Second Edition

field guide to SIMPLE GRAPHS

Volume 1

Peter Steinbach
Albuquerque Technical-Vocational Institute

DESIGN LAB Educational Ideas & Materials Albuquerque

This book was produced on a MacIntosh computer using Freehand software exclusively.

Peter Steinbach is a mathematics educator and former industrial designer, now teaching at the Albuquerque Technical-Vocational Institute. His current interests include graph theory, combinatorial geometry, early mathematics and elementary education, and development of math classroom manipulatives and games.

In loving memory of

Esther Becklinger Hawkins   189? – 1987

who insisted I learn how to be thorough

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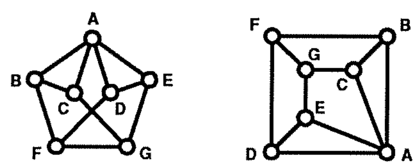
Behold the transcendent in the ordinary!

Preface

The origin of this book requires explanation. (The following discussion may confuse the beginner. Newcomers to graph theory should read the Introduction first.)

One of the most formidable obstacles to the process of generating graphs — especially when generating pictorially rather than numerically — is known as the isomorphism problem: How can one quickly decide whether or not two graphs depict the same set of relationships? If time is not a factor, then match the two sets of vertices in every possible combination until there appears a labeling that preserves all adjacencies or until the combinations are exhausted.



But listing these one-to-one correspondences by the thousands and inspecting every adjacency is perfect tedium. So isomorphism testing, though necessary for the inevitable duplications produced in graph generation, involves too many computing steps to allow easy generation of large graphs.

Ideally, a program designed to generate a set of graphs should produce one copy of each — no oversights, no duplicates. But this ideal (though approximated in the matrix-string algorithms) has not been achieved for the arbitrary graph set, even for the arbitrary degree sequence. Oversights, resulting from our poor understanding of the avalanching complexity of these dots and lines, are met with new programs which, by increased thoroughness, create enough redundancy to require excessive isomorphism testing. The redundancy then inspires attempts to formulate shortcuts, and shortcuts usually create oversights because they do not generalize to larger graphs.

There is another way. A glance of human inspection can often settle a matter otherwise requiring a formal computational nightmare. Human inspection, though capable of mechanical one-to-one correspondence, is much more comfortable *and reliable* in recognizing and working with gestalts such as a polygon, a road map, a face.

Look again at the figures drawn above and notice in both a pair of 3-cycles (triangles) joined at a vertex. Notice also that the vertices marked **A** are the only ones with four

incident edges (degree 4) and must form the anchor of any possible isomorphic labeling. Then after calling them **A** and the other four vertices any other letters arranged pairwise, notice finally that the last two vertices, ultimately to be labeled **F** and **G** in some order, neighbor each other as well as disjoint pairs of vertices on opposite 3-cycles. The graphs are isomorphic. (Another even more direct kind of recognition involves mentally picking up one image, distorting or unfolding it, and laying it precisely upon the other.)

How many computing steps does it take a program to recognize a 3-cycle, then to discover that there are two of them, and so on? The well-trained mind's eye finds the isomorphic labeling in three or four gestalts, while the formal algorithm makes blind comparisons until snow falls on the Amazon. For our small example the algorithm may electronically outrun the mind's eye, but for a slightly larger denser graph it may take more than a lifetime to find what the mind sees in one minute. So the consummate graph generating algorithm, if it exists, should be written for a human operator instead of an electronic comparison device.

The nearly 2500 graphs depicted in the following pages are a by-product of years of study of the isomorphism problem from a gestalt perspective. They were generated by eye using tedious but utterly thorough methods which eliminate oversight and rely, for recognizing duplication, on the development of unusual skill in codified manipulative observation.

Why to print them all is another matter. First, I intended to establish a set of *standard readable forms* for the portrayal of simple graphs — forms of standard proportions that highlight, as nearly as possible, the most useful properties. One representation of a graph may conceal its important cycles and cells, another may hide its paths and its connectivities, a third may suggest the wrong symmetry group. But there is usually a form or two that makes most essential information quickly legible to the eye. Occasionally a graph requires four pictures to lay bare its identity, and for the largest densest graphs of Chapter 4 no amount of visual aid will open the depths.

Second, tables serve theorists experimentally. For pure mathematicians the *theorems* are the powerful tools of graph theory. But theorems were once conjectures which sprang from experience with graphs. The decision whether to attempt to prove a conjecture or construct counterexamples or reformulate can often be based on experimental evidence, and such evidence comes from tables.

Finally, publications and interactive programs that teach graph-theoretic concepts are all-too-often found only in university math departments, and therefore only marginally available to other educators, designers, and engineers. Yet in the hands of these neglected applied scientists and strategists the dot and the line have already become *particle and bond, station and conduit, player and match, service and distribution, organism and predation, personality and relationship, condition and option, account and cash flow, state and process,* even *number and operation.* Graphs depict any relationships between any entities whatever. As such these figures form a true *lexicon* — an inventory of iconic law — and therefore rightly belong to all mappers, modellers, teachers, organizers, and researchers.

I wish to express my heartfelt thanks to R C Entringer of the University of New Mexico for lucid and enthusiastic instruction, to T M Steinbach Sr of Design Lab for help with aesthetic decisions and for the original problem, and to Carolyn for patience and support.

September 6, 1989

Preface to the Second Edition

Thanks to everyone who expressed their needs for more information and everyone who pointed out errors in the first edition, here is the Second Edition of the Field Guide.

There are four main additions. Readers asked for more information on trees, and I have supplied five theorems and an extra set of drawings that show centers and centroids. Some readers wanted to see more cubic (3-regular) graphs, and this was an easy task thanks to the work of Bussemaker et al. The new Chapter 3 on subgraphs was my own idea since I needed this data and anticipated that others would need it as well.

But the overwhelming response was: "We need the 8-point graphs! How soon can you draw them?" It has taken a very long time to draw just half of them for Volume 2, and from the outset I could see that the other half was not going to happen — and not because there are so many of them. It would be impossible to choose just one or two forms to portray most of these dense graphs, and even given a best form, its legibility (therefore its usefulness) is severely limited. As R C Entringer says, any graph that is large enough is solid black. But one does not need pictures of dense graphs if their sparse complements are available. For instance, a search for triangles in a dense graph is equivalent to a search for non-triangles in its complement.

I have Ronald Read of the University of Waterloo to thank for graciously supplying a list of degree sequences of the 8-point graphs and the numbers of graphs with each sequence. From there I modified the existing drawings of 7-point graphs, and relatively few had to be drawn from scratch.

June 3, 1995

The Second Revised Edition of 2004 corrects errors, expands Table 1 (pp. 180 – 181), and adds two properties to the regular graphs of Chapters 5 & 6. Most of the errors occurred in the subgraphs of Chapter 3. Special thanks to Peter Adams of University of Queensland, Roger Eggleton of Illinois State University, James MacDougall of University of Newcastle, and Ebad Mahmoodian of Sharif University of Technology for correcting Chapter 3.

The same year also marks the actual (rather than the expected) completion of Volume 2, the most difficult job I have ever done.

May 6, 2004

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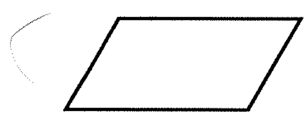
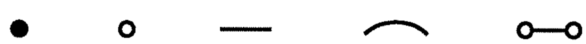
Introduction to Visual and Conceptual Conventions



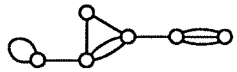
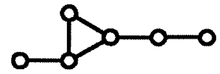
We need to arrive at some agreement about how to read and interpret dots and lines printed on paper. In classical geometry we draw a parallelogram, call it a plane, and agree to see it not as a skewed figure that lives in the page but as a rectangle that is leaning away from us out the back of the book. This is not automatic. We have to learn to see it with depth, then agree always to see it that way, except when we do intend to record a parallelogram that lives in the page. Similarly, graph theory has its own bizarre conventions, both visual and conceptual.



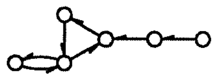
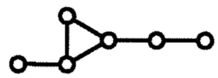
A *vertex* is depicted as a small dot or circle and, when applied to a real world task, represents anything at all. An *edge* is depicted as a line segment or arc, never alone but connecting exactly two vertices, and stands for any conceivable or inconceivable relationship between the two entities represented by the vertices. A *graph* then is any collection of vertices and edges, provided of course that all the edges are capped with two vertices. At this point some restrictions come into play.

A *connected graph*, as the name implies, is all in one piece. Disconnected graphs, which appear frequently in Chapter 2, will always be subdued in halftone for easy identification. The connected parts of a disconnected graph are its *components*.

As already mentioned, an edge must have end vertices. But there is no restriction keeping several edges from spanning the same two vertices or keeping an edge from looping back on itself.

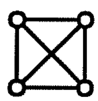
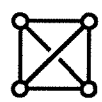
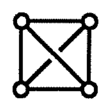



Graphs with multiple edges or loops are called *multigraphs*, while those with single edges are the *Simple Graphs*, the subject of this book. If a graph is to portray a network of flows or routes between stations, one might need to restrict directions of travel and indicate them with a *directed graph* or directed multigraph, otherwise known as a flowchart.

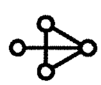
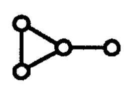



Directed graphs and multigraphs, both fleshed out on an underlying simple graph, play no part in this *Field Guide*.

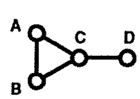
With these definitions out of the way, let's look at pictures that require a visual interpretation. These three figures are portraits of a graph with a disputed central region:

In the one on the left the diagonal edges seem to intersect, while the other two portraits suggest a viaduct, where one edge passes over another. The graph in question is conventionally drawn as shown at left, but its interpretation is understood to resemble both of the other drawings. In other words: where edges appear to intersect they do not. Edges intersect only at vertices (endpoints) or not at all. This means that edges never tangle. One can often mentally untangle a visually uncomfortable drawing by picking up a free end and rearranging it.

These observations point out that the real graph *is not a drawing*. Actually, in the most abstract sense, a graph is two related sets — the first a set of any number of elements (the vertex set, visually), and the second a set of pairings of those same elements (the edges). For instance:

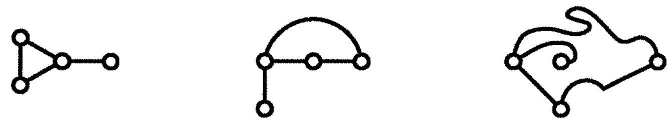


is a picture of

V: {A, B, C, D}
E: {AB, AC, BC, CD}

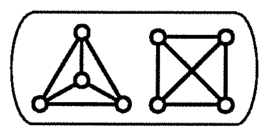
The fact that the pairing **AD** does not appear in the second set corresponds to the fact that no edge connects vertex **A** and vertex **D**.

Now since the only function of an edge is to pair two vertices, its shape has no meaning. Thus these three graphs are identical:

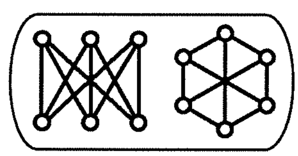


In each case three vertices and three edges form a circuitous path, known as a 3-cycle, and one of those vertices neighbors a fourth. Technically we call them *isomorphic* ("equivalent form") rather than identical. If these glyphs represented a postal route, acquaintances at a dinner party, and a treasure map, then we would need to consider them three isomorphic graphs rather than three drawings of one graph. This is genuine hairsplitting but it is necessary. Although "the map is not the territory," it must resemble the territory in order to be a map. Maps of three different territories, even if isomorphic, should never be considered one map, thus avoiding the suggestion that the territories are *one place*.

After having clarified the conceptual dangers of simultopeous territories, I now contradict myself slightly by saying that for the blank unapplied innocent meaningless tinkertoys in the following pages the simplest way of thinking about a set of isomorphs is with the mind's eye — as different views of one graph.



Isomorphic sets will always be encapsulated in a cartouche. This graph can be thought of as a tetrahedron seen from above and then edge-on, or as a flat map of electrical wiring in two orientations, or of course as a set of abstract elements and a set of pairs of those elements. A graph is not meant to be a geometric solid or a planar region, though it can represent both. A graph is its own space — a space of connection — and originally implies no other space. But working in that space of connection, the comfortable mind's eye has severe limitations, as demonstrated by this graph (these isomorphs):



In your mind, fold or unfold the left figure without disconnecting it, until it looks like its partner. If you prefer, turn the right one into the left instead. Here we meet learned geometric biases that scold the mind's eye for bending angles and stretching lengths and ignoring tangles. But biases can be overcome and graphs freely manipulated through practice. And once you have solved the little isomorphism exercise above, you may be inclined to rename the operative faculty *mind's touch.* We feel our way through graphs as we feel our way through the real networks they represent.

To emphasize this feeling quality of graphs, look at another example that illustrates not only the fluid nature of edges but the impersonal nature of vertices as well. Are these two graphs isomorphic or essentially different?

The key to a decision like this lies in words. Describe the graphs using whatever numerical or serial terms are necessary but avoid mentioning shapes: *One vertex has eight neighbors,* *and two of those neighbors each have one distinct further neighbor not adjacent to the originally mentioned vertex.* This sentence describes both graphs completely. We know exactly what is connected to what and we could draw either graph from the description. Suppose the vertices represented eleven people and an edge connected acquaintances. If we draw the situation with dots and lines, how could our placement of the elements affect who knows whom? We could label the vertices with the names of our eleven subjects, thereby immobilizing the edges and leaving just one possible portrait. But if the application recognizes no individuality among vertices (if we are interested only in the structure of a relationship), then the graphs above are isomorphic. Their difference is only in angle, and angle has no meaning here. Just grab a couple of appendages and twist: the left graph is the right graph. Vertices all look alike. Edges, being abstract bonds, never tangle. They are not sticks or ropes but relationships, conversations, light beams.

The concept of distance or length may be the one most fundamental to our understanding of our physical surroundings. The concept of direction or angle is independently basic to our physical understanding. So far in this Introduction we have eliminated both direction and distance from our sphere of attention, leaving one idea that precedes both because we use it to interpret not only physical space but conceptual space. That is connection. Why is connection important enough to warrant the development of an exact science to study it exclusively? To answer this question fully the reader is encouraged to look into any reputable graph theory text — especially one featuring applications to a familiar job — because this *Field Guide* describes graphs and their properties (valuable alone), but mentions few theorems. So in order to illustrate the importance of connection as a conceptual tool, I offer here a small but powerful theorem and a few other theorems in chapters to come. This example relies heavily on arithmetic; other theorems may or may not.

Draw 7 vertices and imagine that they are 7 players in a tournament. Furthermore, suppose that an edge connects players in a match and that each of the 7 will play exactly 3 matches, meaning that we must draw this graph so that 3 edges end at each vertex. Make a drawing of such a graph before reading on.

To make it easier to talk about this we'll invent some props. Let the letter **v** stand for the number of vertices in a graph and let the letter **e** represent the number of edges. In this case **v** = 7 and **e** is unknown. Also, the number of edges incident on a vertex we'll call the *degree* of that vertex. In this case the degree of every vertex is going to be 3. Such graphs with uniform degrees are called *regular*, this one 3-regular.

Now calculate **e**. If we add the degrees of all 7 vertices, we will be counting both ends of all the edges. And since every edge has two ends, we can divide our result by 2 and arrive at **e**, the number of edges. In this case the sum of degrees is $7 \cdot 3 = 21$, which should have been an even number, allowing us to divide by 2. Instead we have **e** = 10.5. Is there an edge with only one end? No. The trouble is that odd times odd is odd. Therefore no graph of this description exists. So seven players will never make a tournament of three matches each, and seven towns will never be connected by roads, three leading to each.

Theorem 1 In any graph, simple or not, the sum of all vertex degrees is twice the number of edges (2**e**), and thus an even number.

Corollary 1 In any regular graph, **v** and the common degree are not both odd.