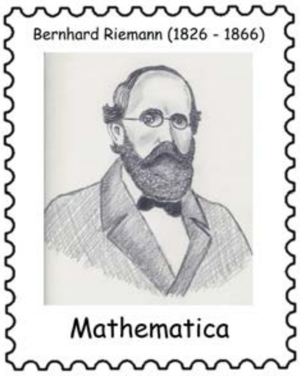
BERNHARD RIEMANN (September 17, 1826 – July 20, 1866)

by HEINZ KLAUS STRICK, Germany



Although GEORG FRIEDRICH BERNHARD RIEMANN died at the young age of 39, his influence on the development of modern mathematics was considerable. There are more than a few historians of science who consider him among the most important mathematicians of all time.

Nevertheless, the German postal authorities have not yet memorialized him on a postage stamp.

BERNHARD RIEMANN was born in the village of Breselenz in the district of Lüchow-Dannenberg, in Lower Saxony, Germany, the second of six children. Later, his father took a post as a Lutheran pastor in Quickborn. Till the age of 13, BERNHARD was educated primarily by his father. Then he moved to Hanover, where he lived with his grandmother and attended a gymnasium. A shy young man, he suffered acute homesickness for his parents and siblings, whom he now saw only during the school holidays.

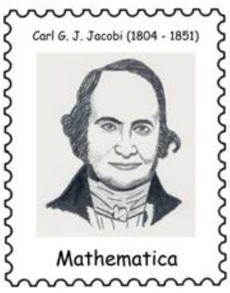
After the death of his grandmother, the by now fifteen-year-old transferred to the *Johanneum*, a gymnasium in Lüneburg, where he completed the last four years of his secondary education. He visited his family as often as possible, but it became increasingly difficult for the sickly youth to make the journey of 90 kilometres (on foot).

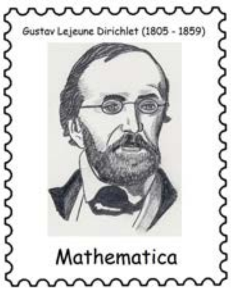
His extraordinary mathematical talent soon became apparent, and it motivated the school's headmaster to introduce him to the works of LEONHARD EULER as well as ADRIEN-MARIE LEGENDRE's *Théorie des Nombres*. When BERNHARD returned the books, the headmaster was astonished that not only had RIEMANN clearly understood the material, he could also recapitulate it.

In the spring of 1846, RIEMANN began, at his father's wish, to study philosophy and theology at the University of Göttingen; but in addition to the lectures in those subjects, he also attended lectures on mathematics and physics, and in the following semester, he attended a course of lectures given by GAUSS on the method of least squares.

Following BERNHARD's insistent pleading, his father finally allowed him to devote himself solely to mathematics and physics.

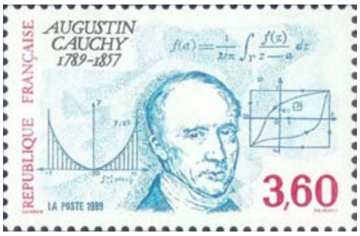
BERNHARD RIEMANN soon observed that the lecture courses at Göttingen offered him nothing new. Despite the family's straitened financial circumstances, his parents allowed him to transfer to Berlin. There, the lectures on mathematics and physics of CARL GUSTAV JACOBI, PETER GUSTAV LEJEUNE DIRICHLET, JAKOB STEINER, and GOTTHOLD EISENSTEIN provided him with the resources to begin his own research.





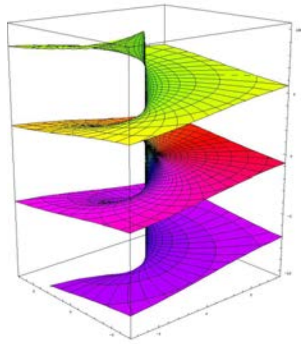
(drawings © Andreas Strick)

He returned to Göttingen to take up the position of assistant to the experimental physicist WILHELM WEBER. At the same time, he worked on his dissertation, *Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen komplexen Größe* (foundations of a general theory of functions of a complex variable), under the supervision of GAUSS.



Building on the work of CAUCHY, RIEMANN worked on the theory of complex variables. He introduced the definition that a function w with $w(z) = u(z) + i \cdot v(z)$ is *differentiable* at the point $z = x + i \cdot y$ if it satisfies what are today known as the CAUCHY-RIEMANN differential equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

He also solved a problem that can arise in the case of functions of a complex variable: if, for example with the logarithm function, one moves from a given point of the complex plane by various paths back to that same point, it can happen that one will end up with values of the function on different “branches.” RIEMANN solved this problem of multivaluedness by stacking as many complex planes as necessary one on top of the other and then “gluing” them together along suitable cuts. The figure to the right shows what is today known as a RIEMANN surface of the complex logarithm function. (source: Wikipedia CC BY-SA 3.0 Leonid2)



RIEMANN needed more than two years to finish his habilitation thesis *Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe* (on the representation of a function by a trigonometric series). The reasons for this were his time-consuming participation in the newly established mathematics-physics seminar and his poor health, which frequently left him bedridden. In addition, he was hindered by an almost pathological diligence and thoroughness in completing the work.

While earlier researchers had dealt with the problem of necessary conditions for a function to be representable by a FOURIER series, RIEMANN considered the question from the opposite point of view, namely, the properties exhibited by a function if it can be represented as a trigonometric series.

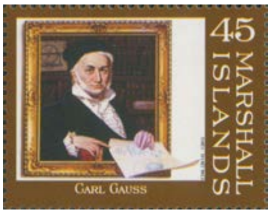
In this work, he considered what it meant for a function to be integrable. Beginning with a partition of the interval $[\, a ; b \,]$ into (not necessarily equal) subintervals, and points t_i in the subintervals $[\, x_{i-1} ; x_i \,]$, he considered sums of the form $\sum_{i=1}^{n} f(t_i) \cdot (x_i - x_{i-1})$ (today called RIEMANN sums), which give the area of the staircase figure associated with the decomposition and choice of intermediate points.

He declared a function f to be *integrable* on the interval $[\, a ; b \,]$ if the sums for arbitrary choices of decomposition and intermediate points all converge to the same fixed number, which is called the integral (today RIEMANN integral) of f over $[\, a ; b \,]$.

For his public habilitation lecture, RIEMANN was required to offer the faculty three topics from which they could choose. In parallel to his habilitation thesis, he had worked on an article on the subject of electricity, galvanism, light, and gravitation, and he offered that as one of the topics. The second topic was also related to questions in physics.

GAUSS, however, wanted to hear about the third topic, *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (on the hypotheses underlying geometry), because he was curious to learn what sorts of insights into the subject so young a man as RIEMANN had come up with.

RIEMANN had the ambition of presenting a talk that would be accessible to a general audience, even nonmathematicians. However, it is likely that GAUSS was the only one present who could truly follow the exposition. He was deeply impressed, and he praised – something he did not often do – the intellectual depth of RIEMANN's ideas. It was only sixty years later that the importance of RIEMANN's approach was recognized, when EINSTEIN relied on RIEMANN's work in the development of his general theory of relativity.



RIEMANN's lecture dealt with the *mathematical possibility* of the existence of non-Euclidean geometries; whether they actually *exist* is something for physicists to worry about. RIEMANN defined space as an *n*-dimensional manifold with a metric that allows the determination of the distance between points. In this case, the "shortest" points joining pairs of lines will not necessarily be lines, but curves (geodesics), whose curvature can depend on the location of the points in space. The angle sums of triangles can equal 180°, but they can also be less than or greater than 180°.

After being granted the *venia legendi*, or right to give lectures, he was able to teach as a privatdozent beginning in the winter semester of 1854–1855. At first, he had great difficulty in adjusting to the pace at which his students could master new material, but gradually he succeeded in reading from their reactions whether an explanation needed to be repeated in different words and whether a proof had to be explained in greater detail.

Following the death of GAUSS, DIRICHLET was called to Göttingen as his successor. He tried in vain to have RIEMANN, whom he valued highly, to be named an associate professor. Nevertheless, DIRICHLET did manage to obtain for him an annual stipend of two hundred thalers as a lecturer. This ensured for RIEMANN a modest living, for otherwise, the income of dozents depended on the number of students who attended their lectures, and that number in RIEMANN's case was small: the number of students who registered for RIEMANN's first lecture was only eight, and for RIEMANN, that was a large number.

RIEMANN worked tirelessly on his publications, which appeared in CRELLE's journal as well as in the publications of the *Göttingen Academy of Sciences*. (The latter had published only the works of GAUSS in the previous fifty years.) RIEMANN developed his ideas further by leaps and bounds. In one case, it was an article on elliptic integrals, WEIERSTRASS withdraw his own article on the subject, having recognized that RIEMANN was much further advanced in the topic.

Finally, at the end of 1857, RIEMANN was appointed associate professor, which included an increase in annual salary to three hundred thalers. However, at this time, he had to take into his household his three unmarried sisters, who had previously lived with a brother, who recently had died.

When DIRICHLET died in May 1859, RIEMANN was called as his successor and elected a member of the *Göttingen Academy of Sciences*. The *Berlin Academy of Sciences* named him a corresponding member. On the occasion of his being so named, he wrote his famous article *Über die Anzahl der Primzahlen unter einer bestimmten Größe* (on the number of primes less than a given number).

In it, he improved on the estimates of GAUSS and LEGENDRE based on the fact that the prime number function $\pi(x)$, which gives the number of primes less than x, is asymptotic to $\frac{x}{\ln(x)}$ and to $Li(x) = \int_2^x \frac{1}{\ln(t)}\,dt$. Beginning with the zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-\frac{1}{p^s}}$ introduced by EULER, he introduced a function that is defined for all complex numbers s and agrees for real numbers $s > 1$ with EULER's zeta function. (Thus RIEMANN's zeta function is an *analytic continuation* of EULER's.)

RIEMANN mentioned, rather in passing, that it seemed a reasonable conjecture that all the zeros of the function in what is called the *critical strip* $0 \le \text{Re}(s) \le 1$ lie on the line with $\text{Re}(s) = ½$.

It is quite likely, he wrote, *that all roots* [of ζ] *are real. However, one would like a rigorous proof of this; I have, however, in searching for such a proof, put it aside for now after a few brief, vain attempts*.

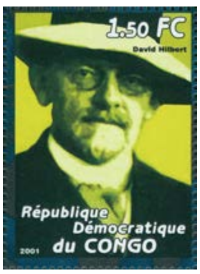
RIEMANN's conjecture, known as the RIEMANN hypothesis, was named a *millennium problem* in the year 2000, with a prize of one million dollars for its solution. So far, no one has been able to prove or disprove the conjecture. When DAVID HILBERT was asked what his first question would be if he could meet again with mathematicians one hundred years after his death, he is said to have answered, "I would ask whether the RIEMANN hypothesis has been proved."

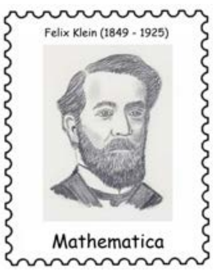
In June 1862, RIEMANN married ELISE KOCH, a friend of his sister; however, any hope of personal happiness was marred by his being ill with tuberculosis. He hoped for a cure through a journey to a warmer climate. A stay of several months in Italy improved his shattered health. But on the return trip, he became so ill that a second, even longer, stay in the south was required.

In Pisa, where a daughter was born to the couple, he declined the offer of a professorship of mathematics, since he felt that he had an obligation to the University of Göttingen. At the end of 1865, he returned temporarily to Göttingen. Yet he was so weakened that he had to postpone several appointments with DEDEKIND, with whom he wished to discuss how to proceed with his unfinished articles.

On June 15, 1866, he left one last time for Italy. On June 28, he reached his destination, Selasca (near Varbania), on Lago Maggiore, where he was able for a few days to work on his articles.
On July 20, his brief life came to a close.

A few days before his death, he was elected a corresponding member of the French *Académie des Sciences* and of the British *Royal Society*. RICHARD DEDEKIND and HEINRICH WEBER assumed responsibility for the publication of his unpublished work. FELIX KLEIN promulgated RIEMANN's approach to complex analysis through his lectures in Leipzig and Göttingen.





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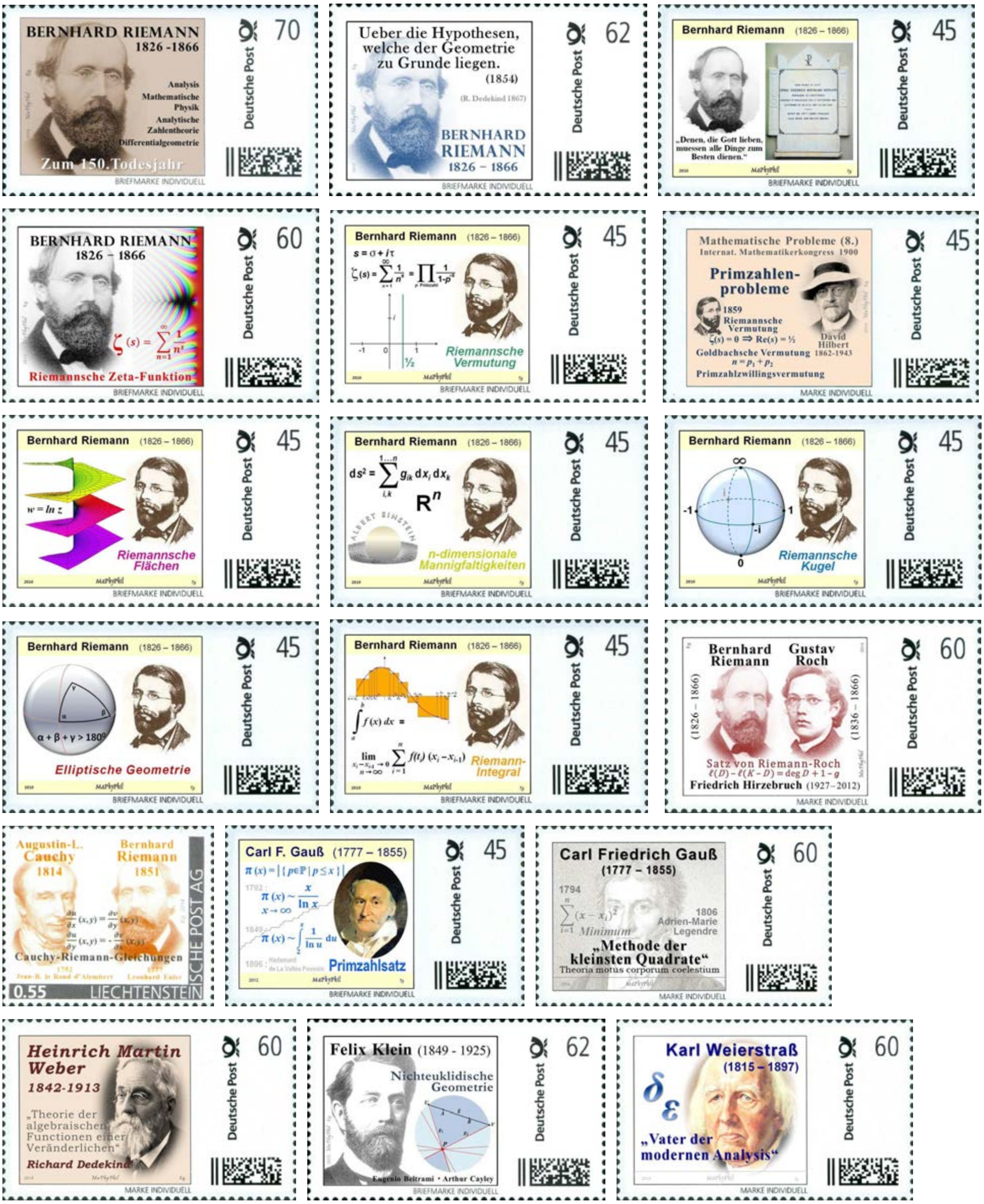

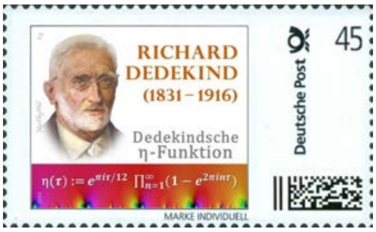

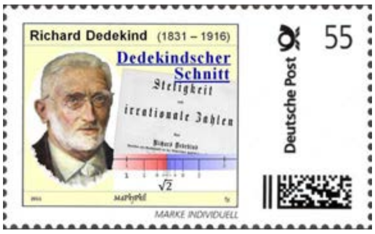

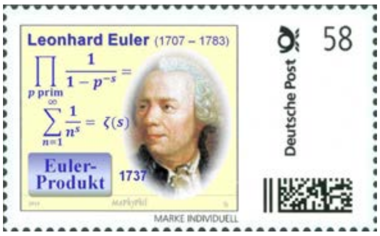

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