Conformal differential geometry

and its interaction with representation theory

The X-ray transform on projective space

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Some useful facts

Curvature on $\mathbb{RP}_n$

$$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad}$$

Curvature on $\mathbb{CP}_n$

$$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad} + J_{ac}J_{bd} - J_{bc}J_{ad} + 2J_{ab}J_{cd}$$

Model embeddings $\mu : \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ totally geodesic

BGG On $\mathbb{RP}_n$ for $n \geq 2$, $\boxed{\omega_a = \nabla_a\phi \Leftrightarrow \nabla_{[a}\omega_{b]} = 0}$

$$\boxed{\omega_{ab} = \nabla_{(a}\phi_{b)} \Leftrightarrow \pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}) = 0} \quad \&c$$

2-form lemma $\mu^*\psi_{ab} = 0\,\forall\mu \quad \Leftrightarrow \quad \psi^{\perp}_{ab} = 0$

Curvature lemma $\mu^*\psi_{abcd} = 0\,\forall\mu \quad \Leftrightarrow \quad \psi^{\perp}_{abcd} = 0$

Curvature lemma

$$\left.\begin{array}{l}\psi_{abcd} = \psi_{[ab][cd]} \\ \psi_{[abc]d} = 0\end{array}\right\} \equiv \text{Riemann tensor symmetries}$$

$$\psi_{abcd} \in \Gamma(\overset{0}{\bullet}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}) \quad (\text{case } n = 4).$$

Branch to $\mathrm{SL}(8,\mathbb{R}) \supset \mathrm{Sp}(8,\mathbb{R})$

$$\begin{array}{ccccccc}
\overset{0}{\bullet}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} & = & \overset{0}{\bullet}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} & \oplus & \overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} & \oplus & \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \\
\psi_{abcd} & = & \psi^{\perp}_{abcd} & + & \phi_{ab} \bowtie J_{cd} & + & \theta J_{ab} \bowtie J_{cd} \\
 & & \text{cf. Weyl} & & \text{cf. Ricci} & & \text{cf. Scalar}
\end{array}$$

$$\boxed{\text{Lemma: } \mu^*\psi_{abcd} = 0 \;\forall \text{ models } \mu \quad \Leftrightarrow \quad \psi^{\perp}_{abcd} = 0}$$

BGG detail

Range of Killing operators on $\mathbb{RP}_n$ for $n \geq 2$

$$\omega_a = \nabla_a \phi \Leftrightarrow \pi(\nabla_a \omega_b) = \nabla_{[a}\omega_{b]} = 0$$

$$\omega_{ab} = \nabla_{(a}\phi_{b)} \Leftrightarrow \pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}) = 0$$

$$\omega_{abc} = \nabla_{(a}\phi_{bc)} \Leftrightarrow \pi(\nabla_{(a}\nabla_c\nabla_{e)}\omega_{bdf} + 4g_{(ac}\nabla_{e)}\omega_{bdf}) = 0$$

Combinatorics

$$\begin{array}{lllllll}
1 & & & & \ddots & \ddots & \\
1 & 1 & & & & \ddots & \\
1 & 4 & & & & & \\
1 & 10 & 9 & & & & \\
1 & 20 & 64 & & & & \\
1 & 35 & 259 & 225 & & & \\
1 & 56 & 784 & 2304 & & & \\
1 & 84 & 1974 & 12916 & 11025 & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots &
\end{array}$$

$\boxed{\text{cf. OEIS}}$

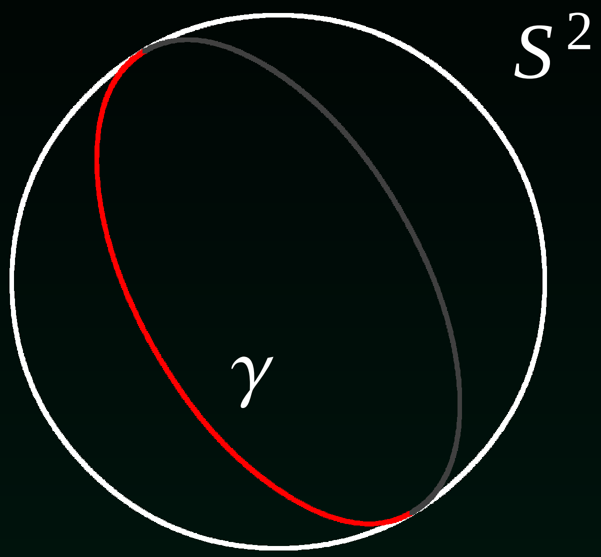
Topics

- Funk or Radon transforms on S^2 or $\mathbb{R}^2$
- X-ray transform on $\mathbb{RP}_3$
- X-ray transform on $\mathbb{CP}_n$
- X-ray transform on functions
- X-ray transform on 1-forms
- Symplectic geometry
- X-ray transform on symmetric tensors
- Complex methods

Funk-Radon

- Funk (1913)

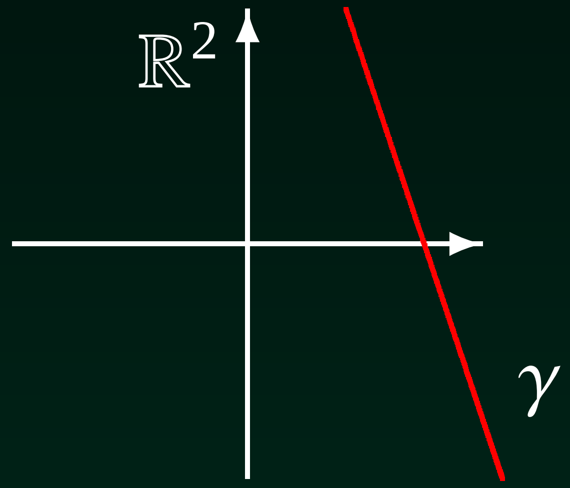
$$f \in \Gamma(S^2, \mathcal{E})$$



$$\phi(\gamma) = \oint_\gamma f$$

- Radon (1917)

$$f \in \Gamma_*(\mathbb{R}^2, \mathcal{E})$$



$$\phi(\gamma) = \int_\gamma f$$

Radon=Funk!

$$\begin{array}{ccc} \mathcal{F} : \Gamma_{\text{even}}(S^2, \mathcal{E}) & \xrightarrow{\cong} & \Gamma_{\text{even}}(S^2, \mathcal{E}) \\ \| & & \| \\ \text{Better:} \quad \Gamma(\mathbb{RP}_2, \mathcal{E}) & & \Gamma(\mathbb{RP}_2, \mathcal{E}) \end{array}$$

Better still: $\mathcal{F} : \Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) \xrightarrow{\cong} \Gamma(\mathbb{RP}_2^*, \widetilde{\mathcal{E}}(-1))$

Usual affine coördinates $\mathbb{R}^2 \hookrightarrow \mathbb{RP}_2 \quad \rightsquigarrow$
projective equivalence!!

$$\left.\begin{array}{ccc} \Gamma_*(\mathbb{R}^2, \mathcal{E}) & \xrightarrow{\mathcal{R}} & \\ \downarrow & & \\ \Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) & \xrightarrow{\mathcal{F}} & \end{array}\right\} \quad \text{agree!}$$

John (1938)

The X-ray transform according to John

$$\Gamma_*(\mathbb{R}^3, \mathcal{E}) \ni f \mapsto \quad \phi(\gamma) = \int_\gamma f$$

Better: $\quad \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \ni f \mapsto \quad \phi(\gamma) = \oint_\gamma f$

NB invariance under $\mathrm{SL}(4,\mathbb{R})$ because

$$\Gamma(\mathbb{RP}_1, \mathcal{E}(-2)) \cong \Gamma(\mathbb{RP}_1, \Lambda^1) \xrightarrow{\int} \mathbb{R}$$

is invariant under $\mathrm{SL}(2,\mathbb{R})$.

X-ray transform on $\mathbb{RP}_3$

$$\mathcal{X} : \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \longrightarrow \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}[-1])$$

Range? Theorem (≈John)

$$\phi = \mathcal{X} f \iff \Box\phi = 0$$

where $\Box : \widetilde{\mathcal{E}}[-1] \to \widetilde{\mathcal{E}}[-3]$

= ultrahyperbolic wave operator ($\mathrm{SL}(4, \mathbb{R})$-invariant).

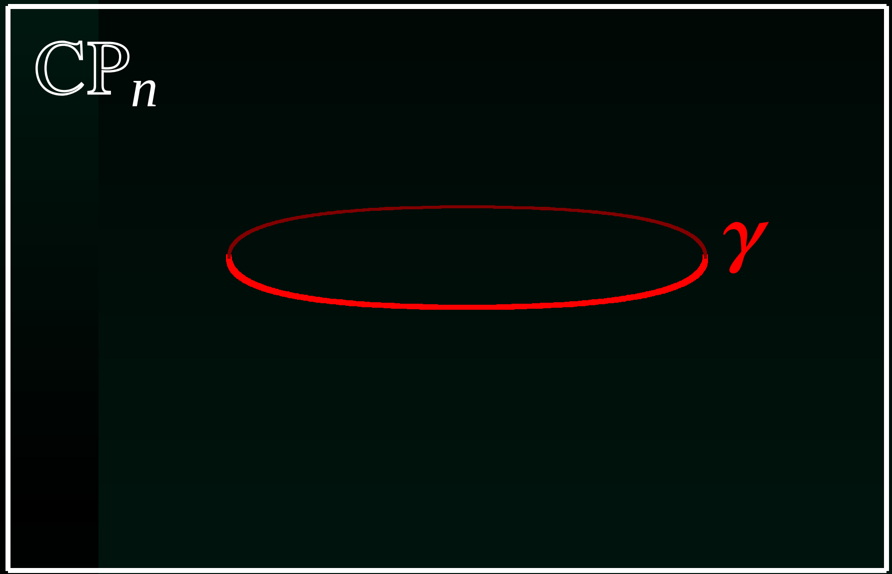
Kernel? $\mathcal{X}$ is injective on $\Gamma(\mathbb{RP}_n, \mathcal{E}(-2))$ for $n \geq 2$

$\|$ Funk

$\Gamma(\mathbb{RP}_n, \Lambda^0)$

As a Riemannian manifold under $\mathrm{SO}(n+1)$

X-ray transform on $\mathbb{CP}_n$



$\mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))$

Fubini-Study metric

f = smooth function on $\mathbb{CP}_n$

γ = geodesic

$$f \xmapsto{\mathcal{X}} \phi(\gamma) = \oint_\gamma f$$

Questions

- Kernel of $\mathcal{X}$?
- What about $\omega \xmapsto{\mathcal{X}} \phi(\gamma) = \oint_\gamma \omega$ for ω a 1-form?
- What about $\omega_{ab\cdots c}$ a symmetric tensor?

X-ray transform on functions

Know $\oint_\gamma f = 0 \ \forall$ geodesics $\gamma \hookrightarrow \mathbb{RP}_n \iff f = 0.$

Suppose Funk (1913)

$$\oint_\gamma f = 0 \quad \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{CP}_n.$$

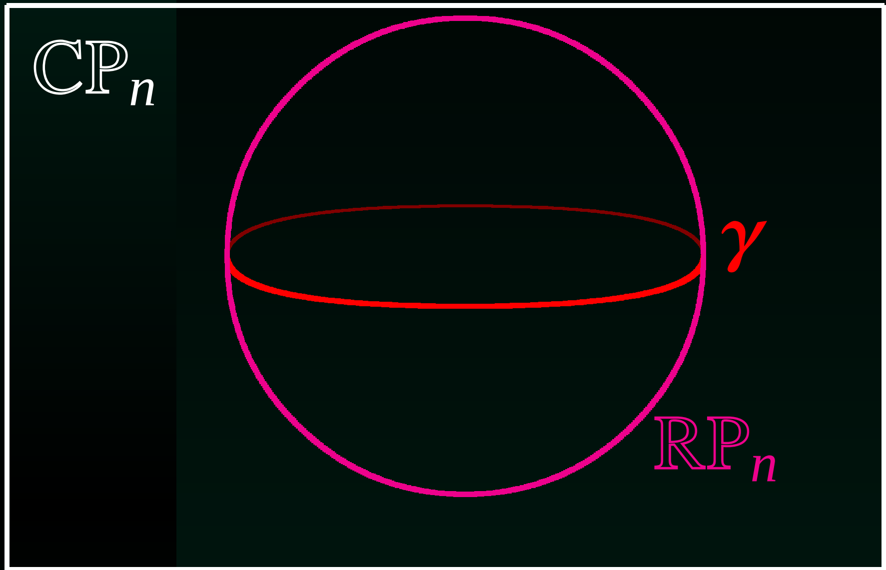
Then $\oint_\gamma f = 0 \ \forall$ geodesics $\gamma \hookrightarrow \mathbb{RP}_n \overset{\mu}{\hookrightarrow} \mathbb{CP}_n$ for any model embedding. Hence,

$$\mu^* f = 0 \ \ \forall \text{ model embeddings } \mu : \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n.$$

Hence $f = 0$, i.e. $\boxed{\mathcal{X} \text{ is injective on functions on } \mathbb{CP}_n}$

cf. Helgason, The Radon Transform, §2 Corollary 2.3

Approach (with Hubert Goldschmidt)



$\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ induced by $\mathbb{R}^{n+1} \hookrightarrow \mathbb{C}^{n+1}$ is totally geodesic.

Translates by $\mathrm{SU}(n+1)$ too!

↑ ‘Model Embeddings’ μ

- The X-ray transform on $\mathbb{RP}_n$ is well-understood.
- Pullback of tensors under μ is well-understood.
- Suitable global techniques on $\mathbb{CP}_n$ are available,
- compatible with similar techniques (BGG) on $\mathbb{RP}_n$.

X-ray transform on 1-forms

Know $\oint_\gamma \omega = 0 \ \forall$ geodesics $\gamma \hookrightarrow \mathbb{RP}_n \iff \omega = d\phi.$

Suppose Michel (1978)

$$\oint_\gamma \omega = 0 \quad \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{CP}_n.$$

Then $\oint_\gamma \omega = 0 \ \forall$ geodesics $\gamma \hookrightarrow \mathbb{RP}_n \overset{\mu}{\hookrightarrow} \mathbb{CP}_n$ for any model embedding. Hence,

$$\mu^* d\omega = 0 \ \ \forall \text{ model embeddings } \mu : \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n.$$

Hence, by 2-form lemma, $\boxed{d\omega = \theta J}$. Claim $\boxed{\omega = d\phi}$

1-forms cont'd

Theorem For $\omega \in \Gamma(\mathbb{CP}_n, \Lambda^1)$, $n \geq 2$, TFAE

(a) $\omega = d\phi$

(b) $d\omega = 0$

(c) $(d\omega)_\perp = 0$

(d) $d\omega = \theta J$

Proof (a)$\Leftrightarrow$(b) because $H^1(\mathbb{CP}_n, \mathbb{R}) = 0$.
(c)$\Leftrightarrow$(d) by definition.
(b)$\Rightarrow$(c) is trivial.
(d)$\Rightarrow$(b) If $d\omega = \theta J$, then

$$0 = d^2\omega = d(\theta J) = d\theta \wedge J \Rightarrow d\theta = 0 \Rightarrow \theta = \text{constant}.$$

But if $\theta \neq 0$, then $d(\omega/\theta) = J$, a contradiction. $\square$

Symplectic geometry

Rumin-Seshadri complex

$$\begin{array}{ccccccccccc}
\boxed{\Lambda^0} & \xrightarrow{d} & \boxed{\Lambda^1} & \xrightarrow{d_\perp} & \Lambda^2_\perp & \xrightarrow{d_\perp} & \Lambda^3_\perp & \xrightarrow{d_\perp} & \cdots & \xrightarrow{d_\perp} & \Lambda^n_\perp \\
 & & & & & & & & & & \downarrow d^{(2)}_\perp \\
\Lambda^0 & \xleftarrow{d_\perp} & \Lambda^1 & \xleftarrow{d_\perp} & \Lambda^2_\perp & \xleftarrow{d_\perp} & \Lambda^3_\perp & \xleftarrow{d_\perp} & \cdots & \xleftarrow{d_\perp} & \Lambda^n_\perp
\end{array}$$

$\boxed{}$ local cohomology $= \mathbb{R}$

- $\Gamma(\mathbb{CP}_n, \Lambda^0) \xrightarrow{d} \Gamma(\mathbb{CP}_n, \Lambda^1) \xrightarrow{d_\perp} \Gamma(\mathbb{CP}_n, \Lambda^2_\perp)$

is exact.

Symplectic geometry cont'd

Suppose ∇ is a connection on $\mathbb{V}$ such that

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\Sigma = J_{ab}\Phi\Sigma \qquad \Phi \in \mathrm{End}\mathbb{V}.$$

Then we can couple the Rumin-Seshadri complex

$$\mathbb{V} \xrightarrow{\nabla} \Lambda^1 \otimes \mathbb{V} \xrightarrow{\nabla_\perp} \Lambda^2_\perp \otimes \mathbb{V} \longrightarrow \cdots$$

It's still a complex and† (if $\mathbb{V}$ is a bundle on $\mathbb{CP}_n$)

$$\Gamma(\mathbb{CP}_n, \mathbb{V}) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_n, \Lambda^1 \otimes \mathbb{V}) \xrightarrow{\nabla_\perp} \Gamma(\mathbb{CP}_n, \Lambda^2_\perp \otimes \mathbb{V})$$

is exact.

† under further mild conditions

Symmetric 2-tensors

Know (Michel (1973))

$$\oint_\gamma \omega_{ab} = 0 \ \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{RP}_n \iff \omega_{ab} = \nabla_{(a}\phi_{b)}.$$

Therefore (BGG),

$$\oint_\gamma \omega_{ab} = 0 \ \forall\gamma \hookrightarrow \mathbb{RP}_n \iff \pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}) = 0.$$

Would like to show (Tsukamoto (1981))

$$\oint_\gamma \omega_{ab} = 0 \ \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{CP}_n \iff \omega_{ab} = \nabla_{(a}\phi_{b)}.$$

Know (BGG & curvature lemma)

$$\oint_\gamma \omega_{ab} = 0 \ \forall\gamma \hookrightarrow \mathbb{CP}_n \Rightarrow (\pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}))^\perp = 0.$$

Therefore, suffices to show, on $\mathbb{CP}_n$

$$\boxed{\omega_{ab} = \nabla_{(a}\phi_{b)} \iff (\pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}))^\perp = 0}$$

Proof of $\boxed{\omega_{ab} = \nabla_{(a}\phi_{b)} \iff \left(\pi\left(\nabla_{(a}\nabla_{b)}\omega_{cd} + g_{ab}\omega_{cd}\right)\right)^{\perp} = 0}$

Consider the connection on $\mathbb{V} \equiv \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^1$

$$\begin{bmatrix} \nabla_a\sigma_b - \mu_{ab} \\ \nabla_a\mu_{bc} + g_{ab}\sigma_c - g_{ac}\sigma_b + J_{ab}\rho_c - J_{ac}\rho_b - J_{bc}\rho_a + J_{bc}J_a{}^d\sigma_d \\ \nabla_a\rho_b + J_a{}^d\mu_{bd} \end{bmatrix}$$

It satisfies† $(\nabla_a\nabla_b - \nabla_b\nabla_a)\Sigma = J_{ab}\Phi\Sigma.$

Now unravel the exactness of

$$\Gamma(\mathbb{CP}_n, \mathbb{V}) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_n, \Lambda^1 \otimes \mathbb{V}) \xrightarrow{\nabla_\perp} \Gamma(\mathbb{CP}_n, \Lambda^2_\perp \otimes \mathbb{V})$$

by Heisenberg Lie algebra cohomology! $\boxed{\leftrightarrow \text{BGG}}$!!

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† and further mild conditions

Further results and summary

Theorem

Suppose $\omega_{ab\cdots c}$ is symmetric on $\mathbb{RP}_n$ or $\mathbb{CP}_n$ for $n \geq 2$.

$$\boxed{\oint_\gamma \omega_{ab\cdots c} = 0 \ \forall \text{ geodesics } \gamma \iff \omega_{ab\cdots c} = \nabla_{(a}\phi_{b\cdots c)},}$$

for some symmetric tensor $\phi_{b\dots c}$.

By combining

- pullback lemmata: curvature &c
- BGG complexes
- symplectic geometry

we can bootstrap from $\mathbb{RP}_n$ to $\mathbb{CP}_n$. Therefore,

It suffices to prove the Theorem for $\mathbb{RP}_n$.

Complex methods

$\boxed{\dim_{\mathbb{R}} = 2n}$

$$\begin{tikzcd}[column sep=small]
& \mathbb{F}_{1,2}(\mathbb{C}^{n+1}) \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & & \supset & & \nu^{-1}(\mathrm{Gr}_2(\mathbb{R}^{n+1})) \equiv F \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\
\mathbb{CP}_n & & \mathrm{Gr}_2(\mathbb{C}^{n+1}) & & \mathbb{CP}_n & \cup & \mathrm{Gr}_2(\mathbb{R}^{n+1}) \\
& & & & & \mathbb{F}_{1,2}(\mathbb{R}^{n+1}) \arrow[dl] \arrow[dr] & \\
& & & & \mathbb{RP}_n & & \mathrm{Gr}_2(\mathbb{R}^{n+1})
\end{tikzcd}$$

$\boxed{\text{Double fibration transform}}$

$\eta : F \to \mathbb{CP}_n$ real blow-up along $\mathbb{RP}_n$

$\Rightarrow$ F acquires an involutive structure (cf. E & Graham)

Further Reading

- T.N. Bailey and M.G. Eastwood, Zero-energy fields on real projective space, Geom. Dedicata 67 (1997) 245–258.
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THANK YOU

THE END