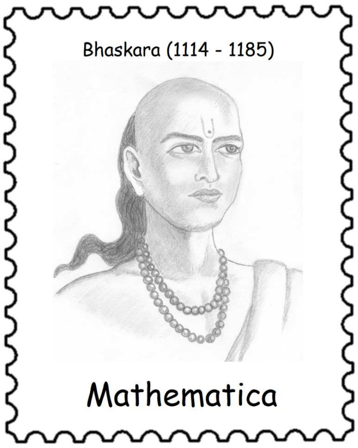
BHASKARA (1114 – 1185)

by HEINZ KLAUS STRICK, Germany



(Drawings: © Andreas Strick)

BHASKARA is considered the most important mathematician of the Indian Middle Ages. He is usually referred to as BHASKARA II – to distinguish him from a 7th century mathematician and astronomer of the same name. Subsequent mathematicians speak of him reverently as BHASKARACHARYA, which means "BHASKARA the teacher" or "BHASKARA the scholar".

He was the son of an astrologer from the South Indian city of Vijayapura (in the state of Karnataka, India) and spent many years of his life in Ujjain (Madhya Pradesh).

There he worked as the head of the astronomical observatory - like BRAHMAGUPTA, the most famous of his predecessors. He wrote (at least) six books with mathematical rules in verse – as was usual in India. Many subsequent generations of students were still taught according to these rules.

The most famous book *Līlāvatī* (The Beauty) contains 277 verses in 13 chapters. It began with the explanation of different units for amounts of money, weight, lengths, areas, volumes and time intervals.

This was followed by explanations of the arithmetical operations for positive and negative numbers, fractions and the number zero, addition, subtraction, multiplication, division, squaring and square roots, cubes and third roots.

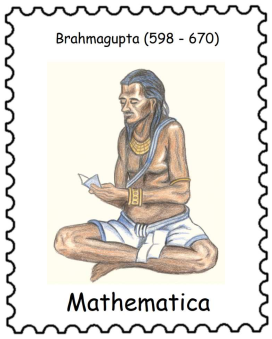
After naming each rule, there was a task for the reader to work on, for example:

- *My friend, tell me quickly what the square of* $3\frac{1}{2}$ *is and what the square root of the square is ....*

Like BRAHMAGUPTA he considered division by zero to be permissible and gave infinity as the result of the division, with the consequence that: $\frac{a}{0} \cdot 0 = a$.

The next verses contained tasks that required backward calculation to solve them, for example:

- *A pilgrim spent half of his money in Prayaga, two ninths of the rest in Kasi, a quarter of the remaining money for fees and six tenths of the rest in Gaya. 63 gold coins were left when he returned home.*
  What amount had he originally taken with him ?



Then followed a problem that BRAHMAGUPTA had already investigated:

- *We are looking for are two rational numbers x and y so that* $x^2 \pm y^2 - 1$ *is a rational square number.*
  (Note: pairs of numbers (*x, y*) with $x = 8a^4 + 1, y = 8a^3$ meet this condition).

After problems to apply the *rule of three* and interest calculations followed various tasks, including the determination of the gold content of an alloy. In connection with simple combinatorial considerations, the question of how many types of different verse forms a poem can have was examined.

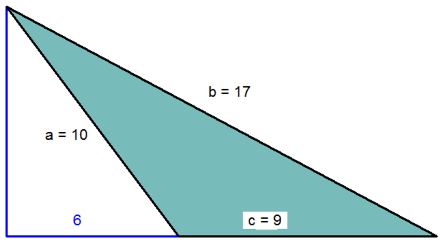
He also dealt with the problem of how many *n*-digit numbers (in the decimal system) with non-zero digits can have a certain sum of digits *S*. And later he determined the number of permutations, using as an example the question of how many different statues of the deity SHIVA could be made, when she holds 10 different objects in her 10 hands. (There are 3 628 800 = 10! possibilities.)

In chapter 5 he dealt with arithmetical and geometrical sequences, for example:

- *On an expedition where a king tries to seize the elephant of his enemy, he marches* 2 *yojanas on the first day. Tell me, wise calculator, by how much does he have to increase the daily distance he covers to reach his destination, the enemy city, which is* 80 *yojanas away, after a week?*

The chapter on geometry began with applications of PYTHAGORAS's theorem.

Here one finds the task of determining the lengths of the segment *q* of a triangle with sides 10, 17 and 9 length units. BHASKARA solved it using a formula already known to BRAHMAGUPTA:



$$q = \frac{1}{2} \cdot \left(c - \frac{b^2 - a^2}{c} \right)$$

With *c* = 9, *b* = 17 and *a* = 10 the result is *q* = -6 which Bhaskara commented as follows:

This is negative, that is, in the opposite direction.

Working with the geometry of circles and spheres, he was the first mathematician in his cultural circle to give the correct formulas: $A = \frac{1}{4} \cdot d \cdot u$ for the area *A*, the circumference *u* and the diameter *d* of a circle, and $O = d \cdot v$, $V = \frac{1}{6} \cdot O \cdot d$ for the surface *O* and the volume *V* of a ball.

For the ratio $\frac{u}{d}$ he used the approximate value $\frac{22}{7}$ and he described the fraction $\frac{3927}{1250} = 3.1416$ as *exact*.

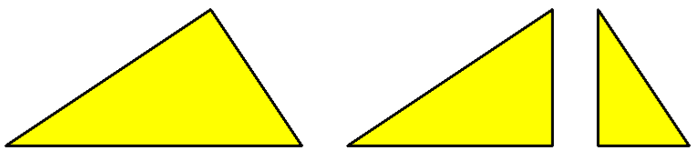
A second work, called *Bīja-ganita* (Fundamentals of Mathematics; *bīja* means Seeds) was aimed at advanced students. It contained mainly algebraic methods and dealt, among other things, with the transformation of roots of equations. The book became famous because it described the *cyclic method* for solving quadratic DIOPHANTINE equations of the type $Nx^2 + k = y^2$ which had already been studied by BRAHMAGUPTA and are now called PELL's equations.

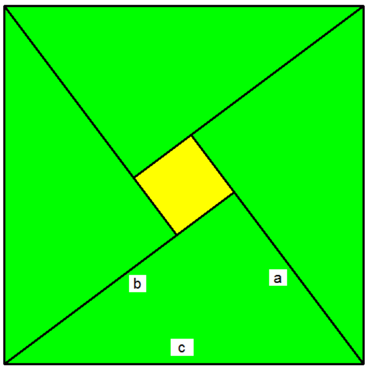
Among the examples with quadratic equations, one finds the following often quoted task:

- *The fifth part of a herd of monkeys, less three, squared, went into a cave. Only one monkey was still to be seen. How many monkeys were there?*

Of the two solutions, 5 and 50, the first was rejected because the term $\frac{x}{5} - 3$ would be negative.

PYTHAGORAS's theorem was based on the similarity between the right-angled triangle and the two smaller triangles created by the altitude H_c .





BHASKARA commented on another proof only with the exclamation:

See!

A square of the side length c is divided into four congruent right-angled triangles with sides *a, b* (where $a < b$) and hypotenuse c together with a square of the side length $a - b$.

From $c^2 - 4 \cdot \frac{1}{2} \cdot a \cdot b + (b-a)^2$ then the relationship $c^2 = a^2 + b^2$ results.

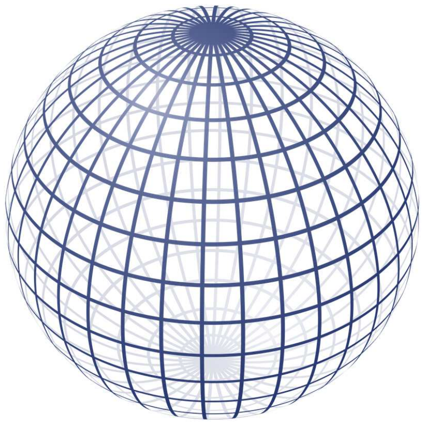
At the end of the book, linear systems of equations with infinitely many solutions were treated, for example:

- *The numbers of horses belonging to four men are* 5, 3, 6 *and* 8. *The camels belonging to the same men are* 2, 7, 4 *and* 1. *The mules belonging to them are* 8, 2, 1 *and* 3 *and the oxen are* 7, 1, 2 *and* 1. *All four men have equal fortunes. Tell me quickly the price of each horse, camel, mule and ox.*

(The smallest possible solution is: horses 85 monetary units, camels 76, mules 31, oxen 4).

In 1150 BHASKARA wrote the work *Siddhānta-śiromani* (Most beautiful jewel of treatises), which dealt mainly with typical astronomical questions such as planetary constellations and lunar and solar eclipses as well as the handling of astronomical instruments. In the context of the investigation of planetary movements he dealt with the question of how to determine the instantaneous speed of a planet. His idea of comparing the positions for ever smaller time intervals for this purpose is regarded by some science historians as an infinitesimal approach. In particular, they see this confirmed by his description that the planets have zero instantaneous speed at the highest point of their daily orbits.

In the mathematical part he presented a method for deriving the volume formula for the sphere. For this purpose, he considered a coordinate network of circles of longitude and latitude.
The surface of the sphere was divided by 48 great circles into 96 "slices" and by 48 parallels of latitude into trapezoidal areas. The areas of the trapezoids were calculated as the arithmetic means of the length of the two sections on the parallel parallels of the parallels of latitude, which were multiplied by the altitudes (= arcs of the great circles).



(source: https://servimg.com/view/13271906/1471)

The length of the sections on the parallels of latitude could be calculated using the sine function. Therefore a sum of sine values was used to calculate the surface of the sphere. BHASKARA did this with the help of a sine table with step size $\frac{1}{24} \cdot 90° = 3°\,45'$ and thus confirmed the validity of the formula $O = d \cdot u$ for the area of the surface.

He then imagined the surface divided into tiny square areas, the corners of which, connected to the centre of the sphere, formed a pyramid-like decomposition of the sphere.

The volume was calculated according to the volume formula for pyramids as $V = \frac{1}{3} \cdot O \cdot d$ so since $d = \frac{1}{2} r$ we have $V = \frac{1}{6} \cdot O \cdot d$.

In the work *Jyotpatti* (Calculation of sines), BHASKARA explained how to obtain the most accurate values possible for the sine function from the known basic values

$$\sin(30°) = \frac{1}{2},\ \sin(45°) = \frac{1}{\sqrt{2}},\ \sin(36°) = \sqrt{\frac{5-\sqrt{5}}{8}}\ .$$

Furthermore, the work contained rules such as $\sin\left(\frac{90° \pm \alpha}{2}\right) = \sqrt{\frac{1 \pm \sin(\alpha)}{2}}$

and useful approximation formulas like: $\sin(\alpha \pm 3.75°) \approx \frac{466}{467} \cdot \sin(\alpha) \pm \frac{100}{1529} \cdot \cos(\alpha)$.

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https://www.spektrum.de/wissen/bhaskara-indischer-mathematiker-mittelalter/1347574

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