A Contraction for Sovereign Debt Models*

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Abstract

Using a dual representation, we show that the Markov equilibria of the one-period-bond Eaton and Gersovitz (1981) incomplete markets sovereign debt model can be represented as a fixed point of a contraction mapping, providing a new proof of the uniqueness and existence of equilibrium in the benchmark sovereign debt model. The arguments can be extended to incorporate re-entry probabilities after default when the shock process is *iid*. Our representation of the equilibrium bears many similarities to an optimal contracting problem. We use this to argue that commitment to budget rules has no value to a benevolent government. We show how the introduction of long-term bonds breaks the link to the constrained planning problem.

1 Introduction

This paper provides a compact characterization of the canonical Eaton and Gersovitz (1981) model of sovereign debt with one-period defaultable, but otherwise noncontingent, bonds. In particular, we show that the Markov equilibrium allocation can be characterized in a single Bellman equation that is the fixed point of a contraction mapping. This immediately yields existence and uniqueness, providing an alternative approach to the recent results of Auclert and Rognlie (2016).

The fact that the equilibrium is the solution to a Bellman equation also sheds light on the economics of the equilibrium. The dynamic programming problem shares many similarities with an optimal contracting problem between a principal and an agent that is risk averse and lacks commitment. In this sense, the equilibrium allocation has a number of efficiency properties. However, the contracting problem includes a constraint that reflects market incompleteness, highlighting

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the inefficiencies that arise due to the ad-hoc constraints on contracts imposed in the Eaton-Gersovitz tradition.

The argument has two crucial steps. The first is to consider a dual representation of the equilibrium. Specifically, we consider maximizing the value of debt subject to delivering a level of utility to the government. This yields a function $B(s, v)$, which specifies the value of debt when the exogenous state is s and the "promised utility" of the government is v. The choice variables in this program are government consumption (or net payments to the lender) in the current state as well as continuation values in successor periods, $v(s')$ for all s' in some state space S. The optimization is subject to the government's lack of commitment.

At this point, the description is a textbook principal-agent problem with complete markets and one-sided limited commitment. Hence, the second step of the argument is to ensure that the optimal allocation is consistent with incomplete markets. In particular, the value of debt next period cannot vary across state realizations in which the government repays. This requires that $B(s', v(s'))$ is invariant across all $s' \in S$ in which the government repays. This is reflected in an additional constraint that features the representative lender's value function.

The equilibrium is then characterized by an operator that maps the space of possible debt values, B, into itself. Differently from a primal approach, our dual representation allows us to show that the function B is the fixed point of a contraction operator. This immediately implies uniqueness of the equilibrium and can be used to show its existence. Uniqueness has recently been established by Auclert and Rognlie (2016), who provide a clever proof of uniqueness based on a replication argument similar to Bulow and Rogoff (1989b). One contribution of this paper is to establish uniqueness using standard recursive techniques that are taught in first-year graduate courses.

As a consequence of the dual-contracting approach, the analysis highlights certain efficiency properties of the equilibrium. The ability to commit to future debt issuances cannot improve upon equilibrium allocation, and thus fiscal rules have no value. In particular, aside from the default decision, there is no loss or gain if the government, as part of the contract, cedes future fiscal deficit choices to the lenders. There is also no coordination failure that can lead to a sub-optimal outcome, highlighting the difference between the Eaton-Gersovitz model and the closely related model of Cole and Kehoe (2000) that does feature self-fulfilling crises. The different timing assumptions in the Cole-Kehoe model are not consistent with our dual formulation.

The use of the dual-contracting approach and the efficiency properties of short-term bonds has a precedent in Aguiar, Amador, Hopenhayn and Werning (2019). That paper demonstrates how short-term financing is efficient in a sovereign debt model where uncertainty affects only the value of default (that is, the only shocks are those that affect the outside option of the government) and consumption is deterministic absent default. A contribution of the present paper is to extend this efficiency result to the standard incomplete markets environment where uncertainty affects the endowment and the equilibrium consumption process. This requires the introduction of a new constraint that restricts how continuation values can vary by state.

The efficiency of short-term bonds stand in contrast to models with long-term bonds.[1] This distinction is made transparent in our dual formulation. The relevant objective for the representative lender is the *market value* of debt. The relevant constraint from incomplete markets is that the *face value* of debt is noncontingent. Conditional on nondefault, with one-period bonds, the market value equals the face value at the time of repayment; hence, the same function B characterizes both the value and the constraint set, and the program searches for a single object that satisfies the dual Bellman equation. With long-term bonds, this is no longer the case. This provides a stark reflection of the fact that long-term bonds are subject to dilution (that is, disagreement about fiscal policy between the lender and government) and can induce multiple equilibria in the Eaton-Gersovitz model (see, for example, Aguiar and Amador (2018) and Stangebye (2018)).

The environment we study hews closely to the canonical one-period Eaton-Gersovitz models popular in the quantitative literature, such as Aguiar and Gopinath (2006) and Arellano (2008). Following the original Eaton-Gersovitz paper, we assume there is no reentry after default, but do allow for arbitrary additional punishments. This makes the deviation utilities primitives of the environment rather than equilibrium objects. The results can be extended to allow for reentry under *iid* endowment shocks, but we do not have results for general Markov processes.

In addition to long-maturity bonds, other deviations from the standard one-period-bond Eaton-Gersovitz model can lead to multiplicity. Lorenzoni and Werning (2018) consider an environment in the spirit of Calvo (1988) where a government following a fiscal rule is vulnerable to self-fulfilling shifts in the interest rate of its debt. Ayres, Navarro, Nicolini and Teles (2018) consider an environment in which lenders offer the sovereign an interest rate before the sovereign commits to a borrowing amount and show how this auction protocol generates multiplicity with one-period bonds.[2] This highlights the role played by the Eaton-Gersovitz assumption that the government is strategic in regard to the impact debt issuance has on prices.

Passadore and Xandri (2018) show that multiplicity can arise in the Eaton-Gersovitz model if the government is prevented from holding assets. We therefore place no bounds on the space of assets. To ensure that B is bounded, we restrict that consumption lies in an arbitrarily large but compact set; this places a de facto upper bound on the government's value achieved in any

[1]See, for example, Chatterjee and Eyigungor (2012), Hatchondo and Martinez (2009), and the already cited Aguiar et al. (2019).

[2]Lorenzoni and Werning (2018) also show how multiplicity arises with long-term bonds when the sovereign endogenously chooses its expenditures but faces constraints in its ability to reduce the deficit when spreads are high. Also in the spirit of Calvo (1988), Ayres et al. (2018) show how the same multiplicity arises when the sovereign commits to financing a given fiscal deficit level *before* the market interest rate is determined.

equilibrium and hence ensures our equilibrium operator maps bounded functions into bounded functions. As is standard, the government has time-consistent preferences. Therefore, the analysis does not extend to models in which the government has quasi-geometric preferences, such as Aguiar and Amador (2011) or Alfaro and Kanczuk (2017).

The paper is organized as follows. Section 2 introduces the environment and provides a basic characterization of equilibria. Section 3 shows that the equilibrium is a fixed point of a contraction mapping. Section 4 discusses the efficiency of the unique equilibrium and why fiscal rules are not useful. We also discuss why introducing long-term bonds breaks the usefulness of our dual approach. Section 6 shows how the analysis can be extended to the case of reentry after default (as long as the shock process is *iid*). Section 7 concludes. Appendix A collects all of the remaining proofs not included in the main text.

2 The Eaton-Gersovitz Model

Environment. Let us consider the standard sovereign debt model with one-period bonds, originally introduced in Eaton and Gersovitz (1981). Time is discrete, indexed by $t = 0, 1, \ldots,$ and there is a single tradable good that is the numeraire. There is a small open economy with a government that makes all consumption, debt-issuance, and default decisions. There is also a international financial market, populated by risk-neutral investors that demand a gross interest rate of $R > 1$.

Let s_t denote the exogenous state vector of the economy at time t, and let s^t denote its history, up to and including period t. The exogenous state vector is $s_t \in \mathbb{S}$ where $\mathbb{S}$ is a finite set. Let $\pi(s'|s)$ denote the probability of state s' next period given s this period.

The timing within a period is as follows. The small open economy starts with an inherited debt level b, all of which matures in the current period. Nature then draws the current state s. After observing s, the government decides whether to repay its maturing debt or default. If it does not default, the government receives an endowment $y(s)$, auctions new bonds, pays off the maturing debt, and consumes. We restrict attention to Markov-perfect equilibria of the game between the government and the international financial markets.

We impose the following assumption on technology:

Assumption 1. *The transition probabilities are such that* $\pi(s'|s) > 0$ *for all* $(s, s') \in \mathbb{S} \times \mathbb{S}$. *In addition,* $\min_{s \in S} y(s) \equiv \underline{y} > 0$ *and* $\max_{s \in S} y(s) \equiv \overline{y} < \infty$.

Let $V^R(s, b)$ denote the equilibrium value to the government if it chooses to repay its debt given (s, b). If it defaults, we assume that the payoff to the government is exogenous and equal to $V^D(s)$. Upon default, we assume that the lenders receive a payoff of zero. The default payoff for the government may depend on the exogenous state, but does not vary with the amount of debt at the time of default. Importantly, in this formulation, the default payoff is a primitive and does not depend on the other aspects of the equilibrium.[3] Strategic default implies that the government defaults if $V^R(s,b) < V^D(s)$. We assume, as it is standard, that the government repays when indifferent, that is, when $V^R(s,b) = V^D(s)$.

Government Optimization. Let us now consider the government's problem. To rule out Ponzi schemes, we impose that $b \le \overline{B}$, where we assume that $\overline{B}$ exceeds the present value of the maximal endowment: $\overline{B} > \overline{y}R/(R-1)$. We let $u(c)$ denote the utility flows received by the government given an associated expenditure level c, and we assume that the government evaluates alternative spending plans discounting future expected utility flows with an exponential factor $\beta < 1$.

We assume consumption is chosen from a compact set: $[0, \overline{c}]$. As will become clear below, an upper bound on consumption allows us to focus on a finite threshold for assets in any equilibrium.[4] The bound on consumption can be set to any arbitrarily large finite number, and in particular we make the natural assumption that it is always possible to consume the present value of the endowment in any period:

Assumption 2. *$\overline{c} > R\overline{y}/(R-1)$.*

Define the upper bound on the value function by $\overline{V} \equiv u(\overline{c})/(1-\beta)$.

We assume that the government always prefers to default rather than consume zero in the current period:

Assumption 3. *There exists a $\underline{c} \in (0, \overline{c})$ such that*

$$u(\underline{c}) + \beta\overline{V} < \min_{s \in \mathbb{S}} V^D(s). \tag{1}$$

This last assumption implies that, conditional on repayment, consumption is bounded away from zero — a feature that is helpful when proving the strict monotonicity of the equilibrium value function.

We now discuss the government's problem conditional on repayment. The government faces an equilibrium price schedule q that maps the current state and newly issued debt $b' \in \mathbb{R}$ into $[0, R^{-1}]$, where R is the world risk-free rate. Given the fact that the government commits to repaying maturing debt b prior to auctioning b', q is not a function of b.[5] Hence, the budget

[3] This is consistent with autarky and exogenous output losses as punishments for default, but it does not admit reentry to financial markets, an extension that we discuss later on.

[4] This assumption helps guarantee that the dual operator we provide below maps bounded function into bounded functions.

[5] See Cole and Kehoe (2000), and the discussion in Aguiar and Amador (2014), for the implications of an alternative timing.

constraint of the government in state (s, b), if the government decides to repay, is $c \leq y(s) - b + q(s, b')b'$.

We let $\mathbb{X}_{feas}(s)$ denote the set of debt levels that are feasible to repay in state $s \in \mathbb{S}$ and let $B^F(s)$ denote its maximum:

Definition 1. *In any equilibrium, we let $\mathbb{X}_{feas}(s)$ to be defined as:*

$$\mathbb{X}_{feas}(s) = \{b | \text{ there exists } b' \leq \overline{B} \text{ such that } y(s) + q(s, b')b' - b \geq 0\}.$$

And we let $B^F(s)$ represents the largest amount of debt that is feasible to repay in state s:

$$B^F(s) \equiv \sup \mathbb{X}_{feas}(s) \tag{2}$$

Note that $\mathbb{X}_{feas}$ and B^F depend on q and hence, are equilibrium outcomes. If the current amount of debt due, b, is such that $b > B^F(s)$, the government has no alternative to default as repayment is not feasible. In that case, we let $V^R(s, b) = V_{NF}$ if $b > B^F(s)$, for some sufficiently low value $V_{NF} < \min_{s \in \mathbb{S}} V^D(s)$.

Given exogenous state s and inherited debt b, the government's problem *conditional on repayment* can be written recursively as follows:

$$\text{If } b \in \mathbb{X}_{feas}(s)\text{: } V^R(s, b) = \sup_{c \in [0, \overline{c}], b'} \left\{ u(c) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) \max \left\{ V^R(s', b'), V^D(s') \right\} \right\} \tag{G}$$

subject to:

$$c \leq y(s) - b + q(s, b')b',$$

$$b' \leq \overline{B}.$$

$$\text{If } b \notin \mathbb{X}_{feas}(s)\text{: } V^R(s, b) = V_{NF}$$

Note that the specific value of V_{NF} does not affect the value function $V^R(s, b)$ for $b < B^F(s)$, as such value is off-equilibrium (that is, it always triggers default).

Lenders' Break-even Condition. Given the risk neutrality of lenders (and infinite collective wealth), we replace the lenders' problem and market clearing in the international financial markets with a break-even condition. In particular, given (s, b) and conditional on the government repaying and auctioning b', let $q(s, b')$ denote the equilibrium price of a bond that pays one next period absent default. The lenders' break-even condition for any $s \in \mathbb{S}$ and $b' \leq \overline{B}$ is[6]

$$q(s,b') = \begin{cases} R^{-1} & \text{if } b' \leq 0 \\ R^{-1} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{1}_{\{V^R(s',b') \geq V^D(s')\}} & \text{otherwise,} \end{cases} \tag{BE}$$

where R^{-1} denotes the international discount factor, as described previously. The first row on the right-hand side states that (positive) assets return the risk-free rate.[7] The second row reflects that debt will be repaid only if doing so is feasible and optimal for the government. A crucial feature of (BE) is that current prices depend on the future only through the associated continuation value V^R; in particular, the debt policy function that generates $V^R(s, b')$ is not directly relevant. This is a consequence of one-period debt.

The Value with No Access to Debt. Before moving on to characterize the equilibrium with borrowing, let us consider a restricted problem, where the government is only allowed to save and cannot borrow, that is, $b \leq 0$. Let $V^{NA}(s, b)$ denote the corresponding government's value function. This value function is the *unique* bounded fixed point of the following Bellman: equation

$$V^{NA}(s,b) = \max_{c \in [0,\overline{c}], b'} \left\{ u(c) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) V^{NA}(s', b') \right\} \tag{NA}$$

subject to:

$$c \leq y(s) - b + R^{-1} b'$$

$$b' \leq 0$$

We impose the following assumption, which implies that default must entail some cost:

Assumption 4. *For all $s \in \mathbb{S}$, $V^{NA}(s, 0) \geq V^D(s)$.*

As established by Auclert and Rognlie (2016), Assumption 4 implies that the government will never default without strictly positive debt (a result we confirm below).

Definition of Equilibrium. Let $\mathbb{X} \equiv \{(s, b)$ such that $s \in \mathbb{S}$ and $b \leq \overline{B}\}$, the set of admissible states. With this, we can now define an equilibrium:

Definition 2. *A* Markov-perfect equilibrium *consists of two functions, $V^R : \mathbb{X} \to \mathbb{R}$ and $q : \mathbb{X} \to [0, 1/R]$ such that: (i) given q, V^R satisfies (G) for (s, b) in $\mathbb{X}$, and the supremum in*

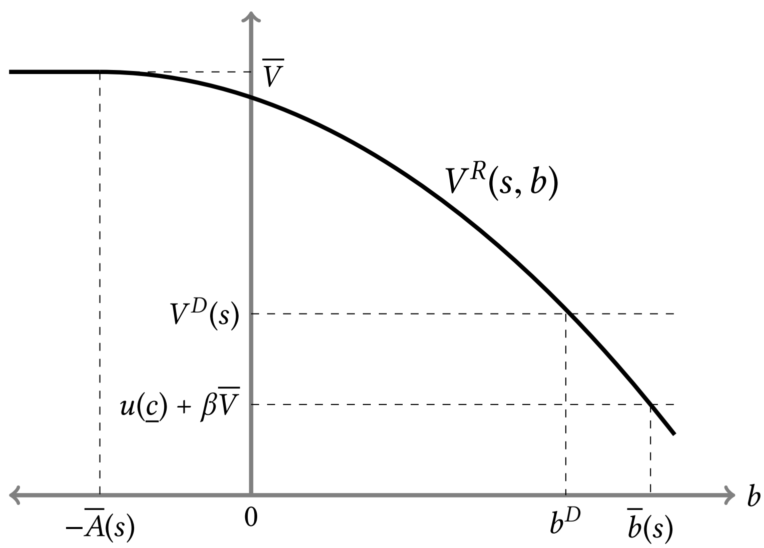
[6]Here, $\mathbb{1}_x$ denotes the indicator function that takes one if x is true and zero otherwise.
[7]We have not explicitly ruled out default when the government holds assets abroad. However, Lemma 1 shows that our Assumption 4 below is sufficient for this.

(G) *is attained for some policy; and (ii) given* V^R, q *satisfies* (BE).

Characterizing Equilibria

We now establish key properties of the government's equilibrium value function. In particular, we establish that the value function, for a given s, is continuous, weakly decreasing, and strictly decreasing whenever $\overline{V}$ is not feasible, as depicted in Figure 1.

Figure 1: The Government's Value Function for a given s



Note: The diagram depicts $V^R(s,b)$ for fixed $s \in \mathbb{S}$ as a function of b. Thresholds $-\overline{A}(s)$ and $\overline{b}(s)$ are defined in Definition 3 and Lemma 3, respectively. Consumption level $\underline{c}$ is defined in Assumption 3. b^D denotes the value such that $V^R(s, b^D) = V^D(s)$ given s, and default occurs in state s for $b > b^D$. Note that Assumption 3 implies that $V^D(s)$ is strictly above $u(\underline{c}) + \beta\overline{V}$ and hence $b^D < \overline{b}(s)$.

Preliminaries

We first establish some preliminary results. The first is a property of the feasible repayment set, $B^F(s)$. Using Definition 1 and that $q(s,b')b' \le R^{-1}\overline{B}$, we have

$$B^F(s) \le \overline{y} + R^{-1}\overline{B} = \overline{B}, \text{ for all } s \in \mathbb{S}.$$

That is, government borrowing is bounded by the present value of the endowment path that features the highest income realization forever.

The next lemma states that it is never optimal to default with a weakly positive net asset position (Part (i)),[8] and it is never optimal to issue debt at a zero price (Part (ii)):

Lemma 1. *In any equilibrium,*

(i) For any state $s \in \mathbb{S}$ and $b \leq 0$, $V^R(s, b) \geq V^D(s)$; and

(ii) For any state $s \in \mathbb{S}$ and $b < B^F(s)$, there exists an optimal debt choice b' and at least one element $s' \in \mathbb{S}$ such that $V^R(s', b') \geq V^D(s')$.

Part (ii) implies that there is always an optimal debt policy such that the government never defaults with probability one next period.

Proof. The proof in the appendix. □

The next set of results concern the feasibility of the maximal consumption, $\bar{c}$, and the maximal government value, $\overline{V}$. Let us define a level of assets that is sufficient to finance $\bar{c}$ forever regardless of future endowment realizations:

Definition 3. *Let $\overline{A}(s)$ be such that*

$$\overline{A}(s) \equiv \bar{c} - y(s) + \frac{\bar{c} - \underline{y}}{R - 1} \text{ for all } s \in \mathbb{S}. \tag{3}$$

Note that $\overline{A}(s) > 0$. We have:

Lemma 2. *In any equilibrium, for any state $s \in S$, $V^R(s, b) = \overline{V}$ if and only if $b \leq -\overline{A}(s)$. Moreover, if $c = \bar{c}$ for any state $(s, b) \in \mathbb{X}$, then $b < 0$.*

Proof. The proof in the appendix. □

Properties of the Value Function

The following establishes continuity and weak monotonicity of V^R on the relevant domain for debt:

Lemma 3. *In any equilibrium,*

(i) For any $s \in \mathbb{S}$, $V^R(s, b)$ is weakly decreasing for $b < B^F(s)$.

[8] Hence, setting $q(s, b') = R^{-1}$ for $b' \leq 0$ is without loss given Assumption 4.

(ii) For any $s \in \mathbb{S}$, there exists a unique *threshold $\overline{b}(s) < B^F(s)$ such that:*

$$V^R(s, \overline{b}(s)) = u(\underline{c}) + \beta\overline{V}.$$

(iii) For any $s \in \mathbb{S}$, $V^R(s, b)$ is continuous for $b \leq \overline{b}(s)$.

Proof. The proof in the appendix. □

We can now strengthen the monotonicity result:

Lemma 4. *In any equilibrium, for all $s \in \mathbb{S}$, $V^R(s, b)$ is strictly decreasing in b for $b \in (-\overline{A}(s), \overline{b}(s)]$.*

Proof. The proof is in the appendix. □

3 An Eaton-Gersovitz Contraction Operator

In this section, we proceed to show that the value function V^R must solve a dual problem, whose solution can be represented as the fixed point of a contraction mapping.

Toward this goal, we first combine the government's problem (G) with the lenders' break-even constraint (BE) to write the equilibrium problem as

$$V^R(s, b) = \max_{c \in [0, \overline{c}], b'} \left\{ u(c) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) \max \left\{ V^R(s', b'), V^D(s') \right\} \right\} \tag{G'}$$

subject to:

$$c \leq y(s) - b + b'R^{-1} \left[\mathbb{1}_{\{b' \leq 0\}} + \mathbb{1}_{\{b' > 0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{1}_{\{V^R(s', b') \geq V^D(s')\}} \right],$$

$$b' \leq \overline{B},$$

where again, we let $V^R(s, b) = V_{NF}$ if the constraint set is empty.

Problem (G′) has the familiar recursive structure that takes a continuation value function and maps it into the current value state by state. Any equilibrium value function V^R is a fixed point of the operator defined by this Bellman equation. The quantitative sovereign debt literature has developed algorithms to find this fixed point numerically. While the operator is monotone and maps the space of bounded functions into itself (if u is bounded), it does not satisfy discounting. Even more, it is possible to show that the operator is not in general a contraction mapping (we provide such an example in Appendix B).

Fortunately, we show below that there is a transformation that delivers an operator that does satisfy all of Blackwell's sufficient conditions. This alternative operator involves the dual problem to (G′).

3.1 The Dual Problem

Given an equilibrium value $V^R(s,b)$, let us define the dual of the optimization problem in (G′):

$$\hat{B}(s,v) \equiv \sup_{c\in[0,\bar{c}],b'} \left\{ y(s) - c + R^{-1} \left[\mathbb{1}_{\{b'\leq 0\}} + \mathbb{1}_{\{b'>0\}} \sum_{s'\in\mathbb{S}} \pi(s'|s) \mathbb{1}_{\{V^R(s',b')\geq V^D(s')\}} \right] b' \right\} \tag{B}$$

subject to:

$$v = u(c) + \beta \sum_{s'\in\mathbb{S}} \pi(s'|s) \max\left\{V^R(s',b'), V^D(s')\right\}, \tag{4}$$

$$b' \leq \overline{B}. \tag{5}$$

We now establish a basic duality result, namely, that the inverse of the government's value function satisfies problem (B). Specifically, Lemmas 2, 3, and 4 imply that in any equilibrium, there exists a continuous, strictly decreasing function $B(s,v)$ such that

$$v = V^R(s, B(s,v))$$

for all $s, v \in \mathbb{S}\times\mathbb{V}$ where $\mathbb{V} \equiv [u(\underline{c}) + \beta\overline{V}, \overline{V}]$, with $B(s,\overline{V}) = -\overline{A}(s)$ and $B(s, u(\underline{c}) + \beta\overline{V})) = \overline{b}(s)$ for all $s \in \mathbb{S}$. That is, $B(s,v)$ is the inverse of equilibrium value function $V^R(s,b)$ with respect to its second argument, b, over its strictly decreasing range $(-\overline{A}(s), \overline{b}(s))$. Then, we have the following duality result:

Lemma 5. *The function $B(s,v) = \hat{B}(s,v)$ for all $(s,v) \in \mathbb{S}\times\mathbb{V}$.*

Proof. The proof is in the appendix. □

In the next subsection, we show how the solution to problem (B) can represented as the fixed point of a contraction mapping operator.

3.2 The Equilibrium Operator

The inverse value function B is a fixed point of an operator implicitly defined in (B). More formally, define the following operator T on functions $f : \mathbb{S} \times \mathbb{V} \to \mathbb{R}$:

$$Tf(s,v) = \sup_{c \in [0,\bar{c}], b', \{w(s')\}_{s' \in \mathbb{S}}} \left\{ y(s) - c + R^{-1} \left[\mathbb{1}_{\{b' \leq 0\}} + \mathbb{1}_{\{b' > 0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{1}_{\{w(s') \geq V^D(s')\}} \right] b' \right\} \qquad \text{(T)}$$

subject to:

$$v \leq u(c) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) \max \left\{ w(s'), V^D(s') \right\} \qquad (6)$$

$$b' \leq f(s', w(s')) \text{ for all } s' \in \mathbb{S} \text{ such that } w(s') \geq V^D(s') \qquad (7)$$

$$w(s') \in \mathbb{V} \text{ for all } s' \in \mathbb{S}.$$

Before discussing useful properties of this operator, we discuss differences with the original dual problem (B). The key alteration is that the government's repayment value V^R no longer appears in the problem. Rather, the problem allows the choice of the government's continuation value state by state, represented by $\{w(s')\}_{s' \in \mathbb{S}}$. In this sense, the problem shares a passing resemblance to a standard contracting problem in which a risk-neutral principal insures a risk-averse agent, subject to limited commitment on the part of the agent. We shall return to this point in Section 4.

However, recall that one crucial friction in the Eaton-Gersovitz model is the lack of state-contingent liabilities. This is accommodated by the presence of b' in the objective and the constraint (7). Specifically, the continuation value in the objective is a scalar, b', rather than a state-contingent vector of values. This is the noncontingent debt carried into the next period.

Moreover, the choice of the government's continuation values must be consistent with the choice of b'. Hence, a new constraint (an *implementability constraint*) is introduced in (7). The equilibrium imposes that debt and the government's payoffs be related by $b' = B(s', w(s'))$ for all s', which is equivalent to $w(s') = V^R(s', b')$ for all s.[9]

In problem (T), we have relaxed this equality constraint to an inequality, imposing it only for levels of the continuation value that do not trigger default. This change allows to restrict attention to continuation values $w(s')$ that lie in $\mathbb{V}$ and thus only values in the domain of the fixed point of (T) need to be considered.[10]

[9] Aguiar et al. (2019) study a sovereign debt model without endowment risk. They also obtain a dual characterization similar to (T), but without the need for constraint (7) as the equilibrium allocations in their model were deterministic conditional on repayment.

[10] In the primal problem, it is in principle possible that for some b' and some realizations of the state s', $V^R(s', b) \notin \mathbb{V}$. Without this change in the dual problem, it would be necessary to assign a debt value to utility levels outside the domain of f, $\mathbb{V}$. As we show in Lemma 6, we can bypass this issue.

Note that, relative to (B), we have also replaced the equal sign in the promise keeping constraint with an inequality in (6) and dropped the no-Ponzi condition, which required $b' \leq \overline{B}$.

Despite these alterations, the equilibrium B that solves (B) is a fixed point of the operator defined by (T):

Lemma 6. *Any equilibrium $B(s, v)$ is a fixed point of T.*

Proof. The proof is in the appendix. □

3.3 A Contraction Mapping

We have shown above that the operator T admits as a fixed point the dual of any equilibrium value function. Interestingly, even though the original operator that defined V^R was not a contraction mapping, the operator T, which works on the dual of V^R, is a contraction mapping. We now proceed to show this.

Toward that goal, we endow the space of functions on which T operates with the sup norm. Our first statement is that T maps bounded functions into bounded functions:

Lemma 7 (Boundedness)**.** *Let $f : \mathbb{S} \times \mathbb{V} \to \mathbb{R}$ be bounded in the sup norm. Then Tf is a bounded function.*

Proof. Consider a bounded f such that $||f|| < M$ for $M > 0$. For any state s_0, v_0, consider the policy of $c = \overline{c}$ and $w(s') = \overline{V}$ for all $s' \in \mathbb{S}$. Let $b'_0 \leq -M$. The policy $c_0, b'_0, \{w_0(s')\}$ satisfies the constraint set of problem (T), and thus

$$(Tf)(s_0, v_0) \geq y(s_0) - \overline{c} - M/R \geq \underline{y} - \overline{c} - M/R,$$

but also

$$(Tf)(s_0, v_0) \leq \overline{y} + M/R.$$

Thus, $||(Tf)(s_0, v_0)|| \leq \max\{\overline{y} + M/R, \overline{c} - \underline{y} + M/R\}$, which is independent of (s_0, v_0), and thus, $||Tf|| \leq \max\{\overline{y} + M/R, \overline{c} - \underline{y} + M/R\}$, which is bounded. □

Note that this is the place where our assumption that consumption of the government has an upper bound has really been used — it guarantees that the dual operator T maps bounded functions into bounded functions. Note also that the particular value of $\overline{c}$ is irrelevant for all of the analysis above, as long as it is large enough so that Assumption 2 is satisfied.

We next show that the operator T is monotone, a property also shared with the original operator implicit in (G′):

Lemma 8 (Monotonicity). *Let f, g be bounded functions mapping $\mathbb{S} \times \mathbb{V}$ to $\mathbb{R}$, with $f(s, v) \leq g(s, v)$ for all $(s, v) \in \mathbb{S} \times \mathbb{V}$. Then, $(Tf)(s, v) \leq (Tg)(s, v)$ for all $s, v \in \mathbb{S} \times \mathbb{V}$.*

Proof. Note that $(Tg)(s_0, v_0)$ only differs from $(Tf)(s_0, v_0)$ because of constraint (7). It follows that any choice available at $(Tf)(s_0, v_0)$ is also feasible at $(Tg)(s_0, v_0)$ and delivers the same objective. Hence, $(Tf)(s, v) \leq (Tg)(s, v)$ for all $s, v \in \mathbb{S} \times \mathbb{V}$. □

The final step, and the one where the dual representation is exploited, is to show that the operator T satisfies the discounting property with module R^{-1}:

Lemma 9 (Discounting). *Let $a \geq 0$ and let $f : \mathbb{S} \times V \to \mathbb{R}$ be bounded. Then,*

$$[T(f + a)](s, v) \leq (Tf)(s, v) + R^{-1}a$$

for all $s, v \in \mathbb{S} \times \mathbb{V}$.

Proof. Let $a > 0$ and we have

$$\begin{aligned}
[T(f + a)](s, v) = \max_{c, v(s'), b'} \Big\{ & y(s) - c + R^{-1} \max\{0, b'\} \sum_{s' \in S} \pi(s'|s) \mathbb{I}_{\{v(s') \geq V^D(s')\}} \\
& + R^{-1} \min\{0, b'\} \Big\}
\end{aligned}$$

subject to:

$$\begin{aligned}
& v \leq u(c) + \beta \sum_{s' \in S} \pi(s'|s) \max\{v(s'), V^D(s')\} \\
& b' \leq f(s', v(s')) + a \text{ for all } s' \in S \text{ such that } w(s') \geq V^D(s'), \\
& c \in [0, \bar{c}] \\
& w(s') \in \mathbb{V} \text{ for all } s' \in \mathbb{S},
\end{aligned}$$

where we have just replaced $b' \mathbb{1}_{b' > 0}$ with $\max\{0, b'\}$ and $b' \mathbb{1}_{\{b' \leq 0\}}$ with $\min\{0, b'\}$.

We can rewrite the final two terms in the objective as

$$
\begin{aligned}
&R^{-1}\max\{0,b'\}\sum_{s'\in S}\pi(s'|s)\mathbb{I}_{\{v(s')\geq V^D(s')\}} + R^{-1}\min\{0,b'\} = \\
&R^{-1}\max\{-a,b'-a\}\sum_{s'\in S}\pi(s'|s)\mathbb{I}_{\{v(s')\geq V^D(s')\}} + R^{-1}\min\{-a,b'-a\} \\
&+aR^{-1}\left(\sum_{s'\in S}\pi(s'|s)\mathbb{I}_{\{v(s')\geq V^D(s')\}}+1\right)\leq \\
&R^{-1}\max\{0,b'-a\}\sum_{s'\in S}\pi(s'|s)\mathbb{I}_{\{v(s')\geq V^D(s')\}} + R^{-1}\min\{0,b'-a\} \\
&+aR^{-1}\left(\mathbb{I}_{\{b'>a\}}\sum_{s'\in S}\pi(s'|s)\mathbb{I}_{\{v(s')\geq V^D(s')\}}+\mathbb{I}_{\{b'\leq a\}}\right)\leq \\
&R^{-1}\max\{0,b'-a\}\sum_{s'\in S}\pi(s'|s)\mathbb{I}_{\{v(s')\geq V^D(s')\}} + R^{-1}\min\{0,b'-a\} \\
&+aR^{-1}.
\end{aligned}
$$

Defining $\hat{b}\equiv b'-a$, this implies

$$
\begin{aligned}
[T(f+a)](s,v)\leq \max_{c,v(s'),\hat{b}}\ & y(s)-c+R^{-1}\max\{0,\hat{b}\}\sum_{s'\in S}\pi(s'|s)\mathbb{I}_{\{v(s')\geq V^D(s')\}} \\
&+R^{-1}\min\{0,\hat{b}\}+R^{-1}a \\
&\text{subject to:} \\
&v\leq u(c)+\beta\sum_{s'\in S}\pi(s'|s)\max\{v(s'),V^D(s')\} \\
&\hat{b}\leq f(s',v(s'))\text{ for } s'\in\mathbb{S}.
\end{aligned}
$$

Note that this problem is identical to the original, save for $R^{-1}a$ in the objective. In particular, $[T(f+a)](s,v)\leq [Tf](s,v)+R^{-1}a$. Thus, T discounts with modulus R^{-1}. □

The intuition behind the discounting result in Lemma 9 is simple. Suppose that for all possible values to the government *tomorrow*, the payoffs to foreigners in case of repayment increases by an amount $a>0$. Absent default tomorrow, this change would lead to an increase in the present value of payments to foreigners *today* of $R^{-1}a$. The possibility of default only reduces this value. Hence, the addition of a to the (dual) continuation value increases the expected present value of repayments today by at most $R^{-1}a<a$.[11]

Lemmas 7, 8, and 9 imply that the operator T satisfies Blackwell's sufficient conditions. Hence, the operator T is a contraction with modulus R^{-1}. The contraction mapping theorem states that there is a unique fixed point of T in the space of bounded functions. Recall that we have shown that an equilibrium dual value function $B(s,v)$ is a fixed point of T, and thus there is at most one equilibrium in the Eaton-Gersovitz model.

Note that the contraction mapping theorem guarantees the existence of a fixed point of T. It

[11]Note that the dual value function only appears in the constraint (7). Because this constraint is linear in the continuation value and binds with equality, this generates an increase in promised payoffs (b') by an amount a.

is possible to invert this fixed point and construct a primal value function and a price function that satisfies the conditions in Definition 2; proving the existence of an equilibrium.[12]

We summarize these results in the following proposition:

Proposition 1. *There exists exactly one Markov-perfect equilibrium.*

Proof. The proof of uniqueness follows directly from the uniqueness of the fixed point and the fact that we have already shown that an equilibrium is a fixed point of T.

The proof of existence is in the appendix. □

3.4 Discussion of Uniqueness

Auclert and Rognlie (2016) is the first proof of uniqueness in the Eaton-Gersovitz model. Auclert and Rognlie (2016) use a different approach to establish the result. In particular, they prove uniqueness by contradiction. Assuming a second equilibrium, the authors construct portfolios that mimic the allocation in the original equilibrium. Thus, the government's welfare is pinned down by the best equilibrium. As prices depend only on the government's value, this uniquely determines prices. There is a link to our proof in that we both exploit the fact that the government commits to pay all of its outstanding debt before taking on any new debt, in which case bond prices depend on the government's values next period (which depend on the amount borrowed), but crucially not on future fiscal policies. As we shall see, longer maturity debt does not have this feature, and the equilibrium is not necessarily unique.

If the government is restricted from holding assets, then the equilibrium is not unique. Passadore and Xandri (2018) discuss multiplicity in an environment without assets. If we restrict $b' \geq 0$, we need an additional constraint in (T). In that case, our proof that T satisfies discounting is not valid. Auclert and Rognlie (2016) show that allowing for an arbitrarily small amount of assets is sufficient to restore uniqueness. The key is that there is some level of assets (or debt) for which default is never optimal regardless of creditor expectations.

On the Irrelevance of Sunspots. Suppose that we were to enlarge the state space $\mathbb{S}$ by including an additional state variable $z \in \mathbb{Z}$, unrelated to any payoff relevant state. This could represent, for example, a sunspot random variable or something related to the history of actions taken by the government. Let $(s, z) \in \mathbb{S} \times \mathbb{Z}$ represent an element of this new state space. The same arguments as above tell us that there is a unique Markov equilibrium, with a value function $V^R((s, z), b)$ that is the unique fixed point of operator (G′) under the enlarged state space. Now suppose $V^R(s, b)$ is the fixed point of (G′) under the original state space that restricts attention to payoff relevant states $s \in S$. It immediately follows that $V^R((s, z), b) = V^R(s, b)$ is a fixed point under the enlarged

[12] Auclert and Rognlie (2016) provide an alternative proof of existence using the monotonicity of the primal operator.

state space. Given that the fixed point is unique, there are no other equilibria. Thus, sunspots or payoff irrelevant state variables have no impact on the equilibrium of the model.

Bulow and Rogoff (1989)'s argument. In their 1989 paper, Bulow and Rogoff show in a complete-markets environment that no strictly positive level of debt is sustainable if the government can save after default and if there are no other direct costs of default besides the inability to borrow again. Their result is based on an arbitrage argument. It is possible to use the uniqueness of the Markov-perfect equilibrium to show that the same result holds in the Eaton and Gersovitz (1981) environment, where markets are incomplete.[13]

Toward this end, let $V^D(s) = V^{NA}(s, 0)$. That is, the government, after default, can save but cannot borrow again. The Bulow-Rogoff claim is that, given this outside option, borrowing is not sustainable in equilibrium. To prove the Bulow-Rogoff claim, we posit that it is true and construct the associated equilibrium value function. If the associated value is a fixed point of (G′), then zero borrowing is the only possible equilibrium outcome.

Specifically, conjecture the following equilibrium price schedule:

$$q^{BR}(s,b) = \begin{cases} R^{-1} & \text{for } b \le 0, \\ 0 & \text{for } b > 0, \end{cases}$$

and value function, $V^{BR}(s, b)$, defined for $b \le y(s)$ as

$$V^{BR}(s,b) = \max_{c\in[0,\bar{c}],b'} \left\{ u(c) + \beta \sum_{s'\in\mathbb{S}} \pi(s'|s) V^{NA}(s,b') \right\}$$

$$\text{subject to:}$$

$$c \le y(s) - b + R^{-1}b',$$

$$b' \le 0,$$

and as $V^{BR}(s, b) = V_{NF}$ for $b > y(s)$ (as before).

Note that $V^{BR}(s, b) = V^{NA}(s, b)$ for $b \le 0$ and $V^{BR}(s, b) < V^{NA}(s, 0) = V^D(s)$ for $b > 0$ (this last following from strict monotonicity of the problem above). This value function justifies the conjectured price q^{BR} and is a fixed point of (G′). Given that there is only one fixed point of (G′), it follows then that $\{q^{BR}, V^{BR}\}$ is the *unique* Markov equilibrium. This equilibrium entails immediate default for any $b > 0$: no level of borrowing can be sustained.

[13] Auclert and Rognlie (2016) use their replication argument to show that the Bulow and Rogoff (1989b) result holds in the Eaton-Gersovitz model. See also Bloise, Polemarchakis and Vailakis (2017) for a general argument under incomplete markets.

Fluctuations in the risk-free rate. It is straightforward to see that we can extend the environment to allow for a stochastic or time varying risk free rate. In particular, suppose we replace R with a function of the exogenous state $R(s)$. If $\underline{R} \equiv \min_{s \in \mathbb{S}} R(s) > 1$, Lemma 9 still holds. Hence, the dual operator remains a contraction mapping, with modulus $\underline{R}^{-1}$, and the uniqueness result is preserved.

In addition, the Bulow and Rogoff (1989) argument described above holds as well for this case, as long as $\underline{R} > 1$. That is, without additional punishment for default other than zero access to debt markets, but with the ability to save in a risk free bond after default, the unique Markov equilibrium cannot sustain any borrowing. This result is closely related to Bloise et al. (2017), who have obtained more general conditions for it. It also connects with the recent work by Bloise, Polemarchakis and Vailakis (2018), who have constructed interesting examples with no punishments and where borrowing is sustained, but where the real interest rates is required to be recurrently *below* 1; that is, $\underline{R} < 1$ (or more generally, below the rate of growth of the economy).

4 Constrained Efficiency: Why Fiscal Rules Add No Value

In this section, we show that the equilibrium of the Eaton and Gersovitz (1981) model with one-period bonds is *constrained efficient*, when the incompleteness of the markets and the government's inability to commit to repayment are both taken into account.

To understand this point, consider a situation where the government at time $t = 0$ commits to a sequence of debt issuances as a function of the history of shocks: $\boldsymbol{b} = \{\boldsymbol{b}(s^t)\}_{t,s^t}$, where $s^t = (s_0, s_1, \ldots, s_t)$ denotes the history of exogenous shocks through time t and $\boldsymbol{b}(s^t)$ is the amount of debt issued at history s^t and due in period $t+1$. We can think of such a state-contingent debt-issuance policy as arising from a constitutional fiscal rule. The government, however, is still able to default if its equilibrium value of following this rule lies below the corresponding outside option for that state.

The potential value to the government of committing to such a rule is that it potentially affects equilibrium prices. That is, as the government is large in its own debt market, it recognizes that there is a corresponding sequence of equilibrium prices associated with a particular fiscal rule.

Define the sequence $\{c(s^t), v(s^t)\}$ associated with a fiscal rule, given an equilibrium price $\boldsymbol{q} = \{\boldsymbol{q}(s^t)\}$, by the following recursion:

$$
\begin{aligned}
c(s^t) &\equiv \min\left\{y(s^t) - \boldsymbol{b}(s^{t-1}) + \boldsymbol{q}(s^t)\boldsymbol{b}(s^t), \bar{c}\right\} \\
v(s^t) &\equiv u(c(s^t)) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s^t) \max\left\{v(\{s^t, s'\}), V^D(s')\right\},
\end{aligned}
\tag{8}
$$

where we let $v(s^t) = V^{NF}$ if $c(s^t) < 0$, as before. The price must keep lenders indifferent, and thus,

$$\boldsymbol{q}(s^t) \equiv \begin{cases} R^{-1} \sum_{s'} \pi(s'|s)\mathbb{1}\{v(\{s^t, s'\}) \geq V^D(s')\} & \text{if } \boldsymbol{b}(s^t) > 0 \\ R^{-1} & \text{if } \boldsymbol{b}(s^t) \leq 0. \end{cases} \tag{9}$$

Thus, associated with any fiscal rule is a sequence $\{c(s^t), v(s^t), \boldsymbol{q}(s^t)\}_{t,s^t}$. The fiscal rule design problem is then to choose $\boldsymbol{b}$ to maximize initial value, $v(s_0)$ given $\boldsymbol{b}(s^{-1}) = b_0$:

$$V^\star(s_0, b_0) \equiv \sup_{\{\boldsymbol{b},\boldsymbol{q},\{v(s^t),c(s^t)\}\}} v(s_0) \text{ subject to } \boldsymbol{b}(s^{-1}) = b_0, (8) \text{ and } (9). \tag{10}$$

In this fiscal design rule, we are assuming that the designer can choose both the debt sequence and its associated price (as long as the latter satisfies the break-even condition for the lenders). In this way, the designer is allowed to choose the best price (if there were many consistent with a given fiscal rule). That is, the designer can coordinate the lenders' expectations. As we will argue next, there is no value to the fiscal rule even in this case. As a result, there will be no value either when the designer cannot coordinate the lenders' expectations.

Using the dynamic programming principle, it follows that $V^\star$ must solve

$$V^\star(s_0, b_0) = \sup_{c_0, b_1, q_1, \{\boldsymbol{b}(s^t), \boldsymbol{q}(s^t), v(s^t)\}_{t\geq 1},} \left\{ u(c_0) + \sum_{s'\in\mathbb{S}} \pi(s'|s_0) \max\{v(\{s_0, s'\}), V^D(s')\} \right\} \tag{11}$$

such that

$$c_0 = \min\{y(s_0) - b_0 + q_1 b_1, \bar{c}\} \tag{12}$$

$$q_1 = \begin{cases} R^{-1} \sum_{s'} \pi(s'|s)\mathbb{1}\{v(\{s_0, s'\}) \geq V^D(s')\} & \text{if } b_1 > 0 \\ R^{-1} & \text{if } b_1 \leq 0 \end{cases} \tag{13}$$

$$\{v(s^t), \boldsymbol{b}(s^t), \boldsymbol{q}(s^t)\}_{t\geq 1} \text{ satisfy } \boldsymbol{b}(\{s_0, s'\}) = b_1 \text{ for all } s' \in \mathbb{S}, (8) \text{ and } (9). \tag{14}$$

Note that it is optimal to choose a continuation sequence $\{v(s^t), \boldsymbol{b}(s^t), \boldsymbol{q}(s^t)\}_{t\geq 1}$ such that $v(\{s_0, s'\}) = V^\star(s', b_1)$ for all $s' \in \mathbb{S}$. Replacing $v(\{s_0, s'\})$ by $V^\star(s', b_1)$ in the above, we have an operator that maps the space of potential $V^\star$ into itself. The value associated with the optimal fiscal rule is a fixed point of this operator. This operator is identical to that defined by the equilibrium in problem (G′). Given that we have shown that there is a unique fixed point to this operator, it follows that $V^R(s, b) = V^\star(s, b)$. *Thus, the ability to commit to a fiscal rule offers no scope to increase the government's value over the Markov-perfect equilibrium value.*

A *critical* feature of the fiscal rule design problem above is that the value of autarky, $V^D(s)$, is not affected by the rule. This is natural under the assumption that, once the country defaults, it cannot access financial markets again, and as a result, it is restricted to consuming its (reduced) endowment. If we were to change the environment and allow the designer to affect, through the fiscal rule, the value of default (and hence, equilibrium prices), then it is possible to construct examples where a fiscal rule generates a value higher than the Markov-perfect equilibrium. It is not surprising that it may be desirable to manipulate the outside option of an agent in this limited commitment model – as this potentially relaxes a main friction in the environment.[14] Our point here is that, beyond this (that is, given the value of default to the government), the equilibrium allocation cannot be further improved once the incompleteness of markets is taken into account.

5 Long-term Bonds: Why the Contraction Argument Fails

Let us now briefly extend the model to incorporate long-duration bonds as in Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012). As is now well known, long-duration bonds generate an inefficiency into the environment, a point analyzed in detail in Aguiar et al. (2019).[15]

The environment is modified in the following way. Rather than issuing a one-period bond, the government instead issues a perpetual claim to an exponentially declining coupon. Specifically, a perpetuity issued at time t offers to pay a coupon 1 in period $t+1$, $(1-\delta)$ in period $t+2$, $(1-\delta)^2$ in period $t+3$, and so on. The parameter δ controls the speed at which the coupon decays: $\delta = 1$ corresponds to the one-period bond, and $\delta = 0$ corresponds to a perpetuity that never decays. Define b as the stock of debt entitled to a coupon 1 today; hence, absent issuance, b decays at the rate δ.

In a Markov-perfect equilibrium of the long-duration bond model, the government solves the following problem:

$$V^R(s,b) = \sup_{\{c\in[0,\bar{c}], b\leq \overline{B}\}} \left\{ u(c) + \beta \sum_{s'} \pi(s'|s) \left\{ V^R(s',b'), V^D(s') \right\} \right\}$$
$$\text{subject to } c \leq y - b + q(s,b')\left(b' - (1-\delta)b\right),$$

where $b' - (1-\delta)b$ represents the amount of new bond issuances and $q(s,b')(b'-(1-\delta)b)$ the amount of revenue raised from them. Let $\mathcal{B}(s,b)$ denote an associated equilibrium debt policy

[14] For other examples where a policy that affects the outside/default option of the agent in a limited commitment model is beneficial see Aguiar, Amador and Gopinath (2009) in a context with investment, and Arellano and Heathcote (2010) in a sovereign default model with dollarization. See Kehoe and Levine (1993) for a general discussion of efficiency in limited commitment models.

[15] This inefficiency is sometimes refered to as "dilution" in the sovereign debt literature. See Hatchondo, Martinez and Sosa-Padilla (2016) and Hatchondo, Martinez and Roch (2015) for explorations of this inefficiency and its interactions with alternative fiscal rules. Arellano and Ramanarayanan (2012) quantitatively explore its implications for maturity choice.

function.

Risk-neutral pricing from the perspective of the lenders leads to the following break-even condition:

$$q(s,b') = \begin{cases} R^{-1} & \text{if } b' < 0 \\ R^{-1} \sum_{s'} \pi(s'|s)\mathbb{1}_{[V^R(s',b') \geq V^D(s')]} \left[1 + (1-\delta)q\left(s', \mathcal{B}(s',b')\right)\right] & \text{if } b' \geq 0. \end{cases}$$

In case of no default next period, the bondholders receive both the coupon as well as the market value of the remaining bond: $1 + (1-\delta)q(s', \mathcal{B}(s', b'))$.

The important element to highlight here is the presence of the equilibrium debt policy function, evaluated at the subsequent state: the price of the long-duration bond depends not only on the debt policy chosen today (b'), but also on the debt policy that the government will choose in subsequent periods. Even if $V^R(s,b)$ is strictly decreasing over some domain, implying that there is a well-defined inverse $B(s,v)$ that maps the government's value to the face value of debt, *this mapping is conditional on a policy* $\mathcal{B}$. Hence, the equilibrium cannot be written as the fixed point of a contraction mapping, as was the case for the one-period bond model. Indeed, as shown in Aguiar and Amador (2018), there exists parameter values such that the long-duration model features multiple equilibria — each of them featuring different issuance policies.

6 Reentry after Default

In our previous analysis, we have assumed that default entails permanent exclusion from financial markets. The quantitative literature, however, usually assumes that exclusion is a transitory state: a government eventually reaccesses the international financial markets. In this section, we show that, under the assumption that shock process s is *iid* across time, it is possible to extend our dual approach to show uniqueness when reentry subsequent to default is possible.[16]

Toward this, let $V^D(s)$ denote the value of default under no reentry. The assumption is that as long as the government is in the default state, the endowment is $y^D(s) \leq y(s)$, where a strict inequality represents the output lost after default. Specifically, let

$$V^D(s) = u\left(y^D(s)\right) + \beta \mathbb{E} V^D(s'),$$

Note that $V^D(s) \leq V^{NA}(s,0)$ for all $s \in \mathbb{S}$.

Now suppose that default is punished by the same lost endowment, but with constant hazard θ, the government's liabilities are forgiven and it regains access to bond markets.[17] Let $\tilde{V}^D$ denote

[16] Auclert and Rognlie (2016) also extend their uniqueness proof to encompass reentry under an *iid* shock process.

[17] The initial quantitative work of Aguiar and Gopinath (2006) and Arellano (2008) both assume such a stochastic

the associated default value conditional on an equilibrium repayment value function V^R:

$$\tilde{V}^D(s) = u\left(y^D(s)\right) + \beta(1-\theta)\mathbb{E}\tilde{V}^D(s') + \theta\beta\mathbb{E}V^R(s',0).$$

Let us define by v_0 the expected gain from reentry:

$$v_0 \equiv \mathbb{E}\left[V^R(s,0) - V^D(s)\right] \geq 0.$$

Manipulating the expressions for V^D and $\tilde{V}^D$, we find that

$$\tilde{V}^D(s) = V^D(s) + \gamma v_0, \qquad \text{where } \gamma \equiv \frac{\theta\beta}{1-\beta(1-\theta)}.$$

In order to show uniqueness, we proceed as follows. As a first step, we take v_0 as a primitive of the environment and show that for a given v_0, there is a unique equilibrium of the model. This step follows the same arguments as in the previous analysis. This implies a mapping from v_0 to an equilibrium value function. Consistency requires that $v_0 = \mathbb{E}[V^R(s,0|v_0) - V^D(s)]$, where $V^R(s,b|v_0)$ is the equilibrium value of repayment conditional on the posited v_0. The final step is to show there is a unique v_0 that satisfies this equation.

Given a value of v_0, we can write the problem of the government as follows:

$$\begin{aligned} V^R(s,b|v_0) = \max_{c\in[0,\overline{c}],b'} & \left\{u(c) + \beta\sum_{s'\in\mathbb{S}}\pi(s')\max\left\{V^R(s',b'), V^D(s') + \gamma v_0\right\}\right\} \\ & \text{subject to:} \\ & c \leq y(s) - b + b'R^{-1}\left[\mathbb{1}_{\{b'\leq 0\}} + \mathbb{1}_{\{b'>0\}}\sum_{s'\in\mathbb{S}}\pi(s')\mathbb{1}_{\{V^R(s',b')\geq V^D(s')+\gamma v_0\}}\right], \\ & b' \leq \overline{B}, \end{aligned}$$

where, as in the benchmark (G′), we have substituted prices using the break-even condition.

Conditional on v_0, this problem is isomorphic to the benchmark (G′); the only difference is that V^D is translated by a constant γv_0. It is helpful to define $\tilde{V}^R(s,b|v_0) \equiv V^R(s,b|v_0) - \gamma v_0$. Using

reentry process.

the above, we can write that

$$
\begin{aligned}
\tilde{V}^R(s,b|v_0) = \max_{c\in[0,\overline{c}],b'} & \left\{ u(c) - (1-\beta)\gamma v_0 + \beta \sum_{s'\in\mathbb{S}} \pi(s') \max\left\{\tilde{V}^R(s',b'|v_0), V^D(s')\right\} \right\} \\
& \text{subject to:} \\
& c \le y(s) - b + b'R^{-1}\left[\mathbb{1}_{\{b'\le 0\}} + \mathbb{1}_{\{b'>0\}} \sum_{s'\in\mathbb{S}} \pi(s') \mathbb{1}_{\{\tilde{V}^R(s',b'|v_0)\ge V^D(s')\}} \right], \\
& b' \le \overline{B},
\end{aligned}
$$

In this translated notation, the consistency condition is $(1-\gamma)v_0 = \mathbb{E}\left[\tilde{V}^R(s',0|v_0) - V^D(s')\right]$. The payoff of the translated problem is that $\tilde{V}^R(s,b|v_0)$ is decreasing in v_0, a feature we now prove.

As in our analysis before, $\tilde{V}^R(s,b|v_0)$ is strictly decreasing in b for $-(\overline{A}(s), \overline{b}(s)]$, where $\overline{A}(s)$ is as defined before, and $\overline{b}(s)$ is such that $\tilde{V}^R(s,\overline{b}(s)|v_0) = u(\underline{c}) + \beta\overline{V} - \gamma v_0$.

We exploit the dual representation to show that $\tilde{V}^R$ is decreasing in v_0. Let $\tilde{B}(s,v|v_0)$ be the inverse of $\tilde{V}^R(s,b|v_0)$ on the translated domain $\tilde{\mathbb{V}} \equiv [u(\underline{c}) + \beta\overline{V} - \gamma v_0, \overline{V} - \gamma v_0]$. Assumption 3 still implies that the continuation value of $u(\underline{c}) + \beta\overline{V} - \gamma v_0$ triggers default, as $v_0 \ge 0$. Thus, all of our conditions from the previous analysis apply, and $\tilde{B}$ is a fixed point of the following operator:

$$
\begin{aligned}
(Tf|v_0)(s,v) = \max_{c\in[0,\overline{c}],b',\{w(s')\}_{s'\in\mathbb{S}}} & \left\{ y(s) - c + R^{-1}\left[\mathbb{1}_{\{b'\le 0\}} + \mathbb{1}_{\{b'>0\}} \sum_{s'\in\mathbb{S}} \pi(s') \mathbb{1}_{\{w(s')\ge V^D(s')\}} \right] b' \right\} \\
& \text{subject to:} \\
& v \le u(c) - (1-\beta)\gamma v_0 + \beta \sum_{s'\in\mathbb{S}} \pi(s') \max\left\{w(s'), V^D(s')\right\} \\
& b' \le f(s', w(s')) \text{ for all } s' \in \mathbb{S} \text{ such that } w(s') \ge V^D(s') \\
& w(s') \in \tilde{\mathbb{V}} \text{ for all } s' \in \mathbb{S}.
\end{aligned}
$$

As in the benchmark environment, this operator is a contraction, given v_0. Hence, it provides a mapping from v_0 to a set of *unique* values, $x(s|v_0) = \tilde{V}^R(s,0|v_0)$ for all $s \in \mathbb{S}$. If v_0 satisfies $\mathbb{E}\left[x(s|v_0) - V^D(s)\right] = (1-\gamma)v_0$, then we have an equilibrium. The question is whether there are multiple values of v_0 that satisfy this consistency condition. To answer this, we first note that $\tilde{B}(s,v|v_0)$ is monotonic in v_0:

Lemma 10. *$\tilde{B}(s,v|v_0)$ is decreasing in v_0.*

Proof. Consider two values of v_0: a, b, where $a < b$, and let B_a and B_b be the corresponding fixed points of $T(\cdot|a)$

and $T(\cdot|b)$. Then

$$\tilde{B}_a = T(\tilde{B}_a|a) \geq T(\tilde{B}_a|b).$$

Given that $T(\cdot|b)$ is a monotone operator (and a contraction), iterating on the above expression implies that

$$\tilde{B}_a \geq \lim_{n\to\infty} T^n(\tilde{B}_a|b) = \tilde{B}_b.$$

□

Recall that equilibrium consistency requires that

$$(1-\gamma)v_0 = \mathbb{E}\left[x_0(s|v_0) - V^D(s)\right], \tag{15}$$

where $x_0(s|v_0)$ are values such that $\tilde{B}(s, x_0(s|v_0)|v_0) = 0$ for all $s \in S$. The monotonicity of $\tilde{B}$ with respect to v and v_0 implies that, as v_0 increases, $x_0(s|v_0)$ must decrease to maintain $\tilde{B}(s, x_0(s|v_0)|v_0) = 0$. Hence, the right-hand side of equation (15) is decreasing in v_0. The left-hand side is, however, strictly increasing in v_0. Hence, there is a unique v_0 that is consistent with equation (15). Thus, there is a unique Markov perfect equilibrium in the model with *iid* reentry.

7 Conclusion

We have shown that a dual approach to characterizing the Markov-perfect equilibria of the Eaton-Gersovitz incomplete markets sovereign debt model implies that the inverse of the equilibrium value function is a fixed point of a contraction mapping. This result implies the uniqueness and existence of equilibrium in the Eaton-Gersovitz model. It may potentially be useful in numerical analysis. Given that the equilibrium can be characterized with a contraction mapping operator, iterating the operator guarantees monotonic convergence with modulus R^{-1}.

The fact that the operator resembles an optimal contracting problem between lenders and the government, subject to an additional implementability condition capturing the market incompleteness, sheds light on the efficiency properties of the model's unique equilibrium.

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A Proofs

A.1 Proof of Lemma 1

We prove each part of the lemma:

Part (i). Let $C^{NA}(s,b)$ and $B^{NA}(s,b)$ denote the optimal consumption and debt policies of problem (NA) for $b \leq 0$, which exist by standard arguments. Such a policy is feasible in an equilibrium for any $b \leq 0$, as $B^{NA}(s,b) \leq 0$ and the corresponding equilibrium price is R^{-1}. It follows that, for all $b \leq 0$,

$$V^R(s,b) \geq u(C^{NA}(s,b)) + \beta \sum_{s' \in \mathbb{S}} \pi(s'|s) V^R(s', B^{NA}(s,b))$$

Iterating this equation forward, using that $B^{NA}(s,b) \leq 0$, we obtain that $V^R(s,b) \geq V^{NA}(s,b)$ for all $b \leq 0$.

Assumption 4 then implies that $V^R(s,b) \geq V^{NA}(s,b) \geq V^D(s)$ for all $s \in \mathbb{S}$ and $b \leq 0$.

Part (ii). For $b \leq 0$, the result is immediate from $V^R(s,0) \geq V^D(s)$ for all $s \in S$. For $b > 0$, suppose that this is not the case, and $V^R(s',b') < V^D(s')$ for all s'. This implies that default is occurring with probability one next period. As a result, the price of the bonds is $q(s,b') = 0$. Thus, from the budget constraint, we have that

$$c \leq y(s) - b.$$

Now consider the alternative policy of issuing zero bonds, $\hat{b}' = 0$. That policy can attain the same consumption level (as it generates the same budget constraint), and the value under this alternative policy is

$$\begin{aligned}
&u(c) + \sum_{s' \in \mathbb{S}} \pi(s'|s) V^R(s', 0) \\
&\geq u(c) + \sum_{s' \in S} \pi(s'|s) V^D(s') \\
&= u(c) + \sum_{s' \in S} \pi(s'|s) \max\left\{V^R(s',b'), V^D(s')\right\} \\
&= V^R(s,b).
\end{aligned}$$

where the second line follows from Part (i) and the third from the premise that $V^R(s',b') < V^D(s')$ for all $s' \in \mathbb{S}$. Hence, such $\hat{b}' = 0$ is a strict improvement (a contradiction) or also constitutes an optimal policy.

A.2 Proof of Lemma 2

We proceed to prove in each statement individually.

If $b \leq -\overline{A}(s)$, then $V^R(s,b) = \overline{V}$. Start from state (s,b), with $b \leq -\overline{A}(s)$, and consider the strategy of setting $c = \overline{c}$ and $b' = \frac{R}{R-1}(\overline{c} - \underline{y}) = -\max_{s \in \mathbb{S}} \overline{A}(s) < 0$. As $b' < 0$, $q(s,b') = R^{-1}$ and the budget constraint is satisfied:

$$\begin{aligned}
&y(s) - b + R^{-1} b' \\
&\geq y(s) + \overline{A}(s) - (\overline{c} - \underline{y})/(R-1) \\
&= \overline{c},
\end{aligned}$$

where the last equality uses the definition of $\overline{A}(s)$. Hence, $c = \overline{c}$ is feasible. As $b' \leq -\overline{A}(s')$ for all $s' \in \mathbb{S}$, the same policy is feasible the following period. It then follows that consuming $\overline{c}$

indefinitely is feasible and achieves the highest possible utility level, $\overline{V}$.

If $b > -\overline{A}(s)$, then $V^R(s,b) < \overline{V}$. Suppose, to generate a contradiction, that $V^R(s,b) = \overline{V}$. To achieve this value, consumption must equal $\bar{c}$, *independently of the sequence of realized shocks in future periods.* Consider the sequence with $y = \underline{y}$ for the next k periods. Iterating on the budget set with $c = \bar{c}$ implies there exists a $k < \infty$ such that debt exceeds $\overline{B}$, violating the no-Ponzi condition.

If $c = \bar{c}$ for any state $(s,b) \in \mathbb{X}$, then $b < 0$. If $c = \bar{c}$ is feasible, then there exists a $b'_1 \leq \overline{B}$ such that

$$\begin{aligned}\bar{c} &\leq -b + y(s) + q(s,b'_1)b'_1 \\ &\leq -b + y(s) + \sup_{b' \leq \overline{B}} q(s,b')b' \\ &= -b + B^F(s),\end{aligned}$$

where the last equality uses Definition 1. Thus, $b \leq B^F(s) - \bar{c} < 0$, where the last inequality uses Assumption 2.

A.3 Proof of Lemma 3

We proceed to prove each part.

Part (i). Note that the constraint set in (G) is shrinking in b for $b < B^F(s)$, where $B^F(s)$ is the maximal debt level that is feasible to repay. It then follows that, for any s, $V^R(s,b)$ is weakly decreasing in b for $b < B^F(s)$.

Parts (ii) and (iii). Note that $V^R(s,-\overline{A}(s)) = \overline{V} > u(\underline{c}) + \beta\overline{V} > \lim_{b \downarrow B^F(s)} V^R(s,b)$. The last inequality follows from noticing that the feasible consumption choices approach 0 as b approaches $B^F(s)$. Then, there exist thresholds $\overline{b}(s) < B^F(s)$ such that $V^R(s,b) > u(\underline{c}) + \beta\overline{V}$ for $b < \overline{b}(s)$ and $V^R(s,b) < u(\underline{c}) + \beta\overline{V}$ for $b > \overline{b}(s)$.

To establish continuity, consider a point $b_0 \leq \overline{b}(s)$. Let $b_1 = b_0 - \epsilon$ for ϵ such that $\underline{c}/2 > \epsilon > 0$. Let (c_1, b'_1) be an optimal policy for state (s, b_1). As $b_1 < b_0 \leq \overline{b}(s)$, we have $u(\underline{c}) + \beta\overline{V} < V^R(s,b_1)$. This, combined with $V^R(s,b'_1) \leq \overline{V}$, requires $c_1 > \underline{c}$.

Now consider $b_2 = b_0 + \epsilon$. Consider the debt choice $b' = b'_1$ starting from b_2. The associated consumption is $\tilde{c}_1 = c_1 + b_1 - b_2 = c_1 - 2\epsilon > c_1 - \underline{c} > 0$. Note also that $\tilde{c}_1 < c_1 \leq \bar{c}$. Hence, this consumption and debt choice is feasible but may not be optimal. This implies $V^R(s,b_1) + u(c_1 - 2\epsilon) - u(c_1) \leq V^R(s,b_2) \leq V^R(s,b_1)$, where the last inequality follows from weak monotonicity. As u is a continuous function and $c_1 - 2\epsilon$ is bounded away from zero, $V^R(s,b_2) \longrightarrow V^R(s,b_1)$ as $\epsilon \longrightarrow 0$. Thus, $V^R(s,b)$ is continuous for all $b_0 \leq \overline{b}(s)$ and part (iii) is proved. The fact that $V^R(s,\overline{b}(s)) = u(\underline{c}) + \beta\overline{V}$, which is part (ii) of the lemma, follows directly from continuity.

A.4 Proof of Lemma 4

The proof is by contradiction. In particular, in contradiction to the lemma, consider the following premise: for some $s \in S$, there exist b_0, b_1, with $b_0 < b_1 \leq \overline{b}(s)$ such that $V^R(s_0, b_0) = V^R(s_0, b_1)$. We establish a number of results based on this premise:

Claim 1. *The equilibrium policy at (s, b_1) sets consumption to its upper bound: $c_1 = \overline{c}$.*

Proof. Let b_1' denote an optimal debt choice at b_1 associated with c_1. If $c_1 < \overline{c}$, then it is feasible at b_0 to issue b_1' while consuming $c_0 = \min\{c_1 + b_1 - b_0, \overline{c}\} > c_1$. This yields a value strictly greater than $V^R(s, b_1)$, contradicting the premise. □

The next claim is that the continuation value following b_1 is flat in the neighborhood below an optimal debt choice b_1' in states of repayment:

Claim 2. *If b_1' is an optimal debt policy at (s, b_1), then for all $s' \in \mathbb{S}$ such that $V^R(s', b_1') \geq V^D(s')$ and $b' \in (b_1' - R(b_1 - b_0), b_1')$, we have $V^R(s', b') = V^R(s', b_1')$.*

Proof. By weak monotonicity, $V^R(s', b') \geq V^R(s', b_1')$ for all $s' \in \mathbb{S}$ if $b' < b_1'$. Now suppose, contrary to the claim, that there is an $\hat{s} \in \mathbb{S}$ and $b' \in (b_1 - R(b_1 - b_0), b_1)$ such that $V^R(\hat{s}, b') > V^R(\hat{s}, b_1') \geq V^D(\hat{s})$. Consider then the following policy in state (s, b_0): $c = \overline{c}$ and $b_0' = b'$. To see that this is feasible, recall that $\overline{c}$ is the consumption policy for b_1. Hence,

$$\begin{aligned}
\overline{c} &\leq y(s) - b_1 + q(s, b_1')b_1' \\
&\leq y(s) - b_1 + q(s, b')b_1' \\
&= y(s) - b_0 + q(s, b')b' - (b_1 - b_0) + q(s, b')(b_1' - b') \\
&\leq y(s) - b_0 + q(s, b')b' - (b_1 - b_0) + q(s, b')R(b_1 - b_0) \\
&\leq y(s) - b_0 + q(s, b')b',
\end{aligned}$$

where the second line uses the weak monotonicity of $q(s, .)$; the third line adds and subtracts b_0 and $q(s, b')b'$; the fourth line uses the fact that $b' > b_1' - R(b_1 - b_0)$ and $q(s, b') \geq 0$; and the final line uses that $q(s, b') \leq R^{-1}$, implying $(b_1 - b_0)(q(s, b')R - 1) \leq 0$. The policy $\{\overline{c}, b'\}$ generates a value to the government that is strictly higher than $V^R(s, b_1)$:

$$\begin{aligned}
&u(\overline{c}) + \beta \sum_{s' \in S} \pi(s'|s) \max\{V^R(s', b'), V^D(s')\} \\
&> u(\overline{c}) + \beta\pi(\hat{s}|s)V^R(\hat{s}, b_1') + \beta \sum_{s \neq \hat{s}} \pi(s'|s) \max\{V^R(s', b'), V^D(s')\} \\
&\geq u(\overline{c}) + \beta \sum_{s' \in S} \pi(s'|s) \max\{V^R(s', b_1'), V^D(s')\} \\
&= V^R(s, b_1),
\end{aligned}$$

where the first strict inequality uses the premise that $V^R(\hat{s}, b') > V^R(\hat{s}, b_1') \geq V^D(\hat{s})$; the second inequality uses $V^R(s', b') \geq V^R(s', b_1')$ for all $s' \in \mathbb{S}$ given $b' < b_1'$, as well as $V^R(\hat{s}, b_1') \geq V^D(\hat{s})$; and the final line uses the fact that $\overline{c}, b_1'$ is an optimal policy for (s, b_1). As $\{\overline{c}, b'\}$ is feasible for b_0, we have $V^R(s, b_0) > V^R(s, b_1)$, a contradiction of our premise. □

This implies the following:

Claim 3. *An optimal policy for* (s, b_1) *involves consuming* $\bar{c}$ *for all future periods.*

Proof. Suppose b_1' is an optimal debt policy at (s, b_1). Let $\hat{S} \equiv \{s' \in \mathbb{S} | V^R(s', b_1') \geq V^D(s')\}$. From Lemma 1 Part (ii), we can choose a b_1' such that $\hat{S}$ is not empty. From the previous claim, for any $s' \in \hat{S}$, $V^R(s', b_1')$ is flat in the neighborhood below b_1'. Hence, there exists a $b' < b_1'$ such that $V^R(s', b') = V^R(s', b_1')$. This replicates the initial scenario, and hence $c = \bar{c}$ in state (s', b_1') for $s' \in \hat{S}$. From Lemma 2, this implies $b_1' < 0$. Lemma 1 Part (i) states that $V^R(s', b_1') \geq V^D(s')$ for all $s' \in \mathbb{S}$, hence $\hat{S} = \mathbb{S}$. Thus, for all $s' \in \mathbb{S}$, we can repeat the above arguments to establish that at (s', b_1') the government consumes $\bar{c}$ and issues b_1'' such that $V^R(s'', b_1'')$ is flat in any state s'' following s'. Iterating forward, $\bar{c}$ is the optimal consumption plan for all future periods following (s, b_1). □

Collecting results, under the premise, consumption is $\bar{c}$ for all periods following initial state (s, b_1). However, by Lemma 2, this requires $b_1 \leq -A(s)$, which generates a contradiction to the lemma's "if" statement. Hence, for all $b > -\overline{A}(s)$, the function $V^R(s, b)$ is strictly decreasing.

A.5 Proof of Lemma 5

Consider a $(s_0, v_0) \in \mathbb{S} \times \mathbb{V}$.

If $v_0 = \overline{V}$, then to satisfy constraint (4) from problem (B), it is necessary to set $c = \bar{c}$ and $V^R(s', b') = \overline{V}$ for all $s' \in \mathbb{S}$. This requires that $b' \leq -\overline{A}(s') < 0$ for all s'. Given that $b' < 0$, it is then optimal, to set $b' = \min_{s' \in \mathbb{S}}\{-\overline{A}(s')\} = (\bar{c} - \underline{y})R/(R-1)$, where the last equality follows from (3). The objective is then

$$\hat{B}(s_0, \overline{V}) = y(s_0) - \bar{c} + \frac{1}{R-1}(\bar{c} - \underline{y}) = -\overline{A}(s_0) = B(s_0, \overline{V}),$$

where the last equality follows from Lemma 2.

Now consider $v_0 < \overline{V}$. From Lemmas 2 and 4, there exists a unique $B(s_0, v_0) = b_0 < -\overline{A}(s_0)$ such that $V^R(s_0, b_0) = v_0$.

First, we show that $B(s_0, v_0) \leq \hat{B}(s_0, v_0)$. Let c_0 and b_0' be an associated optimal policy to problem (G′). Note that the policy (c_0, b_0') satisfies (4) of problem (B), as it delivers the value v_0. The budget constraint of problem (G′) implies that

$$B(s_0, v_0) = b_0 \leq y(s_0) - c_0 + R^{-1}\left[\mathbb{1}_{\{b_0' \leq 0\}} + \mathbb{1}_{\{b_0' > 0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s_0) \mathbb{1}_{\{V^R(s', b_0') \geq V^D(s')\}}\right] b_0' \leq \hat{B}(s_0, v_0),$$

where the last inequality follows from the fact that (c_0, b_0') is feasible in problem (B).

Second, we show that $B(s_0, v_0) = \hat{B}(s_0, v_0)$. To show this, consider a situation in which $B(s_0, v_0) < \hat{B}(s_0, v_0)$. Then, there exists $(\hat{c}_0, \hat{b}_0')$, a policy in problem (B) that delivers some objective $\hat{b}_0 >$

$B(s_0, v_0) = b_0$. Rearranging the objective in (B) evaluated at the policy, we have

$$\begin{aligned}\hat{c}_0 &= y(s_0) - \hat{b}_0 + R^{-1}\left[\mathbb{1}_{\{\hat{b}_0' \le 0\}} + \mathbb{1}_{\{\hat{b}_0' > 0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s_0) \mathbb{1}_{\{V^R(s', \hat{b}_0') \ge V^D(s')\}}\right] \hat{b}_0' \\ &< y(s_0) - b_0 + R^{-1}\left[\mathbb{1}_{\{\hat{b}_0' \le 0\}} + \mathbb{1}_{\{\hat{b}_0' > 0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s_0) \mathbb{1}_{\{V^R(s', \hat{b}_0') \ge V^D(s')\}}\right] \hat{b}_0', \end{aligned} \tag{16}$$

where the second line follows from $\hat{b}_0 = \hat{B}(s_0, v_0) > b_0$. Note that (16) implies that the budget constraint of problem (G′) holds for state $(s_0, \hat{b}_0)$: $\{\hat{c}_0, \hat{b}_0'\}$ is feasible and delivers value v_0. Hence, $V^R(s_0, \hat{b}_0) \ge v_0 = V^R(s_0, b_0)$. By monotonicity of V^R, $\hat{b}_0 \le b_0$, a contradiction.

A.6 Proof of Lemma 6

Let V^R be an equilibrium value function with inverse B. We need to show that $(TB)(s, v) = B(s, v)$ for all $(s, v) \in \mathbb{S} \times \mathbb{V}$.

First we show that $TB \ge B$. Consider a state $(s_0, v_0) \in \mathbb{S} \times \mathbb{V}$. Note that the constraint set of problem (B) is non-empty (for example, set $b' = \min_{s'}\{-\overline{A}(s')\} \le \overline{B}$ and let $c \in [\underline{c}, \overline{c}]$ be such that (4) is satisfied).

Let (c_0, b_0') be an element of the constraint set in problem (B) given that state. For each $s' \in \mathbb{S}$ such that $V^R(s', b_0') \ge V^D(s')$, define $w_0(s') \equiv V^R(s', b_0')$.

This implies $w_0(s') \in [V^D(s'), \overline{V}]$ and thus $w_0(s') \in \mathbb{V}$ by Assumption 3. For all other s' such that $V^R(s', b_0') < V^D(s')$, we let $w_0(s')$ be arbitrary elements of $(u(\underline{c}) + \beta\overline{V}, V^D(s'))$.

We now argue that the choice $(c_0, b_0', \{w_0(s')\})$ satisfies the constraint set of problem (T) when $f = B$ given state (s_0, v_0).

For constraint (7), note that

$$\begin{aligned} b_0' &= B(s', w_0(s')) \text{ if } b_0' \ge -\overline{A}(s') \\ b_0' &\le B(s', w_0(s')) \text{ if } b_0' < -\overline{A}(s') \end{aligned}$$

for all s' such that $w(s') \ge V^D(s')$; hence, constraint (7) is satisfied. Note that $(c_0, b_0', \{w_0(s')\})$ satisfies constraint (6) with equality.

Hence, for any $(s_0, v_0) \in \mathbb{S} \times \mathbb{V}$, and for any feasible choice, (c_0, b_0') in problem (B), there exists a policy, $(c_0, b_0', \{w_0(s')\})$ that is feasible in problem (T) given the state (s_0, v_0) when $f = B$ and attains the same value for the objective. It follows that $(TB)(s, v) \ge B(s, v)$ for all $s, v \in \mathbb{S} \times \mathbb{V}$.

Next, we show that $TB \le B$. Given $(s_0, v_0) \in \mathbb{S} \times \mathbb{V}$, consider a feasible choice $(c_0, b_0', \{w_0(s')\})$ of problem (T) (that is, it satisfies the constraints of that problem) when $f = B$. Let us consider a policy $(\hat{c}, \hat{b}')$ for problem (B). We now check that we can construct a feasible policy where $\hat{b}' = b_0'$ and $\hat{c} \geq c_0$.

(i) Constraint (5) holds with b_0': $b_0' \leq \overline{B}$.

Note that for any s' such that $w_0(s') \geq V^D(s')$, constraint (7) implies $b_0' \leq B(s', w_0(s')) \leq \overline{B}$. If $w_0(s') < V^D(s')$ for all $s' \in \mathbb{S}$, constraint (7) is not relevant and any $b_0' \geq 0$ delivers the same objective; hence, we can consider an arbitrary $b_0' \leq \overline{B}$.

(ii) There exists a $\hat{c} \in [\underline{c}, c_0]$ such that $(\hat{c}, b_0')$ satisfies (4) with equality.

First, consider $s' \in S$ such that $w(s') \geq V^D(s')$. For all such states, constraint (7) evaluated at $f = B$ implies $\overline{b}(s') \geq B(s', w(s')) \geq b_0'$. Now, $V^R(s', B(s', w(s'))) \leq V^R(s', b_0')$, as V^R is monotonic. Using that B is the inverse of V^R, it follows that $w(s') \leq V^R(s', b_0')$ for $s' \in \mathbb{S}$ such that $w(s') \geq V^D(s')$. This implies

$$\begin{aligned}
&u(c_0) + \sum_{s' \in S} \pi(s'|s) \max\{V^R(s', b_0'), V^D(s')\} \\
&\geq u(c_0) + \sum_{s' \in S} \pi(s'|s) \max\{w(s'), V^D(s')\} \\
&\geq v_0 \\
&\geq u(\underline{c}) + \beta \overline{V} \\
&\geq u(\underline{c}) + \sum_{s' \in S} \pi(s'|s) \max\{V^R(s', b_0'), V^D(s')\}
\end{aligned}$$

where the third line follows from (6) and the fourth line from $v_0 \in \mathbb{V}$. Hence, by continuity of u, there exists $\hat{c} \in [\underline{c}, c_0]$ such that

$$v_0 = u(\hat{c}) + \sum_{s' \in S} \pi(s'|s) \max\{V^R(s', b_0'), V^D(s')\}$$

A similar argument guarantees the existence of such $\hat{c} \in [\underline{c}, c_0]$ when $w(s') < V^D(s')$ for all $s' \in \mathbb{S}$.

Hence, $(\hat{c}, b_0')$ satisfies the constraints in problem (B) for (s_0, v_0). From the optimization in problem (B),

$$\begin{aligned}
B(s_0, v_0) &\geq y(s_0) - \hat{c} + R^{-1} \left[\mathbb{1}_{\{b_0' \leq 0\}} + \mathbb{1}_{\{b_0' > 0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{1}_{\{V^R(s', b_0') \geq V^D(s')\}} \right] b_0' \\
&\geq y(s_0) - c_0 + R^{-1} \left[\mathbb{1}_{\{b_0' \leq 0\}} + \mathbb{1}_{\{b_0' > 0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s) \mathbb{1}_{\{w(s') \geq V^D(s')\}} \right] b_0'.
\end{aligned}$$

The final line is the objective in (T). As $(c_0, b_0', \{w_0(s')\})$ was arbitrary, we have $B(s_0, v_0) \geq (TB)(s_0, v_0)$.

A.7 Proof of Proposition 1: Existence

For the proof of existence, note that because T is a contraction mapping, it has a fixed point; which we denote by $f^\star$. The way we proceed is to use the inverse of $f^\star$ to construct a candidate equilibrium value function, $V_R^\star$, and a price function, $q^\star$. For simplicity, we define $\underline{V} \equiv u(\underline{c}) + \beta \overline{V}$, so that $\mathbb{V} = [\underline{V}, \overline{V}]$.

Let g denote the objective of Problem (T):

$$g(c, b', \{w(s')\}|s) \equiv y(s) - c + R^{-1}\left[\mathbb{1}_{\{b' \leq 0\}} + \mathbb{1}_{\{b'>0\}} \sum_{s' \in \mathbb{S}} \pi(s'|s)\mathbb{1}_{\{w(s') \geq V^D(s')\}}\right] b'.$$

Given a bounded $f : \mathbb{S} \times \mathbb{V} \to \mathbb{R}$, consider the following constraint set $\Gamma(s, f)$:

$$\begin{aligned}\Gamma(s, f) = \Big\{v, c, b', \{w(s')\} \Big| \\ & u(c) + \beta \sum \pi(s'|s) \max\{w(s'), V^D(s')\} - v \geq 0, \\ & \mathbb{1}_{\{w(s') \geq V^D(s')\}} (f(s', w(s')) - b') \geq 0 \text{ for all } s' \in \mathbb{S}, \\ & v \in \mathbb{V}, w(s') \in [V^D(s'), \overline{V}] \cup \{V_L\} \text{ for all } s' \in \mathbb{S}, c \in [0, \overline{c}], b' \in [\underline{f}, \overline{f}] \cup \{0\}\Big\},\end{aligned}$$

for some $V_L \in (\underline{V}, \min_{s'} V^D(s'))$, and where $\underline{f} = \inf_{s \in \mathbb{S}, v \in \mathbb{V}} f(s, v)$ and $\overline{f} = \sup_{s \in \mathbb{S}, v \in \mathbb{V}} f(s, v)$.

The set $\Gamma(s, f)$ represents the the constraint set of Problem (T) with two modifications. The first is that we have replaced the constraint that $w(s') \in \mathbb{V}$ with the tighter constraint $w(s') \in [V^D(s'), \overline{V}] \cup \{V_L\} \subset \mathbb{V}$ for $V_L < V^D(s)$. This modification has no effect on Problem (T), as values of $w(s')$ such that $w(s') < V^D(s)$ do not affect the other constraints nor the objective function. The second modification is to restrict b' to be zero or to lie between the minimum and maximum of $f(s, v)$. This is without loss of generality, as if the promised values are such that constraint (7) needs to hold for some s', then it requires $b' \leq \overline{f}$, and it is never optimal to choose $b' < \underline{f}$. If the promised values are such that constraint (7) does not need to hold for any s', then Problem (T) places no restriction on b' and $b' = 0$ is optimal.

The above implies that problem (T) can be rewritten as

$$Tf(s, v) = \max_{c, b', \{w(s')\}} g(c, b', \{w(s')\}|s) \qquad \text{(T')}$$
$$\text{subject to: } (v, c, b', \{w(s')\}) \in \Gamma(s, f).$$

We can now state the following results:

Claim 4. *For any bounded f, $s \in \mathbb{S}$ and $v_0 \in \mathbb{V}$, the set $\{(v, c, b', \{w(s')\}) \in \Gamma(s, f) | v = v_0\}$ is non-empty.*

Proof. Note that the vector $\{v = v_0, c = \overline{c}, b' = \underline{f}, w(s') = \overline{V}$ for all $s' \in \mathbb{S}\}$ satisfies all the constraints in $\Gamma(s, f)$, proving the claim. □

Claim 5. *If $f : \mathbb{S} \times \mathbb{V} \to \mathbb{R}$ is bounded and upper-semicontinuous, then $\Gamma(s, f)$ is closed and bounded for all $s \in \mathbb{S}$.*

Proof. Let $p_n \in \mathbb{R}^{3+|\mathbb{S}|}$ where $|\mathbb{S}|$ denotes the number of elements in $\mathbb{S}$. We take $p_n = (v_n, c_n, b'_n, \{w_n(s')\}) \in \Gamma(s, f)$. Consider now a convergent sequence $\{p_n\}_{n=1}^{\infty}$ with $p_n \in \Gamma(s, f)$ for all $n \in \{1, 2, 3, \ldots\}$. Note that $\Gamma(s, f)$ is a bounded subset of $\mathbb{R}^{3+|\mathbb{S}|}$ and thus the sequence has a finite limit $p^\star = \lim_{n\to\infty} p_n$. Let $v^\star, c^\star, b^\star, \{w^\star(s')\}$ denote the individual elements of the limit. Our goal is to argue that $p^\star \in \Gamma(s, f)$.
First note that

$$v^\star \in \mathbb{V}, w^\star(s') \in [V^D(s'), \overline{V}] \cup \{V_L\} \text{ for all } s' \in \mathbb{S}, c^\star \in [0, \overline{c}], b^\star \in [\underline{f}, \overline{f}] \cup \{0\},$$

as all the respective sets are compact.
Given that $u(c) + \beta \sum_{s'} \pi(s'|s) \max\{w(s'), V^D(s')\} - v$ is a continuous function of c, $\{w(s')\}$ and v, it follows that

$$u(c^\star) + \beta \sum_{s'} \pi(s'|s) \max\{w^\star(s'), V^D(s')\} - v^\star \geq 0$$

by passing the limit to a continuous function.
Turning to (7), consider now a given $s' \in \mathbb{S}$. There are two possibilities to consider: either $w^\star(s') = V_L$ or $w^\star(s') \in [V^D(s), \overline{V}]$. We treat each case separately.
Case 1: $w^\star(s') = V_L$. In this case, the constraint $\mathbb{1}_{w^\star(s') \geq V^D(s')}(f(s', w^\star(s')) - b^\star) \geq 0$ is automatically satisfied as $w^\star(s') < V^D(s')$.
Case 2: $w^\star(s') = [V^D(s), \overline{V}]$. In this case, there must exists a finite N such that $w_n(s') \in [V^D(s), \overline{V}]$ for all $n > N$ (as $V^D(s) > V_L$). Note that for all $n > N$, we have that

$$f(s', w_n(s')) - b_n \geq 0.$$

Given that f is upper-semicontinuous, the left-hand side of the above is an upper-semicontinuous function in $(b', w(s'))$. It follows that:

$$f(s', w^\star(s')) - b^\star \geq \limsup_n f(s', w_n(s')) - b_n \geq 0.$$

Hence, (7) is satisfied in the limit for $s' \in \mathbb{S}$. This argument can be repeated a finite number of times for all $s' \in \mathbb{S}$. Thus, we have shown that $p^\star \in \Gamma(s, v)$. And hence $\Gamma(s, v)$ is closed. □

Note also that the objective g is upper-semicontinuous, and thus:

Claim 6. *If $f : \mathbb{S} \times \mathbb{V} \to \mathbb{R}$ is bounded and upper-semicontinuous, then the supremum in $Tf(s, v)$ is achieved for all $(s, v) \in \mathbb{S} \times \mathbb{V}$.*

Proof. Note that the subset of $\Gamma(s, f)$ such that v is constant is non-empty, bounded and closed by Claims 4 and 5. Given That g is an upper-semicontinuous function on $c, b', \{w(s')\}$, g attains a maximum in a non-empty compact set. □

Let F be the set of functions such that $f \in F$ if and only if: (i) f is bounded and upper-semicontinuous on $S \times \mathbb{V}$; (ii) f is a weakly decreasing function in $v \in V$; (iii) $f(s, \overline{V}) = -\overline{A}(s)$ for all $s \in \mathbb{S}$; and (iv) $f(s, \underline{V}) \leq \overline{B}$ for all $s \in \mathbb{S}$. Let F_0 denote the subset of F that contains only strictly decreasing functions of v. Note that F is closed. Then,

Claim 7. *The operator T maps F into F_0.*

Proof. Let $f \in F$. We have already established (Lemma 7) that Tf is bounded.

Strict monotonicity. Fix $s \in \mathbb{S}$. Consider v_1 and v_2 such that $\underline{V} \leq v_1 < v_2 \leq \overline{V}$ for $v_2 - v_1 \leq \overline{\delta}$ where $0 < \overline{\delta} < u(\underline{c}) - u(0)$. Let $\{c_i, b_i', \{w_i(s')\}\}$, $i = 1, 2$, denote allocations that achieve the supremum for Problem (T') at (s, v_i), $i = 1, 2$, respectively; the existence of such optimizing allocations was established in Claim 6.
Note that $c_i \in [\underline{c}, \overline{c}]$. The upper bound follows from the constraint set. The lower bound follows from Assumption 3, as if $c_i < \underline{c}$, then $u(c_i) + \beta \sum_{s'} \pi(s'|s) \max\{w_i(s'), V^D(s')\} < u(\underline{c}) + \beta\overline{V} = \underline{V}$, violating promise keeping as $v_i \in \mathbb{V} = [\underline{V}, \overline{V}]$.
Define $\tilde{c}_1$ to be such that $u(\tilde{c}_1) - v_1 + v_2 - u(c_2) = 0$. Let $H(c) \equiv u(c) - v_1 + v_2 - u(c_2)$. The function H is continuous and strictly increasing in $c \in [0, \overline{c}]$. $H(c_2) = v_2 - v_1 > 0$, and $H(0) = u(0) - u(c_2) + v_2 - v_1 < u(0) - u(\underline{c}) + \overline{\delta} < 0$. Hence, there exists a unique $\tilde{c}_1 \in (0, c_2)$ such that $H(\tilde{c}_1) = 0$.
For problem $Tf(s, v_1)$, the allocation $(v_1, \tilde{c}_1, b_2', \{w_2(s')\}) \in \Gamma(s, f)$, and is a feasible choice. Hence, we have:

$$Tf(s, v_1) \geq Tf(s, v_2) + c_2 - \tilde{c}_1 > Tf(s, v_2),$$

where the second inequality follows from $\tilde{c}_1 < c_2$. Thus $Tf(s, v)$ is *strictly monotone* in v, given s.

Upper semi-continuity. Consider a given $s \in \mathbb{S}$ and take a monotonically increasing sequence $\{v_n\}_{n=1}^{\infty}$ that converges to v_0 from below. We want to show that $\lim_{n\to\infty} Tf(s, v_n) = Tf(s, v_0)$. Establishing left-hand continuity is sufficient to establish upper semi-continuity as Tf is strictly decreasing in v.
For each $n \geq 1$, let $(c_n, b_n', \{w_n(s')\})$ denote optimal policies associated with problem (T') evaluated at (s, v_n) given f, and let $p_n \equiv (v_n, c_n, b_n', \{w_n(s')\})$. Note that $p_n \in \Gamma(f, s)$. Given that $\Gamma(f, s)$ is a compact subset of $R^{3+|\mathbb{S}|}$, by the Bolzano-Weierstrass theorem, the sequence $\{p_n\}_{n=1}^{\infty}$ contains a convergent subsequence $\{p_{n_i}\}_{i=1}^{\infty}$ in $\Gamma(s, f)$. Let $p^\star \in \Gamma(s, f)$ denote its limit, and let $v^\star, c^\star, b^\star$ and $\{w^\star(s')\}$ denote the respective elements of $p^\star$. Note that $v^\star = v_0$, as the sequence of $\{v_n\}$ converges to v_0.
Let $g(p_n)$ denote $g(c_n, b_n', \{w_n(s')\})$. Note that $Tf(s, v_0) \geq g(p^\star)$ as $p^\star \in \Gamma(s, f)$ with $v = v_0$ but may be suboptimal. Given that g is upper-semicontinuous, it follows that $g(p^\star) \geq \limsup_{i\to\infty} g(p_{n_i})$. Using that $g(p_n) = Tf(s, v_n)$, we have that

$$Tf(s, v_0) \geq g(p^\star) \geq \limsup_{i\to\infty} Tf(s, v_{n_i}) = \lim_{n\to\infty} Tf(s, v_n)$$

where the last step follows from the monotonicity of the sequence $\{v_n\}$ and the monotonicity of Tf with respect to v.
Also, as $v_n \leq v_0$ and Tf is decreasing, we have that

$$Tf(s, v_n) \geq Tf(s, v_0) \Rightarrow \lim_{n\to\infty} Tf(s, v_n) \geq Tf(s, v_0)$$

Taken together, we have that $\lim_{n\to\infty} Tf(s, v_n) = Tf(s, v_0)$ for any increasing sequence $\{v_n\}$ and thus, Tf is upper-semicontinuous in v.

The value at $\overline{V}$. We now show that if $f(s,\overline{V}) = -\overline{A}(s)$ for all $s \in \mathbb{S}$, then $Tf(s,\overline{V}) = -\overline{A}(s)$ for all $s \in \mathbb{S}$. Fix $s \in \mathbb{S}$, and consider the problem (T') for $v = \overline{V}$. It follows from the (6) that $u(c) = \overline{c}$ and $w(s') = \overline{V}$ for all $s' \in \mathbb{S}$. As a result, we have that

$$Tf(s,\overline{V}) = \sup_{b'}\{y(s) - \overline{c} + R^{-1}b'\}$$
$$\text{subject to } b' \leq f(s',\overline{V}) = -\overline{A}(s') \text{ for all } s' \in \mathbb{S}.$$

Note that the constraint on b' binds with equality at an optimum. Using the definition of $\overline{A}(s)$, we therefore have

$$Tf(s,\overline{V}) = y(s) - \overline{c} + R^{-1}\min_{s'}(-\overline{A}(s')) = -\left[\overline{c} - y(s) + \frac{\overline{c} - \underline{y}}{R-1}\right] = -\overline{A}(s)$$

Repeating this argument for all $s \in \mathbb{S}$, we complete the proof that T maps F into F_0.

The value at $\underline{V}$. Note that, using that f is decreasing in v:

$$Tf(s,v) \leq y(s) + R^{-1}\max\left\{\max_{s' \in \mathbb{S}} f(s',\underline{V}), 0\right\} \leq \overline{y} + R^{-1}\overline{B} = \overline{B}$$

Where the first inequality follows from noticing that if $f(s',\underline{V}) \leq 0$ for all $s' \in \mathbb{S}$, then an upper bound to the value $Tf(s,v)$ is $y(s)$. And, if there is at least one $s' \in \mathbb{S}$ such that $f(s,\underline{V}) > 0$, then an upper bound to the value is $y(s) + R^{-1}\max_{s' \in \mathbb{S}} f(s',\underline{V})$. The second inequality follows from the $\overline{B} > 0$ and $f(s',\underline{V}) \leq \overline{B}$. □

As the contraction T maps elements of the closed set F into its subset F_0, it follows then that $f^\star \in F_0$. Let $f^\star$ denote the fixed point of the dual operator (T). Let

$$SB \equiv \{(s,b) | s \in \mathbb{S}, \text{ and } b \leq f^\star(s,\underline{V})\}.$$

Define the following generalized inverse function $v^\star$:

$$v^\star(s,b) \equiv \max\{v \in \mathbb{V} | f^\star(s,v) \geq b\},$$

for $(s,b) \in SB$. Note that the set $\{v \in \mathbb{V} | f^\star(s,v) \geq b\}$ is non-empty as $f^\star(s,\underline{V}) \geq b$ (given that $f^\star$ is strictly decreasing), is bounded, and is closed (by upper-semicontinuity of $f^\star$). Hence, $v^\star$ is well defined. In addition, it is continuous on SB. Note also that $v^\star(s,b) = \overline{V}$ for all $b \leq -\overline{A}(s)$.

Our candidate equilibrium price schedule is then constructed from the set SB and the inverse $v^\star$:

$$q^\star(s,b) \equiv \begin{cases} R^{-1} & ; \text{ if } b \leq 0, \\ R^{-1}\sum_{s' \in \mathbb{S}} \pi(s'|s)\mathbb{1}_{\{(s',b) \in SB \text{ and } v^\star(s',b) \geq V^D(s')\}} & ; \text{ otherwise,} \end{cases}$$

which is defined for all $(s,b) \in \mathbb{X} = \mathbb{S} \times (-\infty, \overline{B}]$. Note also that $q^\star(s,b) \in [0, 1/R]$.

Given this price function, let $\mathbb{X}_{feas}(s)$ and $B^F(s)$ be as in (2). Let $R(b) \equiv \{s|(s,b) \in SB \text{ and } v^\star(s,b) \geq V^D(s)\}$. Then, we then define the repayment value function, $V_R^\star(s,b)$, as follows:

$$V_R^\star(s,b) \equiv \sup_{c\in[0,\bar{c}],b'\leq\bar{B}} \left\{u(c) + \beta\left[\sum_{s'\in R(b')} \pi(s'|s)v^\star(s',b') + \sum_{s'\notin R(b')} \pi(s'|s)V^D(s')\right]\right\} \tag{17}$$

subject to:

$$c \leq y(s) - b + q^\star(s,b')b'$$

for all $b \in \mathbb{X}_{feas}(s)$; and $V_R^\star(s,b) \equiv V^{NF}$ otherwise.

We have the following result:

Claim 8. *The function $V_R^\star(s,b) \leq \overline{V}$ and is weakly decreasing in $b \in \mathbb{X}_{feas}(s)$ given $s \in \mathbb{S}$*

> *Proof.* The bound follows from maximal consumption $c = \bar{c}$ and maximal continuation value $\overline{V} \geq v^\star(s,b)$ all $(s,b) \in SB$ and $\overline{V} > V^D(s)$ for all $s \in \mathbb{S}$. The fact that $V_R^\star$ is weakly decreasing in b follows from the fact that b only appears in the budget set, and a lower b weakly expands it. □

Claim 9. *The function $V_R^\star(s,b) = v^\star(s,b)$ for $(s,b) \in SB$.*

> *Proof.* We now argue that $V_R^\star(s,b) = v^\star(s,b)$ for $(s,b) \in SB$. Towards a contradiction, consider an $(s_0,b_0) \in SB$ such that $V_R^\star(s_0,b_0) \neq v^\star(s_0,b_0)$.
>
> Note that there exists an optimizing allocation for the dual problem (T') evaluated at $f^\star$. Let $\{c_0, b_0', \{w_0(s')\}\}$ be an optimal consumption, debt choices, and promised values associated with (s_0,v_0) where $v_0 = v^\star(s_0,b_0)$.
>
> We now show that the allocation c_0, b_0' is feasible for problem (17).
>
> - First, we show that we can restrict attention to $b_0' \leq \overline{B}$. If $b_0' > \overline{B}$, then $w_0'(s') < V^D(s')$ for all $s' \in \mathbb{S}$ given that $f^\star \in F$ (that is, there is no $w_0(s')$ such that $f^\star(s_0, w_0(s')) > \overline{B}$). Hence setting $b_0' = 0$ is also optimal. That is, $\{c_0, b' = 0, \{w_0(s')\}\}$ is an optimal allocation.
> - Note that if $b_0' \leq 0$, then
>
> $$b_0 \leq f^\star(s_0,v_0) = y(s_0) - c_0 + R^{-1}b_0'$$
>
>   and thus, the choice of c_0, b_0' is feasible for problem (17) as $q^\star(s,b_0') = R^{-1}$.
> - For $\overline{B} \geq b_0' > 0$:
>   - We first show that $(s',b_0') \in SB$ for all s' such that $w_0(s') \geq V^D(s')$. This follows because $0 \leq b_0' \leq f^\star(s',w_0(s')) \leq f^\star(s',V^D(s')) \leq f^\star(s',\underline{V})$, where the first inequality is (7), and the remaining two follow from monotonicity of $f^\star$. And thus $(s',b_0') \in SB$.
>   - Now note that evaluating the objective at an optimum:
>
> $$b_0 \leq f^\star(s_0,v_0) = g(c_0,b_0',\{w_0(s')\}|s_0) = y(s_0) - c_0 + R^{-1}\sum_{s'\in\mathbb{S}} \pi(s'|s_0)\mathbb{1}_{\{w_0(s')\geq V^D(s')\}}b_0'.$$
>
>   - From (7), we have $f^\star(s',w_0(s')) \geq b_0'$ for $s' \in \mathbb{S}$ such that $w_0(s') \geq V^D(s')$. Therefore, by definition of $v^\star$, we have $v^\star(s_0',b_0') \geq w_0(s')$.

– Next we show that $v^\star(s_0', b_0') = w_0(s')$. To see this, towards a contradiction suppose that $v^\star(s', b_0') > w_0(s') \geq V^D(s')$. Then, in the original dual problem, choosing $\tilde{w}(s') = v^\star(s', b_0')$ satisfies (7), strictly relaxes (6) and weakly increases the objective. This implies that there exists a $v' > v_0$ for which the value in the dual problem, $f^\star(s_0, v')$ is weakly higher than $f^\star(s_0, v_0)$, contradicting strict monotonicity of $f^\star$. Hence, $v^\star(s', b_0') = w_0(s')$ for all s' such that $w_0(s') \geq V^D(s')$, and therefore:

$$\begin{aligned} b_0 \leq f^\star(s_0, v_0) &= y(s_0) - c_0 + R^{-1} \sum_{s' \in \mathbb{S}} \pi(s'|s_0) \mathbb{1}_{\{v^\star(s', b_0') \geq V^D(s')\}} b_0' \\ &= y(s_0) - c_0 + q^\star(s_0, b_0') b_0'. \end{aligned}$$

The above implies that $\{c_0, b_0'\}$ is a feasible choice for Problem (17). Therefore, $V^R(s_0, b_0) \geq v^\star(s_0, b_0)$ for all $(s_0, b_0) \in SB$.

We now argue that $V^R(s_0, b_0) = v^\star(s_0, b_0)$ for all $(s_0, b_0) \in SB$.

- Towards a contradiction, consider an $(s_0, b_0) \in SB$ such that $\overline{V} \geq V_R^\star(s_0, b_0) > v^\star(s_0, b_0)$.

  Let us redefine c_0 and b_0' to denote a feasible allocation for problem (17) that delivers a value strictly higher than $v^\star(s_0, b_0)$. Set $w_0(s') = v^\star(s', b_0')$ for all $s' \in R(b_0')$ and set $w_0(s') = V_L$ for all $s' \notin R(b_0')$.
- The allocation $\{c_0, b_0, \{w_0(s')\}\}$ satisfies (7) as $f^\star(s', w_0(s')) \geq b_0'$ for $w_0(s') \geq V^D(s')$.
- Given that $\{c_0, b_0'\}$ delivers a value in problem (17) strictly higher than $v^\star(s_0, b_0)$, we have that:

$$v^\star(s_0, b_0) < u(c_0) + \beta \sum_{s'} \max\{v^\star(s', b_0'), V^D(s')\} = u(c_0) + \beta \sum_{s'} \max\{w_0(s'), V^D(s')\} \equiv \hat{v} \leq \overline{V}.$$

  Note that the objective of (T') evaluated at this allocation is

$$\begin{aligned} g(c_0, b_0', \{w_0(s')\}|s_0) &= y(s_0) - c_0 + R^{-1} \left[\mathbb{1}_{b_0' \leq 0} + \mathbb{1}_{b_0' > 0} \sum_{s' \in \mathbb{S}} \pi(s'|s_0) \mathbb{1}_{\{v^\star(s', b_0') \geq V^D(s')\}} \right] b_0' \\ &= y(s_0) - c_0 + q^\star(s_0, b_0') b_0' \geq b_0, \end{aligned}$$

  where the last step follows from the budget constraint of problem (17).

  The allocation $\{\hat{v}, c_0, b_0', \{w_0(s')\}\} \in \Gamma(s, f^\star)$ and delivers a value in Problem (T') weakly higher than b_0. Given $\hat{v} \in (v^\star(s_0, b_0), \overline{V}]$, this violates the strict monotonicity of $f^\star$.

Hence $V_R^\star(s, b) = v^\star(s, b)$ for $(s, b) \in SB$. □

We now argue that $\{V_R^\star, q^\star\}$ is an equilibrium:

Claim 10. *The functions $V_R^\star$ and $q^\star$ defined above constitute a Markov equilibrium.*

Proof. First note that if $V_R^\star(s, b) \geq V^D(s)$, then $(s, b) \in SB$ and $v^\star(s, b) \geq V^D(s)$.

This follows from the monotonicity of $V_R^\star(s, b)$ with respect to b and that $V_R^\star(s, f^\star(s, \underline{V})) = v^\star(s, f^\star(s, \underline{V})) = \underline{V} < V^D(s)$; where the first equality follows from Claim 9; and the second equality follows from the strict monotonicity of $f^\star$ and the definition of $v^\star$. Hence, we have that

$$\begin{aligned} R(b) &= \{s | (s, b) \in SB \text{ and } v^\star(s, b) \geq V^D(s)\} \\ &= \{s | V_R^\star(s, b) \geq V^D(s)\}. \end{aligned}$$

This implies that the pricing equation for $q^\star$ solves (BE) given $V_R^\star$. Similarly, notice $V_R^\star$ solves (G) given $q^\star$.
Finally, we show that the supremum is achieved. Note that because $v^\star(s,b)$ is continuous in $b \leq \overline{B}$, the objective function of Problem (17) is continuous in b'.
Given that SB is closed, and that $v^\star$ is continuous, it follows that $q^\star(s,b)\times b$ is an upper-semicontinuous function for $b \leq \overline{B}$. Hence the constraint set of Problem (17) is bounded and closed (if non-empty). Thus, problem (17) admits a maximizer. □

B The Primal is not a Contraction

Here we provide a simple example that shows that the operator defined by the primal problem (G′) is not a contraction mapping.

Consider the original primal problem, (G′). Define the primal operator, T_P to be

$$T_P v(s,b) = \max_{c\in[0,\overline{c}],b'} \left\{ u(c) + \beta \sum_{s'\in\mathbb{S}} \pi(s'|s) \max\left\{v(s',b'), V^D(s')\right\}\right\} \tag{18}$$

subject to:

$$c \leq y(s) - b + b'R^{-1}\left[\mathbb{1}_{\{b'\leq 0\}} + \mathbb{1}_{\{b'>0\}} \sum_{s'\in\mathbb{S}} \pi(s'|s)\mathbb{1}_{\{v(s',b')\geq V^D(s')\}}\right],$$

$$b' \leq \overline{B}.$$

where $T_P v(s,b) = v^{NF}$ if the constraint set is empty. A Markov equilibrium value function is a fixed point of T_P under the sup norm: $\|x\| = \sup_{s,b}|x(s,b)|$.

For the example, we narrow attention to the case with only two states, where $S = \{s_1, s_2\}$ with an iid distribution and where $\pi(s_1) = p$. We impose that the endowment is constant, $y(s) = y$ for all $s \in S$; and $V^D(s_1) \equiv \overline{v}^D > V^D(s_2) \equiv \underline{v}^D$.

Consider two initial value function guesses: $v_1(s,b) = \underline{v}^D - \epsilon$ for all s,b and $v_2(s,b) = \underline{v}^D + \epsilon$ for all s,b; for $\overline{v}^D - \underline{v}^D > \epsilon > 0$. Note that $\|v_2 - v_1\| = 2\epsilon > 0$.

Given that the two value functions are assumed to be independent of the debt level; it is possible to compute the solution to the primal above. In particular, for v_1, we have that $T_P v_1(s,b) = u(\min\{y-b,\overline{c}\}) + \beta(p\overline{v}^D + (1-p)\underline{v}^D)$ for $b \leq y$; and $T_P v_1(s,b) = v^{NF}$ for $b > y$. Similarly, for v_2, we have $T_P v_2(s,b) = u(\min\{y-b+R^{-1}(1-p)\overline{B},\overline{c}\}) + \beta(p\overline{v}^D + (1-p)(\underline{v}^D + \epsilon))$ for $y - b + R^{-1}(1-p)\overline{B} \geq 0$; and $T_P v_2(s,b) = v^{NF}$ otherwise.

Now, consider the difference between $T_P v_1$ and $T_P v_2$ evaluated at $b = 0$:

$$T_P(v_2)(s,0) - T_P(v_1)(s,0) = u(\min\{y + R^{-1}(1-p)\overline{B},\overline{c}\}) - u(y) + \beta(1-p)\epsilon >$$

Note that there exists an $\epsilon > 0$ such that $\epsilon < \frac{1}{2-\beta(1-p)}(u(\min\{y + R^{-1}(1-p)\overline{B},\overline{c}\}) - u(y))$. As a result,

$$T_P(v_2)(s,0) - T_P(v_1)(s,0) > 2\epsilon$$

It follows then that

$$\|T_P(v_2) - T_P(v_1)\| \geq |T_P(v_2)(s,0) - T_P(v_1)(s,0)| > 2\epsilon = \|v_2 - v_1\|$$

Hence, T_P is not a contraction.