DIOPHANTUS OF ALEXANDRIA (about 250 AD)

by HEINZ KLAUS STRICK, Germany

(drawings: © Andreas Strick)

It is doubtful whether DIOPHANTUS actually looked like the illustration on the right. We do not even know when this famous mathematician lived. DIOPHANTUS quoted the definition of a polygonal number from HYPSICLES, a mathematician and astronomer who lived around 175 BC. That would be the earliest point in time that could be used for our life dates. On the other hand, THEON OF ALEXANDRIA, father of the famous mathematician HYPATIA, wrote in 364 AD about *Arithmetica*, the main work of DIOPHANTUS. Since the same symbols are used in a 3rd century papyrus as in *Arithmetica*, it is now assumed that DIOPHANTUS lived in Alexandria around 250 AD.

Even if we have not found out when he was alive, we can still tell how long he lived from an epitaph created around 500 AD. The linear equation coming from this puzzle has the solution 84:

> *This tomb holds Diophantus. Ah, what a marvel! And the tomb tells scientifically the measure of his life. God vouch safed that he should be a boy for the sixth part of his life; when a twelfth was added, his cheeks acquired a beard; He kindled for him the light of marriage after a seventh, and in the fifth year after his marriage He granted him a son. Alas! late-begotten and miserable child, when he had reached the measure of half his father's life, the chill grave took him. After consoling his grief by this science of numbers for four years, he reached the end of his life*.

Alexandria still played a special role in the first centuries of our era as a science centre in the ancient world with a large number of scrolls (perhaps 10,000 or even 100,000). In 389, however, the Christian emperor THEODOSIUS ordered the destruction of all "pagan" writings; the remnants were burned in 642 by order of the Caliph UMAR because they were "superfluous".

It is therefore not surprising that only two of the writings of DIOPHANTUS have survived: the *Arithmetica* and a work about polygonal numbers.

In the 9th century, the surviving writings of the Greeks in Baghdad were collected in the *House of Wisdom* and translated into Arabic, including presumably the first seven of the 13 books (chapters) of DIOPHANTUS's *Arithmetica*. Unfortunately these translations are also lost.

It was not until 1971 that a scientist found a copy of Books IV to VII in Meshed (Iran), which were archived there under the name of the translator, since until then none of the librarians had been able to decipher the name of the real author, written in calligraphic script.

Regardless of the efforts in Islamic culture, Byzantine scholars in the 11th century began to study the writings of their Greek ancestors. The problems in DIOPHANTUS's *Arithmetica*, however, caused them considerable problems; one of them wrote a curse on DIOPHANTUS on the margin of the chapter because he could not understand the solution.

In 1463 JOHANNES MÜLLER from Königsberg, known as REGIMONTANUS (the Latin translation of Königsberg), discovered Byzantine manuscripts of six *Arithmetica* chapters in Venice – chapters I to III as well as three chapters, which are now denoted chapters VIII, IX, X (until 1971 they were erroneously numbered IV to VI). From this point on, numerous European mathematicians dealt with this collection of 189 problems.

In 1575, XYLANDER published the first printed translation in Latin. In 1621 a translation by BACHET DE MÉZIRIAC was published in Paris, on the margin of which PIERRE DE FERMAT noted the famous sentence:

Cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet. (I have found a truly wonderful proof, but this margin is too narrow to contain it.)

In the context of the problem of splitting a given square number into two square numbers, he claimed that the corresponding problem was not solvable for cubic numbers and also not for higher natural exponents:

- The equation $x^n + y^n = z^n$ with $x, y, z \in \mathbb{N}$ has no solution for $n > 2$ (*FERMAT's conjecture*).

FERMAT was fascinated by the abundance of original tasks and tricky solutions in DIOPHANTUS's work, and he dealt with the problems and the question of the generalisation of individual problems.

After his death, FERMAT's collected notes on BACHET's translation were published, which in turn prompted other mathematicians such as EULER and LAGRANGE to deal more intensively with questions arising from the problems of *Arithmetica*.

In the first volume of his *Arithmetica*, DIOPHANTUS defined basic terms and letter symbols that he uses further on: The character ς, the last letter of the word ἀριθμός (*arithmos* = number), is used as a placeholder for the unknown number. He also introduced special signs for powers up to the sixth, as well as for reciprocal values. Terms are noted as a sequence of number symbols (see figure below), between which there are symbols for addition or subtraction; the letters ἴσ (ἴσος; isos = equal) are added to form equations.

α	β	γ	δ	ε	ς	ζ	η	θ
1	2	3	4	5	6	7	8	9
ι	κ	λ	μ	ν	ξ	ο	π	ϟ
10	20	30	40	50	60	70	80	90
ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ
100	200	300	400	500	600	700	800	900

In contrast to EUCLID, DIOPHANTUS did not solve equations by geometrical considerations, but by reshaping and substituting them. Therefore, the rediscovery of *Arithmetica* has had a major impact on the development of algebra. While EUCLID used fractions only in the sense of rational numbers, DIOPHANTUS uses them as if they were natural numbers. Solutions to the equations he considered can be any positive rational numbers (in contrast to the term *DIOPHANTine equations* used today, whose solutions must be integers).

In Book I, DIOPHANTUS dealt with linear and quadratic equations as well as with linear systems of equations, which he solved by skilfully introducing new variables. The methods were developed in Babylonia 2000 years earlier and were obviously considered routine procedures at the time of DIOPHANTUS.

The examples selected below from various books (listed here in their usual spelling) are intended to give an insight into the virtuoso methods with which DIOPHANTUS solved the problems he had put together. In most cases, in addition to the example-based solution, he also specified necessary conditions for the given numbers.

- **Problem 1 from Book I:** *Divide a given number into two numbers if their difference is given.*

DIOPHANTUS uses the example of the starting number 100 and difference 40 as follows:

If x is the smaller number, then $x + 40$ is the larger number and the sum of the two numbers is equal to $2x + 40$; therefore the smaller number x is equal to 30 and the larger number is equal to 70.

- **Problem 16 from Book I**: *Finding three numbers so that the sums of two of the numbers are equal to given numbers.*

For example, if the given sums are: $x + y = 20$, $y + z = 30$ and $z + x = 40$, then DIOPHANTUS sets

$s = x + y + z$, thus: $x = s - 30$, $y = s - 40$, $z = s - 20$. Then the sum $x + y + z$ becomes: $s = 3s - 90$.

Also $s = 45$, therefore $x = 15$, $y = 5$ and $z = 25$.

- **Problem 28 from Book I:** *Find two numbers, where the sum and sum of the squares of the numbers are given.*

Example: Sum of the numbers: 20, Sum of the square numbers: 208.

Approach: Call the difference of the numbers 2x; then the one number $10 + x$, is the other $10 - x$ and the sum of the squares is $(10 + x)^2 + (10 - x)^2 = 200 + 2x^2 = 208$, thus $x^2 = 4$ and $x = 2$. The numbers we are looking for are 8 and 12.

- **Problem 8 from Book II:** *Splitting a given square number into two squares.*

(FERMAT notes his famous margin note alongside the solution to this problem.)

Example: If 16 is the given square number and x^2 one of the two square numbers you are looking for, then $16 - x^2$ must also be a square number.

Approach: Set $(2x - 4)^2 = 16 - x^2$, then $5x^2 = 16x$, thus $x = \frac{16}{5}$. The number 16 can therefore be broken down into the square numbers $\frac{256}{25}$ and $\frac{144}{25}$. The method works for every approach with a natural number m, thus: $(mx - 4)^2 = 16 - x^2$, where the 4 in the parenthesis is the root of the given square number 16.

- **Problem 9 from Book II:** *Divide a given number, which is the sum of two squares, into two other squares.*

DIOPHANTUS explains the solution using the example of the number 13: $2^2 + 3^2 = 13$.

Approach: $x = 2 + t$; $y = 2t - 3$; then $x^2 + y^2 = 4 + 4t + t^2 + 4t^2 - 12t + 9 = 5t^2 - 8t + 13 = 13$, also $t \cdot (5t - 8) = 0$, that is $t = \frac{8}{5}$.

In fact, $x = \frac{18}{5}$, $y = \frac{1}{5}$ is a further decomposition of the number 13 into two squares: $(\frac{18}{5})^2 + (\frac{1}{5})^2 = \frac{324+1}{25} = \frac{325}{25} = 13$. The method works not only with the approach $x = 2 + t$, but for any $a, k \in \mathbb{N}$ with $x = 2 + at$; $y = kt - 3$.

- **Problem 11 from Book II:** *To two given numbers add one and the same number so that each becomes a square.*

Example: The numbers should be 2 and 3, i.e. $2 + t$ and $3 + t$ must then be square numbers.

To solve this problem, DIOPHANTUS uses the property that for any number x, y is:

$(x+y)\cdot(x-y)=x^2-y^2$ and $x=\frac{1}{2}\cdot[(x+y)+(x-y)]$, $y=\frac{1}{2}\cdot[(x+y)-(x-y)]$

The difference $x^2-y^2=(3+t)-(2+t)=1$ can be calculated, for example, as the product of $x+y=4$ and $x-y=\frac{1}{4}$, so $x=\frac{1}{2}\cdot[4+\frac{1}{4}]=\frac{17}{8}$ and $y=\frac{1}{2}\cdot[4-\frac{1}{4}]=\frac{15}{8}$, i.e.

$x^2=\frac{289}{64}$ and $y^2=\frac{225}{64}$; $2+t=\frac{225}{64}$ and $3+t=\frac{289}{64}$ are followed by $t=\frac{97}{64}$.

- **Problem 20 from Book II:** *To find two numbers in such a way that the square of each, increased by the other, results in a square.*

If x, y are the two numbers, then the following should apply: $x^2 + y$ and $y^2 + x$ are square numbers.

Approach: $x = t$ and $y = 2t + 1$; then $x^2 + y = t^2 + 2t + 1 = (t + 1)^2$ is obviously a square number.

So that $y^2 + x = 4t^2 + 5t + 1$ is the square of a number, DIOPHANTUS continues:

$4t^2 + 5t + 1 = (2t - 2)^2 = 4t^2 - 8t + 4$, also $13t = 3$ or $t=\frac{3}{13}$.

Two numbers that meet the above condition are $x=\frac{3}{13}$ and $y=\frac{19}{13}$:

$(\frac{3}{13})^2+\frac{19}{13}=\frac{9+247}{169}=\frac{256}{169}=(\frac{16}{13})^2$; $(\frac{19}{13})^2+\frac{3}{13}=\frac{361+39}{169}=\frac{400}{169}=(\frac{20}{13})^2$

The method is also successful here with other approaches, in which the squares of the variables are ultimately omitted.

- **Problem 10 from Book III:** *Finding three numbers so that the product of two of them, increased by* 12*, each time gives a square number.*

DIOPHANTUS drew the reader's attention to the fact that the number 12 should be used, which can be represented as a product in different ways:

$12=2\cdot 6=(4-2)\cdot(4+2)=4^2-2^2$, also $2^2+12=4^2$;
$12=3\cdot 4=(\frac{7}{2}-\frac{1}{2})\cdot(\frac{7}{2}+\frac{1}{2})=(\frac{7}{2})^2-(\frac{1}{2})^2$, also $(\frac{1}{2})^2+12=(\frac{7}{2})^2$.

If you start with the number $4t$, then the second number is t^{-1}, and its product, increased by 12, is equal to the square number 16. As third number you have to choose $\frac{1}{4}t$, then the product with $4t$ is equal to t^2 and the product with t^{-1} equals $\frac{1}{4}$. For which numbers $t^2 + 12$ is a square number, one finds out with the approach $t + 3$: The equation $t^2 + 12 = (t + 3)^2$ has the solution $t=\frac{1}{2}$. Accordingly, three suitable numbers are 2, 2, $\frac{1}{8}$.

- **Problem 1 from Book VIII:** *Divide a number into two cubic numbers so that the sum of the third roots from the cubic numbers is equal to a given number.*

Example: Given are 370 and 10; then 343 and 27 are the two cubic numbers.

- **Problem 18 from Book VIII:** *Two numbers can be found such that the third power of the first, increased by the second, results in a cubic number and the square of the second, increased by the first, a square number.*

Solution: The numbers are $\frac{1}{16}$ and $\frac{262143}{4096}$.

- **Problem 26 from Book VIII:** *Find two numbers so that their product, multiplied by the two numbers, gives a cubic number.*

Solution: The numbers are $\frac{112}{13}$ and $\frac{27}{169}$.

- **Problem 38 from Book VIII:** *Find three numbers so that their sum multiplied by the first number gives a triangular number, multiplied by the second number a square number and by the third number a cubic number.*

Solution: The numbers are $\frac{153}{81}$, $\frac{6400}{81}$ and $\frac{8}{81}$.

- **Problem 15 from Book IX:** *Find three numbers so that the third power of their sum, increased by each of the numbers, gives a cubic number.*

Solution: The numbers are $\frac{1538}{3375}$, $\frac{18577}{3375}$ and 7.

- **Problem 23 from Book IX:** *Finding three square numbers such that each minus the product of the three numbers gives a square number.*

Solution: The numbers are $\frac{144}{169}$, $\frac{100}{169}$ and $\frac{676}{625}$.

- **Problem 2 from Book X:** *Finding a right-angled triangle so that the hypotenuse, added to the other two sides, gives a cubic number.*

Solution: The side length are $\frac{135}{352}$ and 377 (or multiples).

- **Problem 18 from Book X:** *Finding a right-angled triangle such that the hypotenuse, added to the area, gives a cubic number and the circumference is a square number.*

Solution: The side lengths are 1 257 728, 24 121 185 and 24 153 953 (or multiples).

Whoever works with DIOPHANTUS's *Arithmetica* feels the fascination that arises from the problems and can understand why so many famous mathematicians dealt with them.

The German mathematician CARL GUSTAV JACOB JACOBI (1804 – 1851) summed up the significance of DIOPHANTUS in the words:

> *DIOPHANTUS will always be remembered because he started the investigation of the deep-rooted properties and relations between numbers which have finally been understood by the beautiful research of modern mathematics.*

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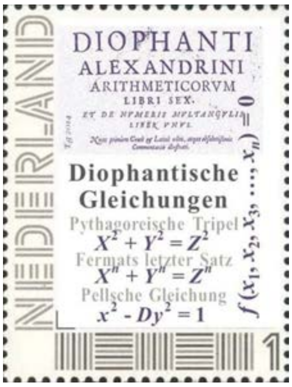
NEDERLAND
DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
Diophantische
Gleichungen
Pythagoreische Tripel
$X^2 + Y^2 = Z^2$
Fermats letzter Satz
$X^n + Y^n = Z^n$
Pellsche Gleichung
$x^2 - Dy^2 = 1$
$f(x_1, x_2, x_3, \dots, x_n) = 0$
1


12. Januar 1665
Zum Gedenken an
den Mathematiker
Pierre de
Fermat
1607-1665
0.55 LIECHTENSTEIN
ISCHE POST AG


Der große fermatsche Satz
Pierre de Fermat
1637
$x^n + y^n \neq z^n$
für alle n>2
Andrew Wiles
1993/95
Leonhard Euler
Sophie Germain
Peter G. Lejeune-Dirichlet
Adrien-Marie Legendre
Gabriel Lamé
Eduard Kummer
Harry Vandiver
Gerd Faltings
Gerhard Frey
Jean-Pierre Serre
Ken Ribet
Richard Taylor
MaPhyPhil
Deutsche Post
45
BRIEFMARKE INDIVIDUELL


Leonhard
Euler
1707
1783
INFINITORUM.
LEONHARDO EULERO.
0.55 LIECHTENSTEIN
ISCHE POST AG


Joseph-Louis
Lagrange
1736-1813
Vier-
Quadrate-
Satz
$n = a^2 + b^2 + c^2 + d^2$
1770
MaPhyPhil
Deutsche Post
62
BRIEFMARKE INDIVIDUELL


CARL GUSTAV JACOB
JACOBI
1804-1851
Thetafunktion
$\vartheta(z,\tau) := \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z}$
$\vartheta_3(\tau)^4 = \vartheta_0(\tau)^4 + \vartheta_2(\tau)^4$
Deutsche Post
62
BRIEFMARKE INDIVIDUELL


Carl Gustav Jacob Jacobi
(1804 – 1851)
dn (z;k)
cn (z;k)
sn (z;k)
Jacobische
elliptische Funktionen
2012
MaPhyPhil
Deutsche Post
45
BRIEFMARKE INDIVIDUELL