Lattice Polytopes and Orbifolds in Quiver Gauge Theories

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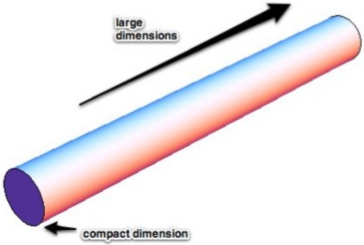
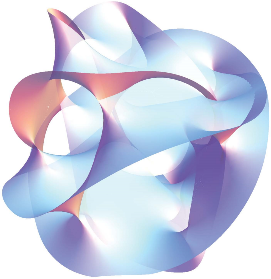
Compactification of Extra Dimensions

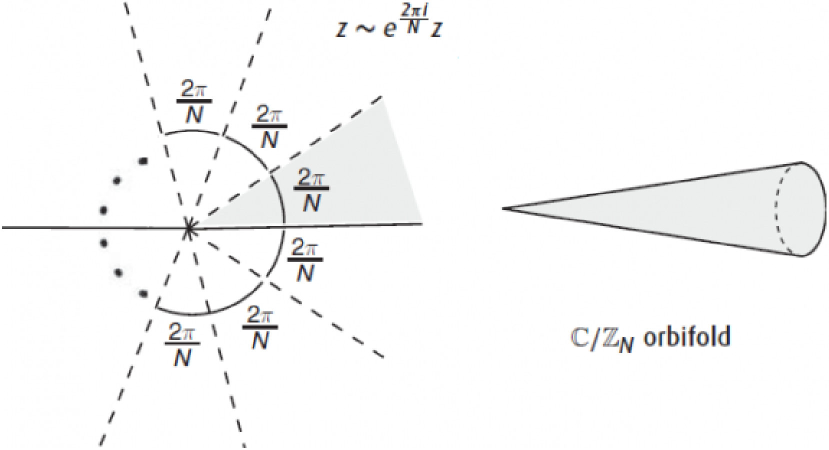
- Superstring theory predicts (9+1) spacetime dimensions $\Longrightarrow$ 6 must be undetectable!
- Calabi-Yau compactifications
- D-branes and gauge theories





Orbifold Compactifications

- Mani*fold* of equivalence classes of *orbi*ts of a finite group (quotient group)



Describing Orbifold Actions

- Most general Abelian action constructible from $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$
- Let g generate $\mathbb{Z}_{n_i}$; action on $\mathbb{C}^3$ given by the following representation:

$$\operatorname{diag}\left(e^{\frac{i2\pi a_1}{n_i}}, e^{\frac{i2\pi a_2}{n_i}}, e^{\frac{i2\pi a_3}{n_i}}\right)$$

- Orbifold action encoded by (a_1, a_2, a_3), with $a_1 + a_2 + a_3 \equiv 0 \pmod{n_i}$.

Describing Orbifold Actions

- However, orbifold actions are not uniquely identified by a single 3-tuple (scaling, permutation)
- Example: $\mathbb{C}^3/\mathbb{Z}_3$ has two unique actions given by: $(0, 1, 2)$ and $(1, 1, 1)$. The former has action $\mathrm{diag}(1, \zeta, \zeta^2)$ for $\zeta^3 = 1$, whereas the second has action $\mathrm{diag}(\zeta, \zeta, \zeta)$.

Toric Diagrams and Orbifolds

- A toric Calabi-Yau orbifold can be represented by a lattice polytope
- Lattice polytopes correspond to the same physical orbifold if and only if they are related to each other by a $\mathrm{GL}(n, \mathbb{Z})$ transformation
  - Diffeomorphism invariance of Polyakov string action
- The area, volume, etc. of the lattice polytope equals the order of the orbifold group.

Hermite Normal Forms and Sublattices

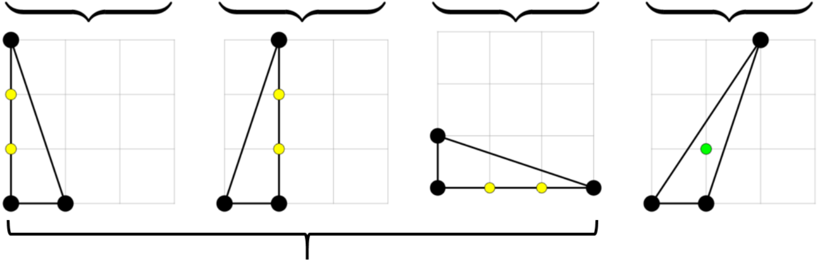
Theorem: every integer-valued 2×2 matrix is the product of a matrix in $GL(2, \mathbb{Z})$ and a *Hermite normal form*:

$$H = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}, \qquad a_{12} < a_{22}$$

- Physically unique toric diagrams can thus be identified with Hermite normal forms
- Hermite normal forms $\iff$ sublattices
  - Given a lattice basis $\Lambda = \{v_1, v_2, \ldots\}$, $H\Lambda$ yields sublattice
  - Can also be obtained from counting lattice points contained in scaled toric diagrams

Example: $\mathbb{C}^3/\mathbb{Z}_3$ Orbifolds

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$



(0,1,2) (1,1,1)

Sublattice Enumeration with Burnside's Lemma

Let g in the permutation representation of G be composed of α_1 1-cycles, α_2 2-cycles, up to α_k k-cycles. We can write:

$$\zeta_g = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k}$$

The *cycle index* of a group G is obtained by averaging the ζ_g:

$$Z_G = \frac{1}{|G|} \sum_g \zeta_g = \frac{1}{|G|} \sum_\alpha c(\alpha_1, \dots, \alpha_k) x_1^{\alpha_1} \dots x_k^{\alpha_k}$$

where $c(\alpha_1, \dots, \alpha_k)$ denotes the degeneracy of each cycle structure and the sum is taken over cycle structures.

Sublattice Enumeration with Burnside's Lemma

Burnside's Lemma: The number $N(G)$ of orbits of G under the group action on X is given by the size of the fixed sets F_g under each element $g \in G$:

$$N(G) = \frac{1}{|G|} \sum_{g \in G} |F_g|$$

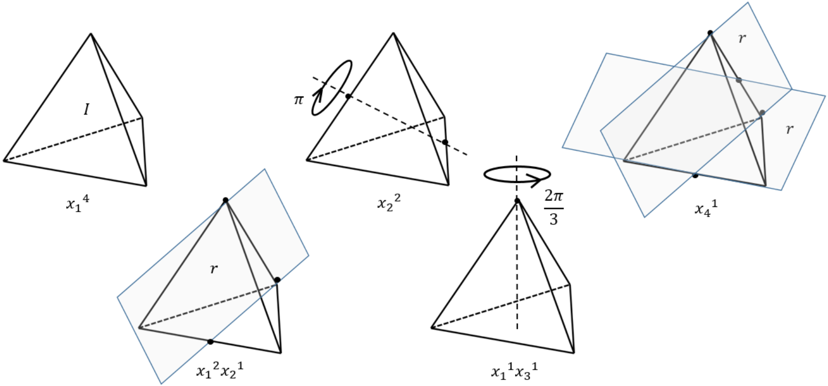
- G symmetry group of toric diagram, X set of sublattices
- Number of inequivalent sublattices of index $n \iff$ number of orbits of G on X
- Only need one $g \in G$ for each cycle structure (conjugacy class)

Motivation for Platonic Solids

- ADE classification of discrete subgroups of $SU(2)$:
  - A_n: binary cyclic group of order $2n$
  - D_n: binary dihedral group of order $4n$
  - E_6: binary tetrahedral group of order 24
  - E_7: binary octahedral group of order 48
  - E_8: binary icosahedral group of order 120
- Unique set of quivers with only finitely many isomorphism classes of indecomposable representations
- Connection to quiver gauge theories: quivers represent matter content of gauge theory for D-branes on orbifolds, where nodes $\iff$ factors of the gauge group and links $\iff$ fields in particular representations

Tetrahedral Symmetries

- Symmetry group given by S_4
  - Permutation action on vertices



Tetrahedron Data

$$Z_{S_4} = \frac{1}{24}\left(x_1^4 + 6x_1^2x_2 + 3x_2^2 + 8x_1x_3 + 6x_4\right)$$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f^{\triangle}_{x_1^4}$	1	7	13	35	31	91	57	155	130	217	133	455	183	399	403	651
$f^{\triangle}_{x_1^2x_2}$	1	3	5	11	7	15	9	31	18	21	13	55	15	27	35	75
$f^{\triangle}_{x_2^2}$	1	3	5	11	7	15	9	31	18	21	13	55	15	27	35	75
$f^{\triangle}_{x_1x_3}$	1	1	1	2	1	1	3	2	4	1	1	2	3	3	1	3
$f^{\triangle}_{x_4^1}$	1	1	1	3	3	1	1	5	2	3	1	3	3	1	3	7
$f^{\triangle}$	1	2	3	7	5	10	7	20	14	18	11	41	15	28	31	58

Table 1: Number of sublattices of index n invariant under representative symmetries from each conjugacy class of S_4.

Analytic Sublattice Enumeration

- *Multiplicative* sequences: $f(nm) = f(n)f(m)$ for n, m coprime.
- Multiplicative sequences form a group under *Dirichlet convolution*:

$$f(n) = (g * h)(n) = \sum_{m|n} g(m)h\left(\frac{n}{m}\right)$$

where the notation $m|n$ means that the sum runs over all the divisors m of n.

Analytic Sublattice Enumeration

Define the unit, number, and square sequences:

$$
\begin{aligned}
u &= \{1, 1, 1, \ldots\} \\
N &= \{1, 2, 3, \ldots\} \\
N^2 &= \{1, 4, 9, \ldots\}
\end{aligned}
$$

The Dirichlet characters $\chi_{k,n}$ of modulus k and index n form an Abelian group of order $\varphi(k)$:

$$
\begin{aligned}
\chi_{1,1} &= u \\
\chi_{2,1} &= \{1, 0, \ldots\} \\
\chi_{3,1} &= \{1, 1, 0, \ldots\} \\
\chi_{3,2} &= \{1, -1, 0, \ldots\}
\end{aligned}
$$

Tetrahedron Analytics

$$f_{x_1^4} = u * N * N^2$$

$$f_{x_1^2 x_2^1} = \{1, -1, 0, 4\} * u * u * N$$

$$f_{x_2^2} = \{1, -1, 0, 4\} * u * u * N$$

$$f_{x_1^1 x_3^1} = \{1, 0, -1, 0, 0, 0, 0, 0, 3\} * u * u * \chi_{3,2}$$

$$f_{x_4^1} = \{1, -1, 0, 2\} * u * u * \chi_{4,2}$$

Tetrahedron Numbers

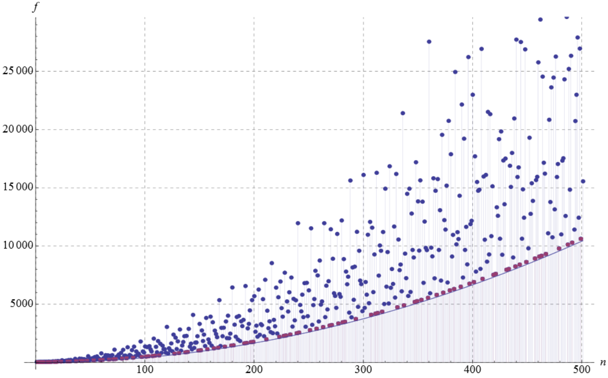
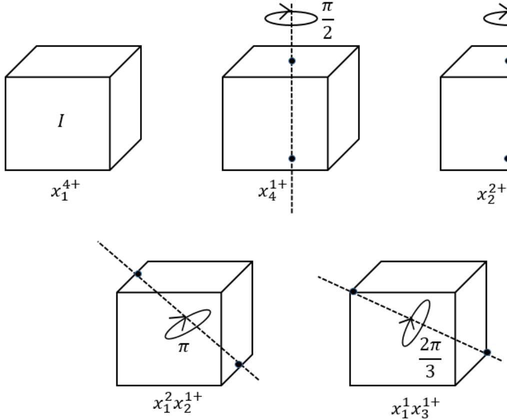


Figure 1: Scatter plot of the number of invariant sublattices of index n for the tetrahedral lattice. Primes are given in purple. The blue line corresponds to $n^2/24$.

Cubic Symmetries

- Symmetry group given by $S_4 \times \mathbb{Z}_2$
  - One copy for each embedded tetrahedron
  - Permutation representation describes action on four space diagonals



Cube Data

$$Z_{S_4\times\mathbb{Z}_2} = \frac{1}{24}\left(x_1^{4+} + 6x_1^2x_2^{1+} + 3x_2^{2+} + 8x_1^1x_3^{1+} + 6x_4^{1+}\right)$$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$f^{\square}_{x_1^{4+}}$	1	7	13	35	31	91	57	155	130	217	133	455	183	399	403	651	307	910
$f^{\square}_{x_1^2x_2^{1+}}$	1	3	5	11	7	15	9	31	18	21	13	55	15	27	35	75	19	54
$f^{\square}_{x_2^{2+}}$	1	7	5	23	7	35	9	59	18	49	13	115	15	63	35	135	19	126
$f^{\square}_{x_1^1x_3^{1+}}$	1	1	1	2	1	1	3	2	4	1	1	2	3	3	1	3	1	4
$f^{\square}_{x_4^{1+}}$	1	3	1	5	3	3	1	7	2	9	1	5	3	3	3	9	3	6
$f^{\square}$	1	3	3	9	5	13	7	24	14	23	11	49	15	33	31	66	21	70

Table 2: Number of sublattices of index n invariant under representative symmetries from each conjugacy class of $S_4 \times \mathbb{Z}_2$.

Cube Analytics

$$f^{\square}_{x_1^{4+}} = u * N * N^2$$

$$f^{\square}_{x_1^2 x_2^{1+}} = \{1, -1, 0, 4\} * u * u * N$$

$$f^{\square}_{x_2^{2+}} = \{1, 3\} * u * u * N$$

$$f^{\square}_{x_1^1 x_3^{1+}} = \{1, 0, -1, 0, 0, 0, 0, 0, 3\} * u * u * \chi_{3,2}$$

$$f^{\square}_{x_4^{1+}} = \{1, 1\} * u * u * \chi_{4,2}$$

Cube Numbers

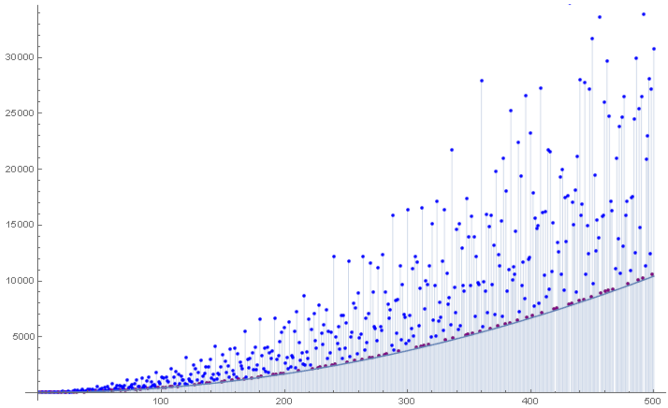
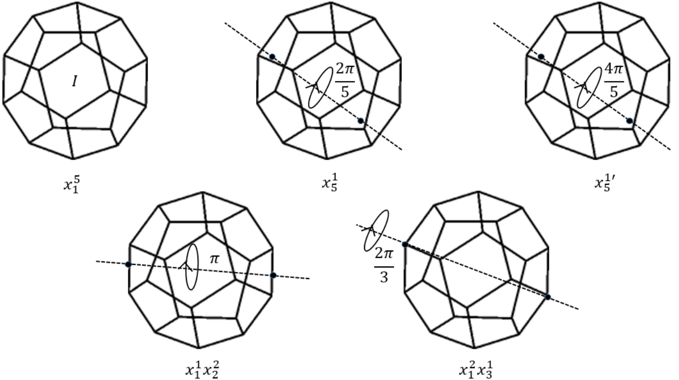


Figure 2: Scatter plot of the number of invariant sublattices of index n for the cubic lattice. Primes are given in purple. The blue line corresponds to $n^2/24$.

- $A_5 \times \mathbb{Z}_2$ symmetry with permutation action on five embedded cubes

$$Z_{A_5 \times \mathbb{Z}_2} = \frac{1}{60}\left(x_1^5 + 12x_5^1 + 12x_5^{1\prime} + 20x + 15x\right)$$



Problems with Dodecahedron Analysis

- Crystallographic restriction in 3D prevents 5-fold lattice symmetry
- Largest $3D$ space group of order 48
- Embedding in higher dimensions

Alternative Route: Projective Geometry

- Surprising correspondence between symmetry groups of lattice polytopes and projective groups over finite fields:

$$S_3 \simeq \mathrm{PGL}(2, \mathbb{F}_2)$$

$$S_4 \simeq \mathrm{PGL}(2, \mathbb{F}_3)$$

$$A_5 \simeq \mathrm{PGL}(2, \mathbb{F}_4)$$

- Rephrase invariant sublattice question in terms of projective spaces

Summary and Future Work

- Number of inequivalent orbifolds on a variety of spaces counted by symmetry analysis
    - Toric diagrams with symmetries of S_3, D_4, D_6, S_4, $S_4 \times \mathbb{Z}_2$
- Projective geometry or higher-dimensional embedding methods
- Representations of Lie algebras, orbifold enumeration, and quiver gauge theories

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