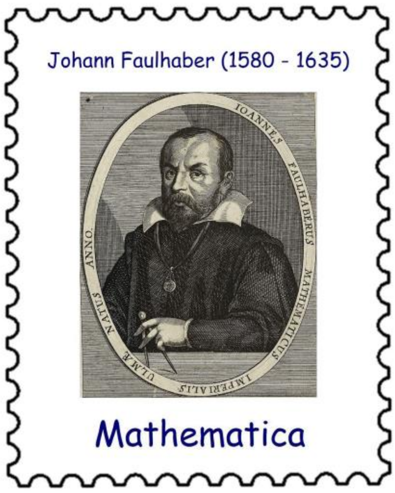
JOHANN FAULHABER (May 5 , 1580 – September 10, 1635)

by HEINZ KLAUS STRICK, Germany



In 1992, the well-known American computer scientist DONALD E. KNUTH succeeded in solving a 360-year-old puzzle: In his paper *JOHANN FAULHABER and sums of powers*, he presented a plausible explanation of how this German arithmetic master found a way to develop formulae for the sums of powers of natural numbers up to the exponent 17 at the beginning of the 17th century.

Since then, the terms *FAULHABER's Formulae* or *FAULHABER's Polynomials*, introduced by KNUTH, have become common in the technical literature.

JOHANN FAULHABER was born in Ulm as the seventh child of a weaver. The father died when the boy was three years old. The mother continued to run the workshop with the support of the journeyman and the boy also learnt his father's trade at an early age.

At the age of 15 he entered the apprenticeship of the master arithmetician DAVID SELZLIN, who is known today mainly as a cartographer, because of his extraordinarily precise map of the Swabian region of South-West Germany.

Following his apprenticeship, FAULHABER worked for one and a half years for the master of arithmetic and calibration JOHANN KRAFFT, who later "wished the plague on him" when his pupil turned out to be more successful than he was.

FAULHABER was extremely inquisitive and obviously gifted in acquiring the necessary technical knowledge. After an unsuccessful application for a post in Biberach, FAULHABER then took up a position as a *German schoolmaster* in Ulm in 1600, married and started a family. Only three of his nine children survived the first years of their lives.

In principle, FAULHABER could live well on the salary paid to him by the Ulm magistrate, and at times he could even hire helpers. Due to the many deaths, however, the number of pupils fluctuated greatly. One of his duties as schoolmaster was to provide board and lodging for the registered children. In order to secure himself financially, FAULHABER also offered private lessons for interested adults.

His first work, *Arithmetischer Cubicossischer Lustgarten* (Arithmetic Cossian pleasure garden) appeared in 1604 as a promotional measure.

On the one hand, the word *Cubicossian* referred to the tradition of the German Cossists (among these first algebraists were ADAM RIES, CHRISTOFF RUDOLFF and MICHAEL STIFEL), and on the other hand to many of the tasks contained in the book, which required methods for solving cubic equations.

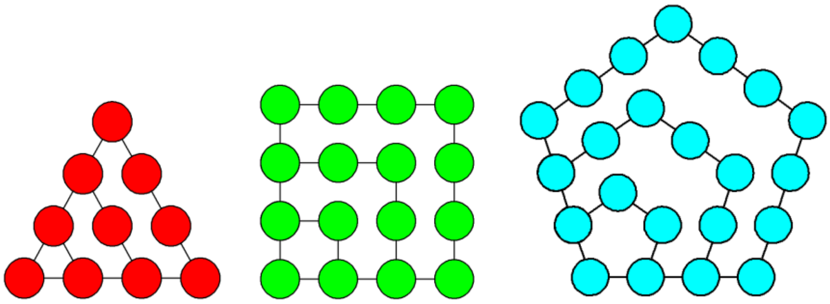
(Drawings © Andreas Strick)

FAULHABER had no problem with this, indicating the sources from which he received suggestions for his exercises. On the contrary, he emphasised his competence when he was able to refer to the fact that he had been able to solve problems from other mathematical masters.

In particular, he referred to the *Ars magna* by GIROLAMO CARDANO, which was published in Nuremberg in 1545. Although he had not had the opportunity to learn Latin, the language of science, and was therefore unable to read CARDANO's work in the original, he found helpers who translated the text for him. Over the years, he acquired a basic knowledge of Latin, French and Italian.

In those days, Nuremberg was a place where numerous mathematical works were printed. FAULHABER endeavoured to acquire as many new German-language publications as possible, and he tapped another source for himself: when he heard of the death of a master of arithmetic, he offered to buy up the notes left behind to the widows.

A special role in FAULHABER's book is also played by the so-called polygonal numbers, i.e. numbers that can also be represented as point patterns in the form of regular polygons, e.g. triangular, square and pentagonal numbers, see the following figures.



Since FAULHABER published his book mainly for advertising purposes, it contained the solutions of all 160 problems, but not the solution methods. This was quite common among the arithmetic masters of the time: anyone who also wanted to be able to solve such problems should take private lessons from the author.

As a member of the weavers' guild, FAULHABER also became a member of the "honourable society" of the *Ulm Meistersinger*s, and wrote songs with religious content. He was so impressed by his own compositional and mathematical abilities that he believed in an inspiration coming directly from God, which enabled him to interpret certain numbers from the Bible (which STIFEL had already dealt with) to mean that the Last Judgement was imminent at the end of 1606. He was temporarily imprisoned for these fantasies and excluded from taking communion.

In 1608 FAULHABER noticed that the competitors were not sleeping, when the arithmetician PETER ROTH from Nuremberg published the work *Arithmetica Philosophica*, which contained, among other things, the solutions to all 160 problems of FAULHABER's *Lustgarten*.

He had various thoughts as to how he should react to this: one idea was to publish the solutions himself as a cheap edition in order to make the competitor's book appear less attractive, or to damage ROTH by publishing solutions to the latter's book, or to come to an agreement with him about the sales territories for the two books.

In the end, however, FAULHABER refrained from taking concrete measures, because in the meantime he was more interested in other topics: perspective drawing, surveying and fortress construction.

In 1610, the book *Newe geometrische und perspektivische Inventiones etlicher sonderbahrer Instrument* (New geometric and perspective inventions of several special instruments) was published. Here, too, he only hinted at many things (e.g. how to use the proportional compass he had developed) in order to attract people interested in private lessons.

Since he had not obtained permission from the Ulm magistrates for the publication, the payment of his salary was temporarily cancelled, but then re-approved and even an additional payment was granted for the new book, since Faulhaber had become known far beyond the city limits through the new book, and this enhanced the importance of the city.

His book *Newer Mathematischer Kunstspiegel* (New mathematical mirror of art) published in 1612, was even translated into Latin.

Beyond the technical content, the new writings also contained adventurous number speculations, which led to disputes with the censorship authorities. Faulhaber was not at all insightful here. He was convinced that God used special, chosen people to interpret the apocalyptic numbers, and only these were granted the necessary enlightenment, and he, Faulhaber, was just such a chosen one. The dispute with the magistrate and with orthodox Protestant theologians at the University of Tübingen intensified in 1613 after the publication of further books: *Himlische gehaime Magia* (Hidden heavenly magic) and *Andeutung einer vnerhorten Wunderkunst* (Unprecedented miracle art).

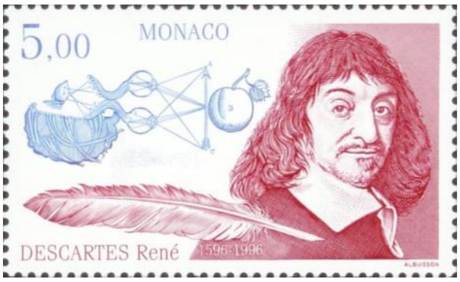
Nevertheless, in 1617 he was commissioned by the magistrate of the city of Ulm to prepare a calendar for the following year.

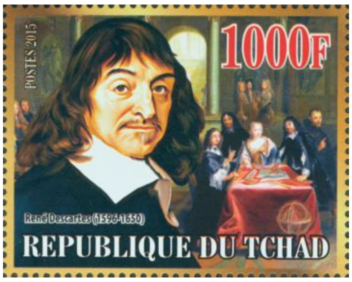
A special constellation of Mars and the Moon calculated by Johannes Kepler prompted him to predict the appearance of a "cruel" comet on 1 September. When two comets were actually seen at the end of 1618 (which Faulhaber mistakenly believed to be identical), he saw himself confirmed and announced the imminent end of the world. His writing *Fama siderea nova* (News of the stars) led to the so-called *Ulm Comet Controversy*, among others with Johann Baptist Hebenstreit, the rector of the Ulm grammar school.

A debate, conducted in German in view of Faulhaber's insufficient knowledge of Latin, on whether comets are natural phenomena or warning signs from God, was judged successful by the supporters of both parties.

Whether a meeting between Faulhaber and René Descartes took place in 1620 and whether the young Frenchman, who was an officer in Bavarian military service, even took lessons from the master arithmetician, can no longer be clarified from the available sources.

It can be assumed that Descartes was familiar with Faulhaber's mathematical writings, but was hardly impressed by his cabbalistic number speculations.



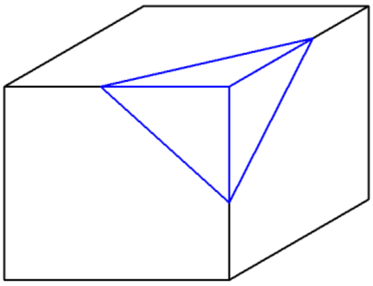


In the beginning of the Thirty Years' War, cities tried to protect themselves from the armies invading Germany by building stronger fortifications. Due to his successful work in Ulm, FAULHABER was called in by several cities as an advisor (Basel, Frankfurt, Meiningen, etc.) and his reputation grew far beyond the region. He also hoped for "privileges" from the respective sovereigns to protect his works from plagiarism.

After renewed disputes with the municipal institutions, FAULHABER fled to Augsburg in 1621, where he published *Miracula Arithmetica* (Miracles of arithmetic) the following year. He explained *in words* how one could generally eliminate the second highest member of a polynomial of the *n*th degree: $x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$ becomes the polynomial $y^n + b_2 y^{n-2} + \ldots + b_{n-1} y + b_n$ by the substitution $x = y - \frac{1}{n} \cdot a_1$ (expressed here in the notation used today).

The fact that 4th degree terms can be represented as the product of two quadratic terms, whose coefficients can be determined by comparison, he highlights as his special "invention". DESCARTES uses a similar approach in the appendix to *La Géométrie of Discours de la méthode* (1637).

In the *Miracula Arithmetica* one also finds a theorem discovered by FAULHABER, the "volume version of the theorem of PYTHAGORAS":



If one cuts off a corner of a cuboid, this corner is a tetrahedron with three right-angled triangles as side faces. There is then a connection between the area *A* of the base of the tetrahedron and the areas A_1, A_2, A_3 of the side faces given by $A^2 = A_1^2 + A_2^2 + A_3^2$.

FAULHABER's work begins with a table that gives an idea of the ideas he used to find formulae for the sums of powers of natural numbers. (FAULHABER's biographer IVO SCHNEIDER calls this the weaver's shuttle technique and he was convinced that FAULHABER's training as a weaver was decisive for his ability to recognise connections with a quick glance at the column contents.)

n	n^2	$s(n^2)$	$s(s(n^2))$	$n \cdot s(n^2)$	n^3	$s(n^3)$	$n \cdot s(n^2) - s(n^3)$	$s(s(n^2) - s(n^2)$
1	1	1	1	1	1	1	0	0
2	4	5	6	10	8	9	1	1
3	9	14	20	42	27	36	6	6
4	16	30	50	120	64	100	20	20
5	25	55	105	275	125	225	50	50
6	36	91	196	546	216	441	105	105
7	49	140	334	980	343	784	196	196
8	64	204	540	1632	512	1296	336	336

Here FAULHABER compared the sequence of the natural numbers *n* and the sequence of the powers n^2 and n^3 with the *n*-fold of the associated sum sequences *s* and again their sum sequences, and thus discovers the connection $n \cdot s(n^2) - s(n^3) = s(s(n^2)) - s(n^2)$, from which a formula for the sum of the first *n* cubic numbers results: $s(n^3) = (n+1) \cdot s(n^2) - s(s(n^2))$.

For higher powers he makes a decomposition approach such as $s(n^4) = s(n^2) \cdot \left(a \cdot s(n) + b\right)$ and he obtains the values of the coefficients by inserting: $a = \frac{6}{5}$ and $b = -\frac{1}{5}$.

Further sum formulae can also be found in the *Academia Algebrae* (1631). In his investigations on power sums, JACOB BERNOULLI referred to FAULHABER's approach and developed representations with the help of the BERNOULLI numbers (later named after him).

The culmination of FAULHABER's mathematical writings was the four-volume work *Ingenieurs-Schul* (Engineering School, 1630 and 1633), in which he dealt in detail with calculating with BRIGGS's logarithms. The work also contained the task of determining the radius of the circumcircle of a heptagon with the side lengths 2300, 1600, 1290, 1000, 666, 1260 and 1335. These were the *apocalyptic numbers* of MICHAEL STIFEL.

The ingenious mathematician JOHANN FAULHABER, often called "the German ARCHIMEDES", died of the plague in Ulm in 1635, together with his wife.

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