Take the Short Route: Equilibrium Default and Debt Maturity*

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January 26, 2018

We study the interactions between sovereign debt default and maturity choice in a setting with limited commitment for repayment as well as future debt issuances. Our main finding is that under a wide range of conditions the sovereign should, as long as default is not preferable, remain passive in long-term bond markets, making payments and retiring long-term bonds as they mature but never actively issuing or buying back such bonds. The only active debt-management margin is the short-term bond market. We show that any attempt to manipulate the existing maturity profile of outstanding long-term bonds generates losses, as bond prices move against the sovereign. Our results hold regardless of the shape of the yield curve. The yield curve captures the average costs of financing at different maturities but is misleading regarding the marginal costs.

*This paper draws on two previously circulated papers: "Equilibrium Default", by the last two coauthors; and "Take the Short Route: How to Repay and Restructure Sovereign Debt with Multiple Maturities" by the first two. We thank comments and suggestions from Fernando Alvarez, Andy Atkeson, Cristina Arellano, V.V. Chari, Raquel Fernandez, Emmanuel Farhi, Doireann Fitzgerald, Stephan Guibaud, Dirk Niepelt, Juan Pablo Nicolini and Chris Phelan as well as several seminar participants. We are grateful to Georgios Stefanidis, who provided excellent research assistance. Manuel Amador acknowledges support from the Sloan Foundation and the NSF (award number 0952816). Hugo Hopenhayn and Iván Werning acknowledge support from NSF (award number 0922461).The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

1 Introduction

Short-term debt is often cast as the villain in sovereign debt crises, exposing the fiscal authority to sharp swings in interest rates and raising the vulnerability to a rollover crisis. Nevertheless, when faced with increased spreads, sovereigns tend to lower debt issuances while tilting the composition of new bonds toward shorter maturities.[1] This favoritism towards short-term debt during periods of crisis appears puzzling.

In this paper, we study a model that captures various essential elements of sovereign debt markets: a risk of default that is affected by borrowing decisions, an inability to commit by the sovereign to both repayment as well as the fiscal trajectory, a dynamic choice over debt maturity, and equilibrium bond prices that reflect and constrain these choices. Our model adopts several important features from the sovereign debt literature, enriching them along some dimensions while simplifying along others to isolate the forces having to do with the commitment problems of the borrower.

A primary contribution of the paper is to characterize equilibrium bond prices and examine the sovereign's budget set, exploring how it responds to the maturity structure. A major result is that refraining from both actively issuing or repurchasing long-term bonds maximizes the equilibrium budget set. Strategies that engage with long-term debt are more expensive, despite the fact that actuarially fair investors price all bonds.

In our model, an infinitely-lived sovereign with concave utility borrows by issuing non-contingent bonds of varying maturity in global financial markets. Investors in these markets are risk neutral. The sovereign makes decisions sequentially, with no commitment to its future actions, as in the canonical Eaton and Gersovitz (1981) model. Importantly, this includes both its decision to repay or default and its fiscal and debt management decisions. When the sovereign is highly indebted, a risk of default arises. We assume default to be costly, to ensure that positive borrowing is possible. We model these costs as stochastic so that default is not typically predictable and its probability, instead, rises smoothly with indebtedness.

A short term one-period bond is always available, but we also allow for a rich and flexible choice over the maturity of debt, allowing the issuance and management of any number of bonds of different maturity. A competitive equilibrium involves a sequence

[1] These facts have been documented for the emerging market debt crises of the 1990s and 2000s. Broner et al. (2013) shows that emerging markets reduce total debt issuances when spreads increase, but the reduction is particularly pronounced for bonds with a maturity greater than 3 years, sharply reducing the average maturity of new issuances. Similarly, Arellano and Ramanarayanan (2012) shows that during crisis periods for four emerging market economies, the average maturity of new debt shortens. Perez (2017) examines a large sample of emerging markets and shows that debt issuance drops when spreads are high, and the maturity profile of debt shortens considerably.

of price functions for each maturity and a description of the sovereign's behavior. Each period, each bond price is a function of the entire distribution of outstanding bonds (the state variable). The sovereign takes these price functions as given to solve a dynamic optimization problem that determines, in each period, whether to default or repay and, in the latter case, how much to issue of each bond. Bond prices, in turn, are pinned down by investors taking future default probabilities into account.

Any given sequence of price functions induces an optimal response by the sovereign, which can be used to define a new set of price functions, consistent with this behavior. An equilibrium is a fixed point of this mapping. Fortunately, we can circumvent this seemingly intractable, high dimensional, fixed-point problem. In particular, we show that the equilibrium outcome solves a planning problem representing a constrained efficient contract between the sovereign and new lenders, with all inherited 'legacy' debt from previous lenders serviced as long as the sovereign does not default. Crucially, our mechanism-design formulation determines a path of transfers without reference to any prices; in this sense, it constitutes a 'primal' approach, involving only the allocation. Moreover, the problem admits a tractable dynamic programming formulation, which we exploit.

Our first result is, thus, a welfare theorem of sorts: if a one-period bond is available, then any competitive equilibrium allocation corresponds to a solution to the mechanism-design problem.[2] Since our representation only requires the presence of a one-period bond, and does not place any additional restrictions on the set of available maturities, it follows that the equilibrium allocation can be achieved by exclusively trading one-period bonds. That is, an equilibrium policy is one where the sovereign services interest payments of existing long-term bonds, pays off any maturing bonds, and all new issuances consist of short-term bonds only.

The above result does not rule out that an alternative strategy involving long-term bonds may also be optimal. Indeed, this occurs when default has zero probability, in which case the sovereign is entirely indifferent to the maturity structure of debt. A second main result shows that the optimum is unique whenever default has non-zero probability. In this sense, the debt maturity choice in our model is entirely determined by considerations having to do with default. We also discuss how the mechanism pinning down this maturity choice works through equilibrium bond prices. As we show, any attempt by the sovereign to change the maturity profile of debt generates losses, as bond prices move against such trades. If the sovereign sells long-term bonds in exchange for short-term bonds, the relative price of long-term bonds falls; if the operation is reversed, the relative

[2] This welfare theorem is a useful tool to characterize equilibria, but, as we discuss below, it does not allow one to conclude that the equilibrium is fully efficient.

price rises. Due to these adverse price reactions, it is optimal to engage only in short-term debt.

What drives these price reactions? The answer lies in the effect of the maturity structure on future default risk. This is not a mechanical effect. Indeed, the maturity profile of debt does not affect default decisions nor prices if one were to hold fixed future government consumption. However, the maturity of debt *does* affect the chosen fiscal trajectory, which, in turn, affects sovereign default decisions. In particular, short-term bonds issuances incentivize the sovereign to choose a fiscal trajectory that reduces default, to economize on the costs of rolling over debt in the future. In the absence of legacy long-term debt, this explains why a sovereign would engage only in short term borrowing, producing a constrained efficient outcome.

Now suppose the sovereign does inherit some legacy long-term debt. Given the virtues of short-term debt exposed above, why is it not optimal to repurchase outstanding long-term debt in exchange for short-term debt? The answer is that doing so creates incentives to reduce future borrowing, which in turn lowers the future default probabilities, raising the relative price of long-term bonds. The sovereign prefers to abstain from this operation since this relative price response makes the short-term debt issuance, which is required for the repurchase, too onerous.

This outcome is clearly not efficient, underscoring the fact that our welfare theorem, useful as it is as a tool for the study of equilibrium, does not establish the overall efficiency for all parties. It solves a constrained efficient problem between the borrower and new lenders, but does not include the legacy creditors. If the sovereign, new lenders and all of its legacy creditors could efficiently bargain and restructure debt, the outcome would be to shorten the maturity of the outstanding debt all the way achieving full efficiency with only one-period bonds. However, this cannot take place in a competitive equilibrium, where individual legacy creditors act as price takers, since these legacy investors would have an incentive to hold out and reap a capital gain.

In summary, our results formalize a rationale for the favoritism of short-term borrowing during times where default is likely. The central insight is that with short-term borrowing the costs of higher default risk, reflected in higher interest rates, are entirely borne by the sovereign. Keeping the sovereign "marked-to-market" creates market discipline. Long-term bonds also embed a default premium at the time of issuance, but from the perspective of later periods this premium is a sunk cost that detaches the sovereign from the market and weakens incentives going forward.

Our results are driven by a dual lack of commitment. If the borrower lacked commitment to repay but could commit to the path of debt issuance, conditional on repayment, then debt maturity would be indeterminate. Likewise, if the borrower lacked commitment to future debt issuances, but could commit to the states in which default versus repayment occur, then, again, the maturity of debt would be immaterial. Thus, our model uncovers an interesting interaction between these two commitment problems.

An implication of our analysis is that the incentive for short-term borrowing is not encoded in the market yield curve. It is sometimes argued in popular accounts that short term borrowing is preferred when the yield curve is steep. In our model, given that all bond prices are actuarially fair, the equilibrium yield curve ends up reflecting the expected evolution of the default probability. However, maturity choice is not directly affected by the shape of this yield curve. Indeed, our results hold for *any* shape of the yield curve. That is, our model is capable of generating upward or downward sloping yield curves, i.e. default probabilities that rise or fall over time. Recall that the sovereign should not engage in trades of long-term bonds to avoid adverse *reactions* of bond prices. Thus, our results highlight that that the yield curve reflects *average* costs at different maturities along the equilibrium path, but does not capture the *marginal* costs of financing different maturities off the equilibrium path.

Related Literature

Our paper relates to an extensive literature on maturity choice, both in corporate finance and macroeconomics. We review key strands of analysis here, highlighting how our contribution differs from and complements the existing literature. For expositional clarity, many of our modeling choices are designed to isolate our mechanism from other forces already established in the literature.

Lucas and Stokey (1983) study optimal fiscal policy with complete markets and discuss at length how maturity choice is a useful tool to provide incentives to a government that lacks commitment to taxes and debt issuance, but cannot default. The government has an incentive to manipulate the risk-free real interest rate, by changing taxes which affects investors' marginal utility, to alter the value of outstanding long-term bonds, something ruled out by our small open economy framework with risk neutral investors. Their main result is that the maturity of debt should be spread out, resembling the issuance of consols. Our model instead emphasizes default risk, something absent from their work. Our main result is also the reverse, providing a force for the exclusive use of short-term debt.

A corporate finance literature initiated by Leland (1994) focuses on the optimal default decision for a fixed and given capital structure, that may or may not be chosen optimally in the initial period. In contrast, our model allows the level and maturity of outstanding debt is a sequential choice. In fact, these dynamics are crucial to our results.

The corporate finance and banking literature frequently builds on the notion that bankruptcy involves partially liquidation of an asset, influencing debt maturity choice in a variety of ways. Calomiris and Kahn (1991) and Diamond and Rajan (2001) emphasize how short-term debt and the threat of liquidation can discipline a manager. A similar mechanism is at play in Jeanne (2009) in an international context.

The fact that existing bondholders hold a claim on liquidated assets also makes them vulnerable to dilution. This is the focus of another vast literature starting from Fama and Miller (1972). Dilution implies that the recovery value is lower because it is divided across a larger number of creditors; this effect is present even when the probability of default is constant and unchanged. Sovereign default differs from private bankruptcies in that there is no direct liquidation of assets; to make the distinction even starker we abstract from partial repayment after default. Thus, in our setting there is no dilution in recovery values for a fixed default probability. Instead, our mechanism works through the incentives for further debt issuances, which ultimately impact default decisions. Partial liquidation also endows short-term bonds with implicit seniority. Brunnermeier and Oehmke (2013) show how this may induce a maturity rat-race that results in a collapse of the maturity structure. However, this mechanism is not at play when the liquidation value in bankruptcy is zero, as in our environment.

Maturity choice determines how the available assets span shocks, a feature which arises in closed-economy models with incomplete markets and perfect commitment, such as in Angeletos (2002) and Buera and Nicolini (2004). These papers show that the maturity structure can be appropriately chosen to exploit changes in the yield curve, providing insurance to the fiscal authority.[3] The international quantitative sovereign debt literature emphasizes instead incentives, lack of commitment and the resulting default risk. Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012) show that restricting the government to issue long-term bonds improves the quantitative fit of sovereign debt models, and discuss the government's incentives to dilute existing bond-holders. Arellano and Ramanarayanan (2012) introduces maturity choice into this framework and shows that a calibrated version of the model features shortening maturity as default risk increases. The model combines both a desire to use the maturity structure to hedge fiscal risks, together with the inability of the government to commit to future actions (this latter been the main focus of our analysis). These papers focus on Markov equilibria,

[3] Whether or not the cyclical shifts in the slope of the yield curve can be exploited to insure the risks facing economies remains questionable. For example, in the context of full commitment, Buera and Nicolini (2004) found the positions required to hedge to be implausibly large (see also Faraglia, Marcet and Scott, 2010, for a more recent analysis showcasing several problems with this approach).

just as we do here. Relying instead on trigger strategies, Dovis (forthcoming) generates a shortening of maturity of the stock of debt through the hedging motive alone. In related work, Niepelt (2014) sets up a tractable framework to study maturity choice and highlights that long-term bond prices are relatively elastic, a feature which is also generated in our framework. Cole and Kehoe (2000) highlighted the potential downsides of short-term borrowing in a sovereign debt model, because of self-fulfilling roll-over crises, an aspect that we ignore. Broner et al. (2013) where among the first to focus attention on the general shift to short-term borrowing during crisis in emerging markets. They proposed an explanation that is based on time varying risk premia, something that we rule out by assuming risk neutral lenders.[4]

Our analysis complements these papers by providing a transparent and tractable framework for analyzing maturity choice. Independently of parameterizations, we identify the role of the maturity structure in the incentives to borrow, and explain why an active use of long-term bonds shrinks the budget set of the sovereign through changes in bond prices. To achieve this, we consciously construct our model to eliminate the hedging motive, focusing solely on incentives. In general, the quantitative literature features a stochastic process that drives both the endowment of the government and the cost of default. The second one arises by modeling default costs as a nonlinear function of the stochastic endowment. We separate the two: the endowment process of the government is deterministic in our model, while the cost of default fluctuates stochastically. In our benchmark model, iid default payoffs guarantee that the maturity composition provides no insurance. In an extension, we introduce a simple form of persistence that creates a trade-off between insurance and incentives.

The sub-optimality of repurchasing long-term bonds on secondary markets is reminiscent of a result in Bulow and Rogoff (1988, 1991). Their analysis turns on a finite amount of resources available to pay bond holders. In such a situation, a bond buyback concentrates the remaining bondholders' claim on this collateral, and so drives up the price of bonds. Indeed, the sovereign would like to dilute existing bond holders by selling additional claims to this fixed recovery amount. The link with Bulow and Rogoff is discussed in more detail in Section 5.3.

An important feature of our analysis is our focus on outside option shocks. Most of the literature has primarily focused on income shocks as the main source of uninsurable risk. We have made this choice to transparently highlight how the incentives for fiscal policy, and the corresponding budgetary implications, are sensitive to maturity choice along the

[4] See also the work of Perez (2017) for a more recent data analysis, covering a larger sample of emerging markets; as well for an alternative explanation based on asymmetric information.

transition. In particular, maturity choice is not used to *hedge* risk in our environment. A hedging motive would arise if shocks affected consumption *absent* default, and if these changes in consumption had a non-zero covariance with bond prices. Our environment abstracts from this hedging motive as shocks do not change equilibrium consumption in periods of no default (that is, consumption conditional on not-defaulting is deterministic). Different maturity bonds in our environment are therefore not useful to hedge the risks we consider. Finally, the desire to hedge is also operative in models of full commitment under incomplete markets, while our results arise exclusively due to limited commitment. We provide an extension with a hedging motive in Section 8.

2 The Environment

Consider a small open economy in discrete time with periods $t = 0, 1, 2, \ldots$ There is a single, freely tradable, consumption good. The economy receives a deterministic sequence of endowments $\{y_t\}$, where $y_t \in (0, \bar{y})$.[5]

Preferences. The sovereign makes economic decisions on behalf of the small open economy. Preferences over consumption streams are characterized by the following utility function:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

where $\beta \in (0,1)$ and u is a continuous, strictly increasing, and strictly concave function defined over the non-negative reals. We denote $\underline{u} \equiv u(0)$ and $\overline{u} \equiv \lim_{c\to\infty} u(c)$, with $\underline{u}, \overline{u} \in (-\infty, +\infty)$. Let $\underline{V} \equiv \underline{u}/(1-\beta)$ and $\overline{V} \equiv \overline{u}/(1-\beta)$.

Financial Markets. The country engages in financial trade with the rest of the world, by issuing bonds of different maturities. The financial market is populated by competitive, risk-neutral investors with discount factor R^{-1}. We assume $\beta R \leq 1$, a natural restriction given the small open economy assumption.

A short-term bond is assumed always available, but we are flexible regarding the assumed availability of long-term bonds. At the beginning of each period, the sovereign

[5] Note that the endowment stream is not subject to unanticipated fluctuations. It is well known that in models with incomplete markets and shocks to output or government expenditure, maturity choice can be used to partially or fully replicate a full set of state contingent assets. To maintain a clear distinction between incentives versus spanning, we abstract from income fluctuations.

inherits a portfolio characterized by a sequence of net liabilities going forward. We denote by $b_t \in \mathbb{R}$ the amount of one-period bonds issued in period $t-1$ and due in period t. The portfolio of long-term liabilities is a sequence $\boldsymbol{l}^t = \{l_0, l_1, \dots\}$, where $l_k \in \mathbb{R}$ represents the net amount due k periods ahead (period $t+k$). When convenient, we drop the t subscript and superscript and let $\{b, \boldsymbol{l}\}$ denote the current period's inherited liabilities. In what follows, we denote by $\boldsymbol{l}_k$ the k-th element of the sequence $\boldsymbol{l}$, and use $\boldsymbol{l}_{\geq k} \equiv \{l_k, l_{k+1}, \dots\}$ to denote the tail of the sequence $\boldsymbol{l}$ starting k periods ahead. Note that $\boldsymbol{l}_0$ represents liabilities that were long-term when issued but are due in the current period; hence, $b + \boldsymbol{l}_0$ represent total debt due in the current period. We restrict $\boldsymbol{l}$ to lie in the set of bounded sequences, i.e. $|\boldsymbol{l}_k| \leq \bar{l} < \infty$ for some $\bar{l}$ for all $k \geq 0$. Let L denote the set of liability sequences that satisfy this boundedness condition.

The government enters a period with liabilities $\{b, \boldsymbol{l}\}$. It has the option to default, which we discuss below; otherwise, it issues new one-period bonds $b' \in \mathbb{R}$ and shuffle its long-term liabilities to a new sequence $\boldsymbol{l}' \in \Gamma(\boldsymbol{l}, t)$. The mapping $\Gamma : L \times \mathbb{N} \to L$ characterizes the available set of maturities at time t given $\boldsymbol{l}$.[6] We do not need to impose any assumptions on this set, except for one natural restriction: it is always feasible to exit the period with the same long-term promises that the government inherited; that is, $\boldsymbol{l}_{\geq 1} \in \Gamma(\boldsymbol{l}, t)$. This notation nests environments which range from an infinite number of maturities that are potentially traded, to cases commonly considered in the quantitative literature in which only a handful of finite maturity bonds or exponentially decaying bonds are available (Hatchondo and Martinez, 2009; Arellano and Ramanarayanan, 2012).

Default. At the beginning of each period, the government has the option to default, in which case all lenders receive a payout of zero. A fundamental issue in sovereign debt markets concerns the limited ability of creditors to enforce contracts with a sovereign government. A large literature has identified reputational and legal mechanisms which can sustain debt repayment. These approaches share the general feature that default is determined by the sovereign comparing the value of repayment to the value achieved by default. We let v_t^D denote the value achieved by default in period t. We model this value directly assuming it follows a stochastic process, with the following properties for $t \geq 1$.

Assumption 1. *The outside option is such that*

(i) v_t^D *is drawn from a continuous c.d.f.* F_t*;*

[6] Note that if Γ does not restrict $\boldsymbol{l}_0'$, then one-period debt issuance b' and $\boldsymbol{l}_0'$ are perfect substitutes and any one of the two is redundant.

It is also possible to make the mapping Γ depend on the one-period bonds, both b and b', without affecting the results. For notational simplicity, we ignore that in what follows.

(ii) v_t^D *is independent across time and independent of liabilities;*

(iii) $v_t^D \in V_t^D \equiv \left[\underline{v}_t^D, \overline{v}_t^D\right]$ *with* $\underline{v}_t^D > \underline{V}$ *and* $\overline{v}_t^D < \overline{V}^D < \overline{V}$ *for some* $\overline{V}^D$*;*

(iv) there exists a $u^{min} > \underline{u}$ *such that* $u^{min} + \beta\overline{V} < \underline{v}_t^D$ *for all* $t \geq 1$*; and*

(v) there exists $u^{max} < \overline{u}$ *such that* $u^{max} + \beta \int v^D \, dF_{t+1}(v^D) > \overline{v}_t^D$ *for all* $t \geq 1$*.*

The continuous distribution assumption in (i) is adopted for expositional convenience. The assumption of independence in (ii) is of more importance and allows us to abstract from the hedging benefit of long-term bonds and focus on incentives. The bounded support restriction in (iii) follows naturally from the notion that v_t^D represents a discounted value of utility achieved after default. Restriction (iv) ensures that receiving zero consumption triggers default with probability one; a simplifying assumption that serves to ensure an interior consumption allocation. The final restriction (v) ensures that default does not occur if debt is sufficiently low.

Timing and Government Problem. At the beginning of the period, the government observes its realized outside option, v_t^D. It then decides whether to take this option and default. If it does not default, it issues new bonds and consumes. When trading, the government takes as given an equilibrium price schedule for its portfolio of bonds. As is standard in the sovereign debt literature, we restrict attention to Markov equilibria where prices are a function only of payoff relevant state variables,[7] in this case, the inherited liabilities and the current period t. Let q denote the one-period bond price function and Q the cost of changing the portfolio of future long-term liabilities from $\boldsymbol{l}$ to $\boldsymbol{l}'$. Given inherited liabilities $(b, \boldsymbol{l})$, the government budget constraint at t is

$$c \leq y_t - b - \boldsymbol{l}_0 + q(b', \boldsymbol{l}', t)b' + Q(\boldsymbol{l}, \boldsymbol{l}', b', t), \tag{BC}$$

where b' and $\boldsymbol{l}'$ denote liabilities brought into to the next period and

$$Q(\boldsymbol{l}, \boldsymbol{l}', b', t) = \sum_{k=1}^{\infty} \rho_k(b', \boldsymbol{l}', t)(\boldsymbol{l}'_{k-1} - \boldsymbol{l}_k), \tag{2}$$

where ρ_k is the price of a promise to pay one unit in k periods. Note that $Q(\boldsymbol{l}, \boldsymbol{l}_{\geq 1}, b, t) = 0$, so there is no cost associated with carrying forward the inherited long-term liabilities unchanged.

[7] As will be clear from the fact that the competitive equilibrium solves a planning problem, the results may hold for non-Markovian equilibria as well.

Let $V(b, \boldsymbol{l}, t)$ denote the value of not defaulting in period t given liabilities $(b, \boldsymbol{l})$. If $V(b, \boldsymbol{l}, t)$ is less than v_t^D, the government defaults; otherwise it solves:

$$V(b, \boldsymbol{l}, t) = \sup_{c \geq 0, b', \boldsymbol{l}' \in \Gamma(\boldsymbol{l}, t)} \left\{ u(c) + \beta \int \max \left\{ V(b', \boldsymbol{l}', t+1), v^D \right\} dF_{t+1}(v^D) \right\} \tag{3}$$

subject to the budget constraint (BC) and the No-Ponzi condition $b' \leq \bar{B}$, for some finite $\bar{B} \geq R(\bar{y} + \bar{l})/(R-1)$.[8] The continuation value incorporates the strategic default decision next period. Denote by $\mathcal{B}(b, \boldsymbol{l}, t)$ and $\mathcal{L}(b, \boldsymbol{l}, t)$ the optimal policies for one-period bonds and long-term liabilities, respectively. If the constraint set for (3) is empty, i.e. there are no way to repay current liabilities, even with zero consumption, we set $V(b, \boldsymbol{l}, t) = \underline{V}$; this ensures that default is triggered at the beginning of this period.

Lenders' Break-Even Condition. Lenders must break even in expectation. To compute prices that are consistent with this condition requires computing default probabilities. If the government enters period t with debt $(b, \boldsymbol{l})$ and exits with portfolio $(b', \boldsymbol{l}')$, the break-even condition for one-period debt is[9]

$$q(b', \boldsymbol{l}', t) = R^{-1} F_{t+1}(V(b', \boldsymbol{l}', t+1)). \tag{4}$$

To compute the break even condition for long debt, we start from an initial state $(b, \boldsymbol{l}, t)$, and iterate on the government policy functions forward, obtaining the probability of default. Let $\{b_k\}_{k=1}^{\infty}$ and $\{\boldsymbol{l}^k\}_{k=1}^{\infty}$ be given by the recursion

$$b_{k+1} = \mathcal{B}(b_k, \boldsymbol{l}^k, t+k), \text{ and } \boldsymbol{l}^{k+1} = \mathcal{L}(b_k, \boldsymbol{l}^k, t+k) \text{ for } k \geq 1,$$

with initial conditions $b_1 = b'$ and $\boldsymbol{l}^1 = \boldsymbol{l}'$. Then $\rho_1(b', \boldsymbol{l}', t) = q(b', \boldsymbol{l}', t)$ and for $k \geq 2$,

$$\rho_k(b', \boldsymbol{l}', t) = \rho_{k-1}(b', \boldsymbol{l}', t) q(b_k, \boldsymbol{l}^k, t+k-1), \tag{5}$$

a version of the expectations hypothesis. Iterating, using equation (4), we have

$$\rho_k(b', \boldsymbol{l}', t) = R^{-k} \prod_{i=1}^{k} F_{t+i}(V_{t+i})$$

[8] The value $R(\bar{y} + \bar{l})/(R-1)$ is an upper bound on the present value of the country's endowment net legacy payments.

[9] The continuous c.d.f assumption allows to ignore the point where the government is indifferent between defaulting or not, as it is a zero probability event.

where $V_{t+i} = V(b_i, \boldsymbol{l}^i, t+i)$.

Equilibrium. We are now ready to define an equilibrium in the usual way:

Definition 1. A *Markov Competitive Equilibrium* (CE) consists of functions $\{V, q, Q, \rho, \mathcal{B}, \mathcal{L}\}$ such that:

(i) $V : \mathbb{R} \times L \times \mathbb{N} \to [\underline{V}, \overline{V}]$ solves the Bellman equation (3);

(ii) $\mathcal{B} : \mathbb{R} \times L \times \mathbb{N} \to \mathbb{R}$ and $\mathcal{L} : \mathbb{R} \times L \times \mathbb{N} \to L$ are policies that attain the maximum in (3);

(iii) $Q : L \times L \times \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ satisfies equation (2);

(iv) $q : \mathbb{R} \times L \times \mathbb{N} \to [0, 1]$ satisfies equation (4);

(v) $\rho_k : \mathbb{R} \times L \times \mathbb{N} \to [0, 1]$ satisfy equation (5) for all integers $k \in \mathbb{N}, k \geq 1$.

As stated, the equilibrium represents a complicated fixed-point problem. There are potentially an infinite number of price schedules (one for each maturity), which depend on the government's fiscal policy going forward. The government's policies, in turn, depend on equilibrium price schedules. However, in the next section, we prove that the competitive equilibrium solves a modified planning problem, which allows a direct characterization of key properties of the equilibrium.

Discussion. Our modeling choices are guided by our focus on scenarios where the risk of default is a first-order concern for both consumption-saving decisions as well as the choice over debt maturity. Of course, in reality there may be many other considerations. During tranquil periods sovereigns issue a range of maturities to smooth tax distortions; to provide a source of safe assets for savers; to facilitate payments systems; and to insure against fluctuations in tax revenues, output or interest rates. However, in the midst of a sovereign debt crisis these considerations are to a large extent dominated by a sovereign's need to issue new debt to skeptical investors, to roll over or buy back outstanding debt, and perhaps to reduce the outstanding stock of debt in a credible (that is, time consistent) manner. Our model is intended to transparently isolates the role of maturity choice under the threat of default.

Before moving on, it is helpful to state a simple result about any Markov equilibrium:

Lemma 1. *Consider a Markov Competitive Equilibrium with value function V. Then, for any $(\boldsymbol{l}, t)$, the value function $V(b, \boldsymbol{l}, t)$ is non-increasing in b. In addition, for any $v \in [\underline{v}_t^D, \overline{V})$, there exists a finite value b such that $V(b, \boldsymbol{l}, t) = v$.*

3 A Planning Problem

We now characterize competitive equilibria by considering a modified planning problem. We show although equilibria are not necessarily efficient in the usual Pareto sense, we can characterize them by solving a planning problem. To motivate the approach, consider the following contracting problem. A government enters period t with legacy liabilities $(b, \boldsymbol{l})$. It then contracts with a new set of lenders, receiving a sequence of consumption $\{c_{t+k}\}_{k\geq 0}$ in exchange for a sequence of payments $\{y_{t+k} - \boldsymbol{l}_k - c_{t+k}\}_{k\geq 0}$ conditional on not defaulting through $t+k$. As long as the government does not default it repays any legacy claims currently due. We consider contracts that maximize the joint surplus between the government and its new lenders. At it turn out, the allocations delivered by such contracts are equivalent to the outcome of competitive equilibria.

This contracting problem considers legacy lenders and new lenders as different agents, and legacy lenders' payoffs are not included in the joint surplus.[10] This is precisely why equilibria are not generally Pareto efficient, a point we discuss in detail in Section 7.

3.1 Efficient Contracts

Starting from any period t where the government has not defaulted and has long-term liabilities denoted by $\boldsymbol{l}^t = \boldsymbol{l}$, consider the Pareto problem of allocating the country's resources from time t onwards between the government and a representative new lender, taking as given that (i) the government will default whenever it receives an outside option shock that is higher than its continuation payoff; (ii) if no default occurs in a given period, previously issued long-term claims must be paid; and (iii) in case of default, all lenders receive zero. In this particular planning problem, we also impose that the outside option shocks are not-observable, and as a result, the resulting consumption allocation cannot be made contingent on the realization of v_t^D, absent default.[11]

An allocation is characterized by a consumption sequence $\{c_{t+k}\}_{k\geq 0}$ which determines the amount the government consumes at time $t+k$ conditional on no default through $t+k$. This sequence implies a corresponding sequence of values, $\{V_{t+k}\}_{k\geq 0}$, where V_{t+k} denotes the expected value of the government conditional on not defaulting through time $t+k$. Incentive compatibility implies that if $v_{t+k}^D > V_{t+k}$, the government defaults in $t+k$;

[10] Of course, in equilibrium, the set of old lenders and new lenders may overlap; the idea of contracting with *new* versus *old* lenders is simply a useful device to characterize competitive equilibrium allocations.

[11] Similarly, we restrict attention to deterministic allocations absent default (that is no randomization on the part of the Planner), as these represent the allocations that are feasible in equilibrium given our asset structure.

if $v^D_{t+k} < V_{t+k}$ the government does not default; and is indifferent if $v^D_{t+k} = V_{t+k}$.[12] Given $\{c_{t+k}\}$, we can then define V_{t+k} recursively:

$$V_{t+k} = u(c_{t+k}) + \beta \int_{\underline{v}^D_{t+k+1}}^{\overline{v}^D_{t+k+1}} \max\left\{V_{t+k+1}, v^D\right\} dF_{t+k+1}(v^D), \tag{6}$$

for all $k \geq 0$. The sequence $\{V_{t+k}\}_{k\geq 0}$ is the unique solution to this difference equation that satisfies $V_{t+k} \in [\underline{V}, \overline{V}]$.

Definition 2. An *incentive-compatible allocation* from time t onwards is a sequence of consumption and associated values to the government $\{c_{t+k}, V_{t+k}\}_{k=0}^{\infty}$, such that $c_{t+k} \geq 0$; and $\{V_{t+k}\}_{k=0}^{\infty}$ solves (6).

Conditional on legacy long-term liabilities inherited in period t, $\boldsymbol{l}$, the resource constraint implies that the net payments (absent default) to the new lender n_{t+k} associated with an allocation satisfies:

$$n_{t+k} = y_{t+k} - \boldsymbol{l}_k - c_{t+k}. \tag{7}$$

Thus the allocation $\{c_{t+k}\}_{k\geq 0}$ and $\boldsymbol{l}$ defines a stream of net payments to the new lender. Let B_t denote the expected present value of these payments conditional on not defaulting in period t:

$$B_t = \sum_{k=0}^{\infty} R^{-k} \left(\prod_{i=1}^{k} F_{t+i}(V_{t+i})\right)(y_{t+k} - \boldsymbol{l}_k - c_{t+k}), \tag{8}$$

where the product in brackets represents the probability of not defaulting through period $t+k$ conditional on not defaulting in period t. This product is evaluated to be one at $k = 0$.

As discussed above, we consider the notion of efficiency that weighs the welfare of new lenders and the government, but disregards the impact on payoffs to existing creditors. More formally:

Definition 3. An incentive-compatible allocation from time t onwards fixing long-term liabilities $\boldsymbol{l} \in L$, $\{c_{t+k}, V_{t+k}\}_{k=0}^{\infty}$, is *efficient at time t* if there does not exist an alternative incentive-compatible allocation from time t onwards $\{\hat{c}_{t+k}, \hat{V}_{t+k}\}_{k=0}^{\infty}$ such that $\hat{V}_t \geq V_t$ and $\hat{B}_t \geq B_t$ with at least one of these inequalities strict, where B_t and $\hat{B}_t$ denote the respective solutions to equation (8).

[12] With a continuous distribution, the indifference point is measure zero, and can be ignored.

Given equation (6), the value to a government that does not default at time t can never fall below $\underline{u} + \beta \int v^D dF_{t+1}(v^D)$, given the non-negativity of consumption and the option to the default in the future. Let us denote by $B^\star(v, \boldsymbol{l}, t) : [\underline{u} + \beta \int v^D dF_{t+1}(v^D), \overline{V}] \times L \times \mathbb{N} \to \mathbb{R}$, the solution to the associated Pareto problem:

$$B^\star(v, \boldsymbol{l}, t) = \sup_{\{c_{t+k}, V_{t+k}\}_{k=0}^{\infty}} \sum_{k=0}^{\infty} \left(\prod_{i=1}^{k} R^{-1} F_{t+i}(V_{t+i}) \right) (y_{t+k} - \boldsymbol{l}_k - c_{t+k}) \tag{9}$$

subject to:

$$\{V_{t+k}\}_{k=0}^{\infty} \text{ solves (6) given } \{c_{t+k}\}_{k=0}^{\infty}$$

$$V_t \geq v. \tag{10}$$

An efficient allocation, as in Definition 3, must solve this problem.

The following lemma will prove useful later on. It shows that we only need to consider incentive compatible allocations where equation (10) holds with equality; and, because of $\beta R \leq 1$, where the sequence $\{V_{t+k}\}$ is bounded.

Lemma 2. *Let the state be $(v, \boldsymbol{l}, t)$ with $v \in [\underline{v}_t^D, \overline{V})$. Then for the maximization in Problem 9, it suffices to consider only incentive compatible allocations such that $V_t = v$ and $V_{t+k} \leq \max\left\{v, \overline{V}^D\right\}$ for all $k \geq 1$; where $\overline{V}^D$ is as in Assumption 1.iii.*

The next subsection establishes an associated First Welfare Theorem; namely, that competitive equilibrium allocations are efficient as in Definition 3.[13]

4 A Welfare Theorem

In this section, we show that competitive equilibria are efficient in the sense of Definition 3. This welfare theorem (Proposition 1 below) will allow us in Section 5 to characterize the competitive equilibrium by analyzing a simple planning problem and identify the costs of trading long-term bonds.

Towards this goal, consider a competitive equilibrium with an associated value function V and price functions q and Q. Given these, let us define the following function

[13] It is also the case that any efficient allocation can be decentralized as a competitive equilibria. Given our interest in characterizing competitive equilibria, we omit discussion of this version of the Second Welfare Theorem.

B:

$$B(v,\boldsymbol{l},t) \equiv \sup_{c\geq 0, b'\leq \bar{B}, \boldsymbol{l}', v'\geq \underline{v}^D_{t+1}} \left\{ y_t - \boldsymbol{l}_0 - c + q(b',\boldsymbol{l}',t)b' + Q(\boldsymbol{l},\boldsymbol{l}',b',t) \right\} \tag{11}$$

$$\text{subject to :}$$

$$v = u(c) + \beta F_{t+1}(v')v' + \beta \int_{v^D \geq v'} v^D dF_{t+1}(v^D)$$

$$\boldsymbol{l}' \in \Gamma(\boldsymbol{l},t) \text{ and } v' = V(b',\boldsymbol{l}',t+1).$$

where the maximization problem represents the dual of problem 3.[14] Not surprisingly, B is the inverse of the equilibrium value function V:

Lemma 3. *Consider a Markov Competitive Equilibrium with value function V and price functions q, Q. Then, $B(V(b,\boldsymbol{l},t),\boldsymbol{l},t) = b$ for any $(b,\boldsymbol{l},t)$ such that $V(b,\boldsymbol{l},t) \geq \underline{v}^D_t$ and where B is defined as in* (11).

A first key result is that the efficient allocation in Problem 9 provides an upper bound on any equilibrium inverse value function B:

Lemma 4. *Consider a Markov Competitive Equilibrium, and let B be as defined in equation* (11)*. Then, for any $(\boldsymbol{l},t)$ and $v \in [\underline{v}^D_t, \overline{V})$, we have $B(v,\boldsymbol{l},t) \leq B^\star(v,\boldsymbol{l},t)$.*

The proof of this result uses the no-arbitrage equilibrium restriction on prices to show that the constraint set of the dual problem 11 is a subset of the constraints for the planning problem 9. Intuitively, take a given equilibrium consumption sequence that may involve trading long-term bonds. In particular, suppose the government trades a portfolio of bonds in the first period. The amount raised must equal the present discounted value of payments to all bondholders net of the pre-committed payments to legacy bondholders. That is, the amount raised must equal the present value of the endowment minus the sum of consumption and payments to legacy bondholders, discounted at the equilibrium probability of default. This equality follows from equilibrium pricing condition, and is the key step in the proof. It is then possible for the planning problem to implement the same consumption allocation, which implies the same default probabilities, the same value to the government, and the same net payments. This stream of payments is the objective in the planning problem and therefore must weakly dominate that of any equilibrium.

We now proceed to show that the upper bound of Lemma 4 is achieved in equilibrium. For this, we first show that we can use equation 11 to generate a lower bound on B by

[14]Note that we have restricted attention to continuation values weakly higher than $\underline{v}^D_{t+1}$ (the lowest possible outside option) as this is without loss.

artificially restricting $\boldsymbol{l}' = \boldsymbol{l}_{\geq 1}$. Under this restriction, $Q = 0$, and from equation 11 we obtain that[15]

$$B(v, \boldsymbol{l}, t) \geq \sup_{c \geq 0, v' \geq \underline{v}^D_{t+1}} \left\{ y_t - \boldsymbol{l}_0 - c + R^{-1} F_{t+1}(v') B(v', \boldsymbol{l}_{\geq 1}, t+1) \right. \tag{12}$$

$$\text{subject to :}$$

$$v = u(c) + \beta F_{t+1}(v') v' + \beta \int_{v^D \geq v'} v^D dF_{t+1}(v^D)$$

Note that if the inequality in (12) were to always hold with equality, then the resulting functional equation would correspond to the Bellman equation that must be solved by $B^\star$. Exploiting this idea, together with the boundedness result of Lemma 2, we obtain the following lemma:

Lemma 5. *Consider a Markov Competitive Equilibrium, and let B be as defined in equation* (11)*. Then, for any* $(\boldsymbol{l}, t)$*, and* $v \in [\underline{v}^D_t, \overline{V})$*,* $B(v, \boldsymbol{l}, t) \geq B^\star(v, \boldsymbol{l}, t)$*.*

Lemma 5 shows that the presence of Q in equation (11) cannot reduce value as compared to an efficient outcome. The proof of the lemma establishes that it is always possible to choose an allocation that replicates the efficient outcome, and sets $Q = 0$. Note that for this lower-bound result, we do not need to know the equilibrium shape of Q, except for the property that $Q = 0$ when no trades in long-term bonds occur. This is different from Lemma 4, were we exploited the equilibrium restrictions that arbitrage imposes on Q, and showed that the equilibrium value cannot do better than the efficient outcome. Putting these wtwo lemmas together implies that, for any Markov equilibria, $B^\star = B$:

Proposition 1. *[Efficiency of CE] Let* $\{V, q, Q, \rho, \mathcal{B}, \mathcal{L}\}$ *be a Markov Competitive Equilibrium. Then* $b = B^\star(V(b, \boldsymbol{l}, t), \boldsymbol{l}, t)$ *for any* $V(b, \boldsymbol{l}, t) \geq \underline{v}^D_t$*; that is, a competitive equilibrium allocation is efficient.*

Because the competitive equilibrium solves a planning problem, the equilibrium value function $V(b, \boldsymbol{l})$ is unique. This immediately implies that the one-period bond price schedule is also uniquely determined. However, there may be multiple allocations that solve the same planning problem, implying that the long-term bond price schedules are not necessarily unique.

[15] We first use the equilibrium condition $q(b', \boldsymbol{l}', t) = R^{-1} F_{t+1}(V(b', \boldsymbol{l}', t+1))$ together with the constraint $v' = V(b', \boldsymbol{l}', t+1)$ to substitute $q(b', \boldsymbol{l}', t)$ by $R^{-1} F_{t+1}(v')$. In addition, $v' = V(b', \boldsymbol{l}', t+1)$ implies that $b' = B(v', \boldsymbol{l}', t+1)$, by Lemma 3, so we can substitute out for b' for $B(v', \boldsymbol{l}', t+1)$, and $V(b', \boldsymbol{l}', t+1)$ for v'. In addition, we would still need to impose the restriction that any value v' considered in the dual problem is attainable in equilibrium, i.e., there exists a b' that delivers $v' = V(b', \boldsymbol{l}', t+1)$. Lemma 1 guarantees that this can always be done. The restriction that $\boldsymbol{l}' = \boldsymbol{l}_{\geq 1}$ provides then a lower bound.

Recall that in the discussion leading up to Lemma 5, we achieved the efficient payoff to new lenders in a competitive equilibrium by not trading long-term bonds. More generally, it is without loss to consider government policies that do not adjust long-term liabilities. We state that as a corollary to the proposition:

Corollary 1. *[Sufficiency of Short-Term Debt] Let $\{V, q, Q, \rho, \mathcal{B}, \mathcal{L}\}$ be a Markov Competitive Equilibrium. Then, there exists a Markov Competitive equilibrium with $\{V, q, \hat{Q}, \hat{\rho}, \hat{\mathcal{B}}, \hat{\mathcal{L}}\}$ where $\hat{\mathcal{L}}$ is such that $\hat{\mathcal{L}}(b, \boldsymbol{l}, t) = \boldsymbol{l}_{\geq 1}$.*

Proposition 1 and Corollary 1 imply that an efficient allocation can be implemented in equilibrium using strategies that actively trade only one-period debt. This raises the question of whether there are equilibria that involve active trading in long-term bonds, and if not, what makes one-period bonds special. These questions are the subject of the next section.

5 The Cost of Trading Long-Term Debt

The previous section showed that an equilibrium allocation is efficient in the sense of Definition 3, and it is sufficient to consider policies such that the government trades only one-period claims. We now discuss why trading long-term bonds may generate strict losses to the government. We begin with an important property of the inverse value function B; namely, that it is convex in long-term liabilities. We use this to demonstrate that issuing or repurchasing long-term bonds is dominated by trading only short-term liabilities, and strictly so under certain conditions. The section concludes with a discussion of this key result.

5.1 Convexity

Recall that competitive equilibria deliver $B^{\star}(v, \boldsymbol{l}, t)$ to holders of one-period bonds absent default, conditional on the government's value v and outstanding long-term debt $\boldsymbol{l}$. An important property of the inverse value function $B^{\star}$ is that it is convex in $\boldsymbol{l}$ (and strictly so under some conditions), and its gradient is given by market prices:

Proposition 2. *[Convexity] Let $\{c_{t+k}, V_{t+k}\}_{k=0}^{\infty}$ be an efficient allocation at time t that delivers $V_t = v \geq \underline{v}_t^D$, given long-term liabilities $\boldsymbol{l} \in L$. Then, for any other $\boldsymbol{l}' \in L$:*

$$B^{\star}(v, \boldsymbol{l}', t) \geq B^{\star}(v, \boldsymbol{l}, t) - \sum_{k=0}^{\infty} p_k \left(\boldsymbol{l}'_k - \boldsymbol{l}_k \right)$$

where $p_k \equiv \prod_{i=1}^{k} R^{-1} F_{t+i}(V_{t+i})$ *with* $p_0 = 1$. *The inequality is strict if there exists* $j > 0$ *such that (i)* $p_{j-1} > 0$*; (ii)* $F_{t+j}(V_{t+j}) \in (0,1)$*; and (iii)* $\sum_{k=j}^{\infty} p_k \left(\boldsymbol{l}'_k - \boldsymbol{l}_k\right) \neq 0$.

The first part of the proposition follows from the fact that a consumption allocation that delivers v under $\boldsymbol{l}$ also delivers it under $\boldsymbol{l}'$. Moreover, the objective function in Problem 9 is linear in consumption, and hence moving from $\boldsymbol{l}$ to $\boldsymbol{l}'$ without changing the consumption allocation has a linear effect on value. It then follows that implementing the $\boldsymbol{l}$-allocation is feasible for $\boldsymbol{l}'$ and represents a linear change in the objective, but re-optimizing may be better.

The second part states conditions where re-optimizing, once the state variable has changed, leads to a strict improvement. Condition (i) says that period $t + j$ is reached without default with positive probability; that is, the allocation in period $t + j$ and beyond is relevant to payoffs. Condition (ii) says that default in period $t + j$ conditional on reaching $t + j$ is interior. This implies a small perturbation in the allocation starting from $t + j$ will affect the default probability in period $t + j$. The final condition states that the two long-term liability sequences differ in period $t + j$ or after. We hold off on the intuition behind this result until Subsection 5.3 below.

5.2 Implications for Cost of Long-Term Bonds

We are now ready to consider the cost of long-term trades. Consider an equilibrium and a situation where the government starts time t with state $(b, \boldsymbol{l})$. We know by Proposition 1 that an optimal strategy for the government at this point would be to issue only one-period debt and remain passive in the long-term markets by setting $\boldsymbol{l}' = \boldsymbol{l}_{\geq 1}$. However, perhaps there is an equivalent-payoff strategy that involves trading long-term debt as well. To explore this, suppose that at time t the government pursues a debt policy of $(b', \boldsymbol{l}')$, where $\boldsymbol{l}'_k \neq \boldsymbol{l}_{k+1}$ for some $k \geq 1$. The latter non-equality implies that the government actively issues or repurchases long-term debt.

The equilibrium payoff to the government from the $(b', \boldsymbol{l}')$ strategy is:

$$u(c) + \beta F_{t+1}(V_{t+1}) V_{t+1} + \beta \int_{v^D > V_{t+1}} v^D dF_{t+1}(v^D), \tag{13}$$

where $V_{t+1} = V(b', \boldsymbol{l}', t+1)$, and from the budget constraint we obtain[16]

$$c = y_t - b - \boldsymbol{l}_0 + R^{-1}F(V_{t+1})\left(B(V_{t+1}, \boldsymbol{l}', t+1) + \sum_{k=0}^{\infty} p'_k(\boldsymbol{l}'_k - \boldsymbol{l}_{k+1})\right),$$

where p'_k are the equilibrium prices in period $t+1$, consistent with state $(b', \boldsymbol{l}', t+1)$, that is, $p'_k = \rho_k(\mathcal{B}(b', \boldsymbol{l}', t+1), \mathcal{L}(b', \boldsymbol{l}', t+1), t+1)$ for $k > 0$, and $p'_0 = 1$.

Consider now an alternative trade that only uses one-period bonds, but achieves the same continuation value. That is, suppose the government issues $\hat{b}'$ such that $V(\hat{b}', \boldsymbol{l}_{\geq 1}, t+1) = V_{t+1}$. The fact that continuation values are identical implies that the one-period bond price, $R^{-1}F(V_{t+1})$, remains the same as under the original $(b', \boldsymbol{l}')$ strategy; importantly, long-term bond prices may change, but these have no budgetary impact as no long-term debt is issued or purchased in this alternative. The budget set implies that the associated consumption is:

$$\hat{c} = y_t - b - \boldsymbol{l}_0 + R^{-1}F(V_{t+1})B(V_{t+1}, \boldsymbol{l}_{\geq 1}, t+1),$$

and the government's utility is:

$$u(\hat{c}) + \beta F_{t+1}(V_{t+1})V_{t+1} + \beta \int_{v^D > V_{t+1}} v^D dF_{t+1}(v^D). \tag{14}$$

Comparing (14) to (13), the alternative strategy dominates if $\hat{c} > c$. Comparing the associated expressions for consumption, this is the case if:

$$B(V_{t+1}, \boldsymbol{l}_{\geq 1}, t+1) > B(V_{t+1}, \boldsymbol{l}', t+1) - \sum_{k=0}^{\infty} p'_k(\boldsymbol{l}_{k+1} - \boldsymbol{l}'_k).$$

Given the fact that $B = B^\star$, and that any competitive equilibrium allocation is efficient, it follows that this expression is the same as that in Proposition 2, with the roles of $\boldsymbol{l}$ and $\boldsymbol{l}'$ reversed. Therefore, Proposition 2 implies that this strict inequality holds if the conditions (i),(ii), and (iii) are satisfied. Hence, to achieve a given continuation value, the government has higher consumption if it remains passive in long-term debt markets: trading in the long-term bond markets can only shrink the government's budget set.

The "shrinking" of the budget set is due to the fact that active trades in long-term bonds have adverse impact on prices. We highlight this feature diagrammatically in Figure 1. The diagram reflects the above scenario, in which the government enters period t with long-term liabilities $\boldsymbol{l}$ and pursues a strategy that yields a continuation value V_{t+1}.

[16] We use the fact that $\rho_{k+1}(b', \boldsymbol{l}', t) = q(b', \boldsymbol{l}', t)\rho_k(b'', \boldsymbol{l}'', t+1)$ where $b'' = \mathcal{B}(b', \boldsymbol{l}', t+1)$ and $\boldsymbol{l}'' = \mathcal{L}(b', \boldsymbol{l}', t+1)$. Substituting this into the definition of Q and re-arranging yields the expression in the text.

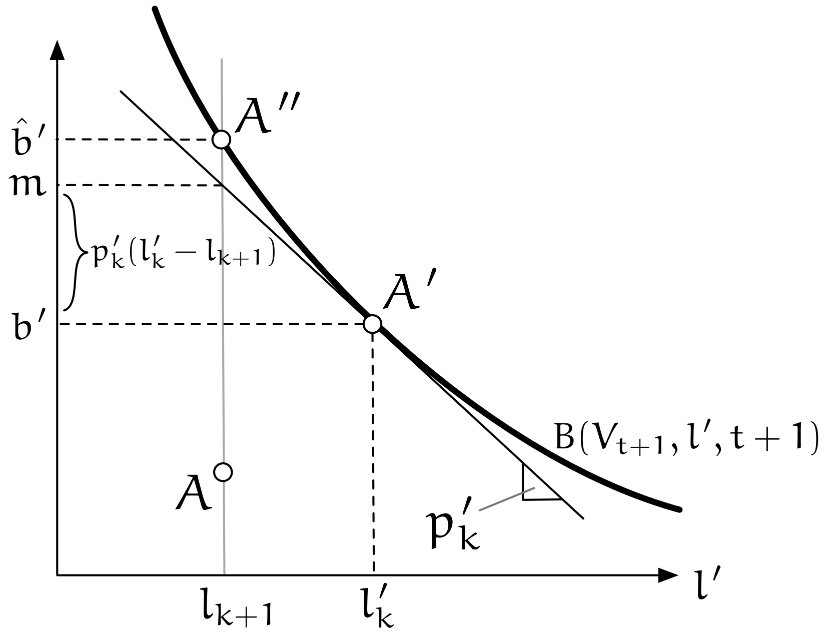

Figure 1: The convexity of the value function and the cost of long-term trades.

The diagram considers two dimensions of the possible debt policy. The diagram's vertical axis, labelled b', represents alternative choices for one-period bonds that will be due next period ($t+1$), and the horizontal axis represents alternative choices for long-term bonds due in period $t+1+k$. Point A represents the inherited one-period debt and period $t+1+k$ liabilities at the start of t: $A = (b, \boldsymbol{l}_{k+1})$. From this point, the government chooses a new portfolio $(b', \boldsymbol{l}'_k)$ to take into period $t+1$.

The bold convex line is the inverse value function, B, evaluated at period $t+1$ states: $b' = B(V_{t+1}, \boldsymbol{l}', t+1)$. Specifically, holding constant V_{t+1}, the function depicts the value of one-period debt associated with alternative choices for liabilities due in period $t+k+1$, $\boldsymbol{l}'_k$. The function is downward sloping as more long-term debt requires less one-period debt to keep the value to the government constant at V_{t+1}. Its convexity is established in Proposition 2, and for the strict convexity that is depicted we assume that the conditions in that proposition hold for period $t+1+k$. From that proposition, the slope of the tangency line at each point of B reflects the $t+1$ price of liabilities due in $t+1+k$, which is denoted p'_k.[17] this price is conditional on the choices $(b', \boldsymbol{l}'_k)$, and thus varies as the government considers alternative policies.

The diagram considers a policy which, starting from point A, shifts the government's

[17]Technically, the price lines are supporting subgradients, as B is not necessarily differentiable everywhere.

end-of-period portfolio to point A'. The period-t consumption associated with this policy is:

$$c_t = y_t - \boldsymbol{l}_0 + R^{-1}F(V_{t+1})\left[b' + p'_k(\boldsymbol{l}'_k - \boldsymbol{l}_{k+1})\right].$$

Note that the term in square brackets is the equation of the line tangent to point A'; thus, as we follow this tangency to $\boldsymbol{l}_{k+1}$, the height of this line (denote by m in the graph) maps into the level of consumption associated with the point-A' policy.

Now consider an alternative policy starting from point A that moves the portfolio to point A''. The vertical height at point A'', denoted by $\hat{b}'$, is strictly higher than m, a fact which follows directly from strict convexity. Correspondingly, the consumption associated with policy-A'' is greater than that of point A'. As the continuation value is the same in both cases, this represents a strict improvement. That is, if the government were to choose A' rather than A'', it will lose resources.

5.3 Discussion

To obtain some intuition for the result, we first discuss the economics behind the convexity of B. Consider the contracting problem between new lenders and the government, given a sequence of long-term liabilities, $\boldsymbol{l}$. The efficient allocation delivers a certain level of utility to the government through a mix of net payments and the values the government achieves through default. Now suppose that we increase the long-term liabilities that are due j periods ahead; $\boldsymbol{l}_j < \boldsymbol{l}'_j$. This reduces the surplus in that period that can be split between new lenders and the government. In an efficient contract, there is an incentive to raise the probability of default in $t + j$ at the margin, as less of the surplus is at stake at that time. Starting from an interior default probability at $\boldsymbol{l}$, the efficient allocation will therefore lower p_j in response to the increased $\boldsymbol{l}'_j$. Note that the long-term debt holder is hurt by this, but the planning problem ignores this loss. Correspondingly, in a competitive equilibrium, there is no price mechanism that makes the government internalize this loss. Similarly, if $\boldsymbol{l}_j > \boldsymbol{l}'_j$, the new lenders and the government now have a greater incentive to avoid default in period $t + j$, and the allocation will adjust accordingly.

This discussion highlights the core inefficiency in the environment. Small perturbations in the probability of default in any given period have a second-order loss for the government, as they are indifferent to default at the margin. However, they represent a first-order loss to existing bond holders. Thus efficiency requires that the government internalize these losses when choosing allocations in the competitive equilibrium. This is possible for one-period bondholders as prices move one-to-one with the probability of default, as is clear from the break-even condition (4). At the margin, the government bears the full cost or benefit of its consumption/savings decisions. This feature of one-period bonds is the reason that the competitive equilibrium is efficient from the perspective of one-period bondholders.

The equilibrium is not efficient from the perspective of long-term bondholders. When issued, the government pays actuarially fair prices, and thus long-term bondholders are compensated for expected default. In the future, the government does not face a price that makes it internalize the consequences of its actions on existing long-term bond holders. If the government could commit to a path of consumption at the time debt is issued, this would not represent a problem. However, the efficient consumption plan from the perspective of long-term bondholders is not generally time consistent, as there is no period-by-period price that aligns incentives. As a result, the default premium in short-term bond prices is akin to a variable cost that must paid each period and moves one-to-one with changes in default probability, *aligning incentives*. Long-term bonds also embed a default premium, at the time of issuance, but from the perspective of later periods, this premium is a sunk cost and provides weaker incentives going forward.

Another perspective on the sub-optimality of long-term debt arises from optimal contracting; namely, the problem with long-term debt is the lack of exclusivity. Recall that a competitive equilibrium allocation corresponds to an efficient contract between a set of new bondholders and the government, conditional on existing liabilities. The contract calls for a sequence of net payments to the lenders. With exclusivity, that sequence could be decentralized as a portfolio of bonds of arbitrary maturity, as the government cannot dilute the creditor by contracting with new bondholders. However, the competitive equilibrium does not admit the possibility of the government committing to future debt issuances. With short-term debt, the contract can be viewed as a sequence of one-period exclusive contracts with the representative lender. Given the assumption that there is only one auction per period, the one-period bond holder is never at risk of the government subsequently contracting with alternative creditors before repayment.

The lack of exclusivity of long-term bonds renders them inefficient. This raises the question of whether repurchasing the long-term bonds and replacing them with one-period bonds can undo this inefficiency and generate a higher value for the government. The above discussion says no. Any change in long-term debt raises the expected payments to bondholders. As the government replaces its long-term liabilities with short-term bonds, its incentives to borrow or save change. Specifically, they change in a manner that raises the price of the repurchased liabilities. However, the new allocation was also feasible without repurchasing long-term bonds, but not optimal. The government does not need to alter its portfolio of long-term debt to contemplate alternative sequences of consumption, holding constant expected net payments. Thus repurchasing debt represents an unnecessary transfer to bondholders that could have been avoided by restricting activity exclusively to one-period bond markets.

Bulow and Rogoff Debt Buybacks

The suboptimality of repurchasing long-term bonds on secondary markets is reminiscent of Bulow and Rogoff (1988, 1991), henceforth BR. BR emphasize an environment in which the lenders collect some resources from the country in case of default. The claim of previous bond-holders on this collateral value can be diluted by issuing new bonds; that is, the government can raise additional resources from new lenders at the expense of the legacy bond holders by selling additional claims to the same collateral value. This mechanism underlies many of the debt-dilution papers in the corporate finance literature.

In an environment with this incentive to dilute legacy bondholders, a government will strictly lose by repurchasing legacy bonds. A buyback does not reduce the amount paid by the government to creditors in *default*. However, the repurchase price includes the legacy bondholders claim on this collateral. Hence, the reduction in future payments from the buyback is less in expected value than the buyback price.[18] This logic holds even if the government repurchases at the pre-buyback market price, a result quite different from our environment, as we show in Section 7.

Our zero long-term trade result shares a similar outcome to BR (repurchases are costly), but via a different mechanism. Note that differently from BR, our model does not have a recovery/collateral value in case of default: there is no fixed amount of resources for foreign lenders to grab in such an event. As a result, the dilution incentive at the heart of BR's result is absent in our framework.[19] The crucial element in our framework is that the maturity structure affects the probability of default. In particular, the incentives to borrow or save depend on how much of the debt must be rolled over along the path. This is absent from BR, but at the core of our result.

[18]In BR's terminology, the reduction in expected future payments is the marginal change to the market value of debt, while the repurchase price is the average market value.

[19]Our example in Figure 1 of issuing long-term bonds represented a loss for existing bondholders but was nevertheless suboptimal. That is, a capital loss for bondholders is not a gain for the government – default in our environment is *not* zero sum.

6 A Stationary Economy with a Perpetuity and a One-Period Bond

To better understand the forces at play in the model, in this section we proceed to characterize the equilibrium in a stationary environment with a simple asset structure. Specifically, we assume the endowment is constant ($y_t = y$), the distribution of outside option shocks is independent of time ($F_t = F$), and the market structure is restricted to a one-period bond and a perpetuity. A one-period bond is, as before, a promise to deliver 1 in the subsequent period and zero thereafter. A perpetuity is a promise to deliver a constant flow of payments of 1 forever. We denote by b the stock of one-period bonds at the beginning of the period, and by l the corresponding stock of perpetuities. Besides previously issued perpetuity claims, there are no other legacy claims.

Let us first focus on the efficient allocation, given an amount of legacy claims, and later on, relate this to the competitive equilibrium using Proposition 1. In this case, the efficient allocation solves:

$$B^{\star}(v,l) = \max_{c\geq 0, v'\in[\underline{v}^D, \overline{V}]} \left\{ y - c - l + R^{-1}F(v')B^{\star}(v',l) \right\}$$

$$\text{subject to :}$$

$$u(c) + \beta F(v')v' + \beta \int_{v^D \geq v'} v^D dF(v^D) = v$$

Although the value function is not necessarily everywhere differentiable, we can show the following marginal characterization:

Proposition 3. *[An Euler equation] Suppose that for some state (v, l), the optimal policy, v', is such that $v' > \overline{v}^D$, then*

$$\frac{1}{u'(c(v',l))} - \frac{\beta R}{u'(c(v,l))} = 0. \tag{15}$$

If instead, $v' \in (\underline{v}^D, \overline{v}^D)$. Then the following holds:

$$\frac{1}{u'(c'(v',l))} - \frac{\beta R}{u'(c(v,l))} = \frac{f(v')}{F(v')} B^{\star}(v',l). \tag{16}$$

Note that equation (15) is the usual Euler equation when debt is risk-free. That is, when the promised value v' is sufficiently large and, as a result, the probability of default is zero, the government behaves (locally) like a standard consumer facing a constant interest rate. If $\beta R = 1$, consumption would be constant. If $\beta R < 1$, consumption would be decreasing with time reflecting relative impatience.

The case when $\beta R = 1$

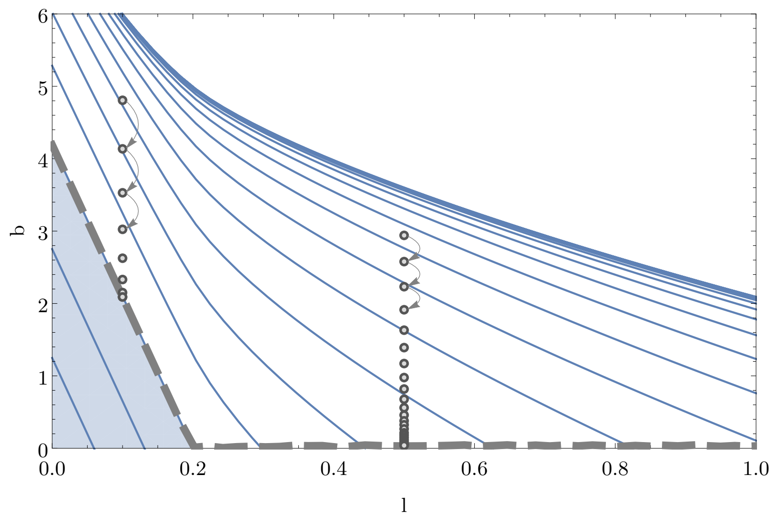

Figure 2: The solid lines are combinations of (l, b) such that $b = B^{\star}(v, l)$ for several values of v. The shaded area is the "no-default" area where $v > \overline{v}^D$, and the ex-ante probability of default in the period is zero. The dashed line represents steady-state points. The figure also plots two possible equilibrium paths (the circles) where the allocation converges to either (i) no default but still positive level of one-period debt, or (ii) zero one-period debt and possible default. The parameters used are $u = \log$, $y = 2$, $R = 1.05$, $\beta = 1/R$, and the outside option v^D equals $u(\tau y)/(1-\beta)$ where τ is uniform in $[0.2, 0.9]$.

Note however that when the promised value v' is within the support of the outside option shocks, then the probability of default is strictly positive. In this case, the Euler equation has a new term given by the right-hand side of equation (16). Suppose for example that $\beta R = 1$ and that $B^{\star}(v', l) > 0$. Then, in this case, $c' > c$; that is, consumption is back-loaded. Intuitively, at the margin, it is optimal to increase future continuation values by postponing consumption in order to reduce the default probability tomorrow by the marginal density $f(v')$. In equilibrium, the government internalizes this change because $f(v')/F(v')$ governs the elasticity of the equilibrium one-period bond price schedule with respect to debt issuances. As long as default has positive probability and there are one-period bonds outstanding, the government has an incentive to save at the margin. Given $\beta R = 1$, this is the only incentive to tilt consumption, and starting from a situation where the government has strictly positive one-period debt, the Euler equations imply that the government will save until either (i) the level of one-period debt is zero, or (ii) default risk completely disappears.

Figure 2 depicts the outcome of a numerical simulation of the model. The downward solid sloping lines are combinations of (l, b) such that v is constant along each line; that is, $b = B^{\star}(v, l)$ for respective values of v. In the Southeast corner of the diagram, debt is low enough (or v is high enough) that default is never optimal. As we move Northeast, v declines and the default probability increases. The figure also plots two possible equilibrium paths depicted by the circles connected by the arrows. The two equilibrium paths are distinguished by the initial debt portfolio. In the example on the left, there is a relatively low level of long-term debt and the allocation converges to the no-default region with strictly positive levels of one-period debt. Once it reaches the no-default region, (15) implies that consumption is stationary thereafter. In the right example, the government pays down its one-period debt but never removes the risk of default posed by long-term debt. Equation (16) implies that consumption is stationary despite the risk of default, as there is no debt being rolled over. However, in this case the government eventually defaults once a high-enough outside option is realized. The fact that the equilibrium paths are vertical lines in this state space reflects the result that the government only actively trades one-period bonds (and the perpetuity never matures).

The equilibrium paths depicted in Figure 2 capture the incentive to save provided by one-period bonds. Recall that bond prices are actuarially fair and are priced at the government's discount rate when $\beta R = 1$. Nevertheless, the government faces inter-temporal prices that induce saving. In particular, equilibrium prices require that the sovereign compensate lenders for the expected creditor losses in default. However, the government does not receive the corresponding equivalent benefit from default. To see this, suppose that the outside option realization is marginally above the government's value of repayment. In this case, the government is nearly indifferent between default and repayment, but the creditors suffer a discrete loss. Ex ante, bond prices are such that the government must compensate creditors for the possibility of the full loss. This provides the government an incentive to raise the price of its bonds at the margin by paying down one-period debt, as indicated by the Euler equation (15). This incentive is operational as long as one-period debt is outstanding or default is a non-zero probability. The perpetuity provides no such incentive, as the benefits of reducing the default probability after issuance are not captured by the government.

When $\beta R < 1$, the incentive to pay down one-period bonds is tempered by relative impatience. From equation (15), the interior of the no-default region is never a stationary point, as there is no countervailing cost to front loading consumption at the margin. Similarly, from equation (16), zero one-period debt is never a stationary point. However, in regions of the state space with $f(v)b > 0$, the elasticity of the bond price schedule provides an incentive for patience that can support a stationary point. This point may be on the boundary of the no-default region or in the region of the state space with strictly positive probability of default, but always with strictly positive one-period debt.

Figure 3 is the counterpart of Figure 2 for the case of $\beta R < 1$. The solid and dotted lines again represent (b, l) loci with constant v. The dashed line represents stationary points of the equilibrium. The left example equilibrium (again depicted by circles connected by the arrows) is similar to that of Figure 2. In particular, the government pays down one-period debt until it reaches the no-default region. Once on the boundary, its impatience is exactly offset by the marginal price decline induced by borrowing into the possible-default region. Differently form Figure 2, in which the entire no-default region contained stationary points, if the initial state had been directly below the no-default region boundary, the government would have borrowed up to the boundary of the region by exclusively issuing one-period bonds. The right-hand path depicts the case with larger initial long-term debt. In this case, the stationary point is in the interior of the region in which default occurs with positive probability. Again, if the initial state had been below the dashed line, the government would borrow to this point by issuing one-period bonds. This is the case shown in the middle path. One other difference between the two figures is that in Figure 3, the convexity of the value function applies to the portions of the dotted lines inside the no-default region. This reflects that if the initial l is large enough, the government will eventually borrow into the interior of the default region. Correspondingly, trades of long-term assets, even though there is currently no risk of default, entail strict losses to the government given that future equilibrium default occurs with positive probability.

6.1 The Yield Curve

The equilibrium trajectories discussed above shed light on the relevance (or irrelevance) of the slope of the yield curve in determining the optimal maturity of debt issuance. Let r_k denote the implied yield of a zero-coupon bond that matures in k periods. In particular, the one-period implied yield is $r_1 = q^{-1} - 1$, where q is the equilibrium price of the one-period bond. Similarly, $r_k = \rho_k^{-\frac{1}{k}} - 1$ for zero-coupon bonds maturing in k periods that trade at price ρ_k.[20] As a result, r_k is therefore the geometric mean of the conditional hazard of default from $t+1$ through $t+k$. Thus $r_{k+1} \gtrless r_k$ depending on whether the $t+k+1$ conditional default probability, $F_{t+k+1}(V_{t+k+1})$, is greater or less than the average of the periods preceding it. In Figure 3, if the government starts below the dashed line

[20] That is, $\rho_k = \rho_k(\mathcal{B}(b, \boldsymbol{l}, t), \mathcal{L}(b, \boldsymbol{l}, t), t)$, where $(b, \boldsymbol{l}, t)$ is the current state.

The case when $\beta R < 1$

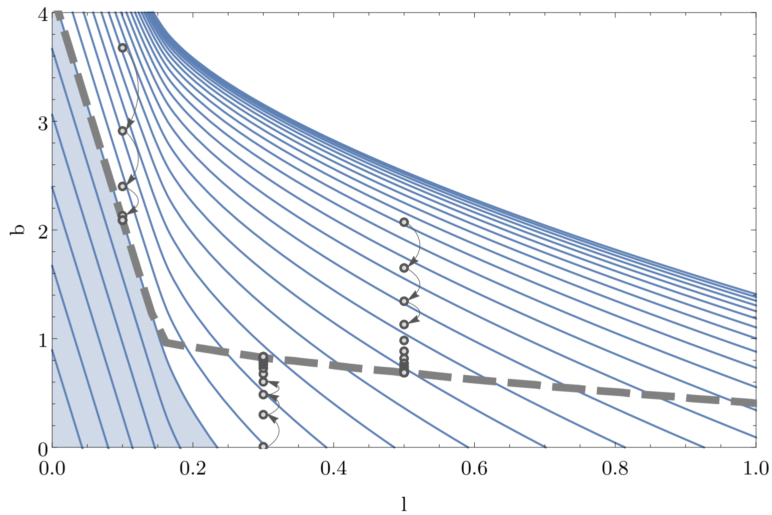

Figure 3: The solid lines are combinations of (l, b) such that $b = B^{\star}(v, l)$ for several values of v. The shaded area is the "no-default" area where $v > \overline{v}^D$, and the ex-ante probability of default is zero. The dashed line represents steady-state points. The figure also plots three possible equilibrium paths (the circles) where the allocation converges to either (i) the boundary of the no-default region, or (ii) the interior of the default region. The parameters used were $u = \log$, $y = 2$, $R = 1.05$, $\beta = .9/R$, and the outside option equals $u(\tau y)/(1-\beta)$ where τ is uniform in $[0.2, 0.9]$.

depicting stationary points, it faces an upward sloping yield curve, as creditors anticipate the higher future debt levels and associated default probabilities. On the other hand, if it starts above the dashed line, the yield curve will slope down. These two situations are plotted in Figure 4. However, in both scenarios, the government can achieve its equilibrium value without trading long-term debt. Moreover, if the strict convexity conditions of Proposition 2 are satisfied, trading long-term bonds generates strict losses independently of whether the yield curve slopes up or down. The irrelevance of the yield curve highlights that what is important for the result is not the relative level of the interest rate at different maturities (as implied by the yield curve or the associated prices), but instead the *elasticity* of bond prices to maturity choice.

7 Pareto Efficient Restructuring

Section 3 introduced a planning problem to analyze competitive equilibria. The notion of efficiency in Definition 3 ignored potential gain or losses to legacy bond holders when comparing different allocations. In this section, we broaden this to a general notion of

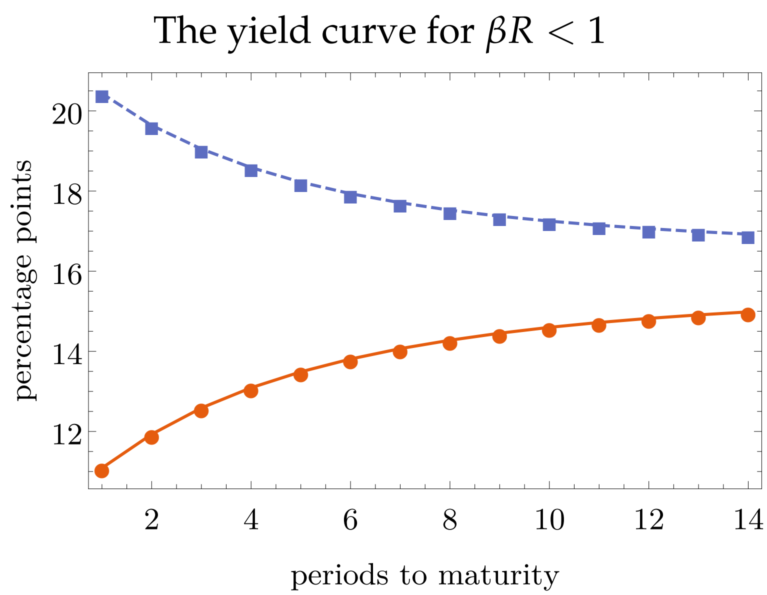

Figure 4: The parameters of the simulation are the same as in Figure 3. The solid line is the yield curve starting from a point above the steady-state line. The dashed line is the yield curve starting from a point below the steady-state line. In both cases $l = 0.25$ and b starts at 0.23 for the solid line and at 1.33 for the dashed line.

efficiency, which would include the expected payoff to long-term bondholders as well.

Specifically, consider making the sequence of payments $\hat{\iota}_k$ a choice variable rather than a state in the contracting problem (9) (that is, the planner can restructure the legacy claims). In addition, we add an additional constraint to that problem that guarantees long-term bondholders a minimal expected payoff w. The problem becomes:

$$B^\star(v, w, t) = \sup_{\{c_{t+k}, V_{t+k}, \hat{\iota}_k\}_{k=0}^\infty} \sum_{k=0}^{\infty} \left(\prod_{i=1}^{k} R^{-1} F_{t+i}(V_{t+i}) \right) (y_{t+k} - \hat{\iota}_k - c_{t+k}) \tag{17}$$

subject to:

$\{V_{t+k}\}_{k=0}^\infty$ solves (6) given $\{c_{t+k}\}_{k=0}^\infty$,

$$V_t \geq v, \tag{18}$$

$$\sum_{k=0}^{\infty} \left(\prod_{i=1}^{k} R^{-1} F_{t+i}(V_{t+i}) \right) \hat{\iota}_k \geq w, \tag{19}$$

This new problem then traces out the maximum payoff to new bondholders conditional on the welfare of the government and long-term bondholders, and thus as we vary v and w we trace out the Pareto frontier.

Note that constraint (19) will bind in the solution to this expanded problem (otherwise the payment to new-bond holders could be increased). Using this constraint to substitute out $\boldsymbol{l}_k$ from the objective, we have a problem that simply subtracts w from the net payments. The problem then is isomorphic to the one considered in Section 3, but with an additional expected payment of w to creditors replacing the legacy liabilities. From the results of that section, the Pareto efficient allocation can then be decentralized by allocating short-term bonds with value w to existing long-term bondholders. As long as the payment w is weakly higher than the market value of their legacy claims in the original competitive equilibrium outcome, long-term bond holders will find this restructuring beneficial.

While such a swap is a Pareto improvement relative to the competitive equlibrium, it cannot be done via market trades. Consider the alternative planning problem where constraint (19) is replaced with the following:

$$\sum_{k=0}^{\infty}\left(\prod_{i=1}^{k} R^{-1}F_{t+i}(V_{t+i})\right)\hat{\boldsymbol{l}}_k \geq \sum_{k=0}^{\infty}\left(\prod_{i=1}^{k} R^{-1}F_{t+i}(V_{t+i})\right)\boldsymbol{l}_k$$

where $\{\boldsymbol{l}_k\}$ represents the original legacy debt. This alternative constraint imposes that legacy lenders have the option to *hold on to their original claims* across alternative allocations. A new allocation that changes the default probabilities from the equilibrium one will change the compensation that legacy bondholders require to give up their original claims. It is easy to see that in the solution to this alternative problem, the hold-out constraint above will hold with equality. Substituting this into the objective function delivers the same problem as in Problem 9; that is, the planner cannot improve upon the equilibrium outcome in the presence of the hold-out constraint. Hence, the friction in the competitive equilibrium is the hold-out problem of legacy bondholders.[21]

The analysis of Section 5 demonstrated that it is never optimal for the government to issue short-term bonds in order to repurchase long-term bonds. The issue here is that such trades give too much of the surplus to legacy lenders for the government to be indifferent. Therefore, such restructurings, if they are to be undertaken, must be implemented via non-market arrangements (such as bargaining between all creditors, as a group, and the government).

We explore visually this result in Figure 5. The figure has the similar elements of Figure 1, but focuses on restructuring from the initial allocation A. The value b on the vertical

[21] This new problem makes clear that the inefficiency of the competitive equilibrium does not rely on the size and behavior of the new lenders, but rather depends on the potential hold-out behavior of the atomistic legacy lenders. For example, consider the scenario in which a large investor buys up all the legacy bonds and negotiates with the government. If this could be done holding fixed equilibrium prices, the solution to problem 17 could be implemented and the large investor would capture the net gain. However, if initial bondholders anticipate this event, prices change and the strategy cannot be profitable.

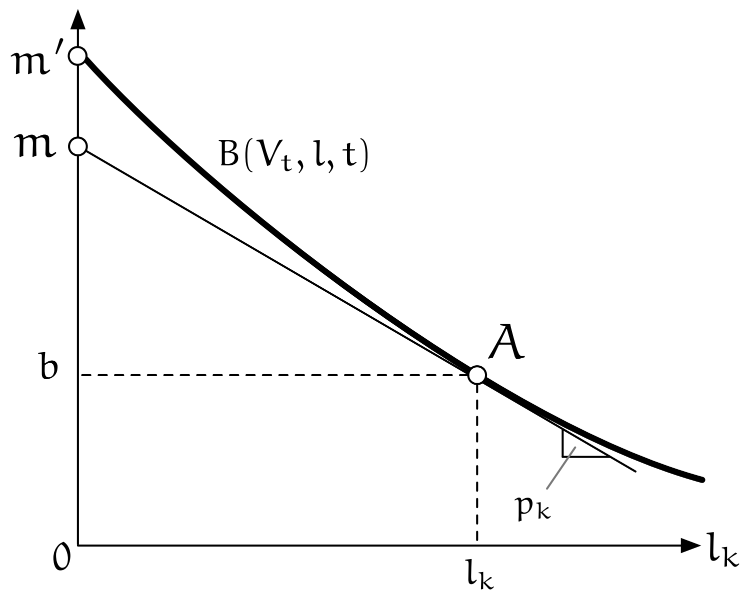

Figure 5: A (Pareto-improving) Restructuring of Legacy Claims.

axis represents the amount of one-period bonds currently due and the value l_k on the horizontal axis, the legacy claims due k periods ahead. The figure is drawn conditional on repayment in the current period. As in Figure 1, a tangency line to the function B presents the price of the long-term claim; in this case, the price assumes equilibrium behavior starting from initial state A, which is our benchmark for restructuring. The vertical intercept of the tangency line denoted m represents the current period market value of all the debt at the initial point A; that is, $m = b + p_k l_k$. Similarly, the intercept of the value function B and the vertical axis, labelled m', represents the amount of one-period bonds that would make the government indifferent between point A and a portfolio composed solely of one-period bonds. Convexity of B implies that m' is weakly greater than m, and strictly so if the conditions for strict convexity are satisfied (which we assume in the diagram). That is, the government is indifferent to a restructuring that swaps all legacy claims $\boldsymbol{l}_k$ in exchange for an amount $m' - b$ of additional one-period bonds. With strict convexity, because $m' > m$ the swap represents a strict improvement for bond-holders, as the market value of outstanding debt increases. More generally, a restructuring to any point on the vertical axis between m and m' represents a Pareto improvement, as these points represent combinations of increases in the market value of debt and increases in the government's value. However, these Pareto-efficient restructurings cannot be implemented through competitive markets. In particular, the convexity of B implies that market prices of long-term bonds will be higher for portfolios on the vertical axis than at the initial point

A. If the government were to propose such a swap to bondholders, legacy bondholders will want to hold-out and reap the resulting capital gains from the restructuring. Therefore, such restructurings require some form of collective bargaining and cannot be done through arms-lengths transactions.

The analysis of this section implies that an efficient restructuring *reduces* the maturity of the government debt portfolio. From this perspective, the lengthening of maturities that accompany actual debt restructurings in practice appears puzzling. However, the conflict between the model and actual practice is less severe than it appears. In the model, one-period debt provides the correct incentives for the government to minimize the net loss from default. In particular, a portfolio composed exclusively of one-period bonds implements the fiscal trajectory that a Pareto planner would choose. Maturity extensions in practice are often motived by providing "breathing room" to the sovereign by mitigating rollover risk, something we abstract from in the present paper. However, our framework suggests such extensions provide perverse incentives for fiscal policy going forward. To address this, many restructurings involving official agencies such as the IMF or EU therefore impose conditionality on the debtor. While private markets lack the ability to impose conditionality, it is also questionable how enforceable official conditionality is in practice, particularly as the IMF and other supra-national lenders may lack the political will to punish the debtor ex post. This issue does not arise in the competitive equilibrium we consider. In particular, bond holders only demand to break even on average, which is always a time consistent disciplining device.

8 An Extension with Hedging

The benchmark analysis features a model that emphasizes the threat of default as the key friction. We deliberately constructed the model to suppress a hedging motive in order to make the analysis as transparent as possible. The preceding discussion emphasized the inefficient properities of long-term debt, yet at the same time established that the government never adjusts the stock of long-term debt in equilibrium. In this section, we extend our baseline model to explore how long-term bonds are used to hedge consumption risk in the presence of default. In particular, we study how the trade-off between incentives and hedging may lay lead a government to issue long-term bonds in the first place.

We extend the previous environment as follows. Consider a new initial period, $t = -1$, where the government needs to raise an amount X of resources from international bond markets. In period $t = 0$, the government faces the following risk. With probability π, a (permanently) risky state is realized. The risky state is our benchmark environment in which default is the dominant consideration. With probability $1 - \pi$, a permanently safe state is realized. In the safe state, the government never defaults and faces risk-free prices for $t \geq 0$. [22]

The important departure from the benchmark is that the states differ in their inherent risk and this difference is persistent. This persistence provides the opportunity to hedge using long-term bonds. One can think of the safe state as a low realization of v^D that persists over time.

We follow the example of Section 6 in which the government trades a one-period bond, b, and a perpetuity, l, that pays 1 every period. The endowment is held fixed at y and the government discounts at the world interest rate: $\beta R = 1$. To simplify expressions, we assume that there is no default in $t = 0$.

The question we address is what is the optimal portfolio of the initial bond issuances. To highlight the hedging motive, we first analyze a complete-markets environment in which the government issues liabilities explicitly contingent on the realization of the period-0 state. We then turn to the incomplete markets model of interest.

8.1 Contingent Debt

Consider a government at $t = -1$ choosing state-contingent consumption sequences. More precisely, the consumption allocation is contingent on whether the safe or risky state is realized at $t = 0$; we continue to assume there is no contingency on the realizations of default values. Let c_t and $\tilde{c}_t$ denote consumption in period t conditional on the risky and safe states, respectively. The risky consumption sequence is also contingent on the government not having defaulted in a previous period. As before, let V_t denote the value in the risky state at time t given the allocation $\{c_t\}$, as defined in equation (6), and $p_t = R^{-t}\Pi_{k=1}^{t}F(V_k)$ for $t > 0$ with $p_0 = 1$. The lenders' break-even condition requires that the price as of time $t = -1$ of a liability promising payment of a unit in period t in the risky state is $\pi R^{-1} p_t$. Period t payments contingent on the safe state are valued by lenders at $(1 - \pi)R^{-(t+1)}$ in period $t = -1$.

[22] The fact that the alternative is risk-free simplifies the analysis but is not crucial for the incentive to hedge.

The government's period $t = -1$ problem is:

$$\max_{\{c_t, \tilde{c}_t, V_t\}} \pi V_0 + (1-\pi) \sum_{t=0}^{\infty} R^{-t} u(\tilde{c}_t) \tag{20}$$

subject to

$$X \leq \pi R^{-1} \sum_{t=0}^{\infty} p_t \left[y - c_t\right] + (1-\pi) \sum_{t=0}^{\infty} R^{-(t+1)} \left[y - \tilde{c}_t\right], \tag{21}$$

where $\{c_t, V_t\}$ is incentive compatible as in equation (6). The budget constraint (21) ensures that the promised payments to lenders exceed X in market value at period-($t = -1$) prices.

Let μR denote the multiplier on the constraint (21). The first-order condition for $\tilde{c}_t$ is

$$u'(\tilde{c}_t) = \mu. \text{ for all } t \geq 0$$

As the government discounts at R^{-1} and there is no risk of default in the safe state, the optimal allocation smooths consumption completely.

The first-order condition for initial consumption in the risky state is:

$$u'(c_0) = p_0 \mu = \mu,$$

where the second equality uses the fact that $p_0 = 1$, given that we have ruled out default in period 0. Thus, the optimal allocation equates consumption in the initial period across the risky and safe states: $c_0 = \tilde{c}_0$.

Because consumption sequences are contingent on the realization of the state, the optimal allocation in the risky state $\{c_t, V_t\}$ must be efficient from time $t = 0$ onwards. Conditional on V_0, the chosen allocation thus coincides with the efficient allocation in our benchmark environment with zero legacy debt. That is, $B^\star(V_0, \mathbf{0}) = \sum_{t=0}^{\infty} p_t \left[y - c_t\right]$. Recalling our benchmark analysis, if $F(V_t) \in (0,1)$ for some $t > 0$, then there is an incentive to backload consumption given $\beta R = 1$. In this case, consumption in the risky state is the same as the safe state at $t = 0$, and then weakly increases over time.

A possible decentralization of the optimal allocation $\{c_t, \tilde{c}_t, V_t\}$ is for the government at $t = -1$ to issue $B^\star(V_0, \mathbf{0})$ units of one-period bonds due in period 0 contingent on the risky state and *zero* long-term liabilities. At the same time, the government issues a one-period bond contingent on the safe state with face value $\tilde{B} = \sum R^{-t} \left[y - \tilde{c}_t\right]$. The fact that

$c_t \geq c_0 = \tilde{c}_0 = \tilde{c}_t$ implies

$$B^\star(V_0, \mathbf{0}) = \sum_{t=0}^{\infty} p_t \left[y - c_t\right] \leq \sum_{t=0}^{\infty} R^{-t} \left[y - \tilde{c}_t\right] = \tilde{B}.$$

This holds with a strict inequality if there is a strictly positive probability of default in the risky state.

The sovereign equates consumption across the states in the initial period; however, after the state is realized, in the event of the risky state, it backloads consumption. Thus, the government shifts its liabilities toward the safe state and consumption towards the risky state in order to reduce the probability of default in the risky state. The complete-markets example provides the benchmark for the incomplete markets environment, setting the stage for trading off insurance and incentives that will determine the optimal portfolio in what follows.

8.2 Incomplete Markets

The government's problem under incomplete markets is to issue in $t = -1$ non-contingent one-period bonds, b, and non-contingent consoles l, using the asset structure and notation of Section (6). Recall that the perpetuities pay 1 every period.

Let $V(b, l)$ denote the value in period 0 conditional on the risky state and the portfolio $\{b, l\}$ chosen in period $t = -1$. From the analysis of Section 4, V is the inverse of $B^\star$; that is, $b = B^\star\left(V(b, l), l\right)$. The corresponding value in the safe state is:

$$\tilde{V}(b, l) = \frac{R}{r} u\left(y - \frac{rb}{R} - l\right),$$

where we use the fact that consumption is perfectly smoothed in the safe state when $\beta R = 1$.

Let $p_t(b, l)$ denote $R^{-t}\Pi_{k=1}^{t} F(V_k)$, with $p_0 = 1$, where $\{V_t\}$ corresponds to the allocation selected in the risky state conditional on inherited debt $\{b, l\}$ in $t = 0$. Let $R^{-1}q_S(b, l)$ and $R^{-1}q_L(b, l)$ denote the prices of the one-period bond and perpetuity, respectively, in period $t = -1$. The lenders' break-even condition requires:

$$\begin{aligned} q_S(b, l) &= 1 \\ q_L(b, l) &= \pi \sum_{t=0}^{\infty} p_t(b, l) + (1 - \pi)\frac{R}{r}, \end{aligned}$$

where the first line reflects that there is no default in $t = 0$.

The government's $t = -1$ problem starting from zero debt is:

$$\max_{\{b,l\}} \pi V(b,l) + (1-\pi)\tilde{V}(b,l) \tag{22}$$

$$\text{subject to } X \leq R^{-1}b + R^{-1}q_L(b,l)l. \tag{23}$$

To provide some intuition for the trade offs involved in the optimal portfolio, first note that it is feasible to equate consumption across all states and time periods. In particular, setting $b = 0$ and issuing only perpetuities implies that $c_t = \tilde{c}_t = y - l$ in every period absent default. Recall from Section 6 that if $b = 0$, the government has no incentive to save and simply pays its perpetuity coupon until it defaults. Thus consumption is constant in the risky state. In the safe state, it is optimal to maintain a constant consumption as well. Given that debt is non-contingent, the budget set implies that consumption is equated across states. While feasible, our contingent-debt analysis suggests that full insurance is not optimal in this environment. This is because it eliminates the incentive to reduce the risk of default in the risky state.

To explore this trade off, we provide a necessary condition for the optimality of the full-insurance portfolio. We do this using a perturbation argument. Suppose that it is optimal for the government to issue zero one-period bonds and l perpetuities. The associated consumption sequence in both states is $c_t = \tilde{c}_t = y - l$ for all t.

Now consider a perturbation in which the government at time $t = -1$ increases one-period bonds Δb and adjusts perpetuities to keep auction revenue constant. In particular,

$$\Delta b + q_L \Delta l + l \Delta q_L = 0, \tag{24}$$

where q_L is short-hand for $q_L(0,l)$ and $\Delta q_L = q_L(\Delta b, l + \Delta l) - q_L(0,l)$ is the change in price associated with the new allocation. To a first-order, the government's objective changes by:

$$\Delta\text{Objective} \approx \left[\pi V_b(b,l) - (1-\pi)u'(c)\right]\Delta b + \left[\pi V_l(b,l) - (1-\pi)\frac{R}{r}\right]\Delta l,$$

where the envelope conditions imply:[23]

$$
\begin{aligned}
V_b &= -u'(c) \\
V_l &= -u'(c) \sum_{t=0}^{\infty} p_t(0,l).
\end{aligned}
$$

Substituting in and rearranging, we have

$$
\begin{aligned}
\Delta \text{Objective} &\approx -u'(c) \left(\Delta b + \left(\pi \sum_{t=0}^{\infty} p_t(0,l) + (1-\pi)\frac{R}{r} \right) \Delta l \right) \\
&= -u'(c) \left(\Delta b + q_L(0,l) \Delta l \right) \\
&= u'(c) l \Delta q_L,
\end{aligned}
$$

where the last line uses (24). Thus, a necessary condition for $(0,l)$ to be an optimal portfolio is that $\Delta q_L = 0$; that is, prices are invariant to revenue-neutral changes in the maturity composition.

This invariance is not true in general. The intuition is that starting from the full-insurance portfolio of only perpetuities, a shortening of maturity has second-order consequences on insurance but alters the ex post incentives to save in the risky state. This alters the ex ante price of insurance, and hence the government has an incentive to issue some one-period debt. The presence of one-period bonds provides the ex post incentive for the government to reduce the probability of default if the risky state is realized. This represents the trade off between insurance and incentives.

A similar perturbation establishes that issuing only one-period bonds is not optimal. While ex post efficient in terms of default, such a portfolio provides no insurance for the government against the realization of the risky state. Lengthening at the margin has second-order efficiency losses but first-order gains in the government's welfare via better insurance.

Using the parameters of Section 6, we solve for the optimal portfolio conditional on π. In Figure 6, we plot the share of total bond revenue raised by issuing one-period bonds for different probabilities of the risky state π, shown in the horizontal axis. The remaining share is that raised via perpetuities. From the benchmark analysis, the share of long-term bonds is zero at $\pi = 1$, as there is no need for insurance and long-term bonds are inefficient in the risky state. For $\pi < 1$, the government issues some long-term debt

[23] Here we are assuming that the probability of default is interior. That is, $F(V(0,l)) \neq \underline{V}^D$ and $F(V(0,l)) \neq \overline{V}^D$. Recall that $V_t = V(0,l)$ for all t in this allocation, so it suffices to only check that the initial value is not at the boundary of the support of v^D.

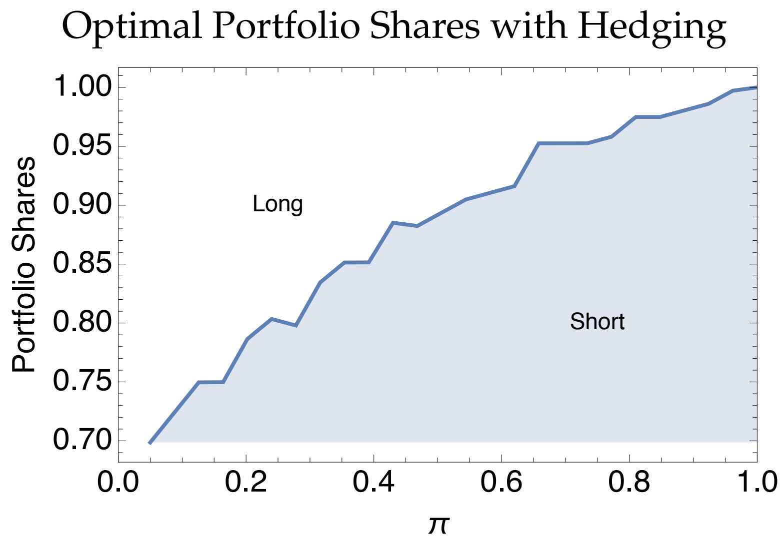

Figure 6: The solid line is the share of total bond revenue raised by issuing one-period bonds for different probabilities of the risky state π, shown on the horizontal axis. The remaining fraction is raised by issuing perpetuities. The parameters used are $u = \log$, $y = 2$, $R = 1.05$, $\beta = 1/R$, $X = 5/R$, and the outside option v^D equals $u(\tau y)/(1-\beta)$ where τ is uniform in $[0.2, 0.9]$.

for insurance purposes, at the cost of reducing incentives in the risky state. Recalling our benchmark analysis, these perpetuities issued in $t = -1$ are never repurchased and no additional perpetuities are issued in subsequent periods. These represent the legacy bonds that we treated as the initial state in the benchmark analysis.

9 Conclusion

In this paper we have shown that actively engaging in the long-term bond market during periods of potential default entails costs for a sovereign. In particular, shifts in the maturity structure imply changes in the equilibrium relative prices of long-term bonds. Such changes are always moving against the borrower; that is, the relative price of long-term bonds rises when the sovereign buys them, while it falls when the sovereign issues more. Quite generally, these actions will tend to shrink the budget set of the government, generating an incentive to use only short-term bonds.

We showed that the competitive equilibrium solved a planning problem, and that the decentralization involves trading one-period bonds exclusively. Interestingly, even though short-term debt is efficient, the sovereign makes no attempt to repurchase existing long-term bonds and replace them with one-period debt. Such a swap will be costly. This holds regardless of whether the equilibrium yield curve slopes up or down. The relevant price is the cost of bonds at the margin, and not the prices of existing bonds on secondary markets.

A Appendix: Proofs

A.1 Proof of Lemma 1

Let us first show a couple of lemmas, that hold for any given equilibrium.

Lemma 6. *Given* $(\boldsymbol{l}, t)$*, there exists a* $\overline{b}$ *such that* $\underline{V} < V(\overline{b}, \boldsymbol{l}, t) < \underline{v}_t^D$*.*

Proof. Let $m \equiv \sup_{\boldsymbol{l}', b' \leq \bar{B}} \{q(b', \boldsymbol{l}', t)b' + Q(\boldsymbol{l}, \boldsymbol{l}', b', t)\}$. Given $\epsilon > 0$, let $\overline{b} \equiv y_t - \boldsymbol{l}_0 + m - \epsilon$. Note that from (BC), for $b = \overline{b}$ it follows that for any choice, $(c_0, b', \boldsymbol{l}')$ we have that

$$0 < c_0 \leq q(b', \boldsymbol{l}', t)b' + Q(\boldsymbol{l}, \boldsymbol{l}', b', t) - m + \epsilon \leq \epsilon$$

From the definition of m and $\epsilon > 0$, there exists a $c_0, b', \boldsymbol{l}'$ that satisfy these inequalities, which guarantees that $V(\overline{b}, \boldsymbol{l}, t) > \underline{u} + \beta \underline{V} = \underline{V}$. In addition, picking the highest possible c_0 and the highest possible continuation value delivers an upper-bound:

$$V(\overline{b}, \boldsymbol{l}, t) \leq u(\epsilon) + \beta \overline{V}$$

Assumption 1.iv in turns guarantees that, for ϵ sufficiently small, $V(\overline{b}, \boldsymbol{l}, t) < \underline{v}_t^D$. And thus, there exists $\overline{b}$ such that $\underline{V} < V(\overline{b}, \boldsymbol{l}, t) < \underline{v}_t^D$. □

Lemma 7. $q(b, \boldsymbol{l}, t) = R^{-1}$ *if* $b < y_t - \boldsymbol{l}_0$ *and* $u(y_t - \boldsymbol{l}_0 - b) + \beta \int v^D dF_{t+1}(v^D) \geq \overline{v}_t^D$*.*

Proof. Note that $V(b, \boldsymbol{l}, t) \geq u(y_t - \boldsymbol{l}_0 - b) + \beta \int v^D dF_{t+1}(v^D)$, as consuming $c = y_t - \boldsymbol{l}_0 - b > 0$ and defaulting in the next period is a feasible strategy. As a result, it follows that $V(b, \boldsymbol{l}, t) \geq \overline{v}_t^D$ and $q(b, \boldsymbol{l}, t) = R^{-1}F(V(b, \boldsymbol{l}, t)) = R^{-1}$. □

Lemma 8. *Given* $(\boldsymbol{l}, t)$*, for any* $v_0 \in [\underline{v}_t^D, \overline{V})$*, there exists a* b_0 *such that* $V(b_0, \boldsymbol{l}, t) > v_0$*.*

Proof. Let $\bar{c}$ be such that $u(\bar{c}) = \max\{u^{max}, (1-\beta)v_0\}$ where u^{max} is as in Assumption 1.v. Note that $\bar{c}$ is finite. Let $b_{t+k} = \sum_{s=0}^{\infty} R^{-s} \min\{(y_{t+k+s} - \boldsymbol{l}_{k+s} - \bar{c}), 0\}$ for all $k \geq 0$. Note that $b_{t+k} \leq 0$ for all $k \geq 0$. It follows also that for all $k \geq 0$,

$$\begin{aligned}
b_{t+k} &= \min\{(y_{t+k} - \boldsymbol{l}_k - \bar{c}), 0\} + \sum_{s=1}^{\infty} R^{-s} \min\{(y_{t+k+s} - \boldsymbol{l}_{k+s} - \bar{c}), 0\} \\
&\leq (y_{t+k} - \boldsymbol{l}_k - \bar{c}) + \sum_{s=1}^{\infty} R^{-s} \min\{(y_{t+k+s} - \boldsymbol{l}_{k+s} - \bar{c}), 0\} \\
&= (y_{t+k} - \boldsymbol{l}_k - \bar{c}) + R^{-1} b_{t+k+1} \leq y_{t+k} - \boldsymbol{l}_k - \bar{c}
\end{aligned}$$

Using this, we have that for any $k \geq 0$,

$$u(y_{t+k} - \boldsymbol{l}_k - b_{t+k}) \geq u(\bar{c}) \geq u^{max} > \overline{v}^D_{t+k} - \beta \int v^D dF_{t+k+1}(v^D)$$

where the last inequality follows from Assumption 1.v. Thus, $V(b_{t+k}, \boldsymbol{l}_{\geq k}, t+k) \geq u(y_{t+k} - \boldsymbol{l}_k - b_{t+k}) + \beta \int v^D dF_{t+k+1}(v^D) \geq \overline{v}^D_{t+k}$ and $q(b_{t+k}, \boldsymbol{l}_{\geq k}, t+k) = R^{-1}$. We then have that for all $k \geq 0$, given state variable $b_{t+k}, \boldsymbol{l}_{\geq k}$, setting consumption equal to $\bar{c}$ and debt $b' = b_{t+k+1}$ is feasible. It follows then that

$$V(b_{t+k}, \boldsymbol{l}_{\geq k}, t+k) \geq u(\bar{c}) + \beta V(b_{t+k+1}, \boldsymbol{l}_{\geq k}, t+k+1)$$

and thus $V(b_t, \boldsymbol{l}, t) \geq (1-\beta)u(\bar{c}) \geq v_0$. □

Proof of Lemma 1. The fact that V is non-increasing in b follows directly from the fact that a reduction in b only relaxes the budget constraint, and thus the value function must weakly increase.

For the second part, we proceed by contradiction. Suppose then that there exists a $v \in [\underline{v}^D_t, \overline{V})$ such that there is no b with$V(b, \boldsymbol{l}, t) = v$. By Lemmas 6 and 8, we can find b_0 and b_1 such that $V(b_0, \boldsymbol{l}, t) < v < V(b_1, \boldsymbol{l}, t)$. The fact that $V(b, \boldsymbol{l}, t)$ is non-increasing in b, implies that there exists a $b_2 \in (b_0, b_1)$ and a $\delta > 0$, such that

$$V(b_2 - \epsilon, \boldsymbol{l}, t) > v > V(b_2 + \epsilon, \boldsymbol{l}, t) \text{ for any } \epsilon > 0 \text{ and } \|V(b_2 + \epsilon, \boldsymbol{l}, t) - V(b_2 - \epsilon, \boldsymbol{l}, t)\| > \delta.$$

That is, V must feature a discontinuity at b_2. Take now the equilibrium consumption policy at $b_2 - \epsilon$, and let us denote it by $c(\epsilon)$. From Assumption 1.iv, it follows that $u(c(\epsilon)) > u^{min}$, as $V(b_2 - \epsilon, \boldsymbol{l}, t) > v \geq \underline{v}^D_t$. Note that for sufficiently small ϵ, the consumption policy $c(\epsilon) - 2\epsilon$ with the same b' is feasible at $b_2 + \epsilon$, as $c(\epsilon)$ is bounded away from 0 for all ϵ. It follows then that

$$0 \leq V(b - \epsilon, \boldsymbol{l}, t) - V(b + \epsilon, \boldsymbol{l}, t) \leq u(c(\epsilon)) - u(c(\epsilon) - 2\epsilon)$$

For ϵ sufficiently small, we have that

$$u(c(\epsilon)) - u(c(\epsilon) - 2\epsilon) < \delta$$

generating a contradiction of $\|V(b_2 + \epsilon, \boldsymbol{l}, t) - V(b_2 - \epsilon, \boldsymbol{l}, t)\| > \delta$.

A.2 Proof of Lemma 2

Let H_s be defined as follows:$H_s(c, V, V') \equiv u(c) - V + \beta G_{s+1}(V')$ where $G_s(x) \equiv \int \max\{x, v^D\} dF_{t+s}(v^D)$. An allocation is incentive compatible allocation if and only if $H_s(c_{t+s}, V_{t+s}, V_{t+s+1}) = 0$ for all $s \geq 0$, that is, equation (6) is satisfied.

Let us first argue that we can restrict attention to allocations with $V_t = v^0$. Consider an incentive compatible allocation $\{c_{t+s}, V_{t+s}\}$ with $V_t > v^0$. Given that $H_0(c_t, V_t, V_{t+1}) = 0$, it follows that $H_0(c_t, v^0, V_{t+1}) > 0$. Note also that $H_0(0, v^0, V_{t+1}) = \underline{u} - v^0 + \beta G_{s+1}(V_{t+1}) < \underline{u} - \underline{v}_t^D + \beta \overline{V} < 0$, where the fist inequality follows from $v^0 \geq \underline{v}_t^D$ and $G_{s+1}(V') \leq \overline{V}$; and the second inequality by Assumption 1.iv. Continuity of H_0 guarantees that there is a value of $\tilde{c}_t$ such that $0 < \tilde{c}_t < c_t$ and $H(\tilde{c}_t, v_0, V_{t+1}) = 0$. We can then construct another feasible allocation, $\{\tilde{c}_{t+k}, \tilde{V}_{t+k}\}$ with $\tilde{c}_{t+k} = c_{t+k}$ and $\tilde{V}_{t+k} = V_{t+k}$ for all $k \geq 1$, and $\tilde{c}_t < c_t$ and $\tilde{V}_t = v^0$. This alternative allocation delivers a strictly higher value than the original (as the continuation allocation is the same, but initial consumption is strictly lower). As a result, it is without loss to consider only allocations with $V_t = v^0$.

Let us now argue that we can restrict attention to allocations where V_{t+s} is bounded. Let$w \equiv \max\{V_t, \overline{V}^D\} < \overline{V}$; and consider any incentive compatible allocation $\{c_{t+s}, V_{t+s}\}$. Let K be the set of all $s \geq 0$ such that $V_{t+s} > w$. Let us propose an alternative allocation, $\{\tilde{c}_{t+s}, \tilde{V}_{t+s}\}$ with the property that $\tilde{V}_{t+s} = V_{t+s}$ for all $s \notin K$ and $\tilde{V}_{t+s} = w$ otherwise.

Such an incentive compatible allocation exists. To show this, we need to find $\{\tilde{c}_{t+s}\}$ such that

$$H_s(\tilde{c}_{t+s}, \tilde{V}_{t+s}, \tilde{V}_{t+s+1}, s) = 0 \tag{25}$$

for all $s \geq 0$. There are four cases to consider. First case is where $\tilde{V}_{t+s} = V_{t+s}$ and $\tilde{V}_{t+s+1} = V_{t+s+1}$. In this case, $\tilde{c}_{t+s} = c_{t+s}$ solves (25). The second case is when $\tilde{V}_{t+s} = V_{t+s} \leq w$ but $\tilde{V}_{t+s+1} = w < V_{t+s+1}$. In this case, $H_s(c_{t+s}, V_{t+s}, w) \leq H_s(c_{t+s}, V_{t+s}, V_{t+s+1}) = 0$. In addition, $\lim_{c\to\infty} H_s(c, V_{t+s}, w) = \overline{u} + \beta w - V_{t+s} \geq \overline{u} + \beta w - w = (1-\beta)(\overline{V} - w) > 0$. Continuity of H_s with respect to c guarantees that there must a $\tilde{c}_{t+s} \in [c_{t+s}, \infty)$ such that (25) holds. The third case is when $\tilde{V}_{t+s} = w < V_{t+s}$ and $V_{t+s+1} = \tilde{V}_{t+s+1} \leq w$. For this case, $H_s(c_{t+s}, w, V_{t+s+1}) \geq H_s(c_{t+s}, V_{t+s}, V_{t+s+1}) = 0$. We also know that $H_s(0, w, V_{t+s+1}) = \underline{u} - w + \beta G_{s+1}(V_{t+s+1}) \leq \underline{u} - w + \beta \overline{V} \leq \underline{u} - \overline{V}^D + \beta \overline{V} \leq \underline{u} - \underline{v}_{t+s+1}^D + \beta \overline{V} < 0$ where the last inequality follows from Assumption 1.iv. Again, continuity of H_s with respect to c guarantees that there must a $\tilde{c}_{t+s} \in (0, c_{t+s}]$ such that (25) holds. The final case is when $\tilde{V}_{t+s} = w < V_{t+s}$ and $\tilde{V}_{t+s+1} = w < V_{t+s+1}$. In this case, $\tilde{c}_{t+s}$ must solve $u(\tilde{c}_{t+s}) = (1-\beta)w$, which must exists given that $w \in [\underline{V}, \overline{V})$.

We argue now that the new allocation $\{\tilde{c}_{t+s}, \tilde{V}_{t+s}\}$ is an improvement over the original. Towards this, first note that $F_{t+s}(V_{t+s}) = F_{t+s}(\tilde{V}_{t+s})$ for all $s \geq 0$. The value generated from this new allocation, in comparison with the original, can be written as:

$$\begin{aligned}\tilde{B}_t(\{\tilde{V}_{t+s}\}) - \tilde{B}_t(\{V_{t+s}\}) &= \sum_{s=0}^{\infty} R^{-s} \left(\prod_{i=1}^{s} F_{t+i}(V_{t+i}) \right) (c_{t+s} - \tilde{c}_{t+s}) \\ &= -\sum_{s=0}^{\infty} p_s \left[C(\tilde{V}_{t+s} - \beta G_{s+1}(\tilde{V}_{t+s+1})) - C(V_{t+s} - \beta G_{s+1}(V_{t+s+1})) \right] \\ &= -\sum_{s \in K'} p_s \left[C(\tilde{V}_{t+s} - \beta G_{s+1}(\tilde{V}_{t+s+1})) - C(V_{t+s} - \beta G_{s+1}(V_{t+s+1})) \right]\end{aligned}$$

where C denotes the inverse function of u; $p_s \equiv R^{-s} \prod_{i=1}^{s} F_{t+i}(V_{t+i})$; $K' \equiv \{s | s \in K \text{ or } s + 1 \in K\}$; and where the second equality uses (6), and the last uses that consumption is only potentially different across allocations for $s \in K'$. It follows that

$$\begin{aligned}&\tilde{B}_t(\{\tilde{V}_{t+s}\}) - \tilde{B}_t(\{V_{t+s}\}) \geq \\ &-\sum_{s \in K'} p_s C' \left(\tilde{V}_{t+s} - \beta G_{s+1}(\tilde{V}_{t+s+1}) \right) \left\{ V_{t+s} - \tilde{V}_{t+s} - \beta \left[G_{s+1}(V_{t+s+1}) - G_{s+1}(\tilde{V}_{t+s+1}) \right] \right\} \\ &\quad = -\sum_{s \in K} p_{s-1} R^{-1} \left[(\beta R) C' \left(\tilde{V}_{t+s-1} - \beta w \right) - C' \left(w - \beta G_{s+1}(\tilde{V}_{t+s+1}) \right) \right] (V_{t+s} - w)\end{aligned} \tag{26}$$

where the first inequality uses the convexity of C; and the equality, uses that $G_s(w) = w$, that for all $s \in K$, $p_s = R^{-1} p_{s-1}$, and rearranges terms. We also know that

$$w - \beta G_{s+1}(\tilde{V}^{\star}_{t+s+1}) \geq \tilde{V}_{t+s-1} - \beta w$$

which follows from $\tilde{V}_{t+s-1} \leq w$, and $G_{s+1}(\tilde{V}_{t+s+1}) \leq G_{s+1}(w) = w$ (given that G_s is increasing). By concavity of C, we have

$$0 \leq C'(\tilde{V}_{t+s-1} - \beta w) \leq C'(w - \beta G_{s+1}(\tilde{V}^{\star}_{t+s+1}))$$

for all $s \in K$. Given that $\beta R < 1$, the above implies that

$$(\beta R) C' \left(\tilde{V}_{t+s-1} - \beta w \right) - C' \left(w - \beta G_{s+1}(\tilde{V}_{t+s+1}) \right) \leq 0; \text{ for all } s \in K$$

Together with $V_{t+s} \geq w$ for all $s \in K$, it follows from (26) that

$$\tilde{B}_t(\{\tilde{V}_{t+s}\}) - \tilde{B}_t(\{V_{t+s}\}) \geq 0$$

Thus, the allocation $\{\tilde{V}^{\star}_{t+s}\}$ generates a weak improvement over the original. As a result, we can restrict attention to allocations such that $V_{t+s} \leq \max\{V_t, \overline{V}^D\}$.

A.3 Proof of Lemma 3

Let $v = V(b, \boldsymbol{l}, t)$. Given that $V(b, \boldsymbol{l}, t) \geq \underline{v}_t^D$, the constraint set in problem 3 is non-empty. Consider then the equilibrium policies, $b' = \mathcal{B}(b, \boldsymbol{l}, t) \leq \overline{B}$ and $\boldsymbol{l}' = \mathcal{L}(b, \boldsymbol{l}, t) \in \Gamma(\boldsymbol{l}, t)$ together with $v' = V(b', \boldsymbol{l}', t+1)$, and let c be the associated equilibrium consumption. The equilibrium budget constraint must hold, so $y_t - \boldsymbol{l}_0 - c + q(b', \boldsymbol{l}', t)b' + Q(\boldsymbol{l}, \boldsymbol{l}', b', t) \geq b$. Given that the policy $\{c, b', \boldsymbol{l}', v'\}$ is feasible in problem 11, we have obtained a lower bound to the value: $B(v, \boldsymbol{l}, t) \geq b$. In addition, if $B(v, \boldsymbol{l}, t) > b$, then there must exist a feasible vector of policies $(\tilde{c}, \tilde{b}', \tilde{\boldsymbol{l}}', \tilde{v}')$ such that $v = u(\tilde{c}) + \beta \int \max\{V(\tilde{b}', \tilde{\boldsymbol{l}}', t+1), v^D\} dF_{t+1}(v^D)$ and $b < y_t - \boldsymbol{l}_0 - \tilde{c} + q(\tilde{b}', \tilde{\boldsymbol{l}}', t)b' + Q(\boldsymbol{l}, \tilde{\boldsymbol{l}}', \tilde{b}', t)$. But this implies that we can find an alternative policy $(\hat{c}, \tilde{b}', \tilde{\boldsymbol{l}}')$ with $\hat{c} > \tilde{c}$ that satisfies the equilibrium budget constraint at $(b, \boldsymbol{l})$ and delivers an equilibrium value higher than $v = V(b, \boldsymbol{l}, t)$, a contradiction of optimality in equilibrium. It follows then that $B(v, \boldsymbol{l}, t) = b$.

A.4 Proof of Lemma 4

Let $\{c_{t+k}\}_{k=0}^{\infty}$ denote an equilibrium consumption sequence with associated equilibrium choices $\{b_k, \boldsymbol{l}^k\}_{k=1}^{\infty}$, starting from initial state $b_0 = b$ and $\boldsymbol{l}^0 = \boldsymbol{l}$. Given this equilibrium, let us define p_k^t as the equilibrium discounted survival probability k additional periods, starting from time t. The equilibrium budget constraint requires that

$$b \leq y_t - \boldsymbol{l}_0 - c_t + p_1^t \left(b_1 + \sum_{k=1}^{\infty} p_{k-1}^{t+1} \left(\boldsymbol{l}_{k-1}^1 - \boldsymbol{l}_k \right) \right)$$

In period $t+1$, the equilibrium budget constraint implies

$$b_1 \leq y_{t+1} - \boldsymbol{l}_0^1 - c_{t+1} + p_1^{t+1} \left(b_2 + \sum_{k=1}^{\infty} p_{k-1}^{t+2} \left(\boldsymbol{l}_{k-1}^2 - \boldsymbol{l}_k^1 \right) \right)$$

Substituting this into the initial inequality, we get:

$$
\begin{aligned}
b &\leq y_t - \boldsymbol{l}_0 - c_t + p_1^t \Bigg[y_{t+1} - \boldsymbol{l}_0^1 - c_{t+1} + p_1^{t+1} \left(b_2 + \sum_{k=1}^{\infty} p_{k-1}^{t+2} \left(\boldsymbol{l}_{k-1}^2 - \boldsymbol{l}_k^1 \right) \right) \\
&\qquad + \sum_{k=1}^{\infty} p_{k-1}^{t+1} \left(\boldsymbol{l}_{k-1}^1 - \boldsymbol{l}_k \right) \Bigg] \\
&\leq y_t - \boldsymbol{l}_0 - c_t + p_1^t \Bigg[y_{t+1} - \boldsymbol{l}_0^1 - c_{t+1} + p_1^{t+1} \left(b_2 + \sum_{k=1}^{\infty} p_{k-1}^{t+2} \left(\boldsymbol{l}_{k-1}^2 - \boldsymbol{l}_k^1 \right) \right) \\
&\qquad + \boldsymbol{l}_0^1 - \boldsymbol{l}_1 + \sum_{k=2}^{\infty} p_{k-1}^{t+1} \left(\boldsymbol{l}_{k-1}^1 - \boldsymbol{l}_k \right) \Bigg] \\
&\leq \sum_{k=0}^{2} p_k^t (y_{t+k} - \boldsymbol{l}_k - c_{t+k}) \\
&\qquad + p_1^t \Bigg[p_1^{t+1} \left(b_2 + \sum_{k=1}^{\infty} p_{k-1}^{t+2} \left(\boldsymbol{l}_{k-1}^2 - \boldsymbol{l}_k^1 \right) \right) + \sum_{k=2}^{\infty} p_{k-1}^{t+1} \left(\boldsymbol{l}_{k-1}^1 - \boldsymbol{l}_k \right) \Bigg] \\
&\leq \sum_{k=0}^{2} p_k^t (y_{t+k} - \boldsymbol{l}_k - c_{t+k}) \\
&\qquad + p_1^t \Bigg[p_1^{t+1} \left(b_2 + \sum_{k=2}^{\infty} p_{k-2}^{t+2} \left(\boldsymbol{l}_{k-2}^2 - \boldsymbol{l}_{k-1}^1 \right) \right) + \sum_{k=2}^{\infty} p_{k-1}^{t+1} \left(\boldsymbol{l}_{k-1}^1 - \boldsymbol{l}_k \right) \Bigg]
\end{aligned}
$$

But now, we can use that $p_{k-1}^{t+1} = p_1^{t+1} p_{k-2}^{t+2}$.and that $p_1^t p_1^{t+1} = p_2^t$. So we have that

$$
\begin{aligned}
b &\leq \sum_{k=0}^{1} p_k^t (y_{t+k} - \boldsymbol{l}_k - c_{t+k}) \\
&\qquad + p_1^t \Bigg[p_1^{t+1} \left(b_2 + \sum_{k=2}^{\infty} p_{k-2}^{t+2} \left(\boldsymbol{l}_{k-2}^2 - \boldsymbol{l}_{k-1}^1 \right) \right) + p_1^{t+1} \sum_{k=2}^{\infty} p_{k-2}^{t+2} \left(\boldsymbol{l}_{k-1}^1 - \boldsymbol{l}_k \right) \Bigg] \\
&\leq \sum_{k=0}^{1} p_k^t (y_{t+k} - \boldsymbol{l}_k - c_{t+k}) \\
&\qquad + p_1^t \Bigg[p_1^{t+1} \left(b_2 + \sum_{k=2}^{\infty} p_{k-2}^{t+2} \left(\boldsymbol{l}_{k-2}^2 - \boldsymbol{l}_{k-1}^1 \right) + \sum_{k=1}^{\infty} p_{k-2}^{t+2} \left(\boldsymbol{l}_{k-1}^1 - \boldsymbol{l}_k \right) \right) \Bigg] \\
&\leq \sum_{k=0}^{1} p_k^t (y_{t+k} - \boldsymbol{l}_k - c_{t+k}) + p_1^t \Bigg[p_1^{t+1} \left(b_2 + \sum_{k=2}^{\infty} p_{k-2}^{t+2} \left(\boldsymbol{l}_{k-2}^2 - \boldsymbol{l}_k \right) \right) \Bigg] \\
&\leq \sum_{k=0}^{1} p_k^t (y_{t+k} - \boldsymbol{l}_k - c_{t+k}) + p_2^t \left(b_2 + \sum_{k=2}^{\infty} p_{k-2}^{t+2} \left(\boldsymbol{l}_{k-2}^2 - \boldsymbol{l}_k \right) \right)
\end{aligned}
$$

We can keep substituting and we get to the following:

$$b \leq \sum_{k=0}^{N-1} p_k^t (y_{t+k} - \boldsymbol{l}_k - c_{t+k}) + p_N^t \left(b_N + \sum_{k=N}^{\infty} p_{k-N}^{t+N} (\boldsymbol{l}_{k-N}^N - \boldsymbol{l}_k) \right)$$

But recall that

$$b_N \leq \bar{B}; \text{ and } \left| \boldsymbol{l}_k^N \right| \leq \bar{l}$$

where $\bar{B} > 0$ and $\bar{l} > 0$. Using that $p_{k-N}^{t+N} \leq R^{-k+N}$, we get that

$$b \leq \sum_{k=0}^{N-1} p_k^t (y_{t+k} - \boldsymbol{l}_k - c_{t+k}) + R^{-N} \left[\bar{B} + 2\frac{\bar{l}}{R-1} \right]$$

Taking limits as $N \to \infty$, we have that that last term goes to zero, and the equilibrium allocation must satisfy that

$$b \leq \sum_{k=0}^{\infty} p_k^t (y_{t+k} - \boldsymbol{l}_k - c_{t+k})$$

where $p_k^t = \prod_{i=1}^k R^{-1} F_{t+i}(V_{t+i})$ and where $\{c_{t+i}, V_{t+i}\}_{i=0}^{\infty}$ is an incentive compatible allocation with $V_t = v$.

The efficient problem 9 maximizes $\sum_{k=0}^{\infty} p_k^t (y_{t+k} - \boldsymbol{l}_k - c_{t+k})$ subject to delivering $v = V(b, \boldsymbol{l}, t)$ to the government. It follows then that $b = B(v, \boldsymbol{l}, t) \leq \sum_{k=0}^{\infty} p_k^t (y_{t+k} - \boldsymbol{l}_k - c_{t+k}) \leq B^\star(v, \boldsymbol{l}, t)$.

A.5 Proof of Lemma 5

Consider an incentive compatible allocation $\{c_{t+k}, V_{t+k}\}_{k \geq 0}$ for the planning problem 9 at time t, given $v \in [\underline{v}_t^D, \overline{V})$ and $(\boldsymbol{l}, t)$, and let us suppose that this allocation satisfies the condition in Lemma 2. Note that choosing c_t and promising $v' = V_{t+1}$ is feasible in Problem 12, and we have that:

$$B(v, \boldsymbol{l}, t) \geq y_t - \boldsymbol{l}_0 - c_t + R^{-1} F_{t+1}(V_{t+1}) B(V_{t+1}, \boldsymbol{l}_{\geq 1}, t+1)$$

Repeating the above argument sequentially for subsequent dates, it follows that

$$\begin{aligned} B(v,\boldsymbol{l},t) &\geq \sum_{k=0}^{n-1}\left(\prod_{i=1}^{k} R^{-1}F_{t+i}(V_{t+i})\right)(y_{t+k}-\boldsymbol{l}_k-c_{t+k}) \\ &\qquad + \left(\prod_{i=1}^{n} R^{-1}F_{t+i}(V_{t+i})\right) B(V_{t+n},\boldsymbol{l}_{\geq n},t+n) \\ &\geq \sum_{k=0}^{\infty}\left(\prod_{i=1}^{k} R^{-1}F_{t+i}(V_{t+i})\right)(y_{t+k}-\boldsymbol{l}_k-c_{t+k}) \\ &\qquad + \lim_{n\to\infty}\left(\prod_{i=1}^{n} R^{-1}F_{t+i}(V_{t+i})\right) B(V_{t+n},\boldsymbol{l}_{\geq n},t+n) \end{aligned}$$

Given that we are only considering allocations where $V_{t+k} \leq \max\{v, \overline{V}^D\}$, we have that

$$B(V_{t+n},\boldsymbol{l}_{\geq n},t+n) \geq B(w,\boldsymbol{l}_{\geq n},t+n) \geq \sum_{k=0}^{\infty} R^{-k}\min\{(y_{t+n+k}-\boldsymbol{l}_{n+k}-c),0\}$$

where c is such that $u(c) = (1-\beta)w$ and $w \equiv \max\{v, \overline{V}^D, u^{max}/(1-\beta)\}$. The first inequality follows from monotonicity of B and that $V_{t+n} \leq w$, and the second follows because $V(b_0,\boldsymbol{l}_{\geq n},t+n) \geq u(c)/(1-\beta) = w$, using the same argument as in the proof of Lemma 8.

So we have that

$$\overline{B} \geq B(V_{t+n},\boldsymbol{l}_{\geq n},t+n) \geq \sum_{k=0}^{\infty} R^{-k}\min\{(y_{t+n+k}-\boldsymbol{l}_{n+k}-c),0\} \geq -R\frac{\bar{l}+c}{R-1} \tag{27}$$

where the first inequality follows from the No-Ponzi condition, and the last one from the fact that $\boldsymbol{l}_{n+k} \leq \bar{l}$, and $y_{t+n+k} \geq 0$. Given that both sides of equation (27) are finite, it follows that the $\lim_{n\to\infty}\left(\prod_{i=1}^{n} R^{-1}F_{t+i}(V_{t+i})\right) B(V_{t+n},\boldsymbol{l}_{\geq n},t+n) = 0$, and

$$B(v,\boldsymbol{l},t) \geq \sum_{k=0}^{\infty}\left(\prod_{i=1}^{k} R^{-1}F_{t+i}(V_{t+i})\right)(y_{t+k}-\boldsymbol{l}_k-c_{t+k})$$

for any incentive compatible allocation $\{c_{t+k}, V_{t+k}\}_{k\geq 0}$ that satisfies the condition in Lemma 2. But given that, to compute the efficient value function, those allocations are sufficient, we have that

$$B(v,\boldsymbol{l},t) \geq B^{\star}(v,\boldsymbol{l},t).$$

A.6 Proof of Proposition 1 and Corollary 1

The proof of Proposition 1 follows from Lemmas 4 and 5.

The proof of Corollary 1 is as follows. For any $(b, \boldsymbol{l}, t)$, we construct policy $\hat{\mathcal{B}}(b, \boldsymbol{l}, t)$ to be such that $V\left(\hat{\mathcal{B}}(b, \boldsymbol{l}, t), \boldsymbol{l}_{\geq 1}, t+1\right) = V\left(\mathcal{B}(b, \boldsymbol{l}, t), \mathcal{L}(b, \boldsymbol{l}, t), t+1\right)$ and let the policy $\hat{\mathcal{L}}(b, \boldsymbol{l}, t) = \boldsymbol{l}_{\geq 1}$. The price of the associated one period bond, $\hat{q}(b, \boldsymbol{l}, t) = R^{-1}F_{t+1}(V(b, \boldsymbol{l}, t)) = q(b, \boldsymbol{l}, t)$, remains the same. In addition, we obtain $\hat{\rho}$ and $\hat{Q}$, using equations (5) and (2). Lemmas 4 and 5 guarantee that the policies $\hat{\mathcal{B}}$ and $\hat{\mathcal{L}}$ attain the equilibrium value, given $\hat{q}, \hat{Q}, \hat{\rho}$, and thus constitutes an equilibrium.

A.7 Proof of Proposition 2

The weak inequality follows immediately from problem (9), by noticing that the efficient value at state $(v, \boldsymbol{l}', t)$ must be at least at high that the value attained using the sequence $\{c_{t+k}, V_{t+k}\}_{k\geq 0}$ that solves the problem at $(v, \boldsymbol{l}, t)$.

To prove the strict inequality part, we proceed by contradiction. That is, let us assume that the conditions (i),(ii), (iii) hold for some j, and the inequality holds with equality.

Let us consider the following perturbation to the efficient allocation that delivers $V_t = v$: we will change c_{t+j} by Δc_{t+j} and change c_{t+j-1} by Δc_{t+j-1} such that:

$$u'(c_{t+j-1})\Delta c_{t+j-1} + \beta F_{t+j}(v_{t+j})u'(c_{t+j})\Delta c_{t+j} = 0$$

Note that we can perform this perturbation, as by condition (ii), $F_{t+j}(V_{t+j}) > 0$, and by the same argument used in the proof of Lemma 2.ii, consumption at both dates must be interior (as $V_{t+k} \geq \underline{v}_t^D$).

By construction, this perturbation has no first order impact on the government values V_{t+i} for $i \leq j-1$ nor for $i > j+1$. As a result the perturbation has no first order effect on p_k for $k \leq j-1$ and no first order effect on p_k/p_j for $k > j+1$.

The effect of this perturbation on the objective function is

$$-\left(p_{j-1}\Delta c_{t+j-1} + p_j \Delta c_{t+j}\right) + \sum_{k=j}^{\infty} \left(\frac{p_k}{p_j}\right)\left(\frac{\partial p_j}{\partial c_{t+j}}\Delta c_{t+j}\right)(y_{t+k} - \boldsymbol{l}_k - c_{t+k})$$

Using condition (ii), we know that V_{t+j} is in the interior of the support for F_{t+j}, and as a result

$$\frac{\partial p_j}{\partial c_{t+j}} = R^{-1}p_{j-1}f_{t+j}(V_{t+j})u'(c_{t+j}) > 0$$

In addition, $p_j = R^{-1}F_{t+j}(V_{t+j})p_{j-1}$, and we get that the first order impact on the objective is

$$\Delta Obj_t \equiv -(p_{j-1}\Delta c_{t+j-1} + p_j \Delta c_{t+j}) + \sum_{k=j}^{\infty} p_k \frac{f_{t+j}(V_{t+j})}{F_{t+j}(V_{t+j})} u'(c_{t+j})(y_{t+k} - \boldsymbol{l}_k - c_{t+k})\Delta c_{t+j}$$

A necessary condition for optimality is that this perturbation has no first order effect on the objective, $\Delta Obj_t = 0$. Under the hypothesis (to be contradicted),

$$B(v, \boldsymbol{l}', t) = B(v, \boldsymbol{l}, t) - \sum_{k=0}^{\infty} p_k \left(\boldsymbol{l}'_k - \boldsymbol{l}_k\right)$$

which means that the allocation $\{c_{t+k}\}_{k=0}^{\infty}$ achieves the maximum value at $\boldsymbol{l'}$, and thus is optimal as well. We can then proceed with the same perturbation described above, and obtain that, a necessary condition for optimality at $(v, \boldsymbol{l'}, t)$ is that $\Delta \hat{Obj} = 0$ where

$$\Delta \hat{Obj}_t \equiv -(p_{j-1}\Delta c_{t+j-1} + p_j \Delta c_{t+j}) + \sum_{k=j}^{\infty} p_k \frac{f_{t+j}(V_{t+j})}{F_{t+j}(V_{t+j})} u'(c_{t+j})(y_{t+k} - \boldsymbol{l}'_{(k)} - c_{t+k})\Delta c_{t+j}$$

As a result, $\Delta \hat{Obj}_t - \Delta Obj_t = 0$, which requires that

$$\frac{f_{t+j}(V_{t+j})}{F_{t+j}(V_{t+j})} u'(c_{t+j}) \sum_{k=j}^{\infty} p_k (\boldsymbol{l}'_k - \boldsymbol{l}_k) = 0$$

But this implies that

$$\sum_{k=j}^{\infty} p_k (\boldsymbol{l}'_k - \boldsymbol{l}_k) = 0$$

a contradiction of condition (iii).

A.8 Proof of Proposition 3

From Assumption 1.iv, we know that consumption is interior both today and tomorrow if both v_t and v_{t+1} are bigger than $\underline{v}^D$, respectively. Consider now increasing c_t by an amount Δc_t and c_{t+1} by an amount Δc_{t+1} so that v_t does not change:

$$u'(c_t)\Delta c_t + \beta F(v_{t+1}) u'(c_{t+1})\Delta c_{t+1} = 0$$

The effect on the objective is given by:

$$-\Delta c_t + R^{-}1 \left(\frac{dF(v_{t+1})}{dc_{t+1}} B_{t+1} - F(v_{t+1}) \right) \Delta c_{t+1}$$

Optimality of the original allocation implies that the above must be zero.

Using the previous equation, we have that

$$-\Delta c_t - R^{-1}\left(\frac{dF(v_{t+1})}{dc_{t+1}}B_{t+1} - F(v_{t+1})\right)\frac{u'(c_t)\Delta c_t}{\beta F(v_{t+1})u'(c_{t+1})} = 0$$

Note that in the case $v_{t+1} > \overline{v}^D$, we have that $dF(v_{t+1})/dc_{t+1} = 0$, and (15) follows. For the case where $v_{t+1} \in (\underline{v}^D, \overline{v}^D)$, we have that $dF(v_{t+1})/dc_{t+1} = f(v_{t+1})u'(c_{t+1})$, and (16) follows.

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