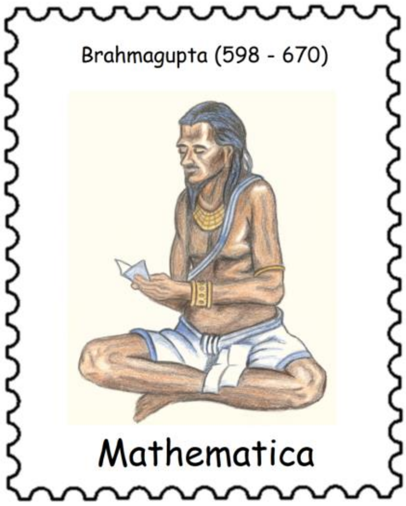
BRAHMAGUPTA (598 – 670)

by HEINZ KLAUS STRICK, Germany



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At the beginning of the 9th century, AL-KHWARIZMI introduced the decimal place value system to the Islamic world using Indian numerals. In his work *Al Kitāb al-muhtasar fi hisāb al-ğabr w-al-muqābala*, he gave different methods for solving quadratic equations, since he only allowed positive numbers as coefficients:

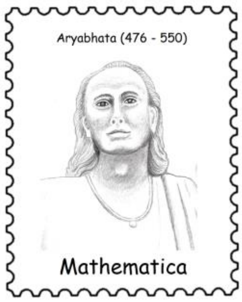
$ax^2 + bx = c$, $ax^2 + c = bx$ and $ax^2 = bx + c$.

This was a "step backwards" with far-reaching consequences for the development of mathematics, since 200 years earlier the Indian mathematician BRAHMAGUPTA had already given a formula for solutions of equations of the type $ax^2 + bx = c$ with arbitrary coefficients: $x = \frac{\sqrt{b^2 + 4ac} - b}{2a}$.

BRAHMAGUPTA was born in 598 in Bhinmal, a town in the northwest of India (today: in the state of Rajasthan). At the age of 30, he wrote a work that has been handed down under the name *Brāhmasphutasiddhānta* (Perfection of the Teachings of Brahma, siddhānta = treatise). This was followed in 665 by *Khandakhādyaka*, another treatise dealing primarily with astronomical calculations.

In the meantime, BRAHMAGUPTA worked as the director of the astronomical observation station in Ujjain. This city, located in the present-day state of Madhya Pradesh, was one of the seven holy cities of India.

Only two of the 25 chapters of *Brāhmasphutasiddhānta* deal with mathematical questions, namely chapter 12 (*Ganitādhyāya*, from gana = to count) and chapter 18 (*Kuttakādhyāya*, from kuttaka = literally: to crush).



Despite a number of comments, some of them very critical, on the work of his predecessor ĀRYABHATA, published 130 years earlier, it was probably no coincidence, but rather a sign of reverence, that the 12th chapter contained exactly twice as many verses as the corresponding *ganita* chapter of the *Āryabhatīya*. With regard to the calculation procedures and the solution of various application tasks, however, one initially finds little more in BRAHMAGUPTA than what ĀRYABHATA had compiled.

Only in verses 10 to 13 of the 12th chapter did BRAHMAGUPTA go beyond the treatment of simple proportional relationships.

He uses two examples to explain the following rule of the five quantities:

➢ *Put the quantities in the columns of a table. The solution is found by swapping two of the entries; then the factors of the numerator and the denominator of a fraction are on top of each other.*

- If a fabric of length a_1 and width b_1 is sold at a price of p_1 monetary units, what should be the width x of a fabric of length a_2 that costs p_2 monetary units?

a_1	a_2
b_1	x
p_1	p_2

→

a_1	a_2
b_1	
p_2	p_1

→ $$x = \frac{a_1 \cdot b_1 \cdot p_2}{a_2 \cdot p_1}$$

- If n_1 pieces of a given item cost a total of p_1 monetary units and n_2 pieces of a second item cost a total of p_2 monetary units, how many pieces y of the second item can be exchanged if q_1 pieces of the first item are available for exchange?

p_1	p_2
n_1	n_2
q_1	y

→

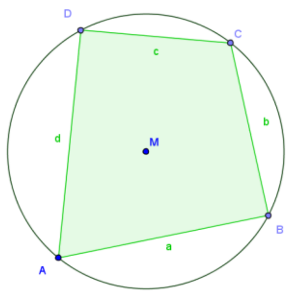
p_1	p_2
n_2	n_1
q_1	

→ $$y = \frac{p_1 \cdot n_2 \cdot q_1}{p_2 \cdot n_1}$$

Verses 21 to 32 of the *Brāhmasphutasiddhānta* deal with calculations of area and side length.

Here we find the remarkable approximation formula for determining the area of quadrilaterals

$$A = \frac{a+c}{2} \cdot \frac{b+d}{2}$$



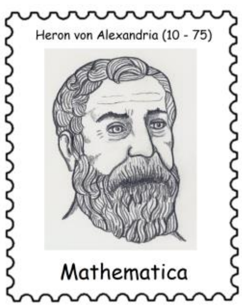
as well as the famous formula of BRAHMAGUPTA for calculating the area of cyclic quadrilaterals

$$A = \sqrt{(s-a) \cdot (s-b) \cdot (s-c) \cdot (s-d)}$$

where $s = \frac{1}{2} \cdot (a+b+c+d)$ denotes half the perimeter of the quadrilateral.

This formula is not proved, but – as was usual in Indian mathematics – only given as a calculation rule (a mnemonic rule in verse form).

In the case *d* = 0 it is the formula already derived by HERON for calculating the area of a triangle. Therefore, the formula given above is also called BRAHMAGUPTA's generalisation of HERON's formula.



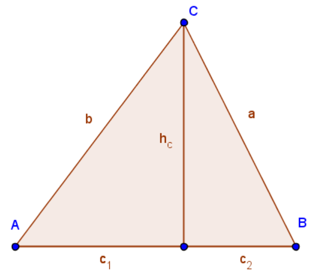
BRAHMAGUPTA does not give any restriction for the validity of the formula. However, it does not apply to arbitrary quadrilaterals, but only to cyclic quadrilaterals. Since the rest of the chapter refers to quadrilaterals whose vertices lay on a circle, it is presumed that BRAHMAGUPTA meant only such quadrilaterals.

Other remarkable formulas are those with which the lengths of distances in triangles and in symmetrical trapezoids could be calculated:

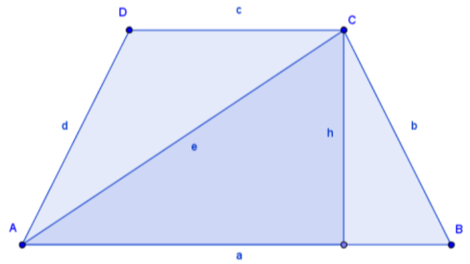
- In any triangle, the following applies to the altitude h_c and the sections c_1 and c_2 of the side *c* (and analogously for the other altitudes and sides in the triangle):

  $c_1 = \frac{1}{2} \cdot \left(c + \frac{b^2 - a^2}{c} \right)$; $c_2 = \frac{1}{2} \cdot \left(c - \frac{b^2 - a^2}{c} \right)$ and

  $h_c = \sqrt{a^2 - c_2{}^2} = \sqrt{b^2 - c_1{}^2}$.



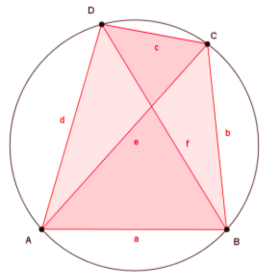
- In isosceles trapezoids: $e = \sqrt{a \cdot c + b \cdot d}$ (from the theorem of PTOLEMY), $h = \sqrt{e^2 - \left(\frac{a+c}{2} \right)^2}$, and moreover, the radius of the circumcircle $r = \frac{b \cdot e}{2h}$.



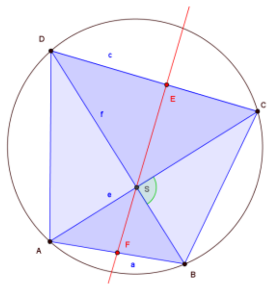
- BRAHMAGUPTA gives formulas for the length of the diagonals *e, f* in an arbitrary cyclic quadrilateral: $\frac{e}{f} = \frac{ad+bc}{ab+cd}$, where

  $e = \sqrt{\frac{(ad+bc) \cdot (ac+bd)}{ab+cd}}$ and $f = \sqrt{\frac{(ab+cd) \cdot (ac+bd)}{ad+bc}}$,

  and for cyclic quadrilaterals with mutually orthogonal diagonals (so-called BRAHMAGUPTA quadrilaterals) he formulates the theorem:



- *A straight line passing through the intersection of the two diagonals and intersecting one of the sides perpendicularly bisects the opposite side.*



In verses 33 to 39 BRAHMAGUPTA deals with the problem of finding triangles, symmetrical trapezoids and cyclic quadrilaterals whose side lengths and areas are rational.

For example, for $u, v, w \in \mathbb{N}$ with $v, w < u$ such rational triangles are obtained when

$$a = \frac{1}{2} \cdot \frac{u^2 + v^2}{v};\; b = \frac{1}{2} \cdot \frac{u^2 + w^2}{w};\; c = \frac{1}{2} \cdot \frac{u^2 - v^2}{v} + \frac{1}{2} \cdot \frac{u^2 - w^2}{w}.$$

The 18th chapter begins with astronomical calculations, such as determining the number of days between two points in time when a planet can be seen at the same place in the sky. Then follows – for the first time in the history of mathematics – calculation rules for positive and negative numbers as well as for the number zero.

Zero was thus regarded as a number and was not just a placeholder for an empty space.

BRAHMAGUPTA refers to positive numbers as *assets*, negative numbers as *debts*. For example, one finds:

- *A debt minus zero is a debt; an asset minus zero is an asset. Zero minus zero is zero. Zero minus a debt is an asset. Zero minus an asset is a debt. The product* (*quotient*) *of a debt and an asset is a debt, of two debts or of two assets is an asset. The product of zero with an asset, a debt or with zero is zero.*

Although he also gives the false rule *zero divided by zero is zero*, he otherwise notes for division by zero that one may write zero in the denominator of a fraction – but without explaining what this means.

Further verses deal with the above-mentioned solution formula for quadratic equations with one variable. Then BRAHMAGUPTA is concerned with equations of the type $N \cdot x^2 + 1 = y^2$ which are later (erroneously) called PELL's equations:

- *Choose any square number* a^2*, multiply it by N and add a suitable number k so that the number* $b^2 = N \cdot a^2 + k$ *is a square number.*

  A solution of the equation $N \cdot (2 \cdot a \cdot b)^2 + k^2 = (N \cdot a^2 + b^2)^2$ *is* $\left(\frac{2 \cdot a \cdot b}{k}; \frac{N \cdot a^2 + b^2}{k} \right)$

  and this also satisfies the initial equation.

Example: One of the solutions of the equation $8x^2 + 1 = y^2$ is (x = 6; y =17).

In DIOPHANTUS's *Arithmetica* we find the theorem that the product of two sums of square numbers can be represented as a sum of square numbers in two ways:

$$(a^2 + b^2) \cdot (c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2 = (ac + bd)^2 + (ad - bc)^2 .$$

FIBONACCI published BRAHMAGUPTA's generalisation in 1225 in the *Liber quadratorum*:

$$(a^2 + n \cdot b^2) \cdot (c^2 + n \cdot d^2) = (ac - n \cdot bd)^2 + n \cdot (ad + bc)^2 = (ac + n \cdot bd)^2 + n \cdot (ad - bc)^2.$$

Finally, a method is given for finding numbers x, y so that $x + y$ as well as $x - y$ and $x - y + 1$ are square numbers:

To do this, choose any numbers a and b with $a > b$ and then calculate

$x = \left(a^2 + b^2\right) \cdot \frac{2a^2}{b^4}$ and $y = \left(a^2 - b^2\right) \cdot \frac{2a^2}{b^4}$.

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https://www.spektrum.de/wissen/brahmagupta-indischer-mathematiker-stellenwertsystem/1317834

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