, so that k is defined by

$$\forall F(kF \equiv \exists y(A!y \ \& \ F = [\lambda z Rzy] \ \& \ \neg yF))$$

Now consider $[\lambda z \ Rzk]$. Assume $\neg k[\lambda z \ Rzk]$. Then, by definition of k,

$$\forall y(A!y \ \& \ [\lambda z \ Rzk] = [\lambda z \ Rzy] \rightarrow y[\lambda z \ Rzk])$$

Instantiate this universal claim to k, and it follows that $k[\lambda z \ Rzk]$, contrary to assumption. So $k[\lambda z \ Rzk]$. So by the definition of k, there is an object, say l, such that

$$A!l \ \& \ [\lambda z \ Rzk] = [\lambda z \ Rzl] \ \& \ \neg l[\lambda z \ Rzk]$$

But since $k[\lambda z \ Rzk]$ and $\neg l[\lambda z \ Rzk]$, $k \neq l$. So

$$\exists x, y(A!x \ \& \ A!y \ \& \ x \neq y \ \& \ [\lambda z \ Rzx] = [\lambda z \ Rzy])$$

⋈

4.4 Indiscernibles Are Not Elements

The problem of indiscernibility for structuralism arises whenever there are non-trivial automorphisms of the domains of mathematical theories. Mathematical practice reveals the existence of a variety of these non-trivial automorphisms: in the points of a dense, linear ordering without endpoints, in the points of Euclidean space, in some of the simplest graphs and groups exhibiting reflectional and rotational symmetries, in the integers under addition, in the complex plane, etc. In each of these cases of non-trivial automorphisms, there appear to be at least two distinct elements of the domain that are absolutely indiscernible, in the sense that they have exactly the same (relational) properties. Burgess 1999, Hellman 2001, and Keränen 2001 clearly describe the problem for the structuralist: if the elements of structures are identical whenever they have the same relational properties, there can't be two or more distinct indiscernibles, contrary to what appear to be the facts from mathematical practice. Shapiro (2006, 112) asks the rhetorical question, 'Indeed, every point of Euclidean space has the same relations to the rest of space as every other point. ... Do I have to say—absurdly—that there is only one point?'.

Ladyman (2005) proposes that the structuralist adopt standards weaker than absolute discernibility in response to the problem. Thus, he notes x and y are 'relatively discernible' just in case there is a formula φ in two free variables such that $\varphi(x, y)$ but not $\varphi(y, x)$ (2005, 220). Moments of time bearing the asymmetrical 'earlier than' relation would be discernible on this standard. But the standard needed for structuralist mathematics, he suggests, is that x and y are 'weakly discernible' whenever there is two-place irreflexive relation R such that Rxy. Thus, 'is the additive inverse of' is an irreflexive relation R that would weakly discern 2 and -2 in the domain of integers under addition (they bear R to each other but not to themselves), and weakly discern i and $-i$ in the complex plane (for the same reason).

MacBride (2006a) offers philosophical considerations against the idea that a structuralist can appeal to weak discernibility. Describing what he takes to be a tacit assumption in Russell 1911–12, he says:

> In order for objects to be eligible to serve as the terms of an irreflexive relation they must be independently constituted as numerically diverse. Speaking figuratively, they must be numerically diverse

'before' the relation can obtain; if they are not constituted independently of the obtaining of an irreflexive relation then there are simply no items available for the relation in question to obtain between. (67)

MacBride thus argues that in order to apply weak discernibility, you must presuppose that you have separate things that are weakly discernible. So a structuralist who relies on weak discernibility will need to explain what constitutes their numerical diversity.

Further, Ketland (2006, 309) offers counterexamples to the standard of weak discernibility by exhibiting mathematical structures with objects that aren't even weakly discernible. He cautions against abandoning the identity relation that comes with a mathematical theory (311). Ladyman, himself, has moved away from arguments based on weak discernibility by providing, with Leitgeb, some additional examples from graph theory (Leitgeb and Ladyman 2008, 392–3).

We present our solution to the issue of indiscernibles through an examination of a few cases in detail. Consider first the case of the theory of dense, linear orderings, without endpoints, as given by the axioms

$$\forall x, y, z(x < y \,\&\, y < z \to x < z) \qquad \text{(Transitivity)}$$
$$\forall x(x \not< x) \qquad \text{(Irreflexivity)}$$
$$\forall x, y(x \neq y \to (x < y \,\lor\, y < x)) \qquad \text{(Connectedness)}$$
$$\forall x, y \exists z(x < z < y) \qquad \text{(Dense)}$$
$$\forall x \exists y \exists z(z < x < y) \qquad \text{(No Endpoints)}$$

Let's call this theory, and the structure that results, D. Isn't everything in D indiscernible?

Ontologically speaking, there are no elements of D; rather, D as a structure encodes the facts about the ordering relation $<_D$ and the identity relation $=_D$ unique to that theory. From such facts, we may conclude that $<_D$ encodes such properties of relations as:

$$[\lambda R \,\forall x \neg x R x]$$
$$[\lambda R \,\forall x, y, z(x R y \,\&\, y R z \to x R z)]$$
etc.

That is, $<_D$ encodes the property of being a relation R that is irreflexive, transitive, etc. Although we can identify $<_D$ and $=_D$ as relations of the structure D (using the definition of Section 3.2), there is no abstract object that qualifies as an element of the structure D.

Consider an analogy. A novel asserts, 'General B advanced upon Moscow with an army of 100,000 men'. We think it is unreasonable to suppose that the analysis of this sentence requires that there be 100,002 characters (General B, Moscow, and 100,000 distinct men) in the novel in question. Instead, there are only 3 characters: General B, Moscow, and the army of 100,000 men. Similarly, in the case of the structure D, the theory D doesn't require that there be an infinite number of indiscernible points; all it requires is that there be two relations (namely $<_D$ and $=_D$) that *encode*—not exemplify—certain properties.

To see why this understanding is justified, note how the closing observation about structuralism in Keränen 2001 fails to find its mark in the present theory:

> In model theory one takes the various domains of discourse as given, and assumes that there is no difficulty in securing reference to objects in these domains. This, we suspect, is the reason why problems of the kind explored in this paper have so far gone virtually unnoticed. While the structuralist purports to be constructing a foundationalist account of mathematical ontology, she nevertheless remains captive to the comforting picture model theory offers. (329)

By contrast, our foundationalist account of mathematical ontology is *not* captive to the picture model theory offers. We reject the model-theoretic definition of what it is to be an object of a theory (i.e., being in the range of a bound variable of the mathematical theory), and of what it is to be an element of a structure (i.e., being in the range of a bound variable in a model-theoretic description of such a structure). But we emphasize that this is *not* to claim that the language of model theory can no longer be used or is somehow illegitimate, but only to claim that we can't draw *ontological* conclusions on the basis of the language and definitions of model theory; that theory is not ontologically basic but rather must itself be analyzed in terms of our philosophically prior system of abstract objects. Our system offers us the definitions about the elements of structures from which we may draw ontological conclusions.

We suggest, therefore, that Shapiro's rhetorical question about the points of Euclidean space should answered by saying that *nothing* exemplifies being a point of Euclidean space, though that is not to say that the property of *being a point in Euclidean space* doesn't itself encode the

properties of properties that such a property must exemplify in the theory of Euclidean spaces.

4.4.1   The Case of i and $-i$

The problem of the indiscernibility of i and $-i$ arises because there exists a non-trivial automorphism of the complex plane in which i is mapped to $-i$. To see how the problem can be precisely stated for the present view, consider first how the theory of complex numbers becomes imported into our theory. Note that usual mathematical practice is to obtain the theory of complex numbers, $\mathbb{C}$, by adding the following axiom to the axioms for $\Re$:

$$i^2 = -1$$

When the theorems of $\mathbb{C}$ are imported into the present theory, one might expect that this last axiom would be represented with such singular, functional, and relational expressions as $i_{\mathbb{C}}$, $2_{\mathbb{C}}$, $1_{\mathbb{C}}$, the exponentiation function $()^{()_{\mathbb{C}}}$, the negative function $-_{\mathbb{C}}()$, and the identity relation $=_{\mathbb{C}}$. However, for reasons that will soon become apparent, we won't index 'i'; we'll simply represent the above axiom as $\mathbb{C} \models i^2 =_{\mathbb{C}} -1_{\mathbb{C}}$, and talk simply of i, $-i$, etc., without indices. We may therefore identify the structure $\mathbb{C}$ as the abstract object that encodes exactly the properties F which are properties of the form $[\lambda x \, p]$ such that $\mathbb{C} \models p$.

Now one might object to our version of structuralism as follows:

> Objection: Your treatment of complex analysis in object theory yields the absurd theorem (in object theory) that $i = -i$.

> Purported Proof: From mathematical practice (i.e., from a known automorphism of the complex plane), we know that any i-free formula $\varphi(x)$ in the language of complex analysis (with only x free) that holds of i also holds of $-i$, and vice versa. Thus, i and $-i$ are indiscernible in complex analysis. During the importation of $\mathbb{C}$ into object theory, each formula $\varphi(x)$ defines a property F (by Comprehension, since the imported formulas have no encoding subformulas). Since i and $-i$ are elements of the structure $\mathbb{C}$, the Reduction Axiom for Individuals ensures that the indiscernibility of i and $-i$ in complex analysis becomes manifest in object theory by the following fact: $\mathbb{C} \models Fi \equiv \mathbb{C} \models F-i$. Independently, by the Equivalence Theorem for Individuals, it follows both that $iF \equiv \mathbb{C} \models Fi$

and $-iF \equiv \mathbb{C} \models F-i$. From all these facts, it then follows that $iF \equiv -iF$. Therefore, $i = -i$, by the definition of identity for abstract objects.

But this purported proof rests on a false premise. In particular, we say:

> Reply: The argument is blocked because neither 'i' nor '$-i$' denote abstract objects—they don't denote elements of the structure $\mathbb{C}$. Thus, neither the Reduction Axiom for Individuals nor the Equivalence Theorem for Individuals can be applied to derive facts expressed by sentences containing 'i' and '$-i$'.

It is well known that indiscernibles arise from symmetries (non-trivial automorphisms) of the structure. Clearly, mathematicians working *with* a structure find it useful to give names to indiscernibles. But these names don't denote elements of the structure. After all, these names are *logically arbitrary* in the same sense as names introduced by the rule of Existential Elimination. So there is nothing (i.e., no property) within the theory that distinguishes the indiscernibles from each other. Indeed, the mathematician doing complex analysis uses 'i' and '$-i$' in a way that is different from their use of '1' and '-1'. The naming of 1 and -1 is not arbitrary—one can't permute 1 and -1 and retain the same structure. So it makes sense to say that 'i' and '$-i$' do not denote objects the way that '1' and '-1' do.

The false premise in the purported proof above is that i and $-i$ are elements of structure $\mathbb{C}$. Indeed, they fail to be elements by the definitions in Section 3.1 and 3.2, where we define: *x is an element of* $\mathbb{C}$ iff $\mathbb{C} \models \forall y(y \neq_{\mathbb{C}} x \rightarrow \exists F(Fx \,\&\, \neg Fy))$. By this definition, i fails to be an element of $\mathbb{C}$, because $\mathbb{C} \models i \neq_{\mathbb{C}} -i$ but there is no property that distinguishes them. And the same reasoning applies to establish that $-i$ is not an element of $\mathbb{C}$.

Indeed, we suggest that the correct procedure for interpreting the language of $\mathbb{C}$ is as follows: before importation, eliminate the logically non-well-defined term 'i' by replacing every theorem of the form $\varphi(\ldots i \ldots)$ by a theorem of the form: $\exists x(x^2 + 1 = 0 \,\&\, \varphi(\ldots x \ldots))$; then import the result. We suggest that this is the right procedure because mathematical practice here really involves two steps: (1) add the axiom that asserts $\exists x(x^2 + 1 = 0)$, and (2) eliminate the quantifier and introduce an arbitrary name for the existentially quantified variable. Though a structuralist should be happy enough with step (1), the use of arbitrary,

non-well-defined names in step (2) is not justified ontologically. Though we are quite happy to allow mathematical practice to carry on in the usual way, our view is that a philosopher may not appeal to that practice of using arbitrary names to generate *ontological* problems.

Under this analysis, then, i and $-i$ disappear and we are left with structural properties of complex addition, complex multiplication, complex exponentiation, etc. For example, in the case of complex addition $+_{\mathbb{C}}$, for each theorem $\exists x(x^2 + 1 = 0\,\&\,\varphi(x, +))$, we can abstract out properties encoded by $+_{\mathbb{C}}$ of the form $[\lambda R\,\exists x(x^2 R1 = 0\,\&\,\varphi(x, R))]$. Similar techniques can be used for complex multiplication $\times_{\mathbb{C}}$, complex exponentiation, etc. The point is that, ontologically speaking, there is no need to worry about what constitutes the numerical diversity of i and $-i$. 'i' and '$-i$' don't denote distinct abstract objects—they are arbitrary names used by mathematicians as labels on a structural symmetry of $\mathbb{C}$.

We note here that the conclusion we've reached about how to understand the mathematician's use of 'i', though consistent with the view described in Shapiro 2008 (300), Brandom 1996 (Section 6), and Menzel (forthcoming), is based both on a theoretical definition of what it is to be an element of a structure and a counterfactual theoretical argument as to why i would fail to be an element of $\mathbb{C}$ if one were to treat it as a singular denoting term in need of an analysis.

4.4.2 Some Other Often Discussed Examples

Our view is that the problems posed for structuralism by simple cardinal structures, simple (e.g., 2-node) graphs, simple groups and the structure $\langle \mathbb{Z}, + \rangle$ are all resolved by recognizing that the model-theoretic perspective has led the discussion astray. In each case, the discussion focuses on existential claims that, within the theory, are multiply-satisfied by indiscernibles. For example, in graph theory, for a 2-node graph with no edges one can assert

$$\exists x, y(Node(x)\,\&\,Node(y)\,\&\,x \neq y))$$

In $\langle \mathbb{Z}, + \rangle$, we assert the existence of additive inverses using

$$\forall m \exists n(n + m = 0)$$

In complex analysis, we assert the existence of an imaginary number using

$$\exists x(x^2 + 1 = 0)$$

In the latter two cases, names are introduced (1 and -1, i and $-i$, etc.), yet as we've seen, these can't be understood as genuine names with unique, distinguishable denotations, but rather must be understood as arbitrary names. The formulas in which those arbitrary names occur have to be understood in expanded notation, where the names in the formula are replaced with variables bound by an existential claim conjoined to the front of the formula. Ignoring the arbitrary names, the latter two cases are handled in the same way as the theory of the 2-node graph with no edges. In all of these cases, our view is that these existential claims are true *only* relative to their respective theories. The symmetries such existential claims introduce are nothing more than higher-order structural (projective) properties of the relations of the structure.

To see this more clearly, recall the case of the structure D (dense, linear orderings without endpoints) and consider another example, namely, the cardinal three-structure ('3S') given by the axiom $\exists x, y, z(x \neq y\,\&\,y \neq z\,\&\,x \neq z\,\&\,\forall u(u = x\,\vee\,u = y\,\vee\,u = z))$. The structural analysis of 3S in object theory yields only an abstract relation $=_{3S}$ and the higher-order projective properties like the following that it encodes:[21]

$$[\lambda R\,\exists x, y, z(\neg Rxy\,\&\,\neg Ryz\,\&\,\neg Rxz\,\&\,\forall u(Rux\,\vee\,Ruy\,\vee\,Ruz))]$$
$$[\lambda R\,\exists x, y(\neg Rxy)]$$
$$[\lambda R\,\neg\exists x_1, x_2, x_3, x_4(\neg Rx_1x_2\,\&\,\neg Rx_1x_3\,\&\,\ldots\,\&\,\neg Rx_3x_4\,\&$$
$$\forall u(Rux_1\,\vee\,Rux_2\,\vee\,Rux_3\,\vee\,Rux_4))]$$

There are no elements of the structure, though there would be individual witnesses in any concrete physical system (i.e., group of concrete objects bearing relations to one another) in which there is a relation R' that exemplifies every (higher-order projective) property that $=_{3S}$ encodes. Only an insistence on the model-theoretic perspective leads one to suggest otherwise, and this, we would argue, is illegitimate when developing a philosophy of mathematics free of ontological danglers.

5 Conclusion

We believe that our solution here actually preserves many of the intuitions of the structuralists *and* the structuralist critics. The relations of

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[21]The following properties are 'higher-order' because they are properties of relations, and they are 'projective' because they are similar to the following properties, both of which may be defined as projections of the relation R: $[\lambda x\,\exists y(Ryx)]$ and $[\lambda x\,\exists y(Rxy)]$.

the theory give a mathematical structure its *structure*. Without relations, there's no structure (even if it is only a relation of identity used in small cardinal structures). An *element* of a structure must be uniquely characterizable in terms of the relations of the structure—it must be discernible. Keränen can't complain that we've used the comforting picture of model theory for doing mathematical ontology. Leitgeb and Ladyman are correct to say that the mathematical theory's identity relation is enough to support the numerical diversity of mathematical objects (2008, 396)—but only when working within the mathematical theory. When you move outside the theory itself, MacBride (2006a) is vindicated when he says that the terms of an (irreflexive) relation must be independently constituted as numerically diverse for the relation to hold.

Given the foregoing, Keränen's conclusion in the following passage is unpersuasive:

> As we have argued, however, as soon as the task of furnishing ontology is taken seriously, such a complacent attitude towards identity and reference must be rejected. And once we do reject it, we come to realize that in the case of systems with structurally indiscernible elements, the idea of treating the structure of such a system as an object in its own right is incoherent. ... Another way of putting the point is to say that Benacerraf was right all along: if mathematical entities have no properties besides the ones relating them to the other elements in the same structure, they are not properly individuated objects at all. We can now see *why* he was right. (Keränen 2001, 329)

We haven't taken a complacent attitude, yet we *can* treat the structures of these systems as objects in their own right, as long as we avoid the 'incoherency' of trying to give ontological weight to the indiscernibles that stem from structural symmetries. Thus, Benacerraf's conclusion, and Russell's conclusion about Dedekind, are incorrect in a setting such as ours, though we acknowledge that, in absence of the distinction between exemplification and encoding predication, those conclusions seemed to be correctly and reasonably drawn.

We also think Leitgeb and Ladyman (2008, 389) must take care when they write:

> Though philosophical claims about mathematical objects cannot actually be derived from descriptions of mathematical practice, the

more closely a position in the philosophy of mathematics resembles the ways in which mathematicians actually talk and reason, the more prima facie plausibility it has.

Prima facie plausibility may lead one astray. We have shown in the foregoing how philosophical claims about mathematical objects *do follow* from descriptions of mathematical practice, once those descriptions are properly analyzed. Indeed, we claim that our analysis is faithful to mathematical practice because we show how to go from the claims mathematicians make within their practice to metaphysical claims that validate the structuralist view. But one can't draw ontological conclusions directly from the mathematicians' actual talk, especially when they take logical shortcuts by using arbitrary names as if they are genuine names.

Our view also provides a way to analyze Yap's deflationist understanding of mathematical ontology. She writes:

> Dedekind, as I have characterised him, is certainly a deflationist about ontology.[14] There are such things as mathematical objects, but they are not things with a 'real existence' in a robust realist sense, being only the intensional objects of mathematics. The objects are created by the axioms, and thus have a rather 'thin' existence, since it only makes sense to talk about them with reference to their background structure. Though, in contrast to an anti-realist view, Benacerraf's first criterion has not entirely been abandoned. Mathematical statements do still refer to mathematical objects, and the properties of these objects determine the truth and falsity of such statements. Although these objects differ from physical ones in their being incomplete, the Dedekindian view does not reject the criterion of a referential semantics. (2009, 170)

On our view, the deflationary conception of mathematical objects arises when one restricts one's attention to the properties they encode.[22]

Our main point has been to provide foundations for structuralism that answer basic questions about this view of mathematics. We have shown

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[22]One need not endorse Yap's suggestion that intensional objects are not real in a robust sense. Yap's view may be a result of the restrictions of the classical logic of exemplification, in which one can't assert the existence of incomplete objects (i.e., objects x such that there is a property F such that neither Fx nor $\bar{F}x$). But, if the quantifiers in the logic of encoding are interpreted in the usual Quinean way, so as to assert real existence, our logic would allow us to assert the real existence of intensional objects that are incomplete with respect to the properties they encode.

how our foundations are grounded in basic insights about abstract objects and the value of theorems and truths over entities. We have shown how philosophical issues are resolved from theoretical considerations instead of appeals to intuitions about a particular concern or intuitions about what constitutes faithfulness to mathematical practice. We emphasize that claims within mathematical practice (existential or otherwise) can't be exported to simple facts. And we are faithful to that practice by showing how to move from descriptions of the practice to philosophical claims about structuralism. We hope to have at least provided a standard for structuralism: the transition from mathematical practice to philosophical claims can be done carefully and precisely.[23]

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