Arndt Compositions

Connections with Fibonacci Numbers, Statistics, and Generalizations

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Matemáticas

UNIVERSIDAD NACIONAL DE COLOMBIA

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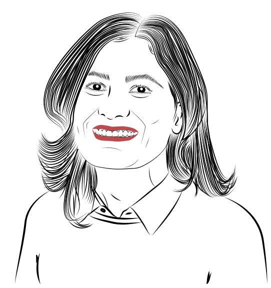
In loving memory of Professor **Amalia Torres**† (1971-2018), who taught me to keep my eyes on the prize, and showed me what a good mathematician should be.



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Sometimes the questions are complicated and the answers are simple.

Dr. Seuss

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Acknowledgements

The development of this project was primarily possible thanks to Professor José Luis Ramírez. I appreciate his relevant feedback, patience, and support of my ideas from the very beginning. I still remember when he first mentioned this problem to me. I must confess that I thought to myself: *"What more can be known about Fibonacci numbers?"* A lot! How wrong I was. I am amazed by the results achieved, and I hope the reader can also be astonished.

I express my gratitude to my alma mater, the National University of Colombia; it is unimaginable to have pursued my career without the support of public education. This work was carried out with financial support from the Faculty of Sciences, QUIPU project No. 2010 1004 0198. I am immensely thankful; they allowed me to dedicate the necessary time. Reasonably, the reproduction of this document is covered by the license CC BY-NC-ND 4.0.

I thank my family and those who supported me throughout and encouraged me to finish. To my parents Ángela and Felipe, for their sacrifices and for giving me everything they never had. To my brother Julián, from whom I learned that talent is wasted without great effort and discipline. To my grandmother Gladys, who has been with me since tying my shoes for school until today. To my cousin Dianny, who has guided me on the path so many times and without conditions. To my closest friends, for all the coffee, pizza, and their listening. To all the professors who ever taught me that the world needs people of integrity, and mathematics cannot be indifferent to their problems.

To whoever reads this document, I also extend my thanks. I want you to know that I did it with great care and this is for you.

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Abstract

Arndt compositions are those integer compositions $(x_1, \ldots, x_k)$ such that $x_{2i-1} > x_{2i}$ for all $i > 0$. Hopkins and Tangboonduangjit—[11]—proved that for each positive integer n, there are f_n Arndt compositions, where f_n is the n-th Fibonacci number. This project aims to provide an alternative proof of this result using generating functions and to analytically explore some statistics and generalizations of this combinatorial object. The studied statistics encompass the number of summands, the size of the last and first summands, the size of the largest and smallest summands, and the number of interior lattice points and semiperimeter associated with the bar graph of each composition. The results are diverse, including an unpublished identity of Fibonacci numbers and new combinatorial interpretations of some sequences.

Keywords: **Arndt, Integer, Compositions, Partitions, Fibonacci, Generating functions, Statistics, Generalizations.**

Resumen

Las composiciones de Arndt son aquellas composiciones de enteros $(x_1, \ldots, x_k)$ que satisfacen $x_{2i-1} > x_{2i}$ para todo $i > 0$. Hopkins y Tangboonduangjit—[11]—probaron que para cada entero positivo n dichas composiciones son contadas por el n-ésimo número de Fibonacci f_n. El objetivo de este proyecto es aportar una prueba alternativa de este resultado empleando funciones generatrices y de forma analítica explorar algunas estadísticas y generalizaciones de este objeto combinatorio. Las estadísticas estudiadas abarcan el número de sumandos, el tamaño del primer y último sumando, el tamaño del sumando más grande y más pequeño, y el número de puntos interiores y semiperímetro del gráfico de barras asociado a cada composición. Los resultados son variados, incluyendo una identidad inédita de los números de Fibonacci y nuevas interpretaciones combinatorias de algunas sucesiones.

Palabras clave: **Arndt, Enteros, Composiciones, Particiones, Fibonacci, Funciones generatrices, Estadísticas, Generalizaciones**

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Declaration of Authenticity

I hereby affirm that I have independently conducted this thesis, with the sole assistance of permissible resources as described in the thesis itself. All passages taken verbatim or figuratively from published and unpublished texts have been duly acknowledged in this work. No part of this work has been employed in any other type of thesis.

Bogotá, D.C., December 1, 2023.

Daniel F. Checa R.

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Notation

$\mathbb{C}$	The set of complex numbers.
$\mathbb{Z}$	The set of integers.
$\mathbb{Z}^+$	The set of positive integers. *No, it is not* $\mathbb{N}$.
$\mathbb{E}(X)$	The expected value—mean—of a random variable X.
$\mathbb{V}(X)$	The variance of a random variable X.
$K[x]$	Polynomials in x with coefficients in K.
$\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$	Font style for a combinatorial class.
$\mathcal{A}_n$	*Set of elements of $\mathcal{A}$ of size n.
a_n	The cardinality of $\mathcal{A}_n$.
$\sim$	We say two sequences $a_n \sim b_n$ $(n \to \infty)$ when $\lim_{n\to\infty} a_n/b_n = 1$.
$A(z)$	Usually, the generating function $\sum_{n\geq 0} a_n z^n$ of a given class $\mathcal{A}$.
$\mathcal{A}^{(k)}$	The set of elements in the class $\mathcal{A}$ with a given parameter of size k.
$\mathcal{A}^{(\geq k)}$	When it makes sense, $\bigcup_{i\geq k} \mathcal{A}^{(i)}$.
$\mathcal{A}^{(\leq k)}$	When it makes sense, $\bigcup_{i\leq k} \mathcal{A}^{(i)}$.
$[z^n]A(z)$	The coefficient of z^n in the Maclaurin series expansion of $A(z)$, usually a_n.
$\cong$	Equivalence or bijection between two combinatorial classes.
$\mathcal{Z}$	The atomic class.
ϵ	The empty class.
SEQ	The sequence operator.
$\lfloor x \rfloor$	Floor of x, i.e., $\max\{n \in \mathbb{Z} \mid n \leq x\}$.
$\lceil x \rceil$	Ceiling of x, i.e., $\min\{n \in \mathbb{Z} \mid n \geq x\}$.
f_n	The n-th Fibonacci number.
g.f.	generating function.
$F(z)$	The g.f. of the Fibonacci numbers, $z/(1-z-z^2)$.
φ	The golden ratio $\frac{1+\sqrt{5}}{2} \approx 1.618\,034$.
$\binom{n}{k}$	The binomial coefficient, n *choose* k.
$(x_1, \ldots, x_k)$	Any integer composition, with summands $x_1, \ldots, x_k$. Sometimes written in the word notation $x_1 x_2 \cdots x_k$.
CV	Coefficient of Variation, i.e., the ratio between the standard deviation and the mean.

**We will also denote by $\mathcal{A}_e$ and $\mathcal{A}_o$ the sets of Arndt compositions whose number of summands is, respectively, even and odd. Do not confuse them with this notation.*

The previous symbols refer to arbitrary combinatorial classes, but starting from Chapter 1, the specific notation we will use for counting sequences related to Arndt compositions is as follows.

a_n	Number of Arndt compositions of n.
$a_n^{(k)}$	Number of k-Arndt compositions of n.
$A^{(k)}(z)$	G.f. of $a_n^{(k)}$.
$b_n^{(k)}$	Number of Arndt compositions of n with k summands.
$B^{(k)}(z)$	G.f. of $b_n^{(k)}$.
$B(z,u)$	Bivariate g.f. of $b_n^{(k)}$.
$c_n^{(k)}$	Number of Arndt compositions of n whose last summand is k.
$C^{(k)}(z)$	G.f. of $c_n^{(k)}$.
$C(z,u)$	Bivariate g.f. of $c_n^{(k)}$.
$d_n^{(k)}$	Number of Arndt compositions of n whose first summand is k.
$D^{(k)}(z)$	G.f. of $d_n^{(k)}$.
$i_n^{(k)}$	Number of Arndt compositions of n with k interior points.
$s_n^{(k)}$	Number of Arndt compositions of n whose semiperimeter is k.
$r_n^{(k)}$	Number of compositions of n such that $\|x_{2i-1} - x_{2i}\| \geq k$.
$G^{(k)}(z)$	G.f. of Arndt compositions whose parts are in $\{1, \ldots, k\}$.
$H^{(k)}(z)$	G.f. of Arndt compositions whose parts are in $\{k, k+1, \ldots\}$.
$L^{(k)}(z)$	G.f. of Arndt compositions whose largest summand is k.
$S^{(k)}(z)$	G.f. of Arndt compositions whose smallest summand is k. *Not related to* $s_n^{(k)}$.
$W^{(k)}(z)$	G.f. of k-block Arndt compositions.

When appropriate, we will also use the superscripts $^{(\geq k)}$ and $^{(\leq k)}$, for example, $b_n^{(\geq k)}$ *is the number of Arndt compositions of* n *with at least* k *parts*, and so on.

Introduction

The study of integer compositions and partitions arises from a very natural question in number theory: In how many ways can an integer n be written as a sum of other integers? Of course, the answer depends on whether we take into account the order of the summands or not. Therefore, this distinction is made when they are defined.

Formally, a *composition* of a positive integer n is a sequence of positive numbers $(x_1, \ldots, x_k)$ such that $\sum_{j=1}^{k} x_j = n$. If we additionally require that the sequence is decreasing, i.e. $x_1 \geq x_2 \geq \cdots \geq x_k$, then we call it a *partition* of n. The parameters n, k are respectively called the *size* and *length* of the composition, and each summand x_i is also known as a *part*.

For example, there are eight compositions of the number 4:

$$4 = 1+1+1+1 = 1+1+2 = 1+2+1 = 2+1+1 = 2+2 = 1+3 = 3+1,$$

and if we do not take order into account, there are only five partitions of this number:

$$4 = 1+1+1+1 = 2+2 = 2+1+1 = 3+1.$$

Clearly, there are fewer partitions than compositions. By hand, you could list the compositions of a given integer, count them, and write down the counting sequence (*OEIS* **A011782**). For the number 1 there is one, for 2 there are two, for 3 there are four, for 4 there are eight, and so on. You might suspect that for each integer n, there are 2^{n-1} compositions, which is indeed true and not difficult to prove.

You can simply represent n as $n = 1+1+1+\cdots+1$, where there are $n-1$ summation symbols, and each composition of n depends on whether each summation is performed or not. The number of ways to choose how many are performed is 2^{n-1}. By the same argument, there are $\binom{n-1}{k-1}$ compositions that use exactly k summands.

Now the same question can be asked in the case of integer partitions, listing the possible partitions and noting down their counting sequence (*OEIS* **A000041**):

$$1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, \ldots$$

In this case, there is no apparent indication of what the next number in the sequence will be, and that is because there is actually no closed formula. It is not as straightforward as in the previous case; we cannot use the same argument. Here, it will be necessary to use generating functions to deduce the next terms. We will see how to do this in Chapter 0.

Due to their complex structure, integer partitions have aroused greater curiosity within the mathematical community. It is known that the first person to formally study them was Leonhard Euler, through generating functions, in Chapter 16 of his work *Introductio in analysin infinitorum*.

One of his results was the famous pentagonal number theorem, published in 1783, which states that if $p(n)$ is the number of partitions of n then the following recurrence holds:

$$p(n) = \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} p(n - \pentagon_k)(-1)^{k-1},$$

where $\pentagon_n = \frac{n(3n-1)}{2}$ is the generalized pentagonal number.

The initial approach by Euler gave rise to the field of additive number theory, in which renowned mathematicians such as Gauss, Legendre, Jacobi, Lagrange, Hardy, and Ramanujan would later work. The latter two are credited with a vast and brilliant array of results. The study of integer partitions has proven useful in various areas, from algorithm analysis to cryptography, and even in statistical mechanics—cf. [7].

This significant interest in integer partitions over compositions should not be misinterpreted; there are also problems related to compositions that cannot be underestimated. The one we will address in this document is one of those.

Arndt compositions are those integer compositions whose pairs of summands are in decreasing order, meaning that the first summand is greater than the second, the third summand is greater than the fourth, the fifth summand is greater than the sixth, and so on. If the length of the composition is odd, we will not impose this condition on the last summand. For example, out of the sixty-four compositions of the number 7, only thirteen of them are Arndt compositions:

$$\begin{aligned} 7 &= 4+3 \\ &= 5+2 \\ &= 6+1 \\ &= 2+1+4 \\ &= 3+1+3 \\ &= 3+2+2 \\ &= 4+1+2 \\ &= 5+1+1 \\ &= 2+1+3+1 \\ &= 3+1+2+1 \\ &= 2+1+2+1+1. \end{aligned}$$

Similarly, we can list the compositions that satisfy this property for each integer, enumerate them, and record the counting sequence. In Table I.1—taken from [11]—, this list appears up to seven. For simplification, each composition is represented without the plus signs, and a_n is the number of Arndt compositions for each positive integer n.

Notice the counting sequence a_n, these appear to be the Fibonacci numbers! This was first observed in 2013 by the German mathematician Jörg Arndt, who posted a brief entry in the *OEIS* **A000045** stating that these compositions are counted by the Fibonacci numbers without providing a formal proof—see Figure I.1.

In 2022, mathematicians Brian Hopkins and Aram Tangboonduangjit published two combinatorial proofs that verify this observation—cf. [11]. One of them consists of an explicit bijection

n	Arndt compositions of n	a_n
1	1	1
2	2	1
3	3, 21	2
4	4, 31, 211	3
5	5, 41, 32, 311, 212	5
6	6, 51, 42, 411, 321, 312, 213, 2121	8
7	7, 61, 52, 511, 43, 421, 412, 322, 313, 3121, 214, 2131, 21211	13

TABLE I.1. Arndt compositions up to $n = 7$.

with the number of compositions with parts 1 or 2. This article describes some statistics, generalizations, and open questions. Subsequently, in 2023, they published another proof through a bijection with compositions that use odd parts—cf. [12].

As a purely interesting tidbit, in 2022, Hopkins gave a short talk about the progress they had made before publishing the first article—cf. [10]. In it, he recounted that before undertaking their research, he reached out to Arndt to inquire if he had a formal proof of this fact. Arndt replied that he had probably proven it many years ago but no longer had those notes and could not remember how to reconstruct the proof, so he invited Hopkins to claim the proof as their own if they managed to find it. Hopkins added, as a moral, *"When you prove something, post it!"*

```
For n >= 1, number of compositions of n where there is a drop between
  every second pair of parts, starting with the first and second
  part; see example. Also, a(n+1) is the number of compositions where
  there is a drop between every second pair of parts, starting with
  the second and third part; see example. - Joerg Arndt, May 21 2013
```

FIGURE I.1. Original entry by Jörg Arndt in the *OEIS* **A000045**.

In this project, we will present alternative proofs of this result. Through Chapter 1 we will demonstrate via the symbolic method presented in [8] that the generating function for its counting sequence corresponds to

$$\sum_{n\geq 0} a_n z^n = \frac{z}{1-z-z^2},$$

which, in turn, is the generating function for the Fibonacci numbers.

Our main objective is to provide information about statistics associated with this combinatorial object, such as the number of summands, the size of the first and the last summands, and the size of the largest and smallest summands. Additionally, there is recent interest in the discrete mathematics community in statistics on bar graphs and polyominoes. In particular, we will count the number of interior points and perimeter in the bar graphs associated with Arndt compositions. This is covered in Chapter 2.

The methods we will employ are advantageous because they will provide us information about the mean and variance, recurrence relations, and asymptotic estimates of these sequences; something unknown to date for this combinatorial object.

Regarding the counting of interior points, I described the counting of this statistic because it proved to be challenging, and I genuinely believe it is a significant contribution to the literature on Fibonacci numbers. But I also do it out of personal interest: Pick's Theorem is my favorite, and it will have a brief appearance as a connection to one of the results presented.

Finally, in Chapter 3, the document will conclude with generalizations that can arise regarding this object, some of which have already been presented in the articles by Hopkins and Tangboonduangjit.

To facilitate the reading of the text, in Appendix A, we present some algorithms in *Wolfram Mathematica*. Ideally, one can verify all theorems with cumbersome computations using a computer rather than by hand. Additionally, we recommend reading Appendix B, which contains the matrices of the sequences that appear throughout the document.

This work led to some of the results shown in [5]. Additionally, it was presented in the homonymous talks at:

- *XXIII Colombian Congress of Mathematics (CCM). June 8, 2023. Tunja, Colombia.*
- *ALTENCOA 9. December 27, 2023. Cali, Colombia.*

Chapter 0

Preliminary Concepts

In this chapter, we will briefly study the most general methods used in analytic combinatorics to deduce counting sequences, statistics, asymptotic estimations, among others.

If you are already familiar with these topics, feel free to proceed to Chapter 1, where we will introduce the combinatorial object that we will emphasize in this monograph, Arndt compositions. On the other hand, if you wish to dive into these topics more thoroughly, I highly recommend[1] the texts [8], [19] and [13]. The first is an essential reference in the field and is where we will employ the majority of the notation, and the other two concisely summarize various concepts while maintaining formality.

0.1. Generating Functions

Generating functions can be seen as a way to encode a sequence of integers into a representing function, and they offer several advantages, which we will explore in this section.

To do this, we first define a combinatorial class $\mathcal{A}$, an enumerable set with an associated size function that satisfies the following conditions: i) the size of each element in $\mathcal{A}$ is a nonnegative integer, ii) the number of elements in $\mathcal{A}$ with a specific size is finite. This size function is usually denoted by $|\cdot|$.

Common examples of combinatorial classes include the aforementioned integer compositions and partitions, subsets and permutations of elements from a finite set, tessellations, graphs, or lattice paths. The size functions can be, in their respective contexts, the number of summands, the number of elements, the number of points, and so on.

Under these conditions, it makes sense to define $\mathcal{A}_n$ as the subset of $\mathcal{A}$ whose elements have size n. Likewise, a_n is defined as the cardinality of $\mathcal{A}_n$. In this way, the generating function $A(z)$ associated with the class $\mathcal{A}$ is defined as

$$A(z) := \sum_{x \in \mathcal{A}} z^{|x|} = \sum_{n \geq 0} a_n z^n.$$

It is worth noting that these sums will be treated as formal series, and their convergence or lack thereof will not be relevant.

[1] I also recommend the online content from [8], available at [9]. There you can find lectures videos, notes and additional exercises.

We say that two classes are equivalent, represented by the symbol $\cong$, if the elements of each size are in bijection. If two classes are equivalent, they will have the same generating function and counting sequence; and vice versa. For example, if $\mathcal{C}$ is the class of integer compositions and $\mathcal{W}$ is the class of binary words starting with zero, we have $\mathcal{C} \cong \mathcal{W}$.

When a sequence is of exponential order, a geometric series is often used to find the generating function. In the case of counting compositions, the generating function for this sequence is simply

$$1 + \sum_{n \geq 1} 2^{n-1} z^n = \frac{1-z}{1-2z}.$$

The utility of using generating functions primarily lies in that it is not always possible to find the general term of a sequence. Let us imagine for a moment that we are unaware of the formula for the number of compositions. The argument for immediately finding its generating function is based on the fact that an ordered sequence of elements from the same combinatorial object has a generating function which is the product of the generating functions of its elements. For example, the generating function for compositions using a single summand is

$$I(z) = \sum_{n \geq 1} z^n = \frac{z}{1-z},$$

since for each positive integer, there is only one composition that satisfies this condition. If we require each composition to have two summands, the corresponding generating function is

$$\frac{z}{1-z} \cdot \frac{z}{1-z} = I(z)^2,$$

as it is an ordered sequence of two elements counted by $I(z)$. Similarly, the generating function for compositions with three summands is $I(z)^3$, and so on. By summing over all possible lengths of compositions—including length zero—, we obtain the same generating function,

$$1 + I(z) + I(z)^2 + I(z)^3 + \cdots = \frac{1}{1 - I(z)} = \frac{1}{1 - \frac{z}{1-z}} = \frac{1-z}{1-2z}. \tag{0.1}$$

What we have done is known as the *symbolic method*, as it allows us to translate the disjoint union of sets into sums of their generating functions, and the concatenation of these sets into products.

This method is not limited to counting compositions or number theory; it also has applications in graph theory, paths, set partitions, tessellations, among others. Of course, we are skipping over several formal details about why this method works. It might make sense why counting a disjoint union translates into a sum, but it is not as immediately obvious why concatenation—Cartesian product of sets—transforms into a product. Everything is explained with the rigor it deserves in [8, §I.1].

The usual process often involves decomposing the combinatorial class under study into smaller parts for which the generating function is known, and then translating this decomposition into products and sums. To aid in this process, a standardized notation of classes and operators has also been established, which facilitates this approach.

For example, in the case of compositions, the class $\{\bullet\}$—usually abbreviated as $\bullet$ or $\mathcal{Z}$—and the operator SEQ are used. These represent, respectively, the class containing a single element of weight zero and the arbitrary concatenation of objects. In this way, integer compositions are symbolically represented by

$$\textsc{Seq}(\mathcal{I}),$$

where $\mathcal{I} = \textsc{Seq}_{\geq 1}(\bullet)$ and the subscripts of the operators refer to the number of possible concatenations. In the case of summing over sequences of length greater than or equal to zero, we simply use SEQ, and we will represent the empty class $\{\}$ as ϵ just for notation purposes, as we do in computer science theory.

This can be extended to any combinatorial class $\mathcal{A}$, with the only condition that $\mathcal{A}$ does not contain elements of size zero, meaning that $a_0 = 0$. In this case, if $\mathcal{A}$ has a generating function $A(z)$, the generating function for $\textsc{Seq}(\mathcal{A})$ is $\frac{1}{1-A(z)}$.

A similar argument is used for integer partitions. Each partition of an integer n can be viewed as a multiset—a finite set where the repetition of elements is allowed—composed of integers. Each multiset, denoted by the operator MSET, is in bijection with a Cartesian product of sequences of each integer:

$$\mathcal{P} = \textsc{MSet}(\mathcal{I}) \cong \textsc{Seq}(\bullet) \times \textsc{Seq}(\bullet\bullet) \times \textsc{Seq}(\bullet\bullet\bullet) \times \cdots,$$

where $\bullet\bullet$ stands for $\{\bullet\} \times \{\bullet\}$ and so on. Hence, the generating function for partitions corresponds to

$$\begin{aligned} P(z) &= \prod_{n\geq 1} \frac{1}{1-z^n} \\ &= 1 + z + 2z^2 + 3z^3 + 5z^4 + 7z^5 + 11z^6 + 15z^7 + 22z^8 + 30z^9 + 42z^{10} + O(z^{11}), \end{aligned}$$

whose coefficients form the sequence we described earlier.

On the other hand, if we require that the parts of each partition be different (*OEIS* **A000009**), it suffices to take the following Cartesian product, represented by the operator powerset, PSET, which we will use whenever we want the possible finite subsets of a combinatorial class; in this case, repetitions are not allowed. Then,

$$\mathcal{P}_{\text{diff}} \cong \textsc{PSet}(\mathcal{I}) = (\epsilon + \bullet) \times (\epsilon + \bullet\bullet) \times (\epsilon + \bullet\bullet\bullet) \times \cdots,$$

and the generating function corresponds to

$$\begin{aligned} P_{\text{diff}}(z) &= \prod_{n\geq 1}(1+z^n) \\ &= 1 + z + z^2 + 2z^3 + 2z^4 + 3z^5 + 4z^6 + 5z^7 + 6z^8 + 8z^9 + 10z^{10} + O(z^{11}). \end{aligned}$$

Euler also demonstrated that partitions of n into odd parts are in bijection with those partitions of n with distinct parts. The generating function for the first class corresponds to

$$P_{\text{odd}}(z) = \prod_{\substack{n=2k-1 \\ k\in\mathbb{Z}^+}} \frac{1}{1-z^n} = \prod_{n\geq 1} \frac{1}{1-z^{2n-1}},$$

and it can be shown to be equal to $P_{\text{diff}}(z)$, because

$$\prod_{n\geq 1}\frac{1}{1-z^{2n-1}}=\prod_{n\geq 1}\frac{1}{1-z^{2n-1}}\cdot\frac{1-z^{2n}}{1-z^{2n}}=\prod_{n\geq 1}\frac{1-z^{2n}}{1-z^{n}}=\prod_{n\geq 1}(1+z^n).$$

As in the case of compositions, it is also possible to restrict the number of parts to use in each partition. Simply note that the set of partitions of n that use at most k parts is in bijection with the set of partitions of n where the parts are in $\{1, 2, \ldots, k\}$. This can be justified by representing each partition using a Ferrer diagram, which involves representing each part of the partition as a row of dots, with as many dots as the size of that part. For example, the partition $8+5+3+3+1$ is represented in the left image of Figure 0.1. In this case, exactly 5 summands are used.

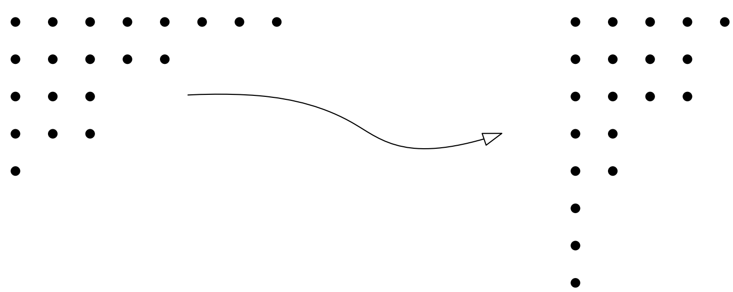

Figure 0.1. Representation of the partition $8+5+3+3+1$.

If we reflect this image across the main diagonal, we obtain a partition with parts of size at most 5, as shown in the right image. In general, if we take a partition of n into at most k parts, by reflecting its diagram across the diagonal, we get an arbitrary partition whose parts are in $\{1, 2, \ldots, k\}$; and the reverse process also holds. Therefore, the generating function for the set of partitions with at most k parts is

$$P^{(\leq k)}(z)=\prod_{n=1}^{k}\frac{1}{1-z^n},$$

and the one for the set of partitions with exactly k parts is

$$P^{(k)}(z)=P^{(\leq k)}(z)-P^{(\leq k-1)}(z)=z^k\prod_{n=1}^{k}\frac{1}{1-z^n}.$$

Several results on partitions can be proven using these graphical arguments. Depending on the convenience of each situation, integer partitions and compositions can be represented as Ferrer diagrams, Young diagrams, as bar graphs (as we will do in Chapter 1), or as a word for notation simplification when a single digit is used for each part. For example, the above partition can be read as the word 85331.

Finally, let us calculate the generating function for the class of partitions that use k distinct parts. This function will become relevant in Chapter 3 when we explore one generalization of Arndt compositions. Once again, we will use Ferrer diagrams. From each partition of n into k distinct parts, we can extract a triangular partition with exactly k parts, as shown in the example in Figure 0.2.

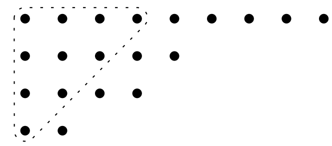

FIGURE 0.2. Triangular partition from 9542.

Note that after this procedure, an arbitrary partition with at most k parts results to the right of this triangle. For each k, there are $\binom{k+1}{2}$ points in the triangle. Therefore, the generating function for partitions with exactly k distinct parts is—cf. [6, §2.5, Th. C]—

$$P_{\text{diff}}^{(k)}(z) = z^{\binom{k+1}{2}} \prod_{n=1}^{k} \frac{1}{1-z^n} = \frac{z^{\binom{k+1}{2}}}{(z;z)_k}, \tag{0.2}$$

where $(a;q)_k = \prod_{n=1}^{k} (1 - aq^{n-1})$ denotes the q-Pochhammer symbol—see [27]. We also have the identity

$$P_{\text{diff}}(z) = \prod_{n\geq 1}(1+z^n) = \sum_{k\geq 1} \frac{z^{\binom{k+1}{2}}}{(z;z)_k}.$$

0.2. Fibonacci Numbers

The Fibonacci numbers may not need an introduction; they are perhaps the most famous sequence, even outside the mathematical community. Formally defined as the sequence with initial values $f_0 = 0, f_1 = 1$, and for nonnegative n, it satisfies the recurrence relation $f_{n+2} = f_{n+1} + f_n$. The first values (*OEIS* **A000045**) of the sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

This sequence frequently appears in number theory. It also has applications in other fields, such as computer science and biology.

Interestingly, it is named after the Italian mathematician Leonardo of Pisa, also known as *Fibonacci*, who studied the sequence in medieval Europe. However, it had been described long before. To date, the earliest known record is in ancient Indian mathematics. In the third century BCE, Pingala described patterns resembling the Fibonacci sequence in his work *Chandahsastra*, which focused on poetic meter—[25].

In these sections, we will prove some of the most well-known results about Fibonacci numbers and, in the process, demonstrate how generating functions can be useful when providing asymptotic estimates for counting sequences.

First, we will calculate the generating function for Fibonacci numbers, which we call $F(z)$. Using the recurrence relation from the definition, we get

$$\begin{aligned} F(z) &= \sum_{n\geq 0} f_n z^n \\ &= z + \sum_{n\geq 2} f_n z^n \end{aligned}$$

$$\begin{aligned}
&= z + \sum_{n\geq 0} f_{n+2} z^{n+2} \\
&= z + z^2 \sum_{n\geq 0} f_{n+2} z^{n} \\
&= z + z^2 \sum_{n\geq 0} f_{n+1} z^{n} + z^2 \sum_{n\geq 0} f_{n} z^{n} \\
&= z + z \sum_{n\geq 0} f_{n+1} z^{n+1} + z^2 F(z) \\
&= z + zF(z) + z^2 F(z),
\end{aligned}$$

therefore,

$$F(z) = \sum_{n\geq 0} f_n z^n = \frac{z}{1 - z - z^2}. \tag{0.3}$$

It is well-known—[24]—that the n-th Fibonacci number counts:

1. The number of compositions of $n-1$ where its parts are either one or two.
2. The number of compositions of n whose parts are odd numbers.
3. The number of compositions of $n+1$ where its parts are greater than one.

Let us see how we can deduce these results using the generating function that we have derived.

1. This class can be described as $\textsc{Seq}_{\geq 1}(\bullet, \bullet\bullet)$, so its generating function corresponds to

   $$\frac{z + z^2}{1 - (z + z^2)} = \frac{1}{1 - z - z^2} - 1.$$

   Therefore, for $n \geq 1$, the number of compositions of n using only one and two as summands is counted by

   $$[z^n]\left(\frac{1}{1 - z - z^2} - 1\right) = [z^n]\left(\frac{F(z)}{z} - 1\right) = f_{n+1}.$$

2. The g.f. for odd numbers corresponds to $\sum_{n\geq 0} z^{2n+1} = \frac{z}{1-z^2}$. Therefore, the generating function for this class is

   $$\frac{\frac{z}{1-z^2}}{1 - \frac{z}{1-z^2}} = \frac{z}{1 - z - z^2}.$$

3. Similarly, the g.f. for numbers greater than one is $\sum_{n\geq 2} z^n = \frac{z^2}{1-z}$. So, the generating function for this class is

   $$\frac{\frac{z^2}{1-z}}{1 - \frac{z^2}{1-z}} = \frac{z^2}{1 - z - z^2} = zF(z),$$

   hence, these compositions are counted by f_{n-1} for $n \geq 1$.

Once again, the symbolic method has provided us with a quick approach to problems that would be arduous if one attempted a bijection proof. Certainly, if two sequences have the same generating function, then they coincide; there is no other possibility. This is known as the Transfer Principle in [13, §2.4].

0.3. Asymptotic Estimations

The formula (0.3) can be decomposed into partial fractions in a convenient manner, so that the denominators can be expanded as a geometric series, giving us

$$F(z) = \frac{z}{(1-\varphi z)(1+\varphi^{-1}z)} = \frac{1/\sqrt{5}}{1-\varphi z} - \frac{1/\sqrt{5}}{1+\varphi^{-1}z},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Finally,

$$f_n = [z^n]F(z) = [z^n]\left(\frac{1/\sqrt{5}}{1-\varphi z} - \frac{1/\sqrt{5}}{1+\varphi^{-1}z}\right) = \frac{1}{\sqrt{5}}\left(\varphi^n - (-\varphi)^{-n}\right). \tag{0.4}$$

This is known as the Binet's formula for Fibonacci numbers. In general, we can achieve formulas like this for linear recurrences doing the same process, which is stated in the following theorem.

Theorem 0.1. *Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be a given sequence of complex numbers, $d \geq 1$, and $\alpha_d \neq 0$. The following conditions on a sequence $(a_n)_{n\geq 0}$ of complex numbers are equivalent:*

i.

$$\sum_{n\geq 0} a_n z^n = \frac{P(z)}{Q(z)},$$

where $Q(z) = 1 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_d z^d$ and $P(z) \in \mathbb{C}[z]$ is a polynomial of degree less than d.

ii. For every $n \geq 0$,

$$a_{n+d} + \alpha_1 a_{n+d-1} + \alpha_2 a_{n+d-2} + \cdots + \alpha_d a_n = 0.$$

iii. For every $n \geq 0$,

$$a_n = \sum_{i=1}^{k} P_i(n)\gamma_i^n,$$

where $1 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_d z^d = \prod_{i=1}^{k}(1-\gamma_i z)^{d_i}$, the γ_i's are distinct and nonzero, and $P_i(n) \in \mathbb{C}[n]$ is a polynomial of degree less than d_i.

Proof. See [26, §4.1, Th. 4.1.1]. □

Furthermore, note that the term $(-\varphi)^{-n}$ tends asymptotically to 0 as $n \to \infty$ since $|-\varphi^{-1}| < 1$. Thus, the n-th term of the Fibonacci sequence has the asymptotic expression

$$f_n \sim \frac{\varphi^n}{\sqrt{5}}. \qquad (n \to \infty)$$

From here, another famous result follows: the quotient of two consecutive Fibonacci numbers tends asymptotically to the golden ratio, that is, $f_{n+1}/f_n \to \varphi$ as $n \to \infty$.

For any sequence defined by a linear recurrence, the asymptotic formula will depend on the root γ_i—from Theorem 0.1—whose modulus is largest, leading us to the following theorem, which also covers functions that are not necessarily rational.

Theorem 0.2. *Let $A(z)$ be a meromorphic function in a disk $|z| \leq R$ with poles $z_1, z_2, \ldots, z_k$. If f is analytic at $|z| = R$ and $z = 0$, there exist polynomials $P_j(n)$—with $j = 1, \ldots, n$—, such that*

$$[z^n]A(z) = a_n = \sum_{j=1}^{k} P_j(n) z_j^{-n} + O(R^{-n}),$$

where the degree of each $P_j(n)$ is one less than the multiplicity of z_j. Furthermore,

$$P_j(n) \sim C_j \frac{n^{v_j - 1}}{(v_j - 1)!}, \qquad (n \to \infty)$$

where v_j is the multiplicity of z_j, and

$$C_j = \lim_{z \to z_j} \left(1 - \frac{z}{z_j}\right)^{v_j} A(z).$$

Proof. See [8, §IV. 5, Th. IV.9 and IV.10]. □

0.4. Multivariate Generating Functions

Lastly, it may be of interest to compute a certain statistic or parameter χ related to some combinatorial class. In the case of compositions, this could be the number of summands. For this purpose, one or more variables can be added to the generating function to perform this counting. In the case of two variables, a bivariate generating function can be expressed as

$$A(z, u) = \sum_{x \in \mathcal{A}} z^{|x|} u^{\chi(x)} = \sum_{n,k \geq 0} a_n^{(k)} z^n u^k,$$

For multiple parameters and variables,

$$A(z, u_1, \ldots, u_m) = \sum_{x \in \mathcal{A}} z^{|x|} u_1^{\chi_1(x)} \cdots u_m^{\chi_m(x)}.$$

For example, let us assume that we want to count the number of summands on the compositions of a given integer n. We must introduce the variable u into formula (0.1) each time a new summand is incorporated. Therefore, the bivariate generating function is

$$\frac{1}{1 - \frac{z}{1-z} u} = \frac{1 - z}{1 - z - uz}. \tag{0.5}$$

A generalized version of Theorem 0.1 tells us that if $t_n^{(k)}$ is the number of compositions of n employing exactly k summands, then the recurrence relation $t_n^{(k)} = t_{n-1}^{(k)} + t_{n-1}^{(k-1)}$ holds for $n, k \geq 1$. This makes sense when we consider that if we subtract one unit from the last summand in a given composition, it can either preserve the number of summands or reduce the number of summands by one if that unit was the only one in the last summand.

We have already mentioned that indeed $t_n^{(k)} = \binom{n-1}{k-1}$ for $n \geq k$, so this recurrence also follows when applying Pascal's rule. However, there might be cases where the combinatorial interpretation of a two-variable recurrence is not immediate. For the same reason, it can be interesting to know the asymptotic behavior of the sequence, as well as information about the mean and variance of these parameters. For this purpose, Theorem 0.3 is introduced.

Theorem 0.3. *Let $\mathcal{A}$ be a combinatorial class and $\chi : \mathcal{A} \to \mathbb{N}$ be a parameter. Let $A(z,u)$ be the generating function of $(\mathcal{A}, \chi)$. The expected value of χ for objects of $\mathcal{A}_n$ satisfies the formula*

$$\mathbb{E}_{\mathcal{A}_n}(\chi) = \frac{[z^n]\frac{\partial}{\partial u}A(z,u)\big|_{u=1}}{[z^n]A(z,1)},$$

and the second moment satisfies

$$\mathbb{E}_{\mathcal{A}_n}(\chi^2) = \frac{[z^n]\frac{\partial^2}{\partial u^2}A(z,u)\Big|_{u=1}}{[z^n]A(z,1)} + \mathbb{E}_{\mathcal{A}_n}(\chi),$$

hence, the variance is given by

$$\begin{aligned}\mathbb{V}_{\mathcal{A}_n}(\chi) &= \mathbb{E}_{\mathcal{A}_n}(\chi^2) - \mathbb{E}_{\mathcal{A}_n}(\chi)^2 \\ &= \frac{[z^n]\frac{\partial^2}{\partial u^2}A(z,u)\Big|_{u=1}}{[z^n]A(z,1)} + \frac{[z^n]\frac{\partial}{\partial u}A(z,u)\big|_{u=1}}{[z^n]A(z,1)}\left(1 - \frac{[z^n]\frac{\partial}{\partial u}A(z,u)\big|_{u=1}}{[z^n]A(z,1)}\right).\end{aligned}$$

Proof. See in [8, §III. 2, Pr. III.2]. □

Applying this theorem to the function (0.5), we obtain that the expected value for the number of summands in a composition of n is $\frac{n+1}{2}$, and the variance is $\frac{n-1}{4}$. Of course, we can also apply it to those compositions we mentioned that are counted by the Fibonacci numbers. However, these will require more algebraic manipulation. We will explain how to handle this in detail when we address the number of summands in Arndt compositions in Chapter 2.

In addition, in Appendix A.1, an example is provided to calculate the mean and variance for a given generating function.

In subsequent chapters, we will continue applying these concepts to further understand Arndt compositions and delve deeper into the study of their properties.

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Chapter 1

Arndt Compositions

Let us explore another proof regarding the counting of Arndt compositions by Fibonacci numbers.

1.1. Definitions

Let us recall that Arndt compositions are those integer compositions that satisfy the condition that each pair of summands is decreasing. Formally, they are defined as follows.

Definition 1.1 (**Arndt compositions**). An *Arndt composition* is a composition $(x_1, \ldots, x_k)$ that satisfies the condition $x_{2i-1} > x_{2i}$ for all $i \geq 1$. If the length k of the composition is odd, this condition is vacuously true for the last summand. ◇

From now on, we will denote by a_n the number of Arndt compositions that exist for a given integer n. We will not consider the empty composition in this definition, so $a_0 := 0$. For example, as we seen in the Introduction, $a_7 = 13$.

We have already provided examples of Arndt compositions in Table I.1. Additionally, you can refer to Appendix A.2, where we provide an example of how to compute Arndt compositions for a given integer and their counting sequence based on those lists.

To prove that these compositions are counted by Fibonacci numbers, it will be crucial to represent them as bar graphs, as in Figure 1.1. This is straightforward; each part of the composition will be represented as a column with a height equal to the size of that part. These bar graphs are defined and deeply explored in [16], but we do not need to be overly explanatory in this regard, the concept is quite intuitive.

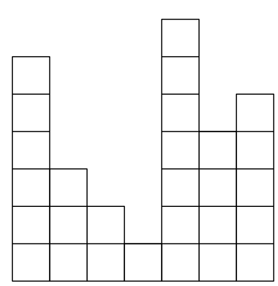


FIGURE 1.1. Representation of the Arndt composition 6321745.

In Appendix A.3, it is explained how to compute Arndt compositions as bar graphs.

On the other hand, it will be relevant to separate the class of Arndt compositions into those of even and odd lengths.

Definition 1.2. Let $\mathcal{A}$ be the combinatorial class of Arndt compositions, and let $\mathcal{A}_e, \mathcal{A}_o \subset \mathcal{A}$ be the classes of those whose length is even and odd, respectively. ◇

1.2. Arndt Compositions with Two Summands

Since the restrictions on the summands in Arndt compositions occur only between pairs of consecutive ones, we will see that it will be easy to enumerate them if we first focus on those of Arndt that use exactly two summands.

Definition 1.3. Let $\mathcal{M}$ be the class of those compositions that use only two summands, and let $\mathcal{M}^{(k)} \subset \mathcal{M}$ be those that also satisfy the property that the difference between the first summand and the second is exactly k—this k could be negative. ◇

Lemma 1.4. *For all $k \in \mathbb{Z}$, the generating function $M^{(k)}(z)$ associated to $\mathcal{M}^{(k)}$ has expression*

$$M^{(k)}(z) = \frac{z^{|k|+2}}{1 - z^2}.$$

Proof. For $k \geq 0$, every composition in $\mathcal{M}^{(k)}$ is the basic composition whose first part is $k+1$ and the second 1; or it is a composition in $\mathcal{M}^{(k)}$ concatenated with two units, one in each part—see Figure 1.2.

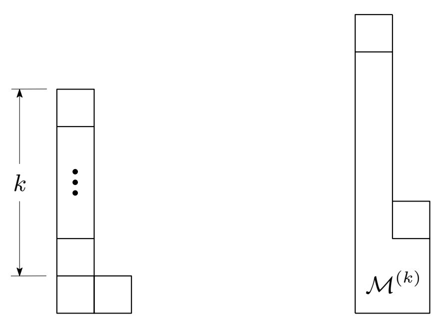


FIGURE 1.2. Decomposition of the elements in $\mathcal{M}^{(k)}$.

Then $M^{(k)}(z)$ satisfies the relation $M^{(k)}(z) = z^{k+2} + z^2 M^{(k)}(z)$. Therefore

$$M^{(k)}(z) = \frac{z^{k+2}}{1 - z^2}.$$

By symmetry, for $k < 0$, $\mathcal{M}^{(k)} \cong \mathcal{M}^{(-k)}$, so $M^{(k)} = \frac{z^{-k+2}}{1-z^2}$ for $k < 0$. We can summarize both results in the expression given in the theorem. □

Let $\mathcal{M}^{(\geq k)} := \bigcup_{i \geq k} \mathcal{M}^{(i)}$.

Lemma 1.5. *The generating function $M^{(\geq 1)}(z)$ associated with $\mathcal{M}^{(\geq 1)}$ has the expression*

$$M^{(\geq 1)}(z) = \frac{z^3}{1 - z - z^2 + z^3}.$$

Proof. From the definition we have

$$M^{(\geq 1)}(z) = \sum_{k\geq 1} M^{(k)}(z) = \sum_{k\geq 1} \frac{z^{k+2}}{1-z^2} = \frac{z^3}{1-z-z^2+z^3}. \qquad \square$$

The following result will be useful when we later count those compositions whose pairs of summands differ by k or more summands.

Lemma 1.6. *The generating function $M^{(\geq k)}(z)$ has expression*

$$M^{(\geq k)}(z) = \begin{cases} \dfrac{z^{k+2}}{1-z-z^2+z^3}, & \text{if } k \geq 0 \\ \dfrac{z^2+z^3-z^{-k+3}}{1-z-z^2+z^3}, & \text{if } k < 0. \end{cases} \tag{1.1}$$

Proof. For $k \geq 0$,

$$M^{(\geq k)}(z) = \sum_{j\geq k} M^{(j)}(z) = \sum_{j\geq k} \frac{z^{j+2}}{1-z^2} = \frac{z^{k+2}}{1-z-z^2+z^3}.$$

For $k < 0$,

$$\begin{aligned} M^{(\geq k)}(z) &= \sum_{j\geq k} M^{(j)}(z) \\ &= \sum_{j\geq 0} M^{(j)}(z) + \sum_{j=k}^{-1} M^{(-j)}(z) \\ &= \frac{z^2}{1-z-z^2+z^3} + \sum_{j=1}^{-k} \frac{z^{j+2}}{1-z^2} \\ &= \frac{z^2}{1-z-z^2+z^3} + \frac{z^3-z^{-k+3}}{1-z-z^2+z^3} \\ &= \frac{z^2+z^3-z^{-k+3}}{1-z-z^2+z^3}. \qquad \square \end{aligned}$$

Another way to prove the first part of Lemma 1.6 is to observe that every composition in $\mathcal{M}$ whose parts differ by k units or more can be decomposed as a summand in the first part of height greater than or equal to k, concatenated with pairs of units; one unit for each part of the composition—see Figure 1.3. Symbolically this translates into the classes

$$\mathcal{M}^{(\geq k)} \cong \textsc{Seq}_{\geq 1}(\mathcal{Z})\, \textsc{Seq}_{\geq k}(\mathcal{Z}^2).$$

Therefore,

$$M^{(\geq k)}(z) = \frac{z^k}{1-z}\frac{z^2}{1-z^2} = \frac{z^{k+2}}{1-z-z^2+z^3}.$$

This construction will make sense in Section 2.1.

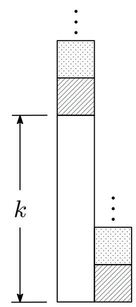


FIGURE 1.3. Alternative proof of the Lemma 1.6 for $k \geq 0$.

1.3. Counting Arndt Compositions with Fibonacci Numbers

Now, we present the main result.

Theorem 1.7. *The generating function $A(z)$ associated with $\mathcal{A}$ has the expression*

$$A(z) = \sum_{n \geq 0} a_n z^n = \frac{z}{1 - z - z^2}.$$

Proof. This is derived from the following observations:

1. It is clear that $\mathcal{A} = \mathcal{A}_e + \mathcal{A}_o$, so

$$A(z) = A_e(z) + A_o(z). \tag{1.2}$$

2. Given that the restrictions of Arndt compositions only occur between consecutive pairs of summands, we can express every composition in $\mathcal{A}_e$ as an arbitrary concatenation of compositions in $\mathcal{M}^{(\geq 1)}$, that is,

$$\mathcal{A}_e = \text{SEQ}_{\geq 1}\left(\mathcal{M}^{(\geq 1)}(\mathcal{Z})\right).$$

Thus,

$$A_e(z) = \frac{M^{(\geq 1)}(z)}{1 - M^{(\geq 1)}(z)}. \tag{1.3}$$

3. As noted in the definition, the restriction $x_{2i-1} > x_{2i}$ vacuously holds for the last summand if the length of the composition is odd. Then every odd composition of length greater than one is an even composition concatenated with an arbitrary part at the end—see Figure 1.4.

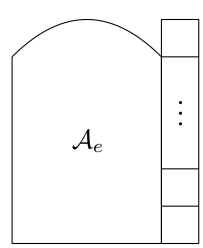


FIGURE 1.4. Compositions of odd length.

Symbolically,

$$\mathcal{A}_o = \textsc{Seq}_{\geq 1}(\mathcal{Z}) + \mathcal{A}_e \, \textsc{Seq}_{\geq 1}(\mathcal{Z}).$$

Therefore,

$$A_o(z) = (1 + A_e(z)) \, \frac{z}{1 - z}. \tag{1.4}$$

By substituting the found expression for $M(z)$ and solving the system given by equations (1.2), (1.3), and (1.4), we get

$$A(z) = \sum_{n \geq 0} a_n z^n = \frac{z}{1 - z - z^2}, \tag{1.5}$$

$$A_e(z) = \frac{z^3}{1 - z - z^2},$$

$$A_o(z) = \frac{z - z^3}{1 - z - z^2}.$$

□

Corollary 1.8. *For $n \geq 0$, there exist f_n Arndt compositions. Moreover, if $n \geq 2$, f_{n-1} of them have odd length and f_{n-2} have even length.*

Proof. Since a_n and f_n have the same generating function, $a_n = f_n$ for all $n \geq 0$. On the other hand,

$$A_e(z) = z^2 A(z) \quad \text{and} \quad A_o(z) = z + zA(z),$$

so for $n \geq 2$ the number of Arndt compositions of even length coincides with f_{n-2}, and the number of those of odd length coincides with f_{n-1}. □

There is an even simpler proof of this result. Note that if we take a composition in $\mathcal{A}_o$ that is different from unity and subtract one unit from its last summand, we either obtain another element in $\mathcal{A}_o$ or one in $\mathcal{A}_e$—see Figure 1.4. Now, if we take any composition in $\mathcal{A}_e$ and subtract two units, one from the last summand and another from the second to the last, we get another element in $\mathcal{A}_e$ or one in $\mathcal{A}_o$—see Figure 1.5.

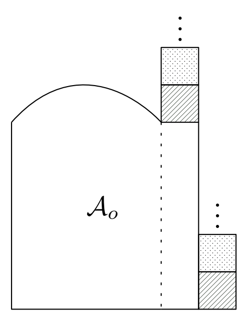


FIGURE 1.5. Compositions of even length.

This translates into the equalities

$$A_o(z) = z + z(A_o(z) + A_e(z)) = z + zA(z),$$
$$A_e(z) = z^2(A_o(z) + A_e(z)) = z^2 A(z).$$

From this, $A(z) = A_e(z) + A_o(z) = z + zA(z) + z^2A(z)$. Solving for $A(z)$ also yields expression (1.5).

One might consider this chapter redundant in view of this concise proof. Do not be mistaken; first, the main proof was the first to take place in the development of this document, and second, it allows us to understand in a deeper way the structure of this class, laying the foundation for the corresponding generalizations in Chapter 3.

Chapter 2

Statistics on Arndt Compositions

We will derive formulas for some of the statistics on $\mathcal{A}$, mainly using Theorems 0.2 and 0.3. The basic statistics on compositions typically include the length and the size of the first and last summands, as well as counts related to the associated bar graph. Although we employ similar methods, each will have its own complexity. Most of these results are novel, and some of them were published in [5] with Professor Ramírez during the development of this thesis.

The intention is not to encompass all possible statistics, as they can be as creative as one proposes. One problem that captivated my attention was the work carried out by Mansour and Shattuck in [18], in which they calculated the area that would be filled with water in a bar graph when water drops on it; outstanding. It could not be ruled out to perform this on Arndt compositions, but that would need to be addressed in another project.

It is possible that in some passages of the text, the symbolic method may become dispensable. Therefore, as we progress, we may omit this step and immediately proceed to find the generating functions for the sequences in question.

2.1. Length (Number of Summands)

To count the composition length, we will use a bivariate generating function in which one variable keeps track of the number of summands. It is sufficient to make some simple modifications to Theorem 1.7.

Definition 2.1. Let $\mathcal{B}^{(k)} \subset \mathcal{A}$ be the combinatorial class of Arndt compositions of length k. For the sake of notation, we define $\mathcal{B} = \bigcup_{k\geq 1} \mathcal{B}^{(k)}$, and $b_n^{(k)}$ as the number of Arndt compositions of n with length k. $\diamond$

Theorem 2.2 (Bivariate generating function). *The bivariate generating function for the class $\mathcal{B}$ is*

$$B(z,u) = \sum_{k\geq 1} B^{(k)}(z)u^k = \frac{uz - uz^3 + u^2z^3}{1 - z - z^2 + z^3 - u^2z^3}.$$

Proof. We just have to multiply the expressions of the functional system of Theorem 1.7 by u each time a new summand appears.

$$B(z,u) = B_e(z,u) + B_o(z,u),$$
$$B_e(z,u) = \frac{M^{(\geq 1)}(z)u^2}{1 - M^{(\geq 1)}(z)u^2},$$

$$B_o(z,u) = (B_e(z,u)+1)\,\frac{uz}{1-z}.$$

Solving the system we obtain the expressions

$$\begin{aligned} B(z,u) &= \frac{uz - uz^3 + u^2z^3}{1-z-z^2+z^3-u^2z^3}, \\ B_e(z,u) &= \frac{u^2z^3}{1-z-z^2+z^3-u^2z^3}, \\ B_o(z,u) &= \frac{uz-uz^3}{1-z-z^2+z^3-u^2z^3}. \end{aligned}$$

□

Corollary 2.3 (First recurrence). *For $n,k \geq 0$, $b_n^{(k)}$ satisfies the recurrence*

$$b_n^{(k)} = \begin{cases} 0, & \text{if } k \geq n \text{ and } nk \neq 1, \text{ or if } k = 0, \\ 1, & \text{if } k = 1 \text{ and } n \geq 1, \text{ or if } n = 3 \text{ and } k = 2, \\ b_{n-1}^{(k)} + b_{n-2}^{(k)} - b_{n-3}^{(k)} + b_{n-3}^{(k-2)}, & \text{if } n \geq 4 \text{ and } k \geq 2. \end{cases}$$

Proof. The initial conditions are immediate, and throughout the document, we will focus solely on the recurrences. The linear recurrence follows from equation

$$B(z,u) = uz - uz^3 + u^2z^3 + (z+z^2-z^3+u^2z^3)B(z,u),$$

this is a general form of Theorem 0.1. □

Verify this result yourself in Table B.1. The combinatorial interpretation of this recurrence is unknown.

Example 2.4. The first terms of $B(z,u)$ are

$$\begin{aligned} B(z,u) = {} & uz + uz^2 + \left(u+u^2\right)z^3 + \left(u+u^2+u^3\right)z^4 + \left(u+2u^2+2u^3\right)z^5 \\ & + \left(u+2u^2+4u^3+u^4\right)z^6 + \left(u+3u^2+6u^3+2u^4+u^5\right)z^7 \\ & + \left(u+3u^2+9u^3+5u^4+3u^5\right)z^8 + \left(u+4u^2+12u^3+8u^4+8u^5+u^6\right)z^9 \\ & + \left(u+4u^2+16u^3+14u^4+16u^5+3u^6+u^7\right)z^{10} + O(z^{11}). \end{aligned}$$

The term $9u^3z^8$ indicates that $b_8^{(3)} = 9$, i.e. there are nine Arndt compositions of 8 that employ exactly three summands. They are shown in Figure 2.1. ◁

Even without yet knowing a formula for $b_n^{(k)}$, it is possible to gather information about its distribution.

Theorem 2.5 (Mean and variance). *For $n \geq 3$, the expected value of the length of Arndt compositions of size n is*

$$\frac{3}{5}(2n-5)\frac{f_{n+1}}{f_n} - \frac{1}{5}(8n-27) \quad \sim \left(\frac{3}{\sqrt{5}} - 1\right)n. \qquad (n \to \infty)$$

On the other hand, the variance is

$$\begin{aligned} & -\left(\frac{3}{5}(2n-5)\frac{f_{n+1}}{f_n}\right)^2 + \frac{1}{25}(36n^2-92n-565)\frac{f_{n+1}}{f_n} \\ & + \frac{2}{25}\left(18n^2-308n+749\right) \quad \sim \frac{4}{25}\left(-75+34\sqrt{5}\right)n. \qquad (n \to \infty) \end{aligned}$$

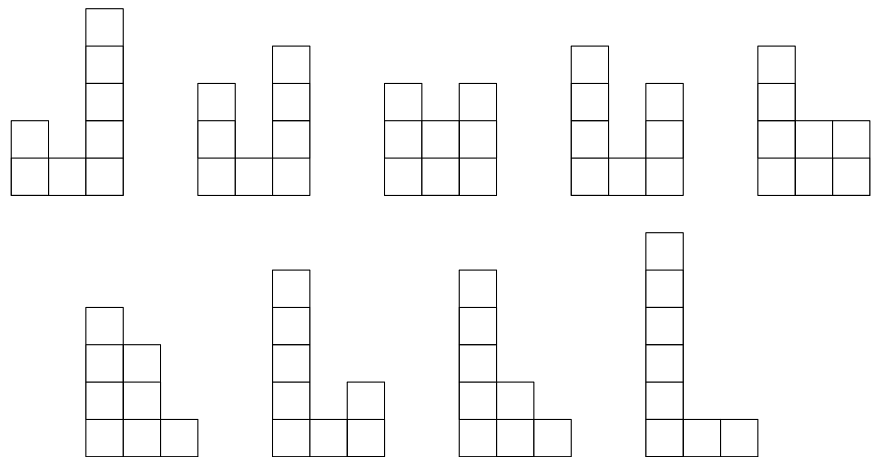


FIGURE 2.1. Compositions in $\mathcal{A}_8$ that employ 3 summands.

Proof. First, we will need the identities—[14, §20.4, Eqs. 20.19 & 20.20]—

$$[z^n]F(z)^2 = \frac{2nf_{n+1} - (n+1)f_n}{5}, \tag{2.1}$$

$$[z^n]F(z)^3 = \frac{-6nf_{n+1} + (n+1)(5n+2)f_n}{50}. \tag{2.2}$$

From Theorem 0.3 the mean of the length is

$$\frac{[z^n]\frac{\partial}{\partial u}B(z,u)\big|_{u=1}}{[z^n]B(z,1)}.$$

The denominator is just f_n, and by equation (2.1) the numerator is

$$\begin{aligned}
[z^n]\frac{\partial}{\partial u}B(z,u)\Big|_{u=1} &= [z^n]\left(\frac{2z^4}{(1-z-z^2)^2} + \frac{z+z^3}{1-z-z^2}\right) \\
&= [z^n]\left(2z^2F(z)^2 + F(z) + z^2F(z)\right) \\
&= f_n + f_{n-2} + 2[z^n]z^2F(z)^2 \\
&= f_n + f_{n-2} + \frac{2}{5}\left(2(n-2)f_{n-1} - (n-1)f_{n-2}\right) \\
&= f_n + \frac{4}{5}(n-2)f_{n-1} + \frac{-2n+7}{5}f_{n-2}.
\end{aligned}$$

From the Fibonacci recurrence formula we obtain

$$[z^n]\frac{\partial}{\partial u}B(z,u)\Big|_{u=1} = \frac{3}{5}(2n-5)f_{n+1} - \frac{1}{5}(8n-27)f_n.$$

Dividing by f_n we get the formula for the mean. From the fact that $\frac{f_{n+1}}{f_n} \sim \frac{1+\sqrt{5}}{2}$ as $n \to \infty$, we also get the asymptotic formula.

In general, we can express any Fibonacci number in terms of f_{n+1} and f_n using the relations

$$f_{n+m} = f_{m-1}f_n + f_mf_{n+1}, \tag{2.3}$$

$$\text{or} \qquad f_{n-m} = (-1)^m \left(f_{m+1} f_n - f_m f_{n+1} \right), \tag{2.4}$$

for all $n \geq m \geq 0$, which can be derived from Binet's formula (0.4). Keep them in mind; we will use them whenever we want to find the mean and variance of any statistic in the document.

For the variance, we have to compute

$$\begin{aligned}
[z^n] \frac{\partial^2}{\partial u^2} B(z,u) \bigg|_{u=1} &= [z^n] \left(\frac{2z^3}{1-z-z^2} + \frac{6z^4+4z^6}{(1-z-z^2)^2} + \frac{8z^7}{(1-z-z^2)^3} \right) \\
&= [z^n] \left(2z^2 F(z) + 2z^2(3+2z^2)F(z)^2 + 8z^4 F(z)^3 \right) \\
&= -\frac{2}{25} \left(30n^2 - 313n + 650 \right) f_{n+1} + \frac{4}{25} \left(25n^2 - 252n + 523 \right) f_n.
\end{aligned}$$

Plugging this expression into the variance formula, we obtain the given expressions. □

The asymptotic formulas for the mean and variance allow us to easily estimate the Coefficient of Variation (CV), which is defined as the ratio of the standard deviation to the mean. This coefficient is a dimensionless percentage and helps estimate how the statistic in question is distributed around the mean. A CV close to zero indicates that the distribution is concentrated around the mean, and very close to one indicates high dispersion—see [8, §III. 2, Pr. III.3]. In our case, the number of parts is asymptotically close to the mean, since

$$CV \sim \frac{\sqrt{\left(\frac{3}{\sqrt{5}} - 1\right) n}}{\frac{4}{25}\left(-75 + 34\sqrt{5}\right) n} \sim 0. \qquad (n \to \infty)$$

Let us continue examining the recurrence properties of $b_n^{(k)}$. It is possible to determine a recurrence formula for $b_n^{(k)}$ that depends only on the variable n. First, it will be necessary to find a closed formula for $B^{(k)}(z)$.

Lemma 2.6. *For all* $k \geq 1$*,*

$$\begin{aligned}
B^{(2k+1)}(z) &= \frac{z}{1-z} B^{(2k)}(z), \\
B^{(2k)} &= \frac{z^2}{1-z^2} B^{(2k-1)}(z),
\end{aligned}$$

with initial condition $B^{(1)}(z) = z/(1-z)$*. Also,*

$$\begin{aligned}
B^{(2k+1)}(z) &= [B^{(2)}(z)]^k B^{(1)}(z), \\
B^{(2k)}(z) &= [B^{(2)}(z)]^k.
\end{aligned}$$

Proof. Note that $\mathcal{B}^{(1)} \cong \textsc{Seq}_{\geq 1}(\mathcal{Z})$, so $B^{(1)}(z) = z/(1-z)$. As shown in Theorem 1.7, any odd composition in $\mathcal{A}$ of length greater than one can be decomposed as an even composition in $\mathcal{A}$ concatenated with an arbitrary summand. Symbolically, $\mathcal{B}^{(2k+1)} \cong \textsc{Seq}_{\geq 1}(\mathcal{Z})\mathcal{B}^{(2k)}$, hence

$$B^{(2k+1)}(z) = \frac{z}{1-z} B^{(2k)}(z).$$

Secondly, any even composition in $\mathcal{A}$ can be decomposed as an odd composition in $\mathcal{A}$ concatenated with pairs of units, in each pair one unit remains in the penultimate part, and the other in the last one—see Figure 1.5.

Therefore $\mathcal{B}^{(2k)} \cong \textsc{Seq}_{\geq 1}(\mathcal{Z}^2)\mathcal{B}^{(2k-1)}$, that is,

$$B^{(2k)}(z) = \frac{z^2}{1-z^2} B^{(2k-1)}(z).$$

You can verify that $B^{(2)}(z) = z^3/((1-z)(1-z^2))$. Substituting the second recurrence into the first and iterating, we obtain

$$\begin{aligned} B^{(2k+1)}(z) &= \frac{z}{1-z} \cdot \frac{z^2}{1-z^2} B^{(2k-1)}(z) \\ &= B^{(2)}(z) B^{(2k-1)}(z) \\ &= \left[B^{(2)}(z)\right]^2 B^{(2k-3)}(z) \\ &= \cdots \\ &= \left[B^{(2)}(z)\right]^j B^{(2k-2j+1)}(z). \end{aligned}$$

Taking $j = k$ in the last expression,

$$B^{(2k+1)}(z) = [B^{(2)}(z)]^k B^{(1)}(z).$$

Similarly,

$$\begin{aligned} B^{(2k)}(z) &= \frac{z^2}{1-z^2} B^{(2k-1)}(z) \\ &= \frac{z^2}{1-z^2} \left[B^{(2)}(z)\right]^{k-1} B^{(1)}(z) \\ &= \left[B^{(2)}(z)\right]^{k-1} B^{(2)}(z) \\ &= \left[B^{(2)}(z)\right]^k. \end{aligned}$$

□

Theorem 2.7 (Generating function). *For $k \geq 1$, the g.f. $B^{(k)}(z)$ has the expression*

$$B^{(k)}(z) = \frac{z^k}{(1-z)^k} \cdot \frac{z^{\lfloor k/2 \rfloor}}{(1+z)^{\lfloor k/2 \rfloor}} = \frac{z^{\lfloor 3k/2 \rfloor}}{(1-z)^k (1+z)^{\lfloor k/2 \rfloor}}. \tag{2.5}$$

Proof. The proof for the formula of $B^{(k)}(z)$ will be done by induction on k. The case $k = 1$ is true. If we assume that the formula is true for some k, we have two cases to consider:

- If $k = 2\ell$ for some $\ell > 0$,

$$\begin{aligned} B^{(k+1)}(z) &= B^{(2\ell+1)}(z) \\ &= \frac{z}{1-z} B^{(2\ell)}(z) \\ &= \frac{z^{\lfloor 3k/2 \rfloor + 1}}{(1-z)^{k+1}(1+z)^{\ell}} \\ &= \frac{z^{\left\lfloor \frac{3(k+1)}{2} \right\rfloor}}{(1-z)^{k+1}(1+z)^{\left\lfloor \frac{k+1}{2} \right\rfloor}}. \end{aligned}$$

◇ Similarly, if $k = 2\ell + 1$,

$$\begin{aligned} B^{(k+1)}(z) &= B^{(2\ell+2)}(z) \\ &= \frac{z^2}{1-z^2} B^{(2\ell+1)}(z) \\ &= \frac{z^{\lfloor 3k/2 \rfloor + 2}}{(1-z)^{k+1}(1+z)^{\lfloor k/2 \rfloor + 1}} \\ &= \frac{z^{\left\lfloor \frac{3(k+1)}{2} \right\rfloor}}{(1-z)^{k+1}(1+z)^{\left\lfloor \frac{k+1}{2} \right\rfloor}}. \end{aligned}$$

□

Example 2.8. Note that by Lemma 2.6

$$\sum_{k \geq 0} B^{(2k+1)}(z) = B^{(1)}(z) + B^{(2)}(z) \sum_{k \geq 0} B^{(2k+1)}(z).$$

Therefore,

$$\sum_{k \geq 0} B^{(2k+1)}(z) = \frac{B^{(1)}(z)}{1 - B^{(2)}(z)} = \frac{\frac{z}{1-z}}{1 - \frac{z}{1-z}\frac{z^2}{1-z^2}} = \frac{z - z^3}{1 - z - z^2} = A_o(z),$$

where $A_o(z)$ is the generating function from Theorem 1.7, as expected. Similarly, $\sum_{k \geq 1} B^{(2k)}(z) = A_e(z)$. What has been done in this section is independent of what is proven in the said theorem, so this result could be considered as an alternative proof. ◁

Corollary 2.9 (Second recurrence). *For $n, k \geq 0$ the following linear recurrence relation holds*

$$b_n^{(k)} = \begin{cases} 0, & \text{if } n < \lfloor 3k/2 \rfloor, \text{ or if } n = 0, \\ 1, & \text{if } n = \lfloor 3k/2 \rfloor \text{ and } n \neq 0, \\ \displaystyle\sum_{i=1}^{\lfloor 3k/2 \rfloor} \sum_{j=0}^{i} \binom{k}{j} \binom{\lfloor k/2 \rfloor}{i-j} (-1)^{j+1} b_{n-i}^{(k)}, & \text{if } n > \lfloor 3k/2 \rfloor. \end{cases}$$

Proof. By the Cauchy product of series, the denominator of $B^{(k)}(z)$ is

$$\begin{aligned} (1-z)^k (1+z)^{\lfloor k/2 \rfloor} &= \left[\sum_{i=0}^{k} \binom{k}{i} (-1)^i z^i \right] \left[\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{\lfloor k/2 \rfloor}{i} z^i \right] \\ &= \sum_{i=0}^{k + \lfloor k/2 \rfloor} \sum_{j=0}^{i} \binom{k}{j} \binom{\lfloor k/2 \rfloor}{i-j} (-1)^j z^i \\ &= \sum_{i=0}^{\lfloor 3k/2 \rfloor} \sum_{j=0}^{i} \binom{k}{j} \binom{\lfloor k/2 \rfloor}{i-j} (-1)^j z^i. \end{aligned}$$

By Theorem 0.1, $b_n^{(k)}$ satisfies the recurrence relation

$$\sum_{i=0}^{\lfloor 3k/2 \rfloor} \sum_{j=0}^{i} \binom{k}{j} \binom{\lfloor k/2 \rfloor}{i-j} (-1)^j b_{n-i}^{(k)} = 0,$$

and solving for the term $b_n^{(k)}$ yields the desired equality. □

Now we find a closed formula for $b_n^{(k)}$.

Theorem 2.10 (Closed formula). *For* $n, k \geq 1$*,* $b_n^{(k)}$ *is*

$$b_n^{(k)} = \sum_{i=\lfloor k/2 \rfloor}^{n-k} \binom{n-i-1}{k-1} \binom{i-1}{\lfloor k/2 \rfloor - 1} (-1)^{i+\lfloor k/2 \rfloor}. \tag{2.6}$$

Proof. We can use the well-known identity—cf. [29, §1.5, Eq. 1.5.5]—

$$\frac{1}{(1-z)^k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} z^n$$

for nonnegative k. Therefore,

$$\begin{aligned}
B^{(k)}(z) &= z^{k+\lfloor k/2 \rfloor} \frac{1}{(1-z)^k} \cdot \frac{1}{(1+z)^{\lfloor k/2 \rfloor}} \\
&= z^{k+\lfloor k/2 \rfloor} \left(\sum_{n\geq 0} \binom{n+k-1}{k-1} z^n \right) \left(\sum_{n\geq 0} \binom{n+\lfloor k/2 \rfloor - 1}{\lfloor k/2 \rfloor - 1} (-1)^n z^n \right) \\
&= z^{k+\lfloor k/2 \rfloor} \sum_{n\geq 0} \sum_{i=0}^{n} \binom{n-i+k-1}{k-1} \binom{i+\lfloor k/2 \rfloor - 1}{\lfloor k/2 \rfloor - 1} (-1)^i z^n \\
&= \sum_{n\geq 0} \sum_{i=0}^{n-k-\lfloor k/2 \rfloor} \binom{n-i-\lfloor k/2 \rfloor - 1}{k-1} \binom{i+\lfloor k/2 \rfloor - 1}{\lfloor k/2 \rfloor - 1} (-1)^i z^n \\
&= \sum_{n\geq 0} \sum_{i=\lfloor k/2 \rfloor}^{n-k} \binom{n-i-1}{k-1} \binom{i-1}{\lfloor k/2 \rfloor - 1} (-1)^{i+\lfloor k/2 \rfloor} z^n.
\end{aligned}$$

□

The term $\binom{i-1}{\lfloor k/2 \rfloor - 1}(-1)^{i+\lfloor k/2 \rfloor}$ can also be expressed in terms of the generalized binomial coefficient $\binom{-\lfloor k/2 \rfloor}{-i}$. However, for practical purposes, we will leave the expression in terms of positive entries.

Adding $b_n^{(k)}$ over $k \geq 1$ we get an identity—probably new—for the Fibonacci numbers.

Corollary 2.11 (Wow!). *For* $n \geq 1$*, the* n*-th Fibonacci number* f_n *is*

$$f_n = \sum_{k=1}^{\left\lfloor \frac{2n+1}{3} \right\rfloor} \sum_{i=\lfloor k/2 \rfloor}^{n-k} \binom{n-i-1}{k-1} \binom{i-1}{\lfloor k/2 \rfloor - 1} (-1)^{i+\lfloor k/2 \rfloor}.$$

We could have just added over $k \geq 1$ in the double sum, but the values are null after $k = \left\lfloor \frac{2n+1}{3} \right\rfloor$ since that is the maximum number of parts one can use to form an Arndt composition[1].

Alternatively, the expression for $b_n^{(k)}$ may appear unwieldy, but its asymptotic formula is simpler.

[1] The maximum number of parts is reached when we use the composition $n = 2+1+2+1+\cdots$. If $n = 3\ell$ for $\ell \geq 1$ such number is 2ℓ. In other cases, the maximum is $2\ell + 1$. We can simplify this maximum as $\left\lfloor \frac{2n+1}{3} \right\rfloor$.

Theorem 2.12 (Asymptotic formula). *For every $k \geq 1$,*

$$b_n^{(k)} \sim \frac{n^{k-1}}{2^{\lfloor k/2 \rfloor}(k-1)!}. \qquad (n \to \infty)$$

Proof. The dominant root of the denominator in (2.5) is $z_1 = 1$, and its multiplicity is k. Plugging this into the formulas of Theorem 0.2 yields the desired result. □

Now, the reader might wonder: is there not a more *elegant* way to express $b_n^{(k)}$? We will address this issue as well. To do so, we will employ the algorithms of Zeilberger and Petkovšek†, which are explained in the book [22] and implemented in Appendix A.4. They are highly useful, since the first, under certain conditions, allows us to find a recursion not necessarily linear for a sum like $b_n^{(k)}$, and from there, Petkovšek's algorithm enables us to decide whether or not this sequence has a hypergeometric form, which basically means it can be expressed as a product of factorials or exponential terms.

So, first, we implement Zeilberger's algorithm with the expression found for $b_n^{(k)}$ to prove the following corollary.

Corollary 2.13 (Third recurrence). *For $n, k \geq 0$, it is satisfied the nonlinear recurrence*

$$b_n^{(k)} = \begin{cases} 0, & \text{if } n = 0 \text{, or if } n = 1 \text{ and } k \neq 1, \\ 1, & \text{if } n = \lfloor 3k/2 \rfloor \text{ and } n \geq 1, \\ \dfrac{1}{n - \lfloor 3k/2 \rfloor}\left((n-2)b_{n-2}^{(k)} + \left\lfloor \frac{k+1}{2} \right\rfloor b_{n-1}^{(k)}\right), & \text{if } n \neq \lfloor 3k/2 \rfloor \text{ and } n \geq 2. \end{cases}$$

Proof. Let $F(n, k|i)$ be the i-th term in the sum of (2.6),

$$F(n, k|i) = \binom{n-i-1}{k-1}\binom{i-1}{\lfloor k/2 \rfloor - 1}(-1)^{i+\lfloor k/2 \rfloor},$$

and $G(n, k|i)$ defined as

$$G(n, k|i) = \frac{(n-i)(\lfloor k/2 \rfloor - i)(k-1)}{(n-k-i+1)(n-k-i+2)} F(n, k|i).$$

Then

$$-nF(n, k|i) + (\lfloor k/2 \rfloor - k)\, F(n+1, k|i) + (n - k - \lfloor k/2 \rfloor + 2)\, F(n+2, k|i) \\ = G(n, k|i+1) - G(n, k|i).$$

Summing over all i, the right-hand part cancels out, and we obtain

$$-nb_n^{(k)} + (\lfloor k/2 \rfloor - k)\, b_{n+1}^{(k)} + (n - k - \lfloor k/2 \rfloor + 2)\, b_{n+2}^{(k)} = 0,$$

which is equivalent to

$$(n-2)b_{n-2}^{(k)} + \left\lfloor \frac{k+1}{2} \right\rfloor b_{n-1}^{(k)} + (\lfloor 3k/2 \rfloor - n)b_n^{(k)} = 0. \qquad □$$

† Marko Petkovšek passed away in March of this year; we extend our condolences to his family, and I am grateful for having had access to the free version of his algorithm.

Subsequently, Petkovšek's algorithm can be employed to demonstrate that, indeed, there is no hypergeometric form for $b_n^{(k)}$. In other words, the expression (2.6) is the best that can be obtained for $b_n^{(k)}$. In any case, asking ourselves this question was not in vain, as it led us to Corollary 2.13, leaving us with the third recurrence to calculate this sequence.

Finally, to conclude this section we mention that during this research, it was discovered that the sequence $b_n^{(k)}$ matches with the described in *OEIS* **A354787**. This sequence counts the number of reduced anti-palindromic compositions of n with length k, explored in [2]. They are defined below.

Definition 2.14 (Anti-palindromic compositions). A composition $(x_1, \ldots, x_k)$ is called *anti-palindromic* if satisfies $x_i \neq x_{k-i+1}$ for all $i \neq \frac{k+1}{2}$.

This definition may result in equivalent compositions by flipping any pair x_i and x_{k-i+1}. Therefore, to avoid double counting, it is defined a *reduced anti-palindromic* composition as the one that satisfies $x_i > x_{k-i+1}$ for all $i < \frac{k+1}{2}$. ◇

For example, all of these are equivalent anti-palindromic compositions

$$(3, 8, 6, 5, 2), \quad (2, 8, 6, 5, 3), \quad (3, 5, 6, 8, 2), \quad (2, 5, 6, 8, 3),$$

but the representative would be $(3, 8, 6, 5, 2)$, the only reduced anti-palindromic composition.

Once we define the reduced version of this object, the connection between Arndt compositions and reduced anti-palindromic compositions is immediate, as in both cases, we are comparing pairs of summands, the first greater than the other. In the case of Arndt compositions, the pairs of summands are ordered consecutively, and in the case of reduced anti-palindromic compositions, they are ordered at each side of the composition.

For example, the representation of $(3, 8, 6, 5, 2)$ as an Arndt composition would be $(3, 2, 8, 5, 6)$. To go from one to the other is straightforward; we take each pair of summands at the sides and reorder them to be adjacent in an Arndt composition. When the length of the reduced anti-palindromic composition is odd, the summand in the middle—the one that does not need to be compared to another—becomes the last in the Arndt composition.

The bijection between both sets is simple, but if formality is desired, we proceed as follows.

Theorem 2.15. *The number of Arndt compositions of n with k summands equals the number of reduced anti-palindromic compositions of n with k summands.*

Proof. Consider π defined as

$$\pi(i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \equiv 1 \pmod 2 \text{ and } 1 \leq i \leq k, \\ k - \frac{i}{2} + 1, & \text{if } i \equiv 0 \pmod 2 \text{ and } 1 \leq i \leq k. \end{cases}$$

π is a permutation on $\{1, \ldots, k\}$. To see this, note that in the first case of π, the elements are mapped to positive numbers less than or equal to $\lceil k/2 \rceil$. In the extreme case $i = 1$, $\pi(1) = 1$. When k is odd, $\pi(k) = \frac{k+1}{2} \leq \lceil k/2 \rceil$, and when k is even, $\pi(k-1) = \frac{k}{2} \leq \lceil k/2 \rceil$. This portion of the function is increasing, so elements are not repeated. Similarly, in the second case of π, all elements are mapped to numbers greater than $\lceil k/2 \rceil$ and less than or equal to k. For this reason, $\pi(i) = \pi(j)$ implies $i = j$, because both cases of π are exclusive. Thus, π is one-to-one.

To prove that π is onto, it suffices to verify that has a right inverse $\mu = \pi^{-1}$ given by

$$\mu(i) = \begin{cases} 2i-1, & \text{if } 1 \leq i \leq \lceil k/2 \rceil, \\ 2(k-i+1), & \text{if } \lceil k/2 \rceil < i \leq k. \end{cases}$$

So, given a reduced anti-palindromic composition $(y_1, y_2, \ldots, y_k)$, we obtain its corresponding Arndt composition by taking $(y_{\pi(1)}, y_{\pi(2)}, \ldots, y_{\pi(k)})$; and vice versa, given an Arndt composition $(x_1, x_2, \ldots, x_k)$, we obtain its reduced anti-palindromic composition through $(x_{\mu(1)}, x_{\mu(2)}, \ldots, x_{\mu(k)})$. To check this, note that the condition $y_i > y_{k-i+1}$ becomes $y_{\pi(2i-1)} > y_{\pi(2i)}$; and that $x_{2i-1} > x_{2i}$ becomes $x_{\mu(i)} > x_{\mu(k-i+1)}$. □

This result is very important, as, with the exception of what is detailed in Section 2.5, the discovered statistics also apply to reduced anti-palindromic compositions.

2.2. Size of the Last Summand

To count those Arndt compositions according to the size of the last summand, we introduce a new class.

Definition 2.16. Let $\mathcal{C}^{(k)} \subset \mathcal{A}$ be the combinatorial class of Arndt compositions whose last summand is k, and $c_n^{(k)}$ its counting sequence. Again, we split it into those of even and odd length with the classes $\mathcal{C}_e^{(k)}$ and $\mathcal{C}_o^{(k)}$, respectively. ◇

Fortunately, this time $c_n^{(k)}$ has a simpler formula.

Theorem 2.17 (Generating function). *For $k \geq 1$, the generating function associated with $\mathcal{C}^{(k)}$ is*

$$C^{(k)}(z) = z^k + z^{2k+1} + (z^{k+2} + z^{2k+1})F(z),$$

where $F(z) = z/(1-z-z^2)$. Therefore,

$$c_n^{(k)} = \begin{cases} 1, & \text{if } n = k \text{ and } k > 0, \\ f_{n-k-2}, & \text{if } k+2 \leq n \leq 2k \text{ and } k > 0, \\ f_{k-1} + 1, & \text{if } n = 2k+1 \text{ and } k > 0, \\ f_{n-k-2} + f_{n-2k-1}, & \text{if } n > 2k+1 \text{ and } k > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Every composition in $\mathcal{C}^{(k)}$ can be decomposed according to the parity of its length as follows:

1. If the length of the composition is even, it can be decomposed as the concatenation of any even composition of length greater than two, a part of size at least $k+1$, and a part of size k. Symbolically,

$$\mathcal{C}_e^{(k)} \cong \mathcal{A}_e \, \text{SEQ}_{\geq k+1}(\mathcal{Z})\mathcal{Z}^k + \text{SEQ}_{\geq k+1}(\mathcal{Z})\mathcal{Z}^k.$$

   Therefore,

$$C_e^{(k)}(z) = (A_e(z) + 1)\, \frac{z^{k+1}}{1-z} z^k$$

$$= z^{2k} A_o(z)$$
$$= z^{2k+1}\left(1 + F(z)\right).$$

2. If the composition length is odd and greater than 1—, it can be decomposed as a concatenation of any even composition and a part of size k. Symbolically,

$$\mathcal{C}_o^{(k)} \cong \mathcal{A}_e \mathcal{Z}^k + \mathcal{Z}^k.$$

Then,

$$C_o^{(k)}(z) = z^k\left(1 + A_e(z)\right)$$
$$= z^k\left(1 + z^2 F(z)\right).$$

Since $C_e^{(k)}(z) + C_o^{(k)}(z)$,

$$C^{(k)}(z) = z^k + z^{2k+1} + \left(z^{k+2} + z^{2k+1}\right) F(z)$$
$$= z^k + z^{2k+1} + \sum_{n \geq k+2} f_{n-k-2} z^n + \sum_{n \geq 2k+1} \left(f_{n-k-2} + f_{n-2k-1}\right) z^n.$$

By grouping the summands we obtain the given formula for $k > 1$,

$$C^{(k)}(z) = z^k + \left(f_{k-1} + 1\right) z^{2k+1} + \sum_{n=k+2}^{2k} f_{n-k-2} z^n + \sum_{n \geq 2k+1} \left(f_{n-k-2} + f_{n-2k-1}\right) z^n.$$

The case $k = 1$ is similar. □

Note that after $n \geq 2k+3$, $c_n^{(k)} = c_{n-1}^{(k)} + c_{n-2}^{(k)}$—see Table B.4.

Example 2.18. For $n \geq 6$, there are $f_{n-4} + f_{n-5} = f_{n-3}$ Arndt compositions whose last summand is 2. ◁

Example 2.19. The first terms of $C^{(3)}(z)$ are

$$C^{(3)}(z) = z^3 + z^6 + 2z^7 + 3z^8 + 4z^9 + 7z^{10} + O\left(z^{11}\right).$$

The term $7z^{10}$ shows that $c_{10}^{(3)} = 7$, i.e. out of the 55 Arndt compositions of 10, seven of them have last part 3—see Figure 2.2. ◁

Corollary 2.20 (Restricted size of last part). *The number of Arndt compositions whose last summand is at most k is*

$$c_n^{(\leq k)} = \begin{cases} f_n, & \text{if } 0 \leq n < k \text{ and } k > 0, \\ f_n - f_{n-k}, & \text{if } k \leq n < k+2 \text{ and } k > 0, \\ f_n - f_{n-k-1}, & \text{if } k+2 \leq n < 2k+2 \text{ and } k > 0, \\ f_n - f_{n-k-1} - f_{n-2k-2}, & \text{if } 2k+2 \leq n \text{ and } k > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The number of those whose last summand is at least k is

$$c_n^{(\geq k)} = \begin{cases} f_{n-k+1}, & \text{if } k-1 \leq n < k+1 \text{ and } k > 0, \\ f_{n-k}, & \text{if } k+1 \leq n < 2k \text{ and } k > 0, \\ f_{n-k} + f_{n-2k}, & \text{if } 2k \leq n \text{ and } k > 0, \\ 0, & \text{otherwise.} \end{cases}$$

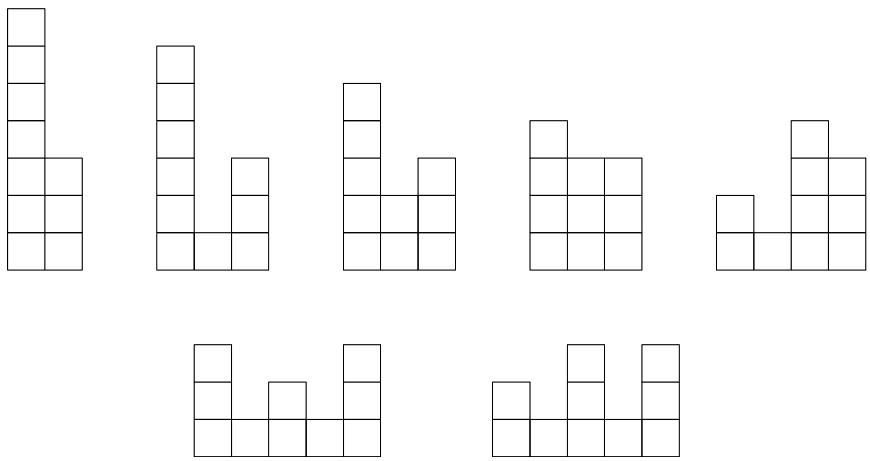


FIGURE 2.2. Compositions in $\mathcal{A}_{10}$ with last part 3.

Proof. For $k \geq 1$, the generating function for $C^{(\leq k)}(z)$ is

$$C^{(\leq k)}(z) = \sum_{j=1}^{k} C^{(j)}(z) = \left(1 - z^k + z^{k+2} - z^{2k+2}\right) F(z).$$

Once again, expanding the sums and grouping all the summands yield the cases for the formula. Similarly, for $C^{(\geq k)}(z)$,

$$C^{(\geq k)}(z) = \sum_{j \geq k} C^{(j)}(z) = \left(z^{k-1} - z^{k+1} + z^{2k}\right) F(z). \qquad \square$$

The sequences above are listed in Tables B.5 and B.6.

Example 2.21. By Theorem 2.17, for $n \geq 4$ there are $2f_{n-3}$ Arndt compositions whose last summand is 1, and by Corollary 2.20 there exist $f_{n-2} + f_{n-4}$ Arndt compositions whose last part is greater than 1. This was pointed out in Corollary 2.2 of [11, §2]. ◁

Theorem 2.22 (Bivariate generating function). *The bivariate generating function for* $\mathcal{C}$ *is*

$$\begin{aligned} C(z,u) &= \sum_{k \geq 1} C^{(k)}(z) u^k \\ &= \frac{u(1-z)z(1+z)(1-z+z^2-uz^2)}{(1-z-z^2)(1-uz)(1-uz^2)} \\ &= \frac{uz - uz^2 - u^2z^3 + uz^4 - uz^5 + u^2z^5}{1 - z - uz - z^2 + 2uz^3 + u^2z^3 + uz^4 - u^2z^4 - u^2z^5}. \end{aligned}$$

Therefore, for $n \geq 6$ *and* $k \geq 3$, $c_n^{(k)} = c_{n-1}^{(k)} + c_{n-1}^{(k-1)} + c_{n-2}^{(k)} - 2c_{n-3}^{(k-1)} - c_{n-3}^{(k-2)} - c_{n-4}^{(k-1)} + c_{n-4}^{(k-2)} + c_{n-5}^{(k-2)}$.

From here, we can study the distribution of this parameter.

Theorem 2.23 (Mean and variance). *For* $n > 0$*, the expected value of the last part of an Arndt composition of size* n *is*

$$2\frac{f_{n+1}}{f_n} - \frac{1+(-1)^n}{2f_n} - 1 \quad \sim \sqrt{5}. \qquad (n \to \infty)$$

The variance is

$$-\left(2\frac{f_{n+1}}{f_n}\right)^2 + 2\left(1+(-1)^n\right)\frac{f_{n+1}}{f_n^2} + 6\frac{f_{n+1}}{f_n} - \left(\frac{1+(-1)^n}{2f_n}\right)^2 - \frac{n+11+(n-1)(-1)^n}{2f_n} + 4 \quad \sim 1+\sqrt{5}. \qquad (n \to \infty)$$

Proof. For $n \geq 1$ we have

$$\begin{aligned}[z^n]\,\frac{\partial}{\partial u}C(z,u)\bigg|_{u=1} &= [z^n]\left(\frac{2-z}{1-z-z^2} - \frac{1/2}{1-z} - \frac{1/2}{1+z} - 1\right)\\ &= 2f_{n+1} - f_n - \frac{1}{2}(1+(-1)^n).\end{aligned}$$

Also,

$$\begin{aligned}[z^n]\,\frac{\partial^2}{\partial u^2}C(z,u)\bigg|_{u=1} &= [z^n]\left(\frac{6z}{1-z-z^2} - \frac{1/2}{(1-z)^2} - \frac{1/2}{(1+z)^2} - \frac{7/2}{1-z} + \frac{5/2}{1+z} + 2\right)\\ &= 6f_n - \frac{1}{2}(n+1)(1+(-1)^n) - \frac{7}{2} + \frac{5}{2}(-1)^n\\ &= 6f_n - \frac{1}{2}\left(n+8+(-1)^n(n-4)\right).\end{aligned}$$

Theorem 0.3 yields the result. □

It is worth noting that the term $2f_{n+1} - f_n - \frac{1}{2}(1+(-1)^n)$ coincides with the sequence *OEIS* **A014217**, i.e. the sum of the last parts of all Arndt compositions of size n is

$$\sum_{k=1}^{n} k c_n^{(k)} = \left\lfloor \left(\frac{1+\sqrt{5}}{2}\right)^n \right\rfloor,$$

thus providing a new combinatorial interpretation of it. This time the $CV \sim \frac{\sqrt{1+\sqrt{5}}}{\sqrt{5}} \approx 0.804\,496$ as $n \to \infty$, i.e. the sizes of the last parts tends to be dispersed.

Finally, we also have an asymptotic formula for $c_n^{(k)}$.

Theorem 2.24 (Asymptotic formula). *For $k \geq 1$,*

$$c_n^{(k)} \sim \frac{\varphi^{k-1}+1}{\varphi^2+1}\varphi^{n-2k}, \qquad (n \to \infty)$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Proof. $C^{(k)}(z)$ has two poles, at $z_1 = \varphi^{-1}$ and $z_2 = -\varphi$, each of multiplicity one. According to Theorem 0.2, when $n \to \infty$, $c_n^{(k)}$ is asymptotically $(\varphi^{-1})^{-n} = \varphi^n$ times a constant, given by

$$\lim_{z \to z_1}\left(1 - \frac{z}{z_1}\right)C^{(k)}(z) = \frac{\varphi^{k-1}+1}{\varphi^2+1}\varphi^{-2k}.$$ □

2.3. Size of the First Summand

Analogously, to count the size of the first part, we introduce the sequence $d_n^{(k)}$ as the number of Arndt compositions of size n whose first part is k, and $\mathcal{D}^{(k)}$ as its combinatorial class.

Theorem 2.25 (Generating function). *For $k \geq 1$, the generating function $D^{(k)}(z)$ is*

$$D^{(k)}(z) = \left(z^{k-1} - z^{2k-1} - z^{2k}\right) F(z).$$

Therefore,

$$d_n^{(k)} = \begin{cases} f_{n-k+1}, & \textit{if } k-1 \leq n < 2k-1 \textit{ and } k > 0, \\ f_k, & \textit{if } n = 2k-1 \textit{ and } k > 0, \\ f_{n-k+1} - f_{n-2k+1} - f_{n-2k}, & \textit{if } 2k \leq n \textit{ and } k > 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. If $k > 1$, every composition in $\mathcal{D}_e^{(k)}$ of length greater than two can be decomposed into a pair of summands whose first part is k, its second part is less than k, along with any composition in $\mathcal{A}_e$. Every composition in $\mathcal{D}_o^{(k)}$ of length greater than one can be decomposed into the same pair of summands, along with any composition in $\mathcal{A}_o$. Therefore,

$$D_e^{(k)}(z) = z^k \frac{z - z^k}{1-z} \left(1 + A_e(z)\right), \qquad D_o^{(k)}(z) = z^k + z^k \frac{z - z^k}{1-z} A_o(z).$$

Hence,

$$D^{(k)}(z) = D_e^{(k)}(z) + D_o^{(k)}(z) = \left(z^{k-1} - z^{2k-1} - z^{2k}\right) F(z). \qquad \square$$

From here onwards, the calculations are quite similar to those in Section 2.2, so details are ommited.

Recurrence relation. From $n \geq 2k+2$, $d_n^{(k)} = d_{n-1}^{(k)} + d_{n-2}^{(k)}$—see Tables B.7, B.8 and B.9.

Restricted size of first part. For $k \geq 1$,

$$D^{(\leq k)}(z) = \frac{z - z^2 - z^{k+1} + z^{2k+2}}{1 - 2z + z^3} \quad \text{and} \quad D^{(\geq k)}(z) = \frac{z^k - z^{2k}}{1 - 2z + z^3}.$$

Hence, the number of Arndt compositions whose first summand is at most k is

$$d_n^{(\leq k)} = \begin{cases} 1, & \text{if } n = 1 \text{ and } k > 0, \\ f_n, & \text{if } 2 \leq n < k+1 \text{ and } k \geq 2, \\ f_n - f_{n-k+2} + 1, & \text{if } k+1 \leq n < 2k+2 \text{ and } k > 0, \\ f_n - f_{n-k+2} + f_{n-2k+1}, & \text{if } 2k+2 \leq n \text{ and } k > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The number of those whose first summand is at least k is

$$d_n^{(\geq k)} = \begin{cases} f_{n-k+3} - 1, & \text{if } k \leq n < 2k \text{ and } k > 0, \\ f_{n-k+3} - f_{n-2k+3}, & \text{if } 2k \leq n \text{ and } k > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Bivariate generating function. The bivariate generating function for $\mathcal{D}$ is

$$\begin{aligned} D(z,u) &= \sum_{k\geq 1} D^{(k)}(z)u^k \\ &= \frac{uz(1-z-z^2+uz^3)}{(1-z-z^2)(1-uz)(1-uz^2)} \\ &= \frac{uz-uz^2-uz^3+u^2z^4}{1-z-uz-z^2+2uz^3+u^2z^3+uz^4-u^2z^4-u^2z^5}. \end{aligned}$$

Therefore, for $n \geq 5$ and $k \geq 3$, $d_n^{(k)} = d_{n-1}^{(k)} + d_{n-1}^{(k-1)} + d_{n-2}^{(k)} - 2d_{n-3}^{(k-1)} - d_{n-3}^{(k-2)} - d_{n-4}^{(k-1)} + d_{n-4}^{(k-2)} + d_{n-5}^{(k-2)}$.

Asymptotic formula. For $k \geq 1$,

$$d_n^{(k)} \sim \frac{\varphi^{k+1}-\varphi-1}{\varphi^2+1}\varphi^{n-2k+1}. \qquad (n\to\infty)$$

Mean and variance. For $n > 0$, the expected value of the first part of an Arndt composition of size n is

$$2\frac{f_{n+1}}{f_n} - \frac{1}{4f_n}\left(2n+7+(-1)^n\right)+1 \qquad \sim 2+\sqrt{5}. \qquad (n\to\infty)$$

The variance is

$$\begin{aligned} &-\left(2\frac{f_{n+1}}{f_n}\right)^2 + \frac{f_{n+1}}{f_n^2}\left(2n+7+(-1)^n\right) + 6\frac{f_{n+1}}{f_n} - \left(\frac{2n+7+(-1)^n}{4f_n}\right)^2 \\ &-\frac{6n^2+30n+55+(-1)^n(2n-7)}{8f_n}+6 \qquad \sim 3+\sqrt{5}. \qquad (n\to\infty) \end{aligned}$$

The CV is asymptotically $\frac{\sqrt{3+\sqrt{5}}}{2+\sqrt{5}} \approx 0.540\,182$. The cumulative sequence matches[2] with *OEIS* **A129696**.

$$\sum_{k=1}^{n} kd_n^{(k)} = [z^n]\,\frac{\partial}{\partial u}D(z,u)\bigg|_{u=1} = 2f_{n+1}+f_n-\frac{1}{4}\left(2n+7+(-1)^n\right) = f_{n+3}-\lfloor n/2\rfloor - 2.$$

Also,

$$[z^n]\,\frac{\partial^2}{\partial u^2}D(z,u)\bigg|_{u=1} = 8f_{n+1}+6f_n-\frac{1}{8}\left(6n^2+34n+69+(-1)^n(2n-5)\right).$$

2.4. Size of the Largest and Smallest Summands

To classify Arndt compositions based on the size of the largest and smallest summands, we can leverage some of the results obtained in Chapter 1. First, we need to find the generating function $G^{(k)}(z)$ for those Arndt compositions that use parts from the set $\{1,\ldots,k\}$ and the function $H^{(k)}(z)$ for those that use parts from $\{k,k+1,\ldots\}$.

[2] See Emeric Deutsch entry.

For the first one, we will construct a function $M_G^{(k)}(z)$ analogous to $M^{(\geq 1)}(z)$ from Lemma 1.5, but this time we have to restrict the parts. The function $M_G^{(k)}(z)$ corresponds to the g.f. of those Arndt compositions that use two summands and, in addition, have parts less than or equal to a positive k. In this case, each pair of summands consists of a first summand less than or equal to k and a second summand smaller than the first. Then,

$$M_G^{(k)}(z) = \sum_{j=1}^{k} z^j \left(\sum_{i=1}^{j} z^i \right) = \sum_{j=1}^{k} z^j \frac{z - z^j}{1 - z} = \frac{z^2(1 - z^k)(z - z^k)}{(1 - z)^2(1 + z)}.$$

Analogously to Theorem 1.7, the generating function for Arndt compositions whose parts are restricted to $\{1, \ldots, k\}$ is obtained from the functional system

$$G_e^{(k)}(z) = \frac{M_G^{(k)}(z)}{1 - M_G^{(k)}(z)}, \qquad G_o^{(k)}(z) = \left(1 + G_e^{(k)}(z)\right) \frac{z - z^{k+1}}{1 - z},$$
$$G^{(k)}(z) = G_e^{(k)}(z) + G_o^{(k)}(z),$$

whose solution is

$$G^{(k)}(z) = \frac{z - z^{k+1}}{1 - z - z^2 + z^{k+1}}.$$

From this function, we have the following result.

Theorem 2.26. *The g.f. $L^{(k)}(z)$ for the number of Arndt compositions whose largest summand is $k \geq 1$, is given by*

$$L^{(k)}(z) = G^{(k)}(z) - G^{(k-1)}(z) = \frac{z^k(1 + z)(1 - z)^2}{(1 - z - z^2 + z^k)(1 - z - z^2 + z^{k+1})}.$$

Proof. This idea comes from [23, Ch. 6]. In Problem 13, they count the number of compositions whose largest summand is k by constructing analogous functions to G and L. The subtraction makes sense because once we take all Arndt compositions of some integer n with parts at most k and then subtract those whose parts are at most $k - 1$, those that survive have parts at most k but at least one summand of size exactly k. □

Similarly, to discriminate by the smallest summand, we construct the function $M_H^{(k)}(z)$. Each pair of summands has a second summand of size at least k, and a first summand larger than the second. Therefore,

$$M_H^{(k)}(z) = \sum_{j \geq k} \left(\sum_{i=j+1}^{\infty} z^i \right) z^j = \sum_{j \geq k} \frac{z^{j+1}}{1 - z} z^j = \frac{z^{2k+1}}{(1 - z)^2(1 + z)}.$$

The functional system is

$$H_e^{(k)}(z) = \frac{M_H^{(k)}(z)}{1 - M_H^{(k)}(z)}, \qquad H_o^{(k)}(z) = \left(1 + H_e^{(k)}(z)\right) \frac{z^k}{1 - z},$$
$$H^{(k)}(z) = H_e^{(k)}(z) + H_o^{(k)}(z),$$

and the solution

$$H^{(k)}(z) = \frac{z^k - z^{k+2} + z^{2k+1}}{1 - z - z^2 + z^3 - z^{2k+1}}.$$

Theorem 2.27. *The g.f. $S^{(k)}(z)$ for the number of Arndt compositions whose smallest summand is $k \geq 1$, is given by*

$$S^{(k)}(z) = H^{(k)}(z) - H^{(k+1)}(z) = \frac{z^k(1+z)(1-z)^2(1-z-z^2+z^3+z^{k+1}-z^{k+3}+z^{2k+2})}{(1-z-z^2+z^3-z^{2k+1})(1-z-z^2+z^3-z^{2k+3})}.$$

When studying the statistics in the previous sections, the abundance of results relied heavily on the ease of finding the roots of the denominators of the generating functions. However, this is not the case, and we can barely conclude the recurrence relation derived from each denominator. Nevertheless, in the author's view, this statistic is equally valuable as all the others.

Example 2.28. The first terms of $L^{(4)}(z)$ are

$$L^{(4)}(z) = z^4 + z^5 + 2z^6 + 4z^7 + 6z^8 + 10z^9 + 16z^{10} + O(z^{11}).$$

Out of the 21 Arndt compositions of 8, six of them have 4 as the largest summand—see Figure 2.3. Both sequences are listed in Tables B.10 and B.12. ◁

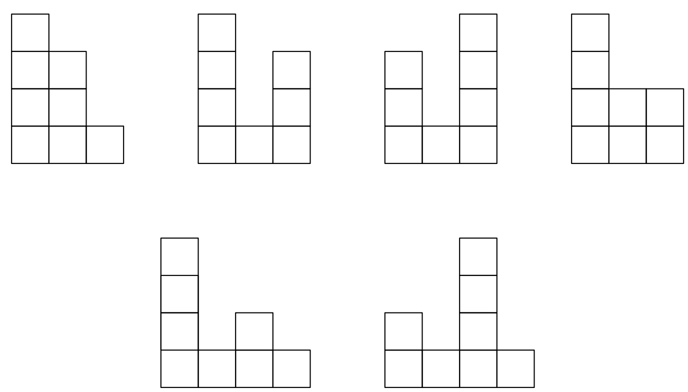


FIGURE 2.3. Compositions in $\mathcal{A}_8$ whose largest summand is 4.

2.5. Interior Lattice Points and Semiperimeter

To finish our study on statistics, we will classify Arndt compositions based on the number of interior points and the semiperimeter of the associated bar graphs for each composition.

This section is motivated by the work done in [17] and [15], where they count, respectively, the number of interior points in bar graphs of compositions and Catalan words.

We define a point to be interior to a bar graph when it is adjacent to four different cells of the bar graph; otherwise, we say it is a boundary point. On the other hand, the perimeter of a bar graph is always even, making it simpler to count the semiperimeter—the half of the perimeter—so as not to unnecessarily increase the number of times we differentiate the generating function to obtain each coefficient.

Let $\text{int}(x)$ be the number of interior points of the associated bar graph for x and $\text{sp}(x)$ be its semiperimeter. For example, $\text{int}(5365437) = 15$ and $\text{sp}(5365437) = 19$—see Figure 2.4.

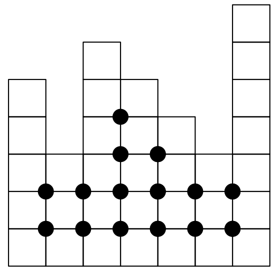


FIGURE 2.4. Interior points of $5365437 \in \mathcal{A}_{33}$.

Let $A^{(k)}(z,p,q)$ be the generating function of Arndt compositions whose last part is k; as usual, the variable z tracks the size of the composition and p and q track the semiperimeter and the number of interior points, respectively. Again, we separate $\mathcal{A}$ into $\mathcal{A}_o$ and $\mathcal{A}_e$.

$$A_o^{(k)}(z,p,q) = \sum_{x\in\mathcal{C}_o^{(k)}} z^{|x|}p^{\mathrm{sp}(x)}q^{\mathrm{int}(x)}, \qquad A_e^{(k)}(z,p,q) = \sum_{x\in\mathcal{C}_e^{(k)}} z^{|x|}p^{\mathrm{sp}(x)}q^{\mathrm{int}(x)},$$
$$A^{(k)}(z,p,q) = A_o^{(k)}(z,p,q) + A_e^{(k)}(z,p,q) = \sum_{x\in\mathcal{C}^{(k)}} z^{|x|}p^{\mathrm{sp}(x)}q^{\mathrm{int}(x)}.$$

For $k \geq 1$, we separate both functions according to the cases that can occur in the last summand. For the case of odd compositions, the last summand can be only one—in the case of having only one summand—and the semiperimeter is one less than the number of cells; there are no interior points. When there is more than one summand, the last summand can be greater than or equal to, or less than the penultimate one. If the former occurs, the number of interior points increases by one less than the size of the penultimate summand, and the semiperimeter increases by one more than the difference between the last and penultimate summands—see Figure 2.5.

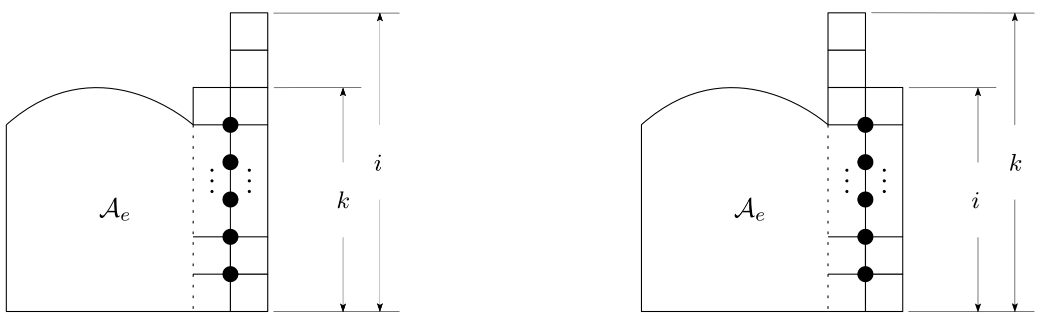


FIGURE 2.5. Cases for interior points and semiperimeter decomposition.

If, on the other hand, the last summand is smaller, the semiperimeter only increases by one unit, and the number of interior points does it by one less than the size of the last summand. This translates into the equation (2.7).

When the composition is odd, the last summand must always be smaller; hence, equation (2.8) is derived.

$$A_o^{(k)}(z,p,q) = z^k p^{k+1} + z^k \sum_{i=1}^{k} A_e^{(i)}(z,p,q)p^{k-i+1}q^{i-1} + z^k p q^{k-1} \sum_{i>k} A_e^{(i)}(z,p,q), \tag{2.7}$$

$$A_e^{(k)}(z,p,q) = z^k p q^{k-1} \sum_{i>k} A_o^{(i)}(z,p,q). \tag{2.8}$$

Define $A(z,p,q|v) := \sum_{k\geq 1} A^{(k)}(z,p,q)v^k$, similarly for $A_o(z,p,q|v)$ and $A_e(z,p,q|v)$. By multiplying (2.7) by v^k and summing over $k \geq 1$ we obtain

$$A_o(z,p,q|v) = \frac{p^2vz}{1-pvz} + \frac{p}{q}\sum_{k\geq 1}\sum_{i=1}^{k} A_e^{(i)}(z,p,q)p^{k-i}q^i z^k v^k + \frac{p}{q}\sum_{k\geq 1}\sum_{i>k} A_e^{(i)}(z,p,q)q^k v^k z^k.$$

Note that $\sum_{i>k} A_e^{(i)}(z,p,q) = A_e(z,p,q|1) - \sum_{i=1}^{k} A_e^{(i)}(z,p,q)$, then

$$\begin{aligned} A_o(z,p,q|v) = {} & \frac{p^2vz}{1-pvz} + \frac{p}{q}\sum_{k\geq 1}\sum_{i=1}^{k} A_e^{(i)}(z,p,q)p^{k-i}q^i z^k v^k \\ & + \frac{p}{q}\left(\frac{qvz}{1-qvz}A_e(z,p,q|1) - \sum_{k\geq 1}\sum_{i=1}^{k} A_e^{(i)}(z,p,q)q^k v^k z^k\right). \end{aligned}$$

Using Cauchy product for series we can simplify these sums to

$$\begin{aligned} A_o(z,p,q|v) &= \frac{p^2vz}{1-pvz} + \frac{p}{q}\left(\sum_{k\geq 1} A_e^{(k)}(z,p,q)q^k v^k z^k\right)\left(\sum_{k\geq 0} p^k v^k z^k\right) \\ &\quad + \frac{p}{q}\left(\frac{qvz}{1-qvz}A_e(z,p,q|1) - \left(\sum_{k\geq 1} A_e^{(k)}(z,p,q)q^k v^k z^k\right)\left(\sum_{k\geq 0} q^k v^k z^k\right)\right) \\ &= \frac{p^2vz}{1-pvz} + \frac{p}{q}A_e(z,p,q|qvz)\frac{1}{1-pvz} \\ &\quad + \frac{p}{q}\left(\frac{qvz}{1-qvz}A_e(z,p,q|1) - A_e(z,p,q|qvz)\frac{1}{1-qvz}\right) \\ &= \frac{p^2vz}{1-pvz} + \frac{pvz}{1-qvz}A_e(z,p,q|1) + \frac{p}{q}\left(\frac{1}{1-pvz} - \frac{1}{1-qvz}\right)A_e(z,p,q|qvz). \end{aligned} \tag{2.9}$$

Analogously, by multiplying (2.8) by v^k and adding over $k \geq 1$ we get

$$A_e(z,p,q|v) = \frac{pvz}{1-qvz}A_o(z,p,q|1) - \frac{p}{q\,(1-qvz)}A_o(z,p,q|qvz). \tag{2.10}$$

Let $|z|, |p|, |q|, |v| < 1$. By iterating[3] equations (2.9) and (2.10) infinitely many times, we obtain

$$\begin{aligned} A_e(z,p,q|v) = & -\sum_{n\geq 1} \frac{p^{2n+1}(q-p)^{n-1}q^{(n-1)^2}v^n z^{n(n+1)}}{\left(\prod_{k=1}^{n-1} 1-q^{2k}vz^{2k}\right)\left(\prod_{k=1}^{n}(1-q^{2k-1}vz^{2k-1})(1-pq^{2k-1}vz^{2k})\right)} \\ & + \sum_{n\geq 1} \frac{p^{2n-1}(q-p)^{n-1}q^{(n-1)^2}v^n z^{n(n+1)-1}}{\left(\prod_{k=1}^{n} 1-q^{2k-1}vz^{2k-1}\right)\left(\prod_{k=1}^{n-1}(1-pq^{2k-1}vz^{2k})(1-q^{2k}vz^{2k})\right)}A_o(z,p,q|1) \end{aligned}$$

[3] Do not worry! You can check this using the code provided in Appendix A.5.

$$-\sum_{n\geq 1}\frac{p^{2n}(q-p)^{n-1}q^{(n-1)^2}v^n z^{n(n+1)}}{\left(\prod_{k=1}^{n-1}1-pq^{2k-1}vz^{2k}\right)\left(\prod_{k=1}^{n}(1-q^{2k-1}vz^{2k-1})(1-q^{2k}vz^{2k})\right)}A_e(z,p,q|1), \tag{2.11}$$

$$\begin{aligned}A_o(z,p,q|v) &= \sum_{n\geq 1}\frac{p^{2n}(q-p)^{n-1}q^{(n-1)(n-2)}v^n z^{n^2}}{\left(\prod_{k=1}^{n}1-pq^{2k-2}vz^{2k-1}\right)\left(\prod_{k=1}^{n-1}(1-q^{2k-1}vz^{2k-1})(1-q^{2k}vz^{2k})\right)}\\ &-\sum_{n\geq 1}\frac{p^{2n}(q-p)^{n}q^{n(n-1)}v^{n+1}z^{n(n+2)}}{\prod_{k=1}^{n}(1-pq^{2k-2}vz^{2k-1})(1-q^{2k-1}vz^{2k-1})(1-q^{2k}vz^{2k})}A_o(z,p,q|1)\\ &+\sum_{n\geq 1}\frac{p^{2n-1}(q-p)^{n-1}q^{(n-1)(n-2)}v^n z^{n^2}}{\left(\prod_{k=1}^{n}1-q^{2k-1}vz^{2k-1}\right)\left(\prod_{k=1}^{n-1}(1-pq^{2k-2}vz^{2k-1})(1-q^{2k}vz^{2k})\right)}A_e(z,p,q|1).\end{aligned} \tag{2.12}$$

The denominators in the sums can be expressed in terms of the q-Pochhammer symbol. Taking $v \to 1^-$ and recalling that $A_o(z,p,q|1) = A_o(z,p,q)$ and $A_e(z,p,q|1) = A_e(z,p,q)$, we have the following theorem.

Theorem 2.29. *The g.f.s for the number of interior points and the semiperimeter of Arndt compositions are given by the formulas*

$$A_e(z,p,q) = \frac{\det\begin{bmatrix}\alpha_1 & \beta_1\\ \alpha_2 & 1+\beta_2\end{bmatrix}(z,p,q)}{\det\begin{bmatrix}\beta_1 & 1+\gamma_1\\ 1+\beta_2 & \gamma_2\end{bmatrix}(z,p,q)}, \qquad A_o(z,p,q) = \frac{\det\begin{bmatrix}\alpha_1 & 1+\gamma_1\\ \alpha_2 & \gamma_2\end{bmatrix}(z,p,q)}{\det\begin{bmatrix}\beta_1 & 1+\gamma_1\\ 1+\beta_2 & \gamma_2\end{bmatrix}(z,p,q)},$$

$$A(z,p,q) = A_e(z,p,q) + A_o(z,p,q) = \frac{\det\begin{bmatrix}\alpha_1 & 1+\beta_1+\gamma_1\\ \alpha_2 & 1+\beta_2+\gamma_2\end{bmatrix}(z,p,q)}{\det\begin{bmatrix}\beta_1 & 1+\gamma_1\\ 1+\beta_2 & \gamma_2\end{bmatrix}(z,p,q)},$$

where $\det\left[\begin{smallmatrix}f_1 & f_2\\ f_3 & f_4\end{smallmatrix}\right](z,p,q)$ *stands for* $\det\left(\begin{smallmatrix}f_1(z,p,q) & f_2(z,p,q)\\ f_3(z,p,q) & f_4(z,p,q)\end{smallmatrix}\right)$ *and* $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$ *are the functions*

$$\alpha_1(z,p,q) = \sum_{n\geq 1}\frac{p^{2n+1}(q-p)^{n-1}q^{(n-1)^2}z^{n(n+1)}}{(q^2z^2;q^2z^2)_{n-1}(qz,pqz^2;q^2z^2)_n}, \qquad \alpha_2(z,p,q) = \sum_{n\geq 1}\frac{p^{2n}(q-p)^{n-1}q^{(n-1)(n-2)}z^{n^2}}{(pz;q^2z^2)_n(qz;qz)_{2n-2}},$$

$$\beta_1(z,p,q) = \sum_{n\geq 1}\frac{p^{2n-1}(q-p)^{n-1}q^{(n-1)^2}z^{n(n+1)-1}}{(q^2z^2,pqz^2;q^2z^2)_{n-1}(qz;q^2z^2)_n}, \quad \beta_2(z,p,q) = \sum_{n\geq 1}\frac{p^{2n}(q-p)^{n}q^{n(n-1)}z^{n(n+2)}}{(pz;q^2z^2)_n(qz;qz)_{2n}},$$

$$\gamma_1(z,p,q) = \sum_{n\geq 1}\frac{p^{2n}(q-p)^{n-1}q^{(n-1)^2}z^{n(n+1)}}{(pqz^2;q^2z^2)_{n-1}(qz;qz)_{2n}}, \qquad \gamma_2(z,p,q) = \sum_{n\geq 1}\frac{p^{2n-1}(q-p)^{n-1}q^{(n-1)(n-2)}z^{n^2}}{(qz;q^2z^2)_n(pz,q^2z^2;q^2z^2)_{n-1}}.$$

Although these expressions may seem initially challenging to handle—there would not even be a possibility of obtaining a closed formula for the counting sequences—there is a lot of information we can derive from them, including the cumulative generating functions and therefore the mean and variance. But first of all, let us look some examples and observations.

Example 2.30. The first terms of $A(z,p,q)$ are[4]

$$A(z,p,q) = p^2z + p^3z^2 + 2p^4z^3 + 3p^5z^4 + \left(4p^6 + p^5q\right)z^5 + \left(6p^7 + 2p^6q\right)z^6$$

[4] See how to manipulate these expressions in Appendix A.6.

$$+\left(9p^8+2p^7q+2p^6q^2\right)z^7+\left(13p^9+3p^8q+5p^7q^2\right)z^8$$
$$+\left(19p^{10}+5p^9q+8p^8q^2+2p^7q^3\right)z^9$$
$$+\left(28p^{11}+7p^{10}q+14p^9q^2+5p^8q^3+p^7q^4\right)z^{10}+O\left(z^{11}\right)$$

The term $5p^8q^3z^{10}$ indicates that there are 5 Arndt compositions of size 10, with a semiperimeter of 8 and 3 interior points. All of them are shown in Figure 2.6. Also, see Tables B.14 and B.17.

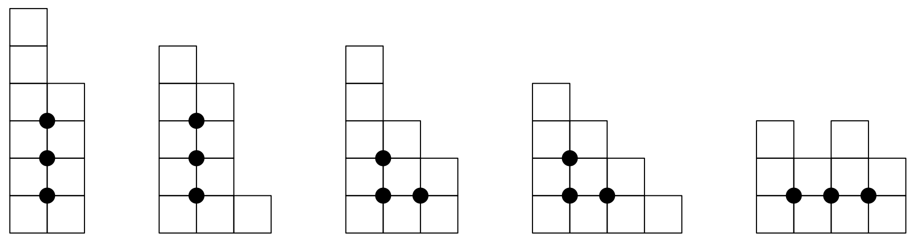


FIGURE 2.6. Bar graphs in $\mathcal{A}_{10}$ with 3 interior points, and semiperimeter of 8.

Note that the multidegree $i+j$ of each term p^iq^j is always one more than the degree in the corresponding variable z; there is no term that does not satisfy this condition. This is not a coincidence, it is the result of Pick's theorem. ◁

Theorem 2.31 (**Pick**). *The area of any—not necessarily convex—simple polygon $Q \subseteq \mathbb{R}^2$ with integral vertices is given by*

$$A(Q)=I+\frac{B}{2}-1,$$

where I and B are respectively the number of integral points in the interior and the boundary of Q.

Proof. Pick's theorem is one of many consequences of Euler's characteristic equation. See all the details in [1, Ch. 13]†. □

In the particular case of bar graphs, the perimeter coincides with the number of points on the boundary. Therefore, the sum of the number of interior points with the semiperimeter always exceeds by one the area of each composition, which is simply its size.

In this way, it becomes interesting to study the cumulative sequences. If $i_n^{(k)}$ denotes the number of Arndt compositions of size n with k interior points, and $s_n^{(k)}$ the number of those with semiperimeter k, we should have

$$\sum_{x\in\mathcal{A}_n}\text{area}(x)=\sum_{x\in\mathcal{A}_n}\text{int}(x)+\sum_{x\in\mathcal{A}_n}\text{sp}(x)-\sum_{x\in\mathcal{A}_n}1,$$
$$nf_n=\sum_{k\geq 0}ki_n^{(k)}+\sum_{m\geq 0}ms_n^{(m)}-f_n. \tag{2.13}$$

This can be verified from the respective generating functions. Although the expressions in Theorem 2.29 may seem unwieldy, we could find the first two moments, as by differentiating and evaluating respectively at $q=1$ or $p=1$, a large portion of the sums cancel out. In this way, we have calculated the generating functions for the total number of interior points and the total semiperimeter for all Arndt compositions of a certain size n.

†We deeply regret the passing of Mr. Aigner in October of this year. We extend our condolences to his family. His legacy as a mathematician and communicator will endure in our community.

Theorem 2.32 (Cumulative g.f.s). *The generating function of the total number of interior points in all Arndt compositions of size n is given by*

$$\sum_{n\geq 1}\sum_{k\geq 0} k i_n^{(k)} z^n = \frac{\partial}{\partial q}A(z,1,q)\bigg|_{q=1} = \frac{z^5\,(1+z^2-z^4-z^6+z^7)}{(1-z^3)(1-z^4)(1-z-z^2)^2}, \tag{2.14}$$

and the g.f. for the total semiperimeter is

$$\sum_{n\geq 1}\sum_{k\geq 0} k s_n^{(k)} z^n = \frac{\partial}{\partial p}A(z,p,1)\bigg|_{p=1} = \frac{z\,(2-z-2z^3-2z^4+z^5-z^6+2z^7+z^{10}-z^{11})}{(1-z^3)(1-z^4)(1-z-z^2)^2}. \tag{2.15}$$

These functions satisfy

$$zF'(z) = \frac{\partial}{\partial q}A(z,1,q)\bigg|_{q=1} + \frac{\partial}{\partial p}A(z,p,1)\bigg|_{p=1} - F(z),$$

which is equivalent to equation (2.13). On the other hand, if we expand the function (2.14) into partial fractions, we obtain the formula for the total number of interior points of compositions in $\mathcal{A}_n$ for $n \geq 2$.

$$\begin{aligned}\sum_{k\geq 0} k i_n^{(k)} &= [z^n]\left(-3+z+\frac{1}{12(-1+z)^2}-\frac{3}{8(-1+z)}+\frac{1}{8(1+z)}+\frac{7+z}{100\,(1+z^2)}\right.\\ &\qquad\left.+\frac{-49+82z}{10\,(-1+z+z^2)^2}+\frac{-733+724z}{100\,(-1+z+z^2)}-\frac{1}{12\,(1+z+z^2)}\right)\\ &= \frac{2(115n-247)f_n-(130n-243)f_{n+1}}{100}+\frac{n}{12}+\frac{11}{24}+\frac{(-1)^n}{8}\\ &\qquad+\frac{1}{100}\left(7\cos\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\right)-\frac{1}{6\sqrt{3}}\sin\left(\frac{2\pi(n+1)}{3}\right) \qquad (2.16)\\ &\sim \frac{-65+33\sqrt{5}}{100}n\varphi^n. \qquad (n\to\infty)\end{aligned}$$

The asymptotic formula can also be deduced from Theorem 0.2. We could do the same on (2.15) to find the cumulative sequence of the semiperimeter, but by using the equation (2.13), we save ourselves all those calculations.

$$\begin{aligned}\sum_{m\geq 0} m s_n^{(m)} &= (n+1)f_n - \sum_{k=0}^{n} k i_n^{(k)}\\ &= \frac{(-130n+594)f_n+(130n-243)f_{n+1}}{100}-\frac{n}{12}-\frac{11}{24}-\frac{(-1)^n}{8}\\ &\qquad-\frac{1}{100}\left(7\cos\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\right)+\frac{1}{6\sqrt{3}}\sin\left(\frac{2\pi(n+1)}{3}\right) \qquad (2.17)\\ &\sim \frac{13\,(5-\sqrt{5})}{100}n\varphi^n. \qquad (n\to\infty)\end{aligned}$$

From here, we can also deduce the mean and variance for both sequences, for which we will need the second derivative of each g.f. However, on this occasion, we prefer to calculate the terms that are significant, since the partial fractions of these are extensive. In the case of interior points,

$$\begin{aligned}\left.\frac{\partial^2}{\partial q^2}A(z,1,q)\right|_{q=1} &= \frac{-1127+1825z}{10\left(1-z-z^2\right)^3}+\frac{288803-322647z}{1100\left(1-z-z^2\right)^2}+\frac{-25128754+18156363z}{121000\left(1-z-z^2\right)} \\ &+58-18z+4z^2+\frac{1}{240(1-z)^5}-\frac{1}{288(1-z)^4}+\frac{61}{360(1-z)^3} \\ &+\frac{19}{1728(1-z)^2}-\frac{14183}{57600(1-z)}-\frac{1}{192(1+z)^3}+\frac{23}{128(1+z)^2} \\ &+\frac{49}{2304(1+z)}+\frac{221+28z}{1000\left(1+z^2\right)^2}+\frac{-1879-119z}{2000\left(1+z^2\right)}+\frac{-2+z}{72\left(1-z+z^2\right)} \\ &-\frac{8+3z}{36\left(1+z+z^2\right)^2}+\frac{5(13-3z)}{108\left(1+z+z^2\right)}+\frac{2\left(96+7z+58z^2-46z^3\right)}{3025\left(1+z+z^2+z^3+z^4\right)}.\end{aligned}$$

The significant terms only come from the first three summands, since the roots of the denominators of the other terms are all roots of unity. By using expressions (2.1) and (2.2), and simplifying with aim of (2.3) and (2.4), the n-th term of the above expression is

$$\begin{aligned}[z^n]\left.\frac{\partial^2}{\partial q^2}A(z,1,q)\right|_{q=1} &= \frac{1}{121000}\Big(\left(11249617-6062166n+844580n^2\right)f_n \\ &-2\left(3498562-1864841n+259545n^2\right)f_{n+1}\Big)+O\left(n^4\right). \qquad (2.18)\end{aligned}$$

For the variance of the semiperimeter, we could do the same, but again, it is not worth it. It can also be deduced from the formula we obtain for the variance of the interior points because, in fact, they are the same.

Theorem 2.33 (Mean and Variance). *For $n \geq 3$, the mean for the number of interior points in Arndt compositions is*

$$\frac{243-130n}{100}\cdot\frac{f_{n+1}}{f_n}+\frac{115n-247}{50}+O\left(n\varphi^{-n}\right)\sim\frac{33-13\sqrt{5}}{20}n, \qquad (n\to\infty)$$

the mean for the semiperimeter is

$$\frac{130n-243}{100}\cdot\frac{f_{n+1}}{f_n}+\frac{-65n+297}{50}+O\left(n\varphi^{-n}\right)\sim\frac{13\left(-1+\sqrt{5}\right)}{20}n, \qquad (n\to\infty)$$

and the variance for both parameters is

$$\begin{aligned}&-\left(\frac{130n-243}{100}\cdot\frac{f_{n+1}}{f_n}\right)^2+\frac{511225n^2+1664300n-9495194}{302500}\cdot\frac{f_{n+1}}{f_n} \\ &+\frac{1022450n^2-15171310n+38495207}{605000}+O\left(n^4\varphi^{-n}\right)\sim\frac{65009\sqrt{5}-141335}{11000}n. \quad (n\to\infty)\end{aligned}$$

Proof. The mean of both parameters is deduced from Theorem 0.3 and equations (2.16) and (2.17). Similarly, the variance of the interior points from (2.18). We only need to answer, why is it the same for the semiperimeter?

To do this, let us abuse the notation a bit from the mentioned theorem. Every time we calculate the variance of either parameter for a fixed n, this n can be taken as a constant. Therefore, by Pick's theorem, we have $\mathbb{V}(\mathrm{sp}) = \mathbb{V}(n + 1 - \mathrm{int})$. Variance is preserved when adding constants, so

$$\mathbb{V}(\mathrm{sp}) = \mathbb{V}(-\,\mathrm{int}) = (-1)^2\mathbb{V}(\mathrm{int}) = \mathbb{V}(\mathrm{int}). \qquad \square$$

The CV asymptotically approaches zero, i.e. both parameters tend to cluster around their respective means.

We could have overlooked Pick's theorem and would have obtained the same results; however, it has saved us a lot of work and allows us to understand the relation between both parameters.

Thus, we conclude this chapter of statistics, where our goal was not only to answer significant questions about Arndt compositions but also to invite the reader to go beyond and hopefully, just as they can take away many answers, also leave with more curious questions. We will answer some in the chapter on generalizations, but, as mentioned before, creativity knows no bounds, and there will still be much to explore.

Chapter 3

Generalizations of Arndt Compositions

In this chapter, we will explore some generalizations of Arndt compositions by modifying the constraint of Definition 1.1 and adjusting the proof of Theorem 1.7. The first two have previously been studied by Hopkins and Tangboonduangjit, and we have verified their results.

3.1. *k*-Arndt Compositions

Definition 3.1. Let $\mathcal{A}^{(k)}$ be the class of compositions whose summands satisfy $x_{2i-1} \geq k + x_{2i}$ for $i > 0$ and any integer k. These are called k-Arndt compositions. $\diamond$

Under this definition $\mathcal{A} = \mathcal{A}^{(1)}$. Note that unlike the statistics we have studied previously, this parameter k is not additive, that is, it is generally not true that the intersection $\mathcal{A}^{(i)} \cap \mathcal{A}^{(j)}$ is empty for some pair of integers i, j. In fact, $A^i \supset A^j$ if $i < j$. Therefore, it is not meaningful to define $A^{(\geq k)}(z)$.

Theorem 3.2. *For $k > 0$, the generating function of $\mathcal{A}^{(k)}$ corresponds to*

$$A^{(k)}(z) = \begin{cases} \dfrac{z - z^3 + z^{k+2}}{1 - z - z^2 + z^3 - z^{k+2}}, & \text{if } k \geq 0 \\[2ex] \dfrac{z + z^2 - z^{-k+3}}{1 - z - 2z^2 + z^{-k+3}}, & \text{if } k < 0. \end{cases}$$

Thus, by Theorem 1.7,

$$a_n^{(k)} = a_{n-1}^{(k)} + a_{n-2}^{(k)} - a_{n-3}^{(k)} + a_{n-k-2}^{(k)}, \tag{3.1}$$

for all $n > k + 2$ and $k > 0$. For $k < 0$ and $n > -k + 3$,

$$a_n^{(k)} = a_{n-1}^{(k)} + 2a_{n-2}^{(k)} - a_{n+k-3}^{(k)}. \tag{3.2}$$

Proof. The proof of this theorem is analogous to Theorem 1.7, we only need to replace the expression of $M^{(\geq 1)}(z)$ with $M^{(\geq k)}(z)$ given in formula (1.1). $\square$

In [11, §3] and [12, §4], Hopkins and Tangboonduangjit explored this sequence under the restriction $x_{2i-1} > k + x_{2i}$, so by substituting $k - 1$ instead of k in the recurrences (3.1) and (3.2), the same formulas are obtained. The reason for choosing our definition in this way is that by doing so, k becomes the smallest difference between each summand, which is more convenient for us. In addition, we have proven the next result, which is equivalent to Corollary 3.3 in the former article—see Table B.20.

Corollary 3.3. *For any k, the number of k-Arndt odd compositions in is $a_{n-1}^{(k)}$, hence the number of k-Arndt even compositions is $a_n^{(k)} - a_{n-1}^{(k)}$.*

Proof. Similarly, one can prove that the generating functions of $\mathcal{A}_o^{(k)}$ and $\mathcal{A}_e^{(k)}$ are, respectively,

$$A_o^{(k)}(z) = \begin{cases} \dfrac{z - z^3}{1 - z - z^2 + z^3 - z^{k+2}}, & \text{if } k \geq 0 \\ \dfrac{z - z^3}{1 - z - 2z^2 + z^{-k+3}}, & \text{if } k < 0, \end{cases}$$

and

$$A_e^{(k)}(z) = \begin{cases} \dfrac{z^{k+2}}{1 - z - z^2 + z^3 - z^{k+2}}, & \text{if } k \geq 0 \\ \dfrac{z^2 + z^3 - z^{-k+3}}{1 - z - 2z^2 + z^{-k+3}}, & \text{if } k < 0. \end{cases}$$

Regardless of the sign of k, $A_o^{(k)}(z) = z + zA^{(k)}(z)$, hence the number of k-Arndt odd compositions is $a_{n-1}^{(k)}$ for $n > 1$. Since $A_e^{(k)}(z) = A^{(k)}(z) - A_o^{(k)}(z)$, the number of k-Arndt even compositions is $a_n^{(k)} - a_{n-1}^{(k)}$. □

3.2. On the Absolute Difference Between Pairs of Summands

As of now, this result has not been formally published, but in the talk [10], Hopkins mentions having found a recurrence formula for the case where we are interested in the absolute difference between the summands. That is, now we count in the same case when $x_{2i-1} - x_{2i} \geq k$ or $x_{2i-1} - x_{2i} \leq -k$ for $k > 0$. What we present next aligns with what they observed.

Theorem 3.4. *Let $\mathcal{R}^{(k)}$ the class of compositions whose summands satisfy $|x_{2i-1} - x_{2i}| \geq k$ for any integer k. Its generating function is*

$$R^{(k)}(z) = \begin{cases} \dfrac{z - z^3 + 2z^{k+2}}{1 - z - z^2 + z^3 - 2z^{k+2}}, & \text{if } k > 0, \\ \dfrac{z}{1 - 2z}, & \text{if } k \leq 0. \end{cases}$$

from where $r_n^{(k)} = r_{n-1}^{(k)} + r_{n-2}^{(k)} - r_{n-3}^{(k)} + 2r_{n-k-2}^{(k)}$ for $k > 0$ and $n > k + 2$.

Proof. Once again, we need to substitute $M^{(\geq 1)}(z)$ from Theorem 1.7 with the generating function of compositions with two summands whose absolute difference is k. For $k > 0$, this expression is derived from Lemma 1.4:

$$M_R^{(k)}(z) = \sum_{j \geq k} M^{(j)}(z) + \sum_{j=-\infty}^{-k} M^{(j)}(z) = \frac{2z^{k+2}}{(1 - z)(1 - z^2)}.$$

Hence,

$$R_e^{(k)}(z) = \frac{M_R^{(k)}(z)}{1 - M_R^{(k)}(z)} \quad \text{and} \quad R_o^{(k)}(z) = \left(1 + R_e^{(k)}(z)\right) \frac{z}{1-z},$$

giving

$$R^{(k)}(z) = R_e^{(k)}(z) + R_o^{(k)}(z) = \frac{z - z^3 + 2z^{k+2}}{1 - z - z^2 + z^3 - 2z^{k+2}}.$$

The case $k \leq 0$ is trivial, we would simply obtain the g.f. of all—non-empty—compositions, without any restrictions. □

When $k = 1$, the counting sequence is related to the Tribonacci numbers (*OEIS* **A000073**). This sequence is defined as $t_n = t_{n-1} + t_{n-2} + t_{n-3}$ for $n > 2$ with the initial conditions $t_0 = t_1 = 0$ and $t_2 = 1$. The generating function for Tribonacci numbers is

$$T(z) = \frac{z^2}{1 - z - z^2 - z^3}.$$

On the other hand,

$$R^{(1)}(z) = \frac{z + z^3}{1 - z - z^2 - z^3} = (z^{-1} + z)T(z),$$

hence, the number of compositions of size n in $\mathcal{R}^{(1)}$ is $r_n^{(1)} = t_{n+1} + t_{n-1}$ for all $n \geq 1$.

This observation is crucial because it serves as a bridge between the research carried out in [10] and [2]. In his talk, Hopkins mentions the same, that these compositions—referred to by him as *pairwise Carlitz compositions*—are counted by Tribonacci numbers. Meanwhile, Andrews et al. count the number of anti-palindromic compositions of any length with the Tribonacci numbers. This follows from the bijection we provided in Theorem 2.15.

If, on the other hand, the condition were $|x_{2i-1} - x_{2i}| \leq k$, we would obtain the generating function

$$\frac{z + z^2 - 2z^{k+3}}{1 - z - 2z^2 + 2z^{k+3}}$$

for all $k \geq 0$ by proceeding similarly—see Tables B.21 and B.22.

3.3. On Multiplying the Restriction by a Constant

We have also been able to determine what happens when either side of the constraint is multiplied by a positive integer—see Tables B.23 and B.24.

Theorem 3.5. *For $a, b \in \mathbb{Z}^+$, the g.f. of the sequence that counts those compositions such $ax_{2i-1} > x_{2i}$ for $i > 0$, has the expression*

$$\frac{z - z^{a+1}}{1 - 2z + z^{a+2}},$$

and the g.f. for those such $x_{2i-1} > bx_{2i}$ is

$$\frac{z}{1 - z - z^{b+1}}.$$

Proof. Once more, we are interested solely in the g.f. of the blocks of pairs of summands. They are, respectively,

$$M_a(z) = \sum_{j\geq 1} z^j \left(\sum_{m=1}^{aj-1} z^m\right) = \frac{z\left(z - z^a + z^{a+1} - z^{a+2}\right)}{(1-z)^2\left(1-z^{a+1}\right)},$$

and

$$M_b(z) = \sum_{j\geq 1} \left(\sum_{m\geq bj+1} z^m\right) z^j = \frac{z^{b+2}}{(1-z)(1-z^{b+1})}. \qquad \square$$

The problem of how to involve simultaneous multiplication by constants and the addition of another integer in the inequality remains open. In other words, the counting problem when the restriction is $ax_{2i-1} \geq bx_{2i} + c$ for $a, b, c \in \mathbb{Z}^+$. This introduces some complexity since there are more cases to consider, depending on whether these constants are coprime or not, and the same for the summands.

3.4. *k*-Block Arndt Compositions

Finally, we will present what may be the most natural generalization of Arndt compositions. Perhaps the reader has wondered: What happens when, instead of comparing pairs of summands, we compare triples or quadruples? What happens when we consider k decreasing summands as blocks of Arndt compositions? To address this, we introduce the following definition.

Definition 3.6. Let $\mathcal{W}^{(k)}$ be the class of—non-empty—compositions that satisfy $x_{ki-k+1} > x_{ki-k+2} > \cdots > x_{ki}$ for $k \in \mathbb{Z}^+$ and $i > 0$. These are called k-block Arndt compositions. $\diamond$

Finding the generating function is straightforward if we consider that a composition of k decreasing summands is just a partition with k distinct parts, whose g.f. was found in equation (0.2). Thus, we can decompose a k-block Arndt composition as an arbitrary—possibly empty—sequence of partitions with k distinct parts, concatenated at the end with a partition with at most k distinct parts. This last partition would be analogous to the last summand of Arndt compositions, since the restriction of Definition 3.6 may be vacuously satisfied for the last summands. Thus, we have the following theorem—see Table B.25.

Theorem 3.7. *The g.f. of the number of k-block Arndt compositions is*

$$W^{(k)}(z) = \frac{1}{1 - P^{(k)}_{\text{diff}}(z)} \sum_{j=1}^{k} P^{(j)}_{\text{diff}}(z) = \frac{\sum_{j=1}^{k} z^{\binom{j+1}{2}}/(z;z)_j}{1 - z^{\binom{k+1}{2}}/(z;z)_k}.$$

We conclude the chapter on generalizations. It may be intriguing for another project to explore how these generalized objects can be studied using the same statistics from Chapter 2.

Conclusions

We have extensively studied Arndt compositions, discovering previously unknown results such as recurrence relations, new combinatorial interpretations of Fibonacci numbers, asymptotic estimates, and more. This was made possible thanks to the symbolic method, as several findings are not trivial to obtain through a purely bijective approach.

In [13], a good analogy is given for what one expects to accomplish when studying a sequence: the *tetrahedron* formed by symbolic sums, generating functions, recurrence relations, and asymptotic estimates. This was completed for several of the presented sequences.

The decomposition of this object into blocks of pairs of summands was essential, allowing us not only to provide an alternative proof to that presented by Hopkins and Tangboonduangjit in [11], but also to obtain information on some statistics as in Section 2.4, as well as generalizations in Chapter 3. All of this while keeping in mind the combinatorial properties of the object.

Despite the different approach to the problem, it largely captures the essence of the object's structure. The bijective interpretation of the presented recurrences and identities, especially those in Section 2.1, remains pending. In the document, we have focused merely on proving these relations, but there is still the task of studying them in depth from a computational perspective, such as the time and efficiency required to execute them.

Furthermore, future research could explore the statistics from Chapter 2 to the generalizations in Chapter 3. One that captivates the author's attention is the counting of interior points and semiperimeter in the k-block Arndt compositions. The determinants of Theorem 2.29 are likely to be generalized to those of square matrices of size k. Although this can be approached similarly, the use of Gröbner bases may lead to simpler answers—see [3]. Another problem that may arise from this object is the counting of *peaks* and *valleys* in the corresponding bar graphs, as well as the occurrence of descending or ascending patterns—see [4].

Finally, further inquiries into restrictions on pairs of summands can be pursued; this work may contribute to other related projects.

Appendix A

Wolfram Mathematica Algorithms

To execute these algorithms we recommend having version 12.1 of *Wolfram Mathematica*, or higher. In case you do not have a license, you can run them on the free version of *Wolfram Cloud*. Find the files on the repository `https://github.com/dfcheca/Arndt-Compositions`.

A.1. Mean and Variance

Filename: `mean-variance-gf.nb`

To calculate the mean and variance, you must first define a bivariate generating function. We take as an example the one that counts compositions of n and the number of summands, but you can replace it with any other function.

```
In[1]:= A[z_,u_]:= (1-z)/(1-z-u z)
```

In the next entry, the functions for the mean, second moment, and variance are defined.

```
In[2]:= Mn[A_,n_]:= SeriesCoefficient[∂_u A[z,u]/. u→1,{z,0,n}] / SeriesCoefficient[A[z,1],{z,0,n}];
        SecM[A_,n_]:= SeriesCoefficient[∂_{u,2} A[z,u]/. u→1,{z,0,n}] / SeriesCoefficient[A[z,1],{z,0,n}] +Mn[A,n]];
        Var[A_,n_]:=SecM[A,n]-Mn[A,n]^2
```

Now, we compute the mean and variance for the distribution given by `A[z,u]` in terms of `n`.

```
In[3]:= Mn[A,n] // FullSimplify
        Var[A,n] // FullSimplify

Out[3]= Piecewise[{{(1+n)/2,n>0},{0,n==0}},Indeterminate]

Out[4]= Piecewise[{{(1/4)*(-1+n),n>0},{0,n==0}},Indeterminate]
```

When running these codes for the generating functions in Chapter 2, you may not obtain expressions as fancy as those shown in the document. However, if you use the built-in function `DiscreteAsymptotic`, you will be able to obtain the asymptotic formulas we have seen. For example, let us find the assymptotic formula for the mean of $b_n^{(k)}$.

```
In[5]:= B[z_,u_]:= (u z-u z^3+u^2 z^3)/(1-z-z^2+z^3-u^2 z^3);
        Mn[B, n] // FullSimplify
```

```
Out[5]= Piecewise[{{1,n==1},{(10*(Sqrt[5]-1)*n
          +(15*(Sqrt[5]+1)*(1-Sqrt[5])^n*(2*n-5))/((Sqrt[5]+1)^n-(1-Sqrt[5])^n)
           -18*Sqrt[5]+60)/(5*(Sqrt[5]+5)),n>1},
            {0,Inequality[0,LessEqual,n,Less,1]}},Indeterminate]
```

```
In[6]:= DiscreteAsymptotic[(10*(Sqrt[5]-1)*n
          +(15*(Sqrt[5]+1)*(1-Sqrt[5])^n*(2*n-5))/((Sqrt[5]+1)^n-(1-Sqrt[5])^n)
           -18*Sqrt[5]+60)/(5*(Sqrt[5]+5)),n→Infinity]//FullSimplify
```

Out[6]= $\left(-1+\frac{3}{\sqrt{5}}\right)n$

A.2. Computing Arndt Compositions

Filename: `arndtcompositions.nb`

This code was provided by Professor José Luis.

Mathematica has the built-in function `IntegerPartitions` to generate integer partitions—see [30]—. We construct the function `IntegerCompositions` to generate integer compositions, which works in a similar manner.

```
In[1]:= IntegerCompositions[n_]:=
          Flatten[Permutations/@IntegerPartitions[n],1];
        IntegerCompositions[n_,{m_}]:=
          Flatten[Permutations/@IntegerPartitions[n,{m}],1]
```

Now, we define the criterion `ArndtCondition`, which compares each pair of summands and decides whether a composition is of Arndt or not; and the `ArndtCompositions` function, which generates all Arndt compositions of a given integer.

```
In[2]:= ArndtCondition[X_]:=
          AllTrue[Flatten[Differences/@Partition[X,2]],Negative];
        ArndtCompositions[n_]:=
          Select[IntegerCompositions[n],ArndtCondition];
        ArndtCompositions[n_,{k_}]:=
          Select[IntegerCompositions[n,{k}],ArndtCondition]
```

For example, these are the Arndt compositions of 5.

```
In[3]:= ArndtCompositions[5]
```

```
Out[3]= {{5},{4,1},{3,2},{3,1,1},{2,1,2}}
```

And this is the counting sequence.

```
In[4]:= Table[Length[ArndtCompositions[n]],{n,1,20}]
```

```
Out[4]= {1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597,2584,4181,6765}
```

A.3. Bar Graphs

Filename: `bargraphs.nb`

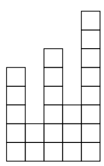
The function `CompositionBarGraph` displays a bar graph from a composition written as a list.

In[1]:=
```
CompositionBarGraph[X_,size_]:=Graphics[{EdgeForm[Thickness[Tiny]],
  Transparent,Flatten[Table[Rectangle[{i-1,j-1}],{i,1,Length[X]},
   {j,1,X[[i]]}]]},ImageSize→size*Length[X]];
CompositionBarGraph[X_]:=CompositionBarGraph[X,8]
```

If you call the function with two arguments, the first one will be the composition to represent, and the second one will be the desired size in printer points per each summand. If you call it with only one argument, the composition will be represented with a default size of 8.

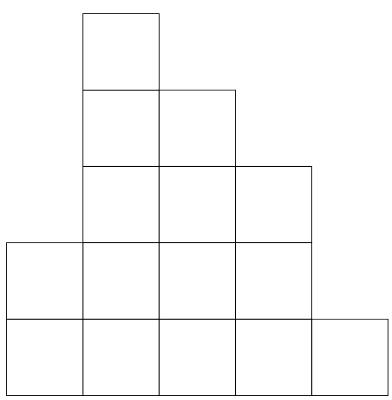
In[2]:=
```
CompositionBarGraph[{5,2,6,3,8}]
```

Out[2]=



In[3]:=
```
CompositionBarGraph[{2,5,4,3,1},20]
```

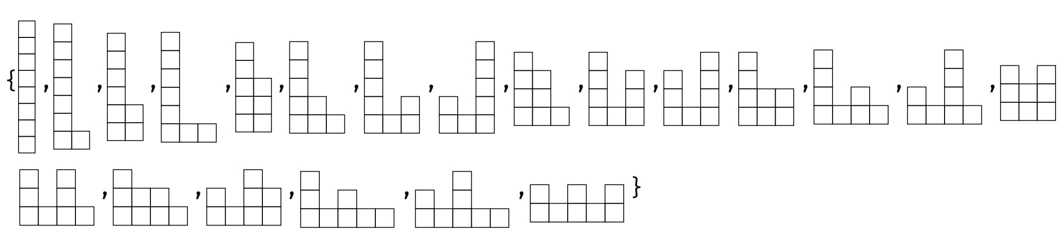
Out[3]=



If you additionally include all the definitions from the `arndtcompositions.nb` file, you can list all Arndt compositions as bar graphs.

In[4]:=
```
CompositionBarGraph/@ArndtCompositions[8]
```

Out[4]=



A.4. Zeilberger's and Petkovšek's Algorithms

Filename: `zeilberger-and-petkovsek.nb`

The Zeilberger algorithm can be implemented using the package `RISCErgoSum`, created by the Research Institute for Symbolic Computation (RISC), available at [21]. This code is password-protected and is not allowed for distribution, but simply sending an email to Professor Peter Paule (Peter.Paule@risc.jku.at) will prompt him to promptly and kindly provide the credentials for access.

Once you have downloaded the `riscergosum-1.2.2.zip` file, execute the following code in *Mathematica*:

```
In[1]:= $UserBaseDirectory
```

This command will output the directory where you should install the package. In my case, as a Windows user, it has been:

```
Out[1]= C:\Users\Daniel\AppData\Roaming\Mathematica
```

Navigate to this folder and extract the contents of the `.zip` file.

Now, to download the Petkovšek algorithm—also known as the Hyper algorithm—, go to [28]. There, you will see a list of algorithms; click on the **Hyper** link. It will redirect you to another page displaying the file's content. Press `Ctrl+S` (or its equivalent on other operating systems) and save the file as `Hyper.m` in the same folder above. It is important to save it with this extension for *Mathematica* to recognize it as a package.

If you do not have the desktop version of *Mathematica*, installing these packages may involve more steps, but it is addressed in question 91194 of the *Mathematica Stack Exchange* forum.

To load and verify that you have installed the first package correctly, execute:

```
In[2]:= <<RISC`fastZeil`
```

```
Fast Zeilberger Package version 3.61
written by Peter Paule, Markus Schorn, and Axel Riese
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria
```

To load the second package:

```
In[3]:= <<Hyper`;
        ?Hyper
```

Out[3]=
Symbol

Hyper[eqn, y[n]] finds at least one hypergeometric
solution of the homogeneous equation eqn over the field of rational numbers Q
(provided any such solution exists). Hyper[eqn, y[n], Solutions -> All] finds
a generating set (not necessarily linearly independent) for the space of
solutions generated by hypergeometric terms over Q. Hyper[eqn, y[n],
Quadratics -> True] finds solutions over quadratic extensions of Q. Solutions
y[n] are given by their rational representations y[n+1]/y[n].
Warning: The worst-case time complexity of Hyper is exponential
in the degrees of the leading and trailing coefficients of eqn.

Let us prove Corollary 2.13. We define the term $F(n,k|i)$:

```
In[4]:= F[n_,i_]:=
          Binomial[n-i-1,k-1] Binomial[i-1,Floor[k/2]-1](-1)^(i+Floor[k/2])
```

Execute Zeilberger's algorithm to find the recurrence formula.

```
In[5]:= Zb[F[n,i],i,n]
```

Out[5]= $\{-n\ F[i,n]+(-k+\text{Floor}[\frac{k}{2}])F[i,1+n]+(2-k+n-\text{Floor}[\frac{k}{2}])F[i,2+n]==\Delta_i[F[i,n]R[i,n]]\}$

It is also possible to show the expression $R(n,k|i) = G(n,k|i)/F(n,k|i)$. This how we defined $G(n,k|i)$ seemingly out of thin air.

```
In[6]:= show[R]
```

Out[6]= $\frac{(-1+k)(-i+n)(-i+\text{Floor}[\frac{k}{2}])}{(1-i-k+n)(2-i-k+n)}$

Finally, we execute `Hyper`—Petkovšek—algorithm.

```
In[7]:= Hyper[n f[n]+(k-Floor[k/2])f[n+1]+(k+Floor[k/2]-n-2)f[n+2]==0,f[n]]
```

Out[7]= {}

This means there is no hypergeometric solution for $b_n^{(k)}$. If we have run these algorithms with a sum that does have a hypergeometric solution, `Hyper` algorithm should output the solution in the form `f[n+1]/f[n]`. For example, it is well-known that $\sum_{i=0}^{n}\binom{n}{i} = 2^n$, and this can be verified with these algorithms.

```
In[8]:= Zb[Binomial[n,i],i,n]
```

Out[8]= $\{2F[i,n]-F[i,1+n]==\Delta_i[F[i,n]R[i,n]]\}$

```
In[9]:= Hyper[2f[n]-f[n+1]==0,f[n]]
```

Out[9]= {2}

A.5. Iteration of the Equations from Section 2.5

Filename: `interior-iteration.nb`

We define the recurrences obtained from equations (2.7) and (2.8). We have added the term `C` to keep track of the constant term.

```
In[1]:= Ae[-1]=Ae;
        Ao[-1]=Ao;
        Ae[i_Integer?NonNegative][z_,p_,q_,v_]:=(p v z)/(1-q v z) Ao[z,p,q,1]-
          p/(q(1-q v z)) Ao[i-1][z,p,q,q v z]
        Ao[i_Integer?NonNegative][z_,p_,q_,v_]:=(p^2 v z)/(1-p v z) C+(p v z)/(1-q v z) Ae[z,p,q,1]+
          p/q (1/(1-p v z)-1/(1-q v z)) Ae[i-1][z,p,q,q v z];
```

If you enter `1` in the first argument, it will iterate the equations for the first time, and so on.

```
In[2]:= Ae[1][z,p,q,v]//Expand
        Ao[1][z,p,q,v]//Expand
```

$$\text{Out[2]=} -\frac{C\ p^3\ v\ z^2}{(1-q\ v\ z)\ (1-p\ q\ v\ z^2)}-\frac{p^2\ v\ z^2\ Ae[z,p,q,1]}{(1-q\ v\ z)\ (1-q^2\ v\ z^2)}-\frac{p^2\ Ae[z,p,q,q^2\ v\ z^2]}{q^2\ (1-q\ v\ z)\ (1-p\ q\ v\ z^2)}+\frac{p^2\ Ae[z,p,q,q^2\ v\ z^2]}{q^2\ (1-q\ v\ z)\ (1-q^2\ v\ z^2)}+\frac{p\ v\ z\ Ao[z,p,q,1]}{1-q\ v\ z}$$

$$\text{Out[3]=} \frac{C\ p^2\ v\ z}{1-p\ v\ z}+\frac{p\ v\ z\ Ae[z,p,q,1]}{1-q\ v\ z}+\frac{p^2\ v\ z^2\ Ao[z,p,q,1]}{(1-p\ v\ z)\ (1-q^2\ v\ z^2)}-\frac{p^2\ v\ z^2\ Ao[z,p,q,1]}{(1-q\ v\ z)\ (1-q^2\ v\ z^2)}-\frac{p^2\ Ao[z,p,q,q^2\ v\ z^2]}{q^2\ (1-p\ v\ z)\ (1-q^2\ v\ z^2)}+\frac{p^2\ Ao[z,p,q,q^2\ v\ z^2]}{q^2\ (1-q\ v\ z)\ (1-q^2\ v\ z^2)}$$

For each function, we need to find: the constant term, the coefficient of $A_o(z,p,q|1)$, and the coefficient of $A_e(z,p,q|1)$. So, in total, we need to find 6 coefficients, as the other terms cancel out as we iterate through the functions.

We will demonstrate how to find the constant term of $A_o(z,p,q|v)$, and we hope that guides the reader to determine the rest. First, note that every two iterations, the constant term increases, and the powers of z are always odd.

```
In[4]:= Table[Coefficient[Expand[Ao[k][z,p,q,v]],C],{k,0,5}]
```

$$\text{Out[4]=} \Big\{\frac{p^2\ v\ z}{1-p\ v\ z},\frac{p^2\ v\ z}{1-p\ v\ z},\frac{p^2\ v\ z}{1-p\ v\ z}-\frac{p^4\ v\ z^3}{(1-p\ v\ z)\ (1-q^2\ v\ z^2)\ (1-p\ q^2\ v\ z^3)}+\frac{p^4\ v\ z^3}{(1-q\ v\ z)\ (1-q^2\ v\ z^2)\ (1-p\ q^2\ v\ z^3)},\frac{p^2\ v\ z}{1-p\ v\ z}-\frac{p^4\ v\ z^3}{(1-p\ v\ z)\ (1-q^2\ v\ z^2)\ (1-p\ q^2\ v\ z^3)}+\frac{p^4\ v\ z^3}{(1-q\ v\ z)\ (1-q^2\ v\ z^2)\ (1-p\ q^2\ v\ z^3)},\frac{p^2\ v\ z}{1-p\ v\ z}-\frac{p^4\ v\ z^3}{(1-p\ v\ z)\ (1-q^2\ v\ z^2)\ (1-p\ q^2\ v\ z^3)}+\frac{p^4\ v\ z^3}{(1-q\ v\ z)\ (1-q^2\ v\ z^2)\ (1-p\ q^2\ v\ z^3)}+\frac{p^6\ v\ z^5}{(1-p\ v\ z)\ (1-q^2\ v\ z^2)\ (1-p\ q^2\ v\ z^3)\ (1-q^4\ v\ z^4)\ (1-p\ q^4\ v\ z^5)}-\frac{p^6\ v\ z^5}{(1-q\ v\ z)\ (1-q^2\ v\ z^2)\ (1-p\ q^2\ v\ z^3)\ (1-q^4\ v\ z^4)\ (1-p\ q^4\ v\ z^5)}-$$

$$\frac{p^6\ v\ z^5}{(1-p\ v\ z)\ (1-q^2\ v\ z^2)\ (1-q^3\ v\ z^3)\ (1-q^4\ v\ z^4)\ (1-p\ q^4\ v\ z^5)}+$$
$$\frac{p^6\ v\ z^5}{(1-q\ v\ z)\ (1-q^2\ v\ z^2)\ (1-q^3\ v\ z^3)\ (1-q^4\ v\ z^4)\ (1-p\ q^4\ v\ z^5)},$$
$$\frac{p^2\ v\ z}{1-p\ v\ z}-\frac{p^4\ v\ z^3}{(1-p\ v\ z)\ (1-q^2\ v\ z^2)\ (1-p\ q^2\ v\ z^3)}+\frac{p^4\ v\ z^3}{(1-q\ v\ z)\ (1-q^2\ v\ z^2)\ (1-p\ q^2\ v\ z^3)}+$$
$$\frac{p^6\ v\ z^5}{(1-p\ v\ z)\ (1-q^2\ v\ z^2)\ (1-p\ q^2\ v\ z^3)\ (1-q^4\ v\ z^4)\ (1-p\ q^4\ v\ z^5)}-$$
$$\frac{p^6\ v\ z^5}{(1-q\ v\ z)\ (1-q^2\ v\ z^2)\ (1-p\ q^2\ v\ z^3)\ (1-q^4\ v\ z^4)\ (1-p\ q^4\ v\ z^5)}-$$
$$\frac{p^6\ v\ z^5}{(1-p\ v\ z)\ (1-q^2\ v\ z^2)\ (1-q^3\ v\ z^3)\ (1-q^4\ v\ z^4)\ (1-p\ q^4\ v\ z^5)}+$$
$$\frac{p^6\ v\ z^5}{(1-q\ v\ z)\ (1-q^2\ v\ z^2)\ (1-q^3\ v\ z^3)\ (1-q^4\ v\ z^4)\ (1-p\ q^4\ v\ z^5)}\}$$

If we factorize any of these, it does not result in anything pleasant. But if we instead group the terms for each power of z, that is where the terms begin to follow a pattern, and we can express the constant coefficient in $A_o(z,p,q|v)$ as a sum over the powers of z.

In[5]:=
```
Table[Coefficient[Ao[10][z,p,q,v]//Expand,C z^(2k+1)]//Factor,{k,0,3}]
```

Out[5]=
$$\Big\{-\frac{p^2\ v}{-1+p\ v\ z},\ -\frac{p^4\ (p-q)\ v^2\ z}{(-1+p\ v\ z)\ (-1+q\ v\ z)\ (-1+q^2\ v\ z^2)\ (-1+p\ q^2\ v\ z^3)},$$
$$-\frac{p^6\ (p-q)^2\ q^2\ v^3\ z^4}{(-1+p\ v\ z)\ (-1+q\ v\ z)\ (-1+q^2\ v\ z^2)\ (-1+p\ q^2\ v\ z^3)\ (-1+q^3\ v\ z^3)\ (-1+q^4\ v\ z^4)\ (-1+p\ q^4\ v\ z^5)},$$
$$-\frac{p^8\ (p-q)^3\ q^6\ v^4\ z^9}{(-1+p\ v\ z)\ (-1+q\ v\ z)\ (-1+q^2\ v\ z^2)\ (-1+p\ q^2\ v\ z^3)\ (-1+q^3\ v\ z^3)\ (-1+q^4\ v\ z^4)\ (-1+p\ q^4\ v\ z^5)\ (-1+q^5\ v\ z^5)\ (-1+q^6\ v\ z^6)\ (-1+p\ q^6\ v\ z^7)}\Big\}$$

From here we can deduct the powers of each coefficient and start deducing equations (2.11) and (2.12) in a similar manner.

A.6. Generating Functions for Interior Points and Semiperimeter

Filename: `interior-gf.nb`

These are the generating functions of Theorem 2.29. The parameter `m` is the top of each sum.

In[1]:=
```
α1[m_][z_,p_,q_]:=
  Sum[p^(2 n+1) (q-p)^(n-1) q^((n-1)^2) z^(n (n+1)) /
    (QPochhammer[q^2 z^2,q^2 z^2,n-1] Times@@QPochhammer[{q z,p q z^2},q^2 z^2,n]), {n,1,m}];
β1[m_][z_,p_,q_]:=
  Sum[p^(2 n-1) (q-p)^(n-1) q^((n-1)^2) z^(n (n+1)-1) /
    (Times@@QPochhammer[{q^2 z^2,p q z^2},q^2 z^2,n-1] QPochhammer[q z,q^2 z^2,n]), {n,1,m}];
γ1[m_][z_,p_,q_]:=
  Sum[p^(2 n) (q-p)^(n-1) q^((n-1)^2) z^(n (n+1)) /
    (QPochhammer[p q z^2,q^2 z^2,n-1] QPochhammer[q z,q z,2 n]), {n,1,m}];
α2[m_][z_,p_,q_]:=
  Sum[p^(2 n) (q-p)^(n-1) q^((n-1) (n-2)) z^(n^2) /
    (QPochhammer[p z,q^2 z^2,n] QPochhammer[q z,q z,2 n-2]), {n,1,m}];
β2[m_][z_,p_,q_]:=
```

```
  m        p^(2 n) (q-p)^n q^(n (n-1)) z^(n (n+2))
  ∑  ─────────────────────────────────────────────────────── ;
 n=1 QPochhammer[p z,q^2 z^2,n] QPochhammer[q z,q z,2 n]
γ2[m_][z_,p_,q_]:=
  m              p^(2 n-1) (q-p)^(n-1) q^((n-1) (n-2)) z^(n^2)
  ∑  ─────────────────────────────────────────────────────────────────────── ;
 n=1 QPochhammer[q z,q^2 z^2,n] Times@@QPochhammer[{p z,q^2 z^2},q^2 z^2,n-1]
A[m_][z_,p_,q_]:=
  Det[{{α1,1+β1+γ1},{α2,1+β2+γ2}}]
  ──────────────────────────────── /.(#→#[m][z,p,q]&)/@{α1,β1,γ1,α2,β2,γ2}
    Det[{{β1,1+γ1},{1+β2,γ2}}]
```

From here, we can find the initial terms of $A(z, p, q)$.

In[2]:=
```
Series[A[10][z,p,q],{z,0,10},{p,0,11},{q,0,10}]//Normal
```

Out[2]=

$$p^2 z+p^3 z^2+2 p^4 z^3+3 p^5 z^4+(4 p^6+p^5 q) z^5+(6 p^7+2 p^6 q) z^6+(9 p^8+2 p^7 q+2 p^6 q^2) z^7+(13 p^9+3 p^8 q+5 p^7 q^2) z^8+(19 p^{10}+5 p^9 q+8 p^8 q^2+2 p^7 q^3) z^9+(28 p^{11}+7 p^{10} q+14 p^9 q^2+5 p^8 q^3+p^7 q^4) z^{10}$$

Likewise, the derivatives to find the first moments.

In[3]:=
```
∂_q FunctionExpand[A[3][z,1,q]]/. q→1//FullSimplify
∂_{q,q} FunctionExpand[A[3][z,1,q]]/. q→1//Apart
```

Out[3]=

$$\frac{z^5 (1+z^2-z^4-z^6+z^7)}{(1+z) (1+z^2) (1+z+z^2) (1-2 z+z^3)^2}$$

Out[4]=

$$58-\frac{1}{240 (-1+z)^5}-\frac{1}{288 (-1+z)^4}-\frac{61}{360 (-1+z)^3}+\frac{19}{1728 (-1+z)^2}+\frac{14183}{57600 (-1+z)}-18 z+4 z^2-\frac{1}{192 (1+z)^3}+\frac{23}{128 (1+z)^2}+\frac{49}{2304 (1+z)}+\frac{221+28 z}{1000 (1+z^2)^2}+\frac{-1879-119 z}{2000 (1+z^2)}+\frac{-2+z}{72 (1-z+z^2)}+\frac{1127-1825 z}{10 (-1+z+z^2)^3}+\frac{288803-322647 z}{1100 (-1+z+z^2)^2}+\frac{25128754-18156363 z}{121000 (-1+z+z^2)}+\frac{-8-3 z}{36 (1+z+z^2)^2}-\frac{5 (-13+3 z)}{108 (1+z+z^2)}-\frac{2 (-96-7 z-58 z^2+46 z^3)}{3025 (1+z+z^2+z^3+z^4)}$$

Appendix B

Matrices

B.1. Number of Summands

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0	0	0	0	0
4	1	1	1	0	0	0	0	0	0	0	0	0	0
5	1	2	2	0	0	0	0	0	0	0	0	0	0
6	1	2	4	1	0	0	0	0	0	0	0	0	0
7	1	3	6	2	1	0	0	0	0	0	0	0	0
8	1	3	9	5	3	0	0	0	0	0	0	0	0
9	1	4	12	8	8	1	0	0	0	0	0	0	0
10	1	4	16	14	16	3	1	0	0	0	0	0	0
11	1	5	20	20	30	9	4	0	0	0	0	0	0
12	1	5	25	30	50	19	13	1	0	0	0	0	0
13	1	6	30	40	80	39	32	4	1	0	0	0	0
14	1	6	36	55	120	69	71	14	5	0	0	0	0
15	1	7	42	70	175	119	140	36	19	1	0	0	0
16	1	7	49	91	245	189	259	85	55	5	1	0	0
17	1	8	56	112	336	294	448	176	140	20	6	0	0
18	1	8	64	140	448	434	742	344	316	60	26	1	0
19	1	9	72	168	588	630	1176	624	660	160	86	6	1
20	1	9	81	204	756	882	1806	1086	1284	376	246	27	7

TABLE B.1. $b_n^{(k)}$, Arndt compositions of n with k parts.

```
In[1]:= b[n_,k_]:=Sum[Binomial[n-i-1,k-1]Binomial[-Floor[k/2],-i],{i,Floor[k/2],n-k}];
        Table[b[n,k],{n,1,20},{k,1,13}]//MatrixForm
```

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1	1
3	1	2	2	2	2	2	2	2	2	2	2	2	2
4	1	2	3	3	3	3	3	3	3	3	3	3	3
5	1	3	5	5	5	5	5	5	5	5	5	5	5
6	1	3	7	8	8	8	8	8	8	8	8	8	8
7	1	4	10	12	13	13	13	13	13	13	13	13	13
8	1	4	13	18	21	21	21	21	21	21	21	21	21
9	1	5	17	25	33	34	34	34	34	34	34	34	34
10	1	5	21	35	51	54	55	55	55	55	55	55	55
11	1	6	26	46	76	85	89	89	89	89	89	89	89
12	1	6	31	61	111	130	143	144	144	144	144	144	144
13	1	7	37	77	157	196	228	232	233	233	233	233	233
14	1	7	43	98	218	287	358	372	377	377	377	377	377
15	1	8	50	120	295	414	554	590	609	610	610	610	610
16	1	8	57	148	393	582	841	926	981	986	987	987	987
17	1	9	65	177	513	807	1255	1431	1571	1591	1597	1597	1597
18	1	9	73	213	661	1095	1837	2181	2497	2557	2583	2584	2584
19	1	10	82	250	838	1468	2644	3268	3928	4088	4174	4180	4181
20	1	10	91	295	1051	1933	3739	4825	6109	6485	6731	6758	6765

TABLE B.2. $b_n^{(\leq k)}$, Arndt compositions of n with at most k parts.

```
In[1]:= b[n_,k_]:=Sum[Binomial[n-i-1,k-1]Binomial[-Floor[k/2],-i],{i,Floor[k/2],n-k}];
        Table[Sum[b[n,i],{i,1,k}],{n,1,20},{k,1,13}]//MatrixForm
```

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0	0
3	2	1	0	0	0	0	0	0	0	0	0	0	0
4	3	2	1	0	0	0	0	0	0	0	0	0	0
5	5	4	2	0	0	0	0	0	0	0	0	0	0
6	8	7	5	1	0	0	0	0	0	0	0	0	0
7	13	12	9	3	1	0	0	0	0	0	0	0	0
8	21	20	17	8	3	0	0	0	0	0	0	0	0
9	34	33	29	17	9	1	0	0	0	0	0	0	0
10	55	54	50	34	20	4	1	0	0	0	0	0	0
11	89	88	83	63	43	13	4	0	0	0	0	0	0
12	144	143	138	113	83	33	14	1	0	0	0	0	0
13	233	232	226	196	156	76	37	5	1	0	0	0	0
14	377	376	370	334	279	159	90	19	5	0	0	0	0
15	610	609	602	560	490	315	196	56	20	1	0	0	0
16	987	986	979	930	839	594	405	146	61	6	1	0	0
17	1597	1596	1588	1532	1420	1084	790	342	166	26	6	0	0
18	2584	2583	2575	2511	2371	1923	1489	747	403	87	27	1	0
19	4181	4180	4171	4099	3931	3343	2713	1537	913	253	93	7	1
20	6765	6764	6755	6674	6470	5714	4832	3026	1940	656	280	34	7

TABLE B.3. $b_n^{(\geq k)}$, Arndt compositions of n with at least k parts.

In[1]:=
```
b[n_,k_]:=Sum[Binomial[n-i-1,k-1]Binomial[-Floor[k/2],-i],{i,Floor[k/2],n-k}];
Table[Sum[b[n,i],{i,k,n}],{n,1,20},{k,1,13}]//MatrixForm
```

B.2. Size of the Last and the First Summands

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	2	2	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	4	2	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	6	3	2	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
8	10	5	3	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
9	16	8	4	3	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0
10	26	13	7	4	2	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0
11	42	21	11	6	4	2	1	1	0	0	1	0	0	0	0	0	0	0	0	0
12	68	34	18	10	6	3	2	1	1	0	0	1	0	0	0	0	0	0	0	0
13	110	55	29	16	9	6	3	2	1	1	0	0	1	0	0	0	0	0	0	0
14	178	89	47	26	15	9	5	3	2	1	1	0	0	1	0	0	0	0	0	0
15	288	144	76	42	24	14	9	5	3	2	1	1	0	0	1	0	0	0	0	0
16	466	233	123	68	39	23	14	8	5	3	2	1	1	0	0	1	0	0	0	0
17	754	377	199	110	63	37	22	14	8	5	3	2	1	1	0	0	1	0	0	0
18	1220	610	322	178	102	60	36	22	13	8	5	3	2	1	1	0	0	1	0	0
19	1974	987	521	288	165	97	58	35	22	13	8	5	3	2	1	1	0	0	1	0
20	3194	1597	843	466	267	157	94	57	35	21	13	8	5	3	2	1	1	0	0	1

TABLE B.4. $c_n^{(k)}$, Arndt compositions of n with last part k.

```
In[1]:= F[z_]:=z/(1-z-z^2);
        c[n_,k_]:=SeriesCoefficient[z^k+z^(2k+1)+(z^(k+2)+z^(2k+1))F[z],{z,0,n}];
        Table[c[n,k],{n,1,20},{k,1,20}]//MatrixForm
```

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
4	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
5	2	4	4	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	4	6	7	7	7	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
7	6	9	11	12	12	12	13	13	13	13	13	13	13	13	13	13	13	13	13	13
8	10	15	18	19	20	20	20	21	21	21	21	21	21	21	21	21	21	21	21	21
9	16	24	28	31	32	33	33	33	34	34	34	34	34	34	34	34	34	34	34	34
10	26	39	46	50	52	53	54	54	54	55	55	55	55	55	55	55	55	55	55	55
11	42	63	74	80	84	86	87	88	88	88	89	89	89	89	89	89	89	89	89	89
12	68	102	120	130	136	139	141	142	143	143	143	144	144	144	144	144	144	144	144	144
13	110	165	194	210	219	225	228	230	231	232	232	232	233	233	233	233	233	233	233	233
14	178	267	314	340	355	364	369	372	374	375	376	376	376	377	377	377	377	377	377	377
15	288	432	508	550	574	588	597	602	605	607	608	609	609	609	610	610	610	610	610	610
16	466	699	822	890	929	952	966	974	979	982	984	985	986	986	986	987	987	987	987	987
17	754	1131	1330	1440	1503	1540	1562	1576	1584	1589	1592	1594	1595	1596	1596	1596	1597	1597	1597	1597
18	1220	1830	2152	2330	2432	2492	2528	2550	2563	2571	2576	2579	2581	2582	2583	2583	2583	2584	2584	2584
19	1974	2961	3482	3770	3935	4032	4090	4125	4147	4160	4168	4173	4176	4178	4179	4180	4180	4180	4181	4181
20	3194	4791	5634	6100	6367	6524	6618	6675	6710	6731	6744	6752	6757	6760	6762	6763	6764	6764	6764	6765

TABLE B.5. $c_n^{(\leq k)}$, Arndt compositions of n with last part of size at most k.

```
In[1]:= F[z_]:= z/(1-z-z^2);
        c[n_,k_]:=SeriesCoefficient[z^k+z^(2k+1)+(z^(k+2)+z^(2k+1))F[z],{z,0,n}];
        Table[Sum[c[n,i],{i,1,k}],{n,1,20},{k,1,20}]//MatrixForm
```

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	5	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	8	4	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	13	7	4	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
8	21	11	6	3	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
9	34	18	10	6	3	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0
10	55	29	16	9	5	3	2	1	1	1	0	0	0	0	0	0	0	0	0	0
11	89	47	26	15	9	5	3	2	1	1	1	0	0	0	0	0	0	0	0	0
12	144	76	42	24	14	8	5	3	2	1	1	1	0	0	0	0	0	0	0	0
13	233	123	68	39	23	14	8	5	3	2	1	1	1	0	0	0	0	0	0	0
14	377	199	110	63	37	22	13	8	5	3	2	1	1	1	0	0	0	0	0	0
15	610	322	178	102	60	36	22	13	8	5	3	2	1	1	1	0	0	0	0	0
16	987	521	288	165	97	58	35	21	13	8	5	3	2	1	1	1	0	0	0	0
17	1597	843	466	267	157	94	57	35	21	13	8	5	3	2	1	1	1	0	0	0
18	2584	1364	754	432	254	152	92	56	34	21	13	8	5	3	2	1	1	1	0	0
19	4181	2207	1220	699	411	246	149	91	56	34	21	13	8	5	3	2	1	1	1	0
20	6765	3571	1974	1131	665	398	241	147	90	55	34	21	13	8	5	3	2	1	1	1

TABLE B.6. $c_n^{(\geq k)}$, Arndt compositions of n with last part of size at least k.

```
In[1]:= F[z_]:=z/(1-z-z^2);
c[n_,k_]:=SeriesCoefficient[z^k+z^(2k+1)+(z^(k+2)+z^(2k+1))F[z],{z,0,n}];
Table[Sum[c[n,i],{i,k,n}],{n,1,20},{k,1,20}]//MatrixForm
```

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	1	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	2	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	3	3	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	5	5	4	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
9	0	8	8	6	5	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0
10	0	13	13	10	7	5	3	2	1	1	0	0	0	0	0	0	0	0	0	0
11	0	21	21	16	11	8	5	3	2	1	1	0	0	0	0	0	0	0	0	0
12	0	34	34	26	18	12	8	5	3	2	1	1	0	0	0	0	0	0	0	0
13	0	55	55	42	29	19	13	8	5	3	2	1	1	0	0	0	0	0	0	0
14	0	89	89	68	47	31	20	13	8	5	3	2	1	1	0	0	0	0	0	0
15	0	144	144	110	76	50	32	21	13	8	5	3	2	1	1	0	0	0	0	0
16	0	233	233	178	123	81	52	33	21	13	8	5	3	2	1	1	0	0	0	0
17	0	377	377	288	199	131	84	53	34	21	13	8	5	3	2	1	1	0	0	0
18	0	610	610	466	322	212	136	86	54	34	21	13	8	5	3	2	1	1	0	0
19	0	987	987	754	521	343	220	139	87	55	34	21	13	8	5	3	2	1	1	0
20	0	1597	1597	1220	843	555	356	225	141	88	55	34	21	13	8	5	3	2	1	1

TABLE B.7. $d_n^{(k)}$, Arndt compositions of n with first part k.

```
In[1]:= F[z_]:= z/(1-z-z^2);
d[n_,k_]:=SeriesCoefficient[(z^(k-1)-z^(2k-1)-z^(2k))F[z],{z,0,n}];
Table[d[n,k],{n,1,20},{k,1,20}]//MatrixForm
```

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	0	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
4	0	1	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
5	0	1	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	0	2	4	6	7	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
7	0	3	6	9	11	12	13	13	13	13	13	13	13	13	13	13	13	13	13	13
8	0	5	10	14	17	19	20	21	21	21	21	21	21	21	21	21	21	21	21	21
9	0	8	16	22	27	30	32	33	34	34	34	34	34	34	34	34	34	34	34	34
10	0	13	26	36	43	48	51	53	54	55	55	55	55	55	55	55	55	55	55	55
11	0	21	42	58	69	77	82	85	87	88	89	89	89	89	89	89	89	89	89	89
12	0	34	68	94	112	124	132	137	140	142	143	144	144	144	144	144	144	144	144	144
13	0	55	110	152	181	200	213	221	226	229	231	232	233	233	233	233	233	233	233	233
14	0	89	178	246	293	324	344	357	365	370	373	375	376	377	377	377	377	377	377	377
15	0	144	288	398	474	524	556	577	590	598	603	606	608	609	610	610	610	610	610	610
16	0	233	466	644	767	848	900	933	954	967	975	980	983	985	986	987	987	987	987	987
17	0	377	754	1042	1241	1372	1456	1509	1543	1564	1577	1585	1590	1593	1595	1596	1597	1597	1597	1597
18	0	610	1220	1686	2008	2220	2356	2442	2496	2530	2551	2564	2572	2577	2580	2582	2583	2584	2584	2584
19	0	987	1974	2728	3249	3592	3812	3951	4038	4093	4127	4148	4161	4169	4174	4177	4179	4180	4181	4181
20	0	1597	3194	4414	5257	5812	6168	6393	6534	6622	6677	6711	6732	6745	6753	6758	6761	6763	6764	6765

TABLE B.8. $d_n^{(\leq k)}$, Arndt compositions of n with first part of size at most k.

In[1]:=
```
F[z_]:=z/(1-z-z^2);
d[n_,k_]:=SeriesCoefficient[(z^(k-1)-z^(2k-1)-z^(2k))F[z],{z,0,n}];
Table[Sum[d[n,i],{i,1,k}],{n,1,20},{k,1,20}]//MatrixForm
```

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	3	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	5	5	4	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	8	8	6	4	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	13	13	10	7	4	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
8	21	21	16	11	7	4	2	1	0	0	0	0	0	0	0	0	0	0	0	0
9	34	34	26	18	12	7	4	2	1	0	0	0	0	0	0	0	0	0	0	0
10	55	55	42	29	19	12	7	4	2	1	0	0	0	0	0	0	0	0	0	0
11	89	89	68	47	31	20	12	7	4	2	1	0	0	0	0	0	0	0	0	0
12	144	144	110	76	50	32	20	12	7	4	2	1	0	0	0	0	0	0	0	0
13	233	233	178	123	81	52	33	20	12	7	4	2	1	0	0	0	0	0	0	0
14	377	377	288	199	131	84	53	33	20	12	7	4	2	1	0	0	0	0	0	0
15	610	610	466	322	212	136	86	54	33	20	12	7	4	2	1	0	0	0	0	0
16	987	987	754	521	343	220	139	87	54	33	20	12	7	4	2	1	0	0	0	0
17	1597	1597	1220	843	555	356	225	141	88	54	33	20	12	7	4	2	1	0	0	0
18	2584	2584	1974	1364	898	576	364	228	142	88	54	33	20	12	7	4	2	1	0	0
19	4181	4181	3194	2207	1453	932	589	369	230	143	88	54	33	20	12	7	4	2	1	0
20	6765	6765	5168	3571	2351	1508	953	597	372	231	143	88	54	33	20	12	7	4	2	1

Table B.9. $d_n^{(\geq k)}$, Arndt compositions of n with first part of size at least k.

In[1]:=

```
F[z_]:=z/(1-z-z^2);
d[n_,k_]:=SeriesCoefficient[(z^(k-1)-z^(2k-1)-z^(2k))F[z],{z,0,n}];
Table[Sum[d[n,i],{i,k,n}],{n,1,20},{k,1,20}]//MatrixForm
```

B.3. Size of the Largest and the Smallest Summands

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	1	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	1	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	1	4	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	1	6	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
9	0	1	8	10	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0
10	0	1	11	16	12	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0
11	0	1	15	25	20	13	7	4	2	1	1	0	0	0	0	0	0	0	0	0
12	0	1	20	39	34	22	13	7	4	2	1	1	0	0	0	0	0	0	0	0
13	0	1	27	60	56	38	23	13	7	4	2	1	1	0	0	0	0	0	0	0
14	0	1	36	92	92	65	40	23	13	7	4	2	1	1	0	0	0	0	0	0
15	0	1	48	140	150	110	69	41	23	13	7	4	2	1	1	0	0	0	0	0
16	0	1	64	212	243	185	119	71	41	23	13	7	4	2	1	1	0	0	0	0
17	0	1	85	320	392	309	203	123	72	41	23	13	7	4	2	1	1	0	0	0
18	0	1	113	481	629	514	345	212	125	72	41	23	13	7	4	2	1	1	0	0
19	0	1	150	721	1006	851	583	363	216	126	72	41	23	13	7	4	2	1	1	0
20	0	1	199	1078	1603	1404	981	619	372	218	126	72	41	23	13	7	4	2	1	1

TABLE B.10. $[z^n]L^{(k)}(z)$, Arndt compositions of n whose largest summand is k.

```
In[1]:= G[z_,k_]:=(z-z^(k+1))/(1-z-z^2+z^(k+1));
        g[n_,k_]:=SeriesCoefficient[G[z,k],{z,0,n}];
        L[z_,k_]:=G[z,k]-G[z,k-1];
        l[n_,k_]:=SeriesCoefficient[L[z,k],{z,0,n}];
        Table[l[n,k],{n,1,20},{k,1,20}]//MatrixForm
```

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	0	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
4	0	1	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
5	0	1	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	0	1	4	6	7	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
7	0	1	5	9	11	12	13	13	13	13	13	13	13	13	13	13	13	13	13	13
8	0	1	7	13	17	19	20	21	21	21	21	21	21	21	21	21	21	21	21	21
9	0	1	9	19	26	30	32	33	34	34	34	34	34	34	34	34	34	34	34	34
10	0	1	12	28	40	47	51	53	54	55	55	55	55	55	55	55	55	55	55	55
11	0	1	16	41	61	74	81	85	87	88	89	89	89	89	89	89	89	89	89	89
12	0	1	21	60	94	116	129	136	140	142	143	144	144	144	144	144	144	144	144	144
13	0	1	28	88	144	182	205	218	225	229	231	232	233	233	233	233	233	233	233	233
14	0	1	37	129	221	286	326	349	362	369	373	375	376	377	377	377	377	377	377	377
15	0	1	49	189	339	449	518	559	582	595	602	606	608	609	610	610	610	610	610	610
16	0	1	65	277	520	705	824	895	936	959	972	979	983	985	986	987	987	987	987	987
17	0	1	86	406	798	1107	1310	1433	1505	1546	1569	1582	1589	1593	1595	1596	1597	1597	1597	1597
18	0	1	114	595	1224	1738	2083	2295	2420	2492	2533	2556	2569	2576	2580	2582	2583	2584	2584	2584
19	0	1	151	872	1878	2729	3312	3675	3891	4017	4089	4130	4153	4166	4173	4177	4179	4180	4181	4181
20	0	1	200	1278	2881	4285	5266	5885	6257	6475	6601	6673	6714	6737	6750	6757	6761	6763	6764	6765

TABLE B.11. $[z^n]G^{(k)}(z)$, Arndt compositions of n whose parts are in $\{1, \ldots, k\}$.

```
In[1]:= G[z_,k_]:=(z-z^(k+1))/(1-z-z^2+z^(k+1));
        g[n_,k_]:=SeriesCoefficient[G[z,k],{z,0,n}];
        Table[g[n,k],{n,1,20},{k,1,20}]//MatrixForm
```

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	3	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	6	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	9	2	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
8	16	3	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
9	26	5	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
10	44	7	2	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
11	73	10	3	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
12	121	15	5	1	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
13	200	22	6	2	1	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
14	329	33	9	3	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0
15	541	49	11	5	1	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0
16	887	73	16	6	2	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0
17	1453	108	21	8	3	1	1	1	0	0	0	0	0	0	0	0	1	0	0	0
18	2376	159	30	10	5	1	1	1	0	0	0	0	0	0	0	0	0	1	0	0
19	3881	234	41	13	6	2	1	1	1	0	0	0	0	0	0	0	0	0	1	0
20	6332	343	58	17	8	3	1	1	1	0	0	0	0	0	0	0	0	0	0	1

TABLE B.12. $[z^n]S^{(k)}(z)$, Arndt compositions of n whose smallest summand is k.

```
In[1]:= H[z_,k_]:=(z^k-z^(k+2)+z^(2k+1))/(1-z-z^2+z^3-z^(2k+1));
h[n_,k_]:=SeriesCoefficient[H[z,k],{z,0,n}];
S[z_,k_]:=H[z,k]-H[z,k+1];
s[n_,k_]:=SeriesCoefficient[S[z,k],{z,0,n}];
Table[s[n,k],{n,1,20},{k,1,20}]//MatrixForm
```

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	5	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	8	2	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	13	4	2	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
8	21	5	2	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
9	34	8	3	2	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
10	55	11	4	2	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
11	89	16	6	3	2	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
12	144	23	8	3	2	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
13	233	33	11	5	3	2	1	1	1	1	1	1	1	0	0	0	0	0	0	0
14	377	48	15	6	3	2	1	1	1	1	1	1	1	1	0	0	0	0	0	0
15	610	69	20	9	4	3	2	1	1	1	1	1	1	1	1	0	0	0	0	0
16	987	100	27	11	5	3	2	1	1	1	1	1	1	1	1	1	0	0	0	0
17	1597	144	36	15	7	4	3	2	1	1	1	1	1	1	1	1	1	0	0	0
18	2584	208	49	19	9	4	3	2	1	1	1	1	1	1	1	1	1	1	0	0
19	4181	300	66	25	12	6	4	3	2	1	1	1	1	1	1	1	1	1	1	0
20	6765	433	90	32	15	7	4	3	2	1	1	1	1	1	1	1	1	1	1	1

TABLE B.13. $[z^n]H^{(k)}(z)$, Arndt compositions of n whose parts are in $\{k, k+1, \ldots\}$.

```
In[1]:= H[z_,k_]:=(z^k-z^(k+2)+z^(2k+1))/(1-z-z^2+z^3-z^(2k+1));
h[n_,k_]:=SeriesCoefficient[H[z,k],{z,0,n}];
Table[h[n,k],{n,1,20},{k,1,20}]//MatrixForm
```

B.4. Interior Points and Semiperimeter

To print these tables, first load the functions from Appendix A.6.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
1	1	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0
3	2	0	0	0	0	0	0	0	0	0	0	0
4	3	0	0	0	0	0	0	0	0	0	0	0
5	4	1	0	0	0	0	0	0	0	0	0	0
6	6	2	0	0	0	0	0	0	0	0	0	0
7	9	2	2	0	0	0	0	0	0	0	0	0
8	13	3	5	0	0	0	0	0	0	0	0	0
9	19	5	8	2	0	0	0	0	0	0	0	0
10	28	7	14	5	1	0	0	0	0	0	0	0
11	41	10	25	8	5	0	0	0	0	0	0	0
12	60	15	42	13	12	2	0	0	0	0	0	0
13	88	22	69	23	23	7	1	0	0	0	0	0
14	129	32	113	39	43	14	7	0	0	0	0	0
15	189	47	183	63	81	25	20	2	0	0	0	0
16	277	69	293	102	147	46	41	10	2	0	0	0
17	406	101	466	165	257	86	80	26	10	0	0	0
18	595	148	737	263	444	155	156	52	29	5	0	0
19	872	217	1159	416	759	270	299	98	70	18	3	0
20	1278	318	1814	656	1279	466	555	188	149	43	18	1

Table B.14. $i_n^{(k)}$, Arndt compositions of n with k interior points in their bar graph.

In[1]:=
```
i[n_,k_]:=SeriesCoefficient[A[5][z,1,q],{z,0,n},{q,0,k}];
Table[i[n,k],{n,1,20},{k,0,11}]//MatrixForm
```

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1
3	2	2	2	2	2	2	2	2	2	2	2	2
4	3	3	3	3	3	3	3	3	3	3	3	3
5	4	5	5	5	5	5	5	5	5	5	5	5
6	6	8	8	8	8	8	8	8	8	8	8	8
7	9	11	13	13	13	13	13	13	13	13	13	13
8	13	16	21	21	21	21	21	21	21	21	21	21
9	19	24	32	34	34	34	34	34	34	34	34	34
10	28	35	49	54	55	55	55	55	55	55	55	55
11	41	51	76	84	89	89	89	89	89	89	89	89
12	60	75	117	130	142	144	144	144	144	144	144	144
13	88	110	179	202	225	232	233	233	233	233	233	233
14	129	161	274	313	356	370	377	377	377	377	377	377
15	189	236	419	482	563	588	608	610	610	610	610	610
16	277	346	639	741	888	934	975	985	987	987	987	987
17	406	507	973	1138	1395	1481	1561	1587	1597	1597	1597	1597
18	595	743	1480	1743	2187	2342	2498	2550	2579	2584	2584	2584
19	872	1089	2248	2664	3423	3693	3992	4090	4160	4178	4181	4181
20	1278	1596	3410	4066	5345	5811	6366	6554	6703	6746	6764	6765

TABLE B.15. $i_n^{(\leq k)}$, Arndt compositions of n with at most k interior points in their bar graph.

In[1]:=
```
i[n_,k_]:=SeriesCoefficient[A[5][z,1,q],{z,0,n},{q,0,k}];
MatInt=Table[i[n,k],{n,1,20},{k,0,11}];
Table[Sum[MatInt[[n,m]],{m,1,k}],{n,1,20},{k,1,12}]//MatrixForm
```

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
1	1	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0
3	2	0	0	0	0	0	0	0	0	0	0	0
4	3	0	0	0	0	0	0	0	0	0	0	0
5	5	1	0	0	0	0	0	0	0	0	0	0
6	8	2	0	0	0	0	0	0	0	0	0	0
7	13	4	2	0	0	0	0	0	0	0	0	0
8	21	8	5	0	0	0	0	0	0	0	0	0
9	34	15	10	2	0	0	0	0	0	0	0	0
10	55	27	20	6	1	0	0	0	0	0	0	0
11	89	48	38	13	5	0	0	0	0	0	0	0
12	144	84	69	27	14	2	0	0	0	0	0	0
13	233	145	123	54	31	8	1	0	0	0	0	0
14	377	248	216	103	64	21	7	0	0	0	0	0
15	610	421	374	191	128	47	22	2	0	0	0	0
16	987	710	641	348	246	99	53	12	2	0	0	0
17	1597	1191	1090	624	459	202	116	36	10	0	0	0
18	2584	1989	1841	1104	841	397	242	86	34	5	0	0
19	4181	3309	3092	1933	1517	758	488	189	91	21	3	0
20	6765	5487	5169	3355	2699	1420	954	399	211	62	19	1

Table B.16. $i_n^{(\geq k)}$, Arndt compositions of n with at least k interior points in their bar graph.

In[1]:=
```
i[n_,k_]:=SeriesCoefficient[A[5][z,1,q],{z,0,n},{q,0,k}];
MatInt=Table[i[n,k],{n,1,20},{k,0,11}];
Table[Sum[MatInt[[n,m]],{m,k,12}],{n,1,20},{k,1,12}]//MatrixForm
```

$n\backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	1	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	2	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	2	2	9	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	5	3	13	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	2	8	5	19	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	1	5	14	7	28	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	5	8	25	10	41	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	2	12	13	42	15	60	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	1	7	23	23	69	22	88	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	7	14	43	39	113	32	129	0	0	0	0	0	0
15	0	0	0	0	0	0	0	2	20	25	81	63	183	47	189	0	0	0	0	0
16	0	0	0	0	0	0	0	2	10	41	46	147	102	293	69	277	0	0	0	0
17	0	0	0	0	0	0	0	0	10	26	80	86	257	165	466	101	406	0	0	0
18	0	0	0	0	0	0	0	0	5	29	52	156	155	444	263	737	148	595	0	0
19	0	0	0	0	0	0	0	0	3	18	70	98	299	270	759	416	1159	217	872	0
20	0	0	0	0	0	0	0	0	1	18	43	149	188	555	466	1279	656	1814	318	1278

TABLE B.17. $s_n^{(k)}$, Arndt compositions of n whose bar graph has semiperimeter k.

In[1]:=
```
ClearAll[s];
s[n_,k_]:=SeriesCoefficient[A[5][z,p,1],{z,0,n},{p,0,k}];
Table[s[n,k],{n,0,20},{k,2,21}]//MatrixForm
```

$n\backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	0	0	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
4	0	0	0	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
5	0	0	0	1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	0	0	0	0	2	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
7	0	0	0	0	2	4	13	13	13	13	13	13	13	13	13	13	13	13	13	13
8	0	0	0	0	0	5	8	21	21	21	21	21	21	21	21	21	21	21	21	21
9	0	0	0	0	0	2	10	15	34	34	34	34	34	34	34	34	34	34	34	34
10	0	0	0	0	0	1	6	20	27	55	55	55	55	55	55	55	55	55	55	55
11	0	0	0	0	0	0	5	13	38	48	89	89	89	89	89	89	89	89	89	89
12	0	0	0	0	0	0	2	14	27	69	84	144	144	144	144	144	144	144	144	144
13	0	0	0	0	0	0	1	8	31	54	123	145	233	233	233	233	233	233	233	233
14	0	0	0	0	0	0	0	7	21	64	103	216	248	377	377	377	377	377	377	377
15	0	0	0	0	0	0	0	2	22	47	128	191	374	421	610	610	610	610	610	610
16	0	0	0	0	0	0	0	2	12	53	99	246	348	641	710	987	987	987	987	987
17	0	0	0	0	0	0	0	0	10	36	116	202	459	624	1090	1191	1597	1597	1597	1597
18	0	0	0	0	0	0	0	0	5	34	86	242	397	841	1104	1841	1989	2584	2584	2584
19	0	0	0	0	0	0	0	0	3	21	91	189	488	758	1517	1933	3092	3309	4181	4181
20	0	0	0	0	0	0	0	0	1	19	62	211	399	954	1420	2699	3355	5169	5487	6765

TABLE B.18. $s_n^{(\leq k)}$, Arndt compositions of n whose bar graph has semiperimeter at most k.

In[1]:=
```
ClearAll[s];
s[n_,k_]:=SeriesCoefficient[A[5][z,p,1],{z,0,n},{p,0,k}];
MatSp=Table[s[n,k],{n,1,20},{k,2,21}];
Table[Sum[MatSp[[n,m]],{m,1,k}],{n,1,20},{k,1,20}]//MatrixForm
```

$n\backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	2	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	3	3	3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	5	5	5	5	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	8	8	8	8	8	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	13	13	13	13	13	11	9	0	0	0	0	0	0	0	0	0	0	0	0	0
8	21	21	21	21	21	21	16	13	0	0	0	0	0	0	0	0	0	0	0	0
9	34	34	34	34	34	34	32	24	19	0	0	0	0	0	0	0	0	0	0	0
10	55	55	55	55	55	55	54	49	35	28	0	0	0	0	0	0	0	0	0	0
11	89	89	89	89	89	89	89	84	76	51	41	0	0	0	0	0	0	0	0	0
12	144	144	144	144	144	144	144	142	130	117	75	60	0	0	0	0	0	0	0	0
13	233	233	233	233	233	233	233	232	225	202	179	110	88	0	0	0	0	0	0	0
14	377	377	377	377	377	377	377	377	370	356	313	274	161	129	0	0	0	0	0	0
15	610	610	610	610	610	610	610	610	608	588	563	482	419	236	189	0	0	0	0	0
16	987	987	987	987	987	987	987	987	985	975	934	888	741	639	346	277	0	0	0	0
17	1597	1597	1597	1597	1597	1597	1597	1597	1597	1587	1561	1481	1395	1138	973	507	406	0	0	0
18	2584	2584	2584	2584	2584	2584	2584	2584	2584	2579	2550	2498	2342	2187	1743	1480	743	595	0	0
19	4181	4181	4181	4181	4181	4181	4181	4181	4181	4178	4160	4090	3992	3693	3423	2664	2248	1089	872	0
20	6765	6765	6765	6765	6765	6765	6765	6765	6765	6764	6746	6703	6554	6366	5811	5345	4066	3410	1596	1278

TABLE B.19. $s_n^{(\geq k)}$, Arndt compositions of n whose bar graph has semiperimeter at least k.

In[1]:=
```
ClearAll[s];
s[n_,k_]:=SeriesCoefficient[A[5][z,p,1],{z,0,n},{p,0,k}];
MatSp=Table[s[n,k],{n,1,20},{k,2,21}];
Table[Sum[MatSp[[n,m]],{m,k,20}],{n,1,20},{k,1,20}]//MatrixForm
```

B.5. *k*-Arndt Compositions

$n\backslash k$	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1
3	4	4	4	4	4	4	4	4	4	4	3	2	1	1	1	1	1	1	1	1	1
4	8	8	8	8	8	8	8	8	8	7	6	3	2	1	1	1	1	1	1	1	1
5	16	16	16	16	16	16	16	16	15	14	10	5	3	2	1	1	1	1	1	1	1
6	32	32	32	32	32	32	32	31	30	26	19	8	5	3	2	1	1	1	1	1	1
7	64	64	64	64	64	64	63	62	58	50	33	13	7	5	3	2	1	1	1	1	1
8	128	128	128	128	128	127	126	122	114	95	61	21	11	7	5	3	2	1	1	1	1
9	256	256	256	256	255	254	250	242	222	181	108	34	16	10	7	5	3	2	1	1	1
10	512	512	512	511	510	506	498	478	435	345	197	55	25	14	10	7	5	3	2	1	1
11	1024	1024	1023	1022	1018	1010	990	946	849	657	352	89	37	20	13	10	7	5	3	2	1
12	2048	2047	2046	2042	2034	2014	1970	1871	1661	1252	638	144	57	29	18	13	10	7	5	3	2
13	4095	4094	4090	4082	4062	4018	3918	3701	3245	2385	1145	233	85	42	24	17	13	10	7	5	3
14	8190	8186	8178	8158	8114	8014	7795	7321	6345	4544	2069	377	130	61	34	22	17	13	10	7	5
15	16378	16370	16350	16306	16206	15986	15505	14481	12400	8657	3721	610	195	88	47	29	21	17	13	10	7
16	32754	32734	32690	32590	32370	31887	30845	28645	24241	16493	6714	987	297	127	67	39	27	21	17	13	10
17	65502	65458	65358	65138	64654	63605	61357	56661	47380	31422	12087	1597	447	183	93	53	34	26	21	17	13
18	130994	130894	130674	130190	129139	126873	122057	112080	92617	59864	21794	2584	679	264	131	73	45	32	26	21	17
19	261966	261746	261262	260210	257937	253073	242801	221701	181032	114051	39254	4181	1024	381	181	100	59	40	31	26	21
20	523890	523406	522354	520079	515197	504805	482997	438540	353866	217286	70755	6765	1553	550	253	137	80	51	38	31	26

TABLE B.20. $a_n^{(k)}$, compositions of n whose pairs of summands satisfy $x_{2i-1} \geq x_{2i} + k$.

```
In[1]:= A[z_,k_]:=If[k≥0,(z-z^3+z^(k+2))/(1-z-z^2+z^3-z^(k+2)),(z+z^2-z^(-k+3))/(1-z-2 z^2+z^(-k+3))];
a[n_,k_]:=SeriesCoefficient[A[z,k],{z,0,n}];
Table[a[n,k],{n,1,20},{k,-10,10}]//MatrixForm
```

B.6. Absolute Difference Between Pairs of Summands

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	4	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	8	5	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	16	9	5	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6	32	17	9	5	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	64	31	13	9	5	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8	128	57	23	13	9	5	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1
9	256	105	37	19	13	9	5	3	1	1	1	1	1	1	1	1	1	1	1	1	1
10	512	193	65	29	19	13	9	5	3	1	1	1	1	1	1	1	1	1	1	1	1
11	1024	355	105	45	25	19	13	9	5	3	1	1	1	1	1	1	1	1	1	1	1
12	2048	653	179	73	37	25	19	13	9	5	3	1	1	1	1	1	1	1	1	1	1
13	4096	1201	293	115	53	33	25	19	13	9	5	3	1	1	1	1	1	1	1	1	1
14	8192	2209	497	181	83	45	33	25	19	13	9	5	3	1	1	1	1	1	1	1	1
15	16384	4063	821	281	125	63	41	33	25	19	13	9	5	3	1	1	1	1	1	1	1
16	32768	7473	1383	437	193	93	55	41	33	25	19	13	9	5	3	1	1	1	1	1	1
17	65536	13745	2293	683	285	137	73	51	41	33	25	19	13	9	5	3	1	1	1	1	1
18	131072	25281	3849	1069	427	205	105	65	51	41	33	25	19	13	9	5	3	1	1	1	1
19	262144	46499	6401	1677	625	299	149	85	61	51	41	33	25	19	13	9	5	3	1	1	1
20	524288	85525	10723	2625	933	433	219	117	77	61	51	41	33	25	19	13	9	5	3	1	1

TABLE B.21. $r_n^{(k)}$, compositions of n whose pairs of summands satisfy $|x_{2i-1} - x_{2i}| \geq k$.

```
In[1]:= R[z_,k_]:=If[k>0,(z-z^3+2 z^(k+2))/(1-z-z^2+z^3-2 z^(k+2)),z/(1-2 z)];
        Table[SeriesCoefficient[R[z,k],{z,0,n}],{n,1,20},{k,0,20}]//MatrixForm
```

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
4	4	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
5	4	12	14	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
6	8	20	28	30	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32
7	8	36	52	60	62	64	64	64	64	64	64	64	64	64	64	64	64	64	64
8	16	64	100	116	124	126	128	128	128	128	128	128	128	128	128	128	128	128	128
9	16	112	188	228	244	252	254	256	256	256	256	256	256	256	256	256	256	256	256
10	32	200	360	444	484	500	508	510	512	512	512	512	512	512	512	512	512	512	512
11	32	352	680	868	956	996	1012	1020	1022	1024	1024	1024	1024	1024	1024	1024	1024	1024	1024
12	64	624	1296	1696	1892	1980	2020	2036	2044	2046	2048	2048	2048	2048	2048	2048	2048	2048	2048
13	64	1104	2456	3312	3740	3940	4028	4068	4084	4092	4094	4096	4096	4096	4096	4096	4096	4096	4096
14	128	1952	4672	6472	7400	7836	8036	8124	8164	8180	8188	8190	8192	8192	8192	8192	8192	8192	8192
15	128	3456	8864	12640	14632	15588	16028	16228	16316	16356	16372	16380	16382	16384	16384	16384	16384	16384	16384
16	256	6112	16848	24696	28944	31008	31972	32412	32612	32700	32740	32756	32764	32766	32768	32768	32768	32768	32768
17	256	10816	31984	48240	57240	61680	63772	64740	65180	65380	65468	65508	65524	65532	65534	65536	65536	65536	65536
18	512	19136	60768	94240	113216	122696	127208	129308	130276	130716	130916	131004	131044	131060	131068	131070	131072	131072	131072
19	512	33856	115392	184096	223912	244064	253736	258276	260380	261348	261788	261988	262076	262116	262132	262140	262142	262144	262144
20	1024	59904	219200	359632	442864	485496	506128	515872	520420	522524	523492	523932	524132	524220	524260	524276	524284	524286	524288

TABLE B.22. Compositions of n whose pairs of summands satisfy $|x_{2i-1} - x_{2i}| \leq k$.

In[2]:= $\texttt{Table[SeriesCoefficient[}\dfrac{z+z^2-2\ z^{k+3}}{1-z-2\ z^2+2\ z^{k+3}}\texttt{,\{z,0,n\}],\{n,1,20\},\{k,0,18\}]//MatrixForm}$

B.7. Multiplying the Restriction by a Constant

$n\backslash a$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	2	3	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
4	3	6	7	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
5	5	11	14	15	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
6	8	20	27	30	31	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32
7	13	37	52	59	62	63	64	64	64	64	64	64	64	64	64	64	64	64	64	64
8	21	68	100	116	123	126	127	128	128	128	128	128	128	128	128	128	128	128	128	128
9	34	125	193	228	244	251	254	255	256	256	256	256	256	256	256	256	256	256	256	256
10	55	230	372	448	484	500	507	510	511	512	512	512	512	512	512	512	512	512	512	512
11	89	423	717	881	960	996	1012	1019	1022	1023	1024	1024	1024	1024	1024	1024	1024	1024	1024	1024
12	144	778	1382	1732	1904	1984	2020	2036	2043	2046	2047	2048	2048	2048	2048	2048	2048	2048	2048	2048
13	233	1431	2664	3405	3777	3952	4032	4068	4084	4091	4094	4095	4096	4096	4096	4096	4096	4096	4096	4096
14	377	2632	5135	6694	7492	7872	8048	8128	8164	8180	8187	8190	8191	8192	8192	8192	8192	8192	8192	8192
15	610	4841	9898	13160	14861	15681	16064	16240	16320	16356	16372	16379	16382	16383	16384	16384	16384	16384	16384	16384
16	987	8904	19079	25872	29478	31236	32064	32448	32624	32704	32740	32756	32763	32766	32767	32768	32768	32768	32768	32768
17	1597	16377	36776	50863	58472	62221	64001	64832	65216	65392	65472	65508	65524	65531	65534	65535	65536	65536	65536	65536
18	2584	30122	70888	99994	115984	123942	127748	129536	130368	130752	130928	131008	131044	131060	131067	131070	131071	131072	131072	131072
19	4181	55403	136641	196583	230064	246888	254989	258817	260608	261440	261824	262000	262080	262116	262132	262139	262142	262143	262144	262144
20	6765	101902	263384	386472	456351	491792	508966	517124	520960	522752	523584	523968	524144	524224	524260	524276	524283	524286	524287	524288

TABLE B.23. Compositions of n whose pairs of summands satisfy $ax_{2i-1} > x_{2i}$.

In[3]:= Table[SeriesCoefficient[$\frac{z-z^{a+1}}{1-2\ z+z^{a+2}}$,{z,0,n}],{n,1,20},{a,1,20}]//MatrixForm

$n\backslash b$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	4	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	8	3	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	16	5	3	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6	32	8	4	3	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	64	13	6	4	3	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8	128	21	9	5	4	3	2	1	1	1	1	1	1	1	1	1	1	1	1	1
9	256	34	13	7	5	4	3	2	1	1	1	1	1	1	1	1	1	1	1	1
10	512	55	19	10	6	5	4	3	2	1	1	1	1	1	1	1	1	1	1	1
11	1024	89	28	14	8	6	5	4	3	2	1	1	1	1	1	1	1	1	1	1
12	2048	144	41	19	11	7	6	5	4	3	2	1	1	1	1	1	1	1	1	1
13	4096	233	60	26	15	9	7	6	5	4	3	2	1	1	1	1	1	1	1	1
14	8192	377	88	36	20	12	8	7	6	5	4	3	2	1	1	1	1	1	1	1
15	16384	610	129	50	26	16	10	8	7	6	5	4	3	2	1	1	1	1	1	1
16	32768	987	189	69	34	21	13	9	8	7	6	5	4	3	2	1	1	1	1	1
17	65536	1597	277	95	45	27	17	11	9	8	7	6	5	4	3	2	1	1	1	1
18	131072	2584	406	131	60	34	22	14	10	9	8	7	6	5	4	3	2	1	1	1
19	262144	4181	595	181	80	43	28	18	12	10	9	8	7	6	5	4	3	2	1	1
20	524288	6765	872	250	106	55	35	23	15	11	10	9	8	7	6	5	4	3	2	1

TABLE B.24. Compositions of n whose pairs of summands satisfy $x_{2i-1} > bx_{2i}$.

```
In[4]:= Table[SeriesCoefficient[z/(1-z-z^(b+1)),{z,0,n}],{n,1,20},{b,0,19}]//MatrixForm
```

B.8. *k*-Block Arndt Compositions

$n\backslash k$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	2	1	1	1	1	1
3	4	2	2	2	2	2
4	8	3	2	2	2	2
5	16	5	3	3	3	3
6	32	8	4	4	4	4
7	64	13	6	5	5	5
8	128	21	8	6	6	6
9	256	34	13	8	8	8
10	512	55	18	10	10	10
11	1024	89	27	13	12	12
12	2048	144	39	17	15	15
13	4096	233	57	23	18	18
14	8192	377	81	31	22	22
15	16384	610	119	43	27	27
16	32768	987	170	59	33	32
17	65536	1597	247	82	40	38
18	131072	2584	357	113	51	46
19	262144	4181	518	156	63	54
20	524288	6765	748	213	81	64

TABLE B.25. k-block Arndt Compositions of n.

```
In[5]:= Table[SeriesCoefficient[Sum[z^Binomial[j+1,2]/QPochhammer[z,z,j],{j,1,k}]/
          (1-z^Binomial[k+1,2]/QPochhammer[z,z,k]),{z,0,n}],{n,1,20},{k,1,6}]//MatrixForm
```

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