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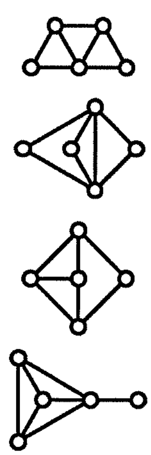
# The Small Connected Simple Graphs ($v \leq 7$)

Facts (theorems), procedures (algorithms), and sources for both (problems) in graph theory have strange names: the Traveling Salesman Problem, the Chinese Postman Problem, several kinds of Dancing and Handshaking Theorems, the Greedy Algorithm, the Utilities Problem, the Seeded Tournament Algorithm, the Feasible Flow Theorem. All of these and many more describe basic properties of graphs that apply directly to problems found in scheduling, storage, communication, and travel networks. But the application of these ideas about graphs to real networks can be made only by virtue of some well-defined correspondence between the real entities and the dots and between the real relationships and the lines. These dots and lines or vertices and edges are shown in this chapter in every possible connected combination up to seven vertices.

The lists in the following pages have been left devoid of all notation so that readers may make their own notations according to their particular needs, and so that the visual and conceptual impact of 996 graphs of connective space will be as plain as possible.

There is, however, a scheme to their arrangement. Graphs of a fixed number of vertices are clustered into a row of columns, each column enumerating a family of graphs of a fixed number of edges. On page 7 the rows of columns are clearly evident.

For instance this column catalogs the connected graphs with 5 vertices and 7 edges.



But in the block of 7-point graphs an edge family can cover as much as a two-page spread.

A few moments inspection reveals the first of a potentially infinite number of patterns found in connected graphs: for each number of vertices the first column consists of trees, connected graphs without cycles (shown separately in Chapter 3), and the second column consists of single-cycled graphs. From the third column on, more than one cycle can be combined in ways yielding more than two cycles.

The final column of each set always consists of a single graph known as the complete graph on **v** vertices and denoted $K_v$. It is complete because every pair of vertices is connected by an edge — all possible connections have been made. How many edges does a complete graph have? Since an edge is essentially a pairing of vertices, this is equivalent to the question: How many pairs are there in a set of **v** elements?

Conventionally denoted by $\binom{v}{2}$, the number of pairs can be calculated by imagining choosing two things out of **v** things. The choice of the first can be made in **v** ways — there are **v** vertices to choose from — and *for each* possible first choice there are **v – 1** left for the second choice, thus **v – 1** ways to choose. The words *for each* suggest multiplying **v** by **v – 1** to obtain the number of ways of making two choices. That done, we have counted every vertex pair twice. For instance if **A** and **B** are names of vertices, we have counted the pairs **AB** and **BA** both. So we divide the result by 2, since edges specify no direction.

*Theorem 2* The number of edges **e** in a complete graph $K_v$ is related to the number of vertices **v** as

$$e = \frac{v(v-1)}{2}$$

The penultimate column also contains only one graph, and the column before that contains two. Since all edges are present in the complete graph, all are abstractly equivalent, and the removal of one is equivalent to the removal of any other. Removing two edges, however, can be done in two ways: either the edges share a vertex or do not. Establishing the number of graphs in *any* given column (the number of graphs with given **v** and **e**) is a more difficult task, involving George Polya's theory of counting. One can see in the following pages that the total number of connected simple graphs on a given number of vertices is growing at an alarming rate. For instance, on eleven vertices there are over a billion of them.

When using these charts, recall that a cartouche encapsulates a collection of views of a single graph.