2

Specifications of Small Simple Graphs

We now define and tabulate twelve invariant properties for the small graphs of Chapter 1. One further property, complementarity, involves some graphs not shown in Chapter 1 — the disconnected graphs. Complementarity is the relationship between two graphs each of which has exclusively the edges that the other does not have. So the complement **~G** of a graph **G** is the graph obtained by edging all the nonadjacent pairs of vertices of **G** and deleting its former edges. For example:

Notice that one of these is disconnected, the other connected. A pair of complements may also both be connected, but both may not be disconnected. To see why this is true, observe first that for any pair of vertices in a connected graph there must be some path connecting them. We can take this as a definition of connected. Now choose any pair of vertices **A** and **B** in a disconnected graph. If they are in different *components* (the connected parts of a disconnected graph), then the complementary graph will connect them with a single edge **AB**. If on the other hand they are in the same component, then the complementary graph will connect them with a path of two edges leading from **A** to a vertex in some other component then back to **B**. Since **A** and **B** are chosen arbitrarily, this satisfies the definition of connected.

Theorem 3 Every disconnected graph has a connected complement.

Included among the following definitions are several more theorems stated here without proof.

DEFINITIONS and SYMBOLS

Two vertices are *ADJACENT* if a single edge connects them.

B The vertices of a *BIPARTITE* graph can be partitioned into two sets, X and Y, so that every edge of the graph has one end in X and the other in Y. Equivalently, the vertices of a *BIPARTITE* graph can be painted with two colors so that no vertex is adjacent to a vertex of the same color.

Theorem 4 A graph is bipartite if and only if it contains no cycles of odd length. (see definition of cycle below)

A *TRAIL* is an alternating sequence of connected vertices and edges, beginning and ending with vertices, in which no edge is repeated. A *closed trail* begins and ends on the same vertex.

E An *EULERIAN* graph has a closed trail that contains all of the edges of the graph.

e A *sub-EULERIAN* graph has a trail with distinct ends that contains all of the edges of the graph, but is not Eulerian.

Theorem 5 A connected graph is Eulerian if and only if it contains no vertices of odd degree, and sub-Eulerian if and only if it contains exactly two vertices of odd degree.

A *PATH* is a trail in which no vertex is repeated — a trail that does not intersect itself. A *CYCLE* is a closed path.

H A *HAMILTONIAN* graph has a cycle that contains all of the vertices of the graph. This cycle is known as a *spanning cycle.*

h A *sub-HAMILTONIAN* graph has only a path that contains all of the vertices of the graph. (spanning path)

P A *PLANAR* graph can be depicted on a planar surface in a way that permits edges to appear to intersect only at vertices. Note that it will not necessarily be drawn this way.

g:f The *GIRTH* of a graph is the length of a smallest cycle of the graph. The *GIRTH FREQUENCY* is the number of smallest cycles. Thus 4:3 indicates that a graph has exactly three 4-cycles and no smaller cycles.

The *DISTANCE* between two vertices is the length (in edges) of a shortest path connecting them.

d The *DIAMETER* of a graph is the maximum distance occuring in it.

ke The *EDGE CONNECTIVITY* of a graph is the smallest number of edges the removal of which can disconnect the graph.

kv The *VERTEX CONNECTIVITY* of a graph is the smallest number of vertices the removal of which can disconnect the graph. A complete graph cannot be disconnected this way.

Theorem 6 Denote the smallest vertex degree in a graph by δ. Then $kv \le ke \le \delta$.

χ The *CHROMATIC NUMBER* of a graph is the minimum number of colors required to paint its vertices so that no vertex is adjacent to a vertex of the same color. Thus for bipartite graphs $\chi = 2$. The *EDGE-CHROMATIC NUMBER* (not given in the specifications) is the analogous concept for edge painting.

Theorem 7 Denote the largest vertex degree in a graph by Δ. Then $\chi \le \Delta + 1$, and the edge-chromatic number equals either Δ or $\Delta + 1$.

~ *COMPLEMENTARITY* — The *COMPLEMENT* (~G) of a graph G is the graph obtained by edging all of the nonadjacent pairs of vertices of G and deleting its former edges.

Γ *SYMMETRY GROUP* — the group of automorphisms (adjacency-preserving vertex permutations) of a graph. Group names are given in standard notation: $\mathbf{Z_n}$ for cyclic groups, $\mathbf{D_n}$ for dihedral, $\mathbf{S_n}$ symmetric, **i** for the identity group, $\mathbf{A \times B}$ for Cartesian product, and **A(B)** for wreath product.

Theorem 8 The group of any graph is isomorphic to the group of its complement.

Theorem 9 The number of distinct ways to assign the labels **1** through **v** to the **v** vertices of a graph G is

$$\frac{v!}{\text{ord }\Gamma(G)}$$

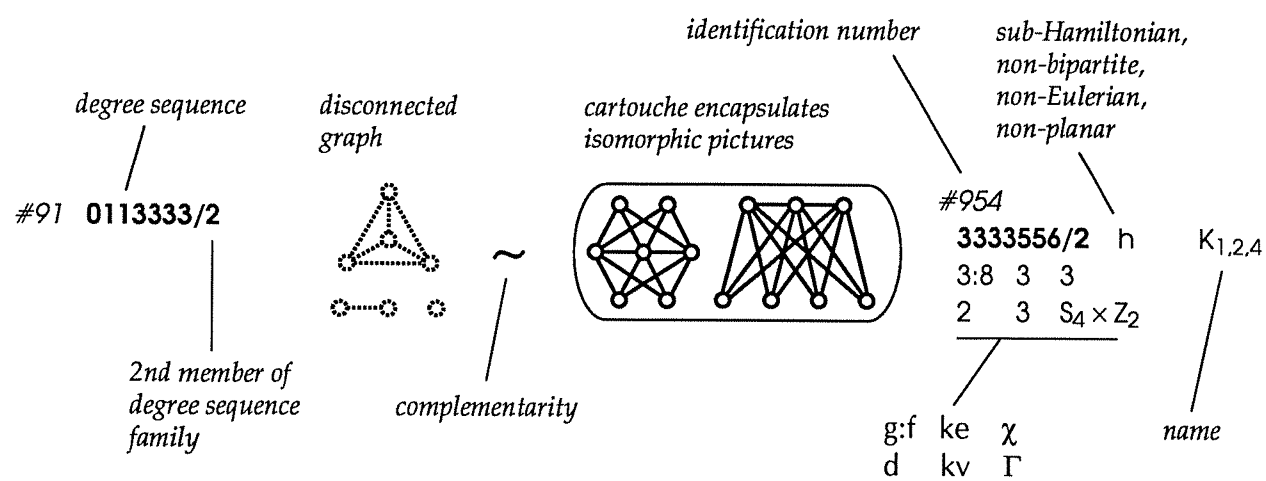
Commonly used names for special graphs are denoted by the following symbols:
$\mathbf{P_n}$ — the path of **n** vertices (**n – 1** edges)
$\mathbf{C_n}$ — the cycle of **n** vertices (or edges)
$\mathbf{E_n}$ — the empty (edgeless) graph on **n** vertices
$\mathbf{K_n}$ — the complete graph on **n** vertices
$\mathbf{K_{m,n}}$ — the complete-bipartite graph on sets of **m** and **n** vertices
$\mathbf{K_{1,n}}$ — the star on **n** edges, **n + 1** vertices
$\mathbf{K_{r,s,t}}$ — the complete tripartite graph on sets of **r**, **s**, and **t** vertices
$\mathbf{W_n}$ — the wheel on **n** vertices (**2n – 2** edges)

Special characteristics of graphs are occasionally noted under or instead of a name.

Besides complementarity and degree sequence, no specifications are given for disconnected graphs because the above definitions either apply to their connected components or do not apply at all.

Note that in a complementary pair the sparser graph is on the left, the denser on the right. Graphs of a fixed number of edges are arranged by degree sequence — right-odometrically for sparse graphs, left-odometrically (write it backwards and use numerical order) for dense graphs. Because of *Theorem 8* the symmetry group is listed only once for a complementary pair.

KEY



v = 1

K_1 the trivial graph

v = 2

e = 0 | e = 1

E_2 **00**

11 B e h P — K_2, P_2, $K_{1,1}$
– 1 2
1 – Z_2

v = 3

e = 0 | e = 3

E_3 *#1* **000**

#4 **222** E H P — K_3, C_3
3:1 2 3
1 – $S_3 = D_3$

e = 1 | e = 2

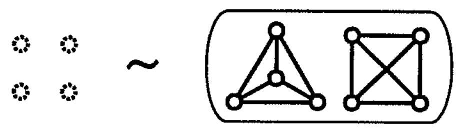
#2 **011**

#3 **112** B e h P — P_3, $K_{1,2}$
– 1 2
2 1 Z_2

v = 4

e = 0 | e = 6

E_4 *#1* **0000**



#11 **3333** H P — K_4, W_4
3:4 3 4
1 – S_4

e = 1 | e = 5

#2 **0011**

#10 **2233** e H P
3:2 2 3
2 2 $Z_2 \times Z_2$

e = 2 | e = 4

#3 **0112**

#9 **1223** e h P
3:1 1 3
2 1 Z_2

#4 **1111**

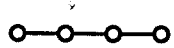
#8 **2222** B E H P — C_4, $K_{2,2}$
4:1 2 2
2 2 $D_4 = Z_2(Z_2)$

e = 3 | e = 3

#5 **0222**

#7 **1113** B P — $K_{1,3}$
– 1 2
2 1 S_3

self-complementary



#6 **1122** B e h P — P_4
– 1 2
3 1 Z_2

g:f ke χ
d kv Γ

v = 5

e = 0 —————— **e = 10**

E_5 *#1* **00000**

#34 **44444** E H
3:10 4 5
1 – S_5

K_5
precedent:
non-P

e = 1 —————— **e = 9**

#2 **00011**

#33 **33444** e H P
3:7 3 4
2 3 $S_3 \times Z_2$

3-bipyramid

e = 2 —————— **e = 8**

#3 **00112**

#32 **23344** e H P
3:5 2 4
2 2 $Z_2 \times Z_2$

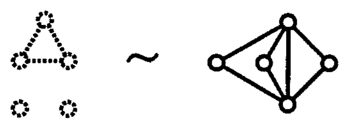
#4 **01111**

#31 **33334** H P
3:4 3 3
2 3 D_4

$K_{1,2,2}$, W_5,
4-pyramid

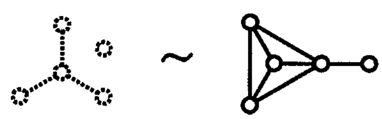
e = 3 —————— **e = 7**

#5 **00222**



#30 **22244** E h P
3:3 2 3
2 2 $S_3 \times Z_2$

#6 **01113**

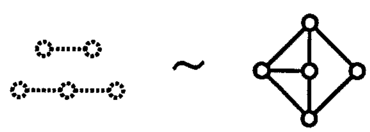


#29 **13334** h P
3:4 1 4
2 1 S_3

#7 **01122**

#28 **22334** e H P
3:3 2 3
2 2 Z_2

#8 **11112**



#27 **23333** H P
3:2 2 3
2 2 $Z_2 \times Z_2$

e = 4 —————— **e = 6**

#9 **01223**

#26 **12234** e h P
3:2 1 3
2 1 Z_2

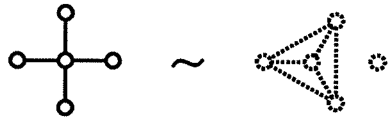
#10 **02222**

#25 **22224** E h P
3:2 2* 3
2 1* D_4

*precedent:
ke ≠ kv

g:f ke χ
d kv Γ

$K_{1,4}$ *#11* **11114** B P
– 1 2
2 1 S_4

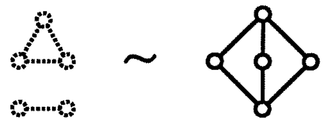


#24 **03333**

#12 **11123** B P
– 1 2
3 1

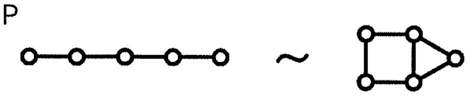
#23 **12333** h P
3:2 1 3
3 1 Z_2

#13 **11222/1**



#22 **22233/1** B e h P $K_{2,3}$
4:3 2 2
2 2 $S_3 \times Z_2$
precedent: non-ordinary polytope

P_5 *#14* **11222/2** B e h P
– 1 2
4 1



#21 **22233/2** e H P
3:1 2 3
2 2 Z_2

e = 5 — **e = 5**

#15 **02233**

#20 **11224** e P
3:1 1 3
2 1 $Z_2 \times Z_2$

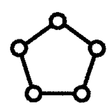
self-complementary

#19 **11233** h P
3:1 1 3
3 1 Z_2

#16 **12223/1** e h P
3:1 1 3
3 1

#18 **12223/2** B e h P
4:1 1 2
3 1 Z_2

self-complementary

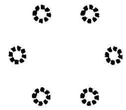
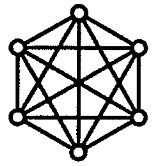


#17 **22222** E H P C_5
5:1 2 3*
2 2 D_5
*precedent without K_3 subgraph

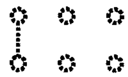
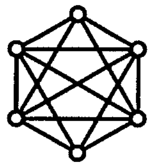
g:f ke χ
d kv Γ

v = 6

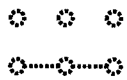
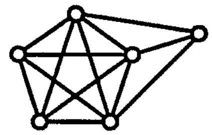
e = 0 — **e = 15**

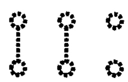
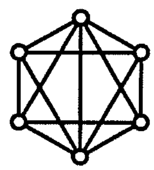
E_6 *#1* **000000**  ~  *#156* **555555** H K_6
3:20 5 6
1 – S_6

e = 1 — **e = 14**

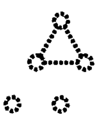
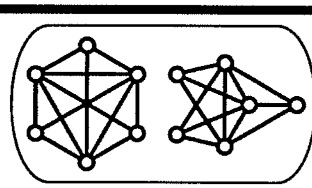
#2 **000011**  ~  *#155* **445555** H
3:16 4 5
2 4 $S_4 \times Z_2$

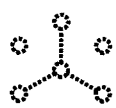
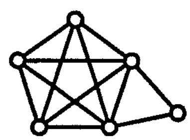
e = 2 — **e = 13**

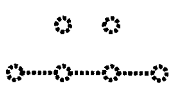
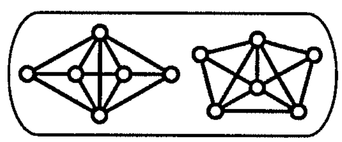
#3 **000112**  ~  *#154* **344555** H
3:13 3 5
2 3 $S_3 \times Z_2$

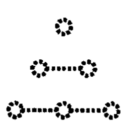
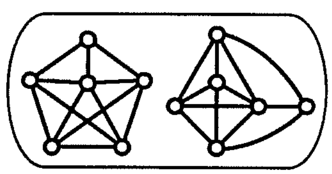
#4 **001111**  ~  *#153* **444455** e H
3:12 4 4
2 4 $D_4 \times Z_2$

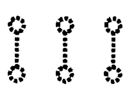
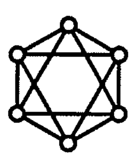
e = 3 — **e = 12**

#5 **000222**  ~  *#152* **333555** H
3:10 3 4
2 3 $S_3 \times S_3$

#6 **001113**  ~  *#151* **244455** e H
3:11 2 5
2 2 $S_3 \times Z_2$

#7 **001122**  ~  *#150* **334455** H P
3:10 3 4
2 3 $Z_2 \times Z_2$

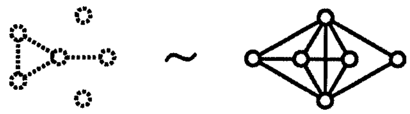
#8 **011112**  ~  *#149* **344445** e H
3:9 3 4
2 3 $Z_2 \times Z_2$

#9 **111111**  ~  *#148* **444444** E H P $K_{2,2,2}$
3:8 4 3 octahedron
2 4 $S_4 \times Z_2 = S_3(Z_2)$ (ord 48)

g:f ke χ
d kv Γ

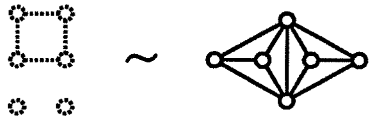
e = 4 — **e = 11**

#10 **001223**



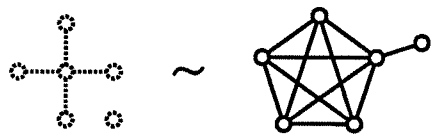
#147 **233455** H P
3:8 2 4
2 2 $Z_2 \times Z_2$

#11 **002222**



#146 **333355** H P
3:8 3 4
2 2 $D_4 \times Z_2$

#12 **011114**

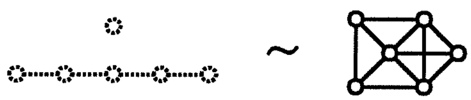


#145 **144445** e h
3:10 1 5
2 1 S_4

#13 **011123**

#144 **234445** e H P
3:8 2 4
2 2 Z_2

#14 **011222/1**

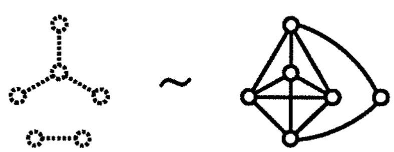


#143 **333445/1** H P
3:7 3 4
2 3 Z_2

#15 **011222/2**

#142 **333445/2** H $K_{1,2,3}$
3:6 3 3
2 3 $S_3 \times Z_2$

#16 **111113**

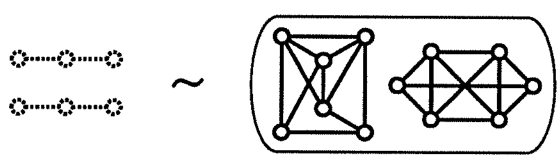


#141 **244444** E H
3:7 2 4
2 2 $S_3 \times Z_2$

#17 **111122/1**

#140 **334444/1** e H P
3:6 3 3
2 3 $Z_2 \times Z_2$

#18 **111122/2**



#139 **334444/2** e H
3:6 3 4
2 3 D_4

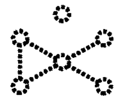
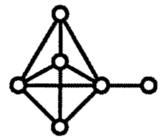
e = 5 — **e = 10**

#19 **002233**

#138 **223355** e h P
3:6 2 4
2 2 $Z_2 \times Z_2 \times Z_2$

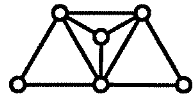
g:f ke χ
d kv Γ

#20 **011224**

 ~ 

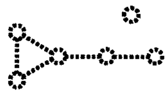
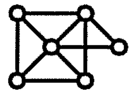
#137 **133445** h P
3:7 1 4
2 1 $Z_2 \times Z_2$

#21 **011233**

 ~ 

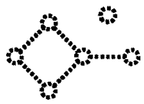
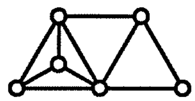
#136 **223445** e H P
3:6 2 4
2 2 Z_2

#22 **012223/1**

 ~ 

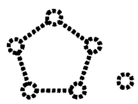
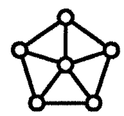
#135 **233345/1** H P
3:5 2 3
2 2 Z_2

#23 **012223/2**

 ~ 

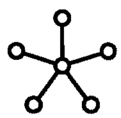
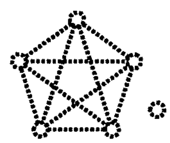
#134 **233345/2** H P
3:6 2 4
2 2 Z_2

#24 **022222**

 ~ 

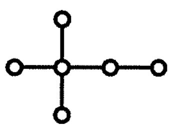
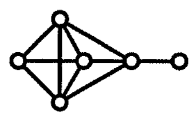
#133 **333335** H P W_6
3:5 3 4* 5-pyramid
2 3 D_5
*precedent without K_4 subgraph

$K_{1,5}$ *#25* **111115** B P
– 1 2
2 1 S_5

 ~ 

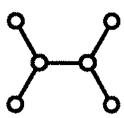
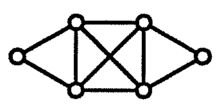
#132 **044444**

#26 **111124** B P
– 1 2
3 1

 ~ 

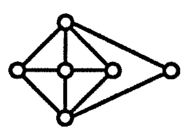
#131 **134444** e h P
3:7 1 4
3 1 S_3

#27 **111133** B P
– 1 2
3 1

 ~ 

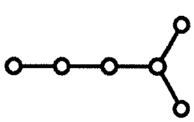
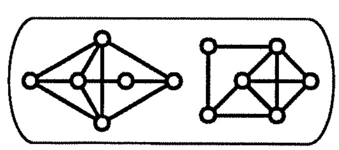
#130 **224444** E H P
3:6 2 4
3 2 D_4

#28 **111223/1**

 ~ 

#129 **233444/1** e H P
3:4 2 3
2 2 $Z_2 \times Z_2$

#29 **111223/2** B P
– 1 2
4 1

 ~ 

#128 **233444/2** e H P
3:5 2 4
2 2 Z_2

g:f ke χ
d kv Γ

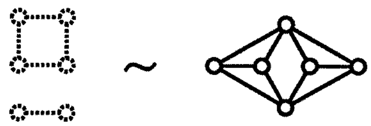
#30 **111223/3** B P
– 1 2
3 1

#127 **233444/3** e H P
3:5 2 3
2 2 Z_2

#31 **112222/1**

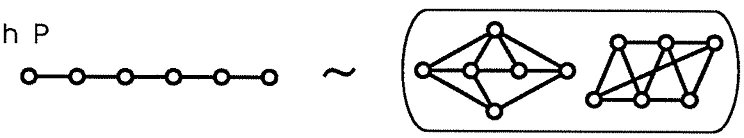
#126 **333344/1** H
3:3 3 3
2 3 $S_3 \times Z_2$

#32 **112222/2**



#125 **333344/2** H P
3:4 3 3
2 2 $D_4 \times Z_2$

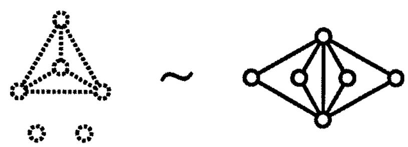
P_6 *#33* **112222/3** B e h P
– 1 2
5 1



#124 **333344/3** H P
3:4 3 3
2 3 Z_2

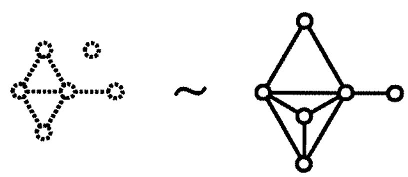
e = 6 —— **e = 9**

#34 **003333**



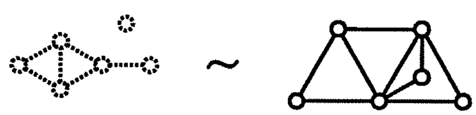
#123 **222255** e P
3:4 2 3
2 2 $S_4 \times Z_2$

#35 **012234**



#122 **123345** h P
3:5 1 4
2 1 Z_2

#36 **012333**

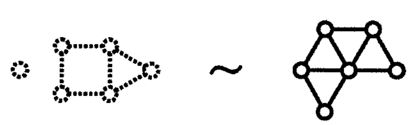


#121 **222345** e h P
3:4 2 3
2 2 Z_2

#37 **022224**

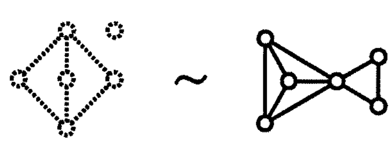
#120 **133335** h P
3:4 1 3
2 1 D_4

#38 **022233/1**



#119 **223335/1** H P
3:4 2 3
2 2 Z_2

#39 **022233/2**



#118 **223335/2** h P
3:5 2 4
2 1 $S_3 \times Z_2$

g:f ke χ
d kv Γ

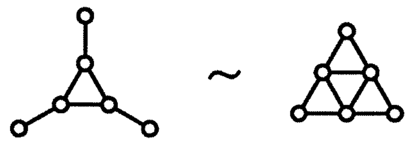
#40 **111225** P
3:1 1 3
2 1 $S_3 \times Z_2$

#117 **033444**

#41 **111234** P
3:1 1 3
3 1

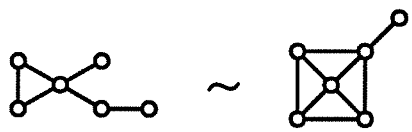
#116 **123444** e h P
3:5 1 4
3 1 Z_2

#42 **111333** P
3:1 1 3
3 1



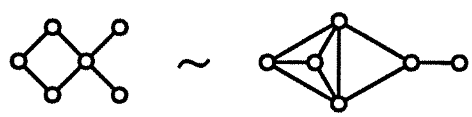
#115 **222444** E H P
3:4 2 3
2 2 S_3

#43 **112224/1** e P
3:1 1 3
3 1



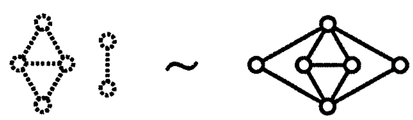
#114 **133344/1** h P
3:4 1 3
3 1 Z_2

#44 **112224/2** B e P
4:1 1 2
3 1



#113 **133344/2** h P
3:5 1 4
3 1 $Z_2 \times Z_2$

#45 **112233/1**

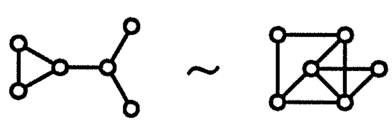


#112 **223344/1** e h P
3:2 2 3
2 2 $Z_2 \times Z_2 \times Z_2$

#46 **112233/2** h P
3:1 1 3
4 1 *i**
*precedent for v > 1

#111 **223344/2** e H P
3:3 2 3
2 2 *i*

#47 **112233/3** P
3:1 1 3
3 1

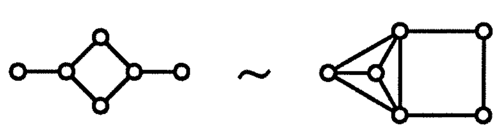


#110 **223344/3** e H P
3:3 2 3
3 2 $Z_2 \times Z_2$

#48 **112233/4** B h P
4:1 1 2
3 1

#109 **223344/4** e H P
3:4 2 3
3 2 Z_2

#49 **112233/5** B P
4:1 1 2
4 1



#108 **223344/5** e H P
3:4 2 4
2 2 $Z_2 \times Z_2$

g:f ke χ
d kv Γ

#50 **122223/1** e h P
3:1 1 3
4 1

#107 **233334/1** H P
3:2 2 3
2 2 Z_2

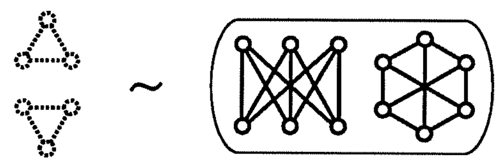
#51 **122223/2** B e h P
4:1 1 2
4 1

#106 **233334/2** H P
3:3 2 3
2 2 Z_2

#52 **122223/3** e h P
5:1 1 3
3 1

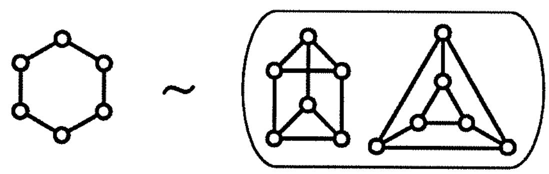
#105 **233334/3** H P
3:3 2 3
2 2 Z_2

#53 **222222/1**



#104 **333333/1** B H $K_{3,3}$
4:9 3 2 9-rhomb
2 3 $Z_2(S_3)$ (ord 72)

C_6 *#54* **222222/2** B E H P
6:1 2 2
3 2



#103 **333333/2** H P 3-prism
3:2 3 3
2 3 $D_6 = S_3 \times Z_2$

e = 7 **e = 8**

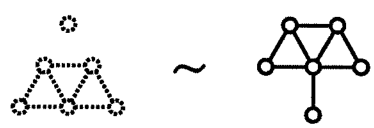
#55 **013334**

#102 **122245** e P
3:3 1 3
2 1 S_3

#56 **022244**

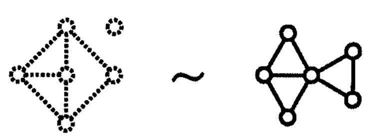
#101 **113335** P
3:4 1 4
2 1 $S_3 \times Z_2$

#57 **022334**



#100 **122335** h P
3:3 1 3
2 1 Z_2

#58 **023333**



#99 **222235** e h P
3:3 2 3
2 1 $Z_2 \times Z_2$

#59 **112235** P
3:2 1 3
2 1 $Z_2 \times Z_2$

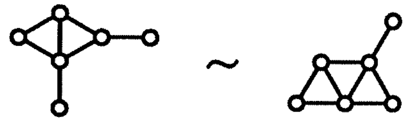
#98 **023344**

g:f ke χ
d kv Γ

#60 **112244** e P
3:2 1 3
3 1

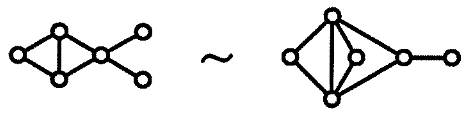
#97 **113344** h P
3:4 1 4
3 1 $Z_2 \times Z_2$

#61 **112334/1** h P
3:2 1 3
3 1



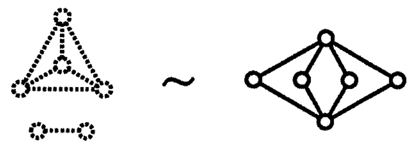
#96 **122344/1** e h P
3:3 1 3
3 1 *i*

#62 **112334/2** P
3:2 1 3
3 1



#95 **122344/2** e h P
3:3 1 3
3 1 $Z_2 \times Z_2$

#63 **113333/1**



#94 **222244/1** B E P $K_{2,4}$
4:6 2 2
2 2 $S_4 \times Z_2$

#64 **113333/2** h P
3:2 1 3
4 1

#93 **222244/2** E h P
3:2 2 3
2 2 $Z_2 \times Z_2$

#65 **122225** e P
3:2 1 3
2 1 D_4

#92 **033334**

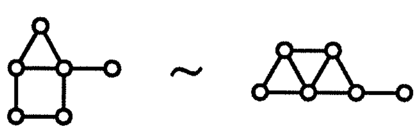
#66 **122234/1** e h P
3:2 1 3
3 1

#91 **123334/1** h P
3:2 1 3
3 1 Z_2

#67 **122234/2** e h P
3:2 1 3
3 1

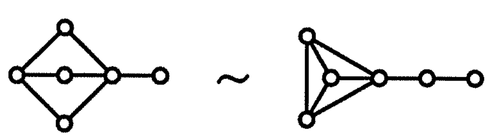
#90 **123334/2** h P
3:2 1 3
3 1 Z_2

#68 **122234/3** e h P
3:1 1 3
3 1



#89 **123334/3** h P
3:3 1 3
3 1 *i*

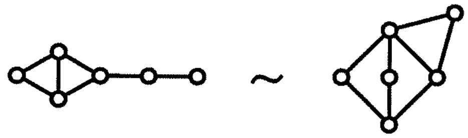
#69 **122234/4** B e P
4:3 1 2
3 1



#88 **123334/4** h P
3:4 1 4
3 1 S_3

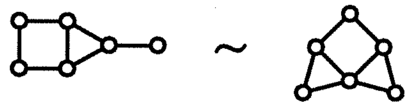
g:f ke χ
d kv Γ

#70 **122333/1** h P
3:2 1 3
4 1



#87 **222334/1** e h P
3:1 2 3
2 2 Z_2

#71 **122333/2** h P
3:1 1 3
3 1



#86 **222334/2** e H P
3:2 2 3
2 2 Z_2

#72 **122333/3** h P
3:1 1 3
3 1

#85 **222334/3** e H P
3:2 2 3
3 2 *i*

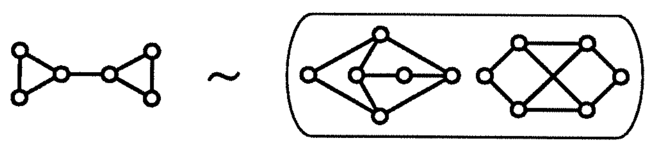
#73 **122333/4** B h P
4:3 1 2
3 1

#84 **222334/4** e h P
3:3 2 3
3 1 $Z_2 \times Z_2$

#74 **222224** E h P
3:1 2 3
3 1

#83 **133333** h P
3:2 1 3
3 1 $Z_2 \times Z_2$

#75 **222233/1** e h P
3:2 1* 3
3 1
*precedent: ke $\neq \delta$



#82 **223333/1** B H P
4:5 2 2
3 2 D_4

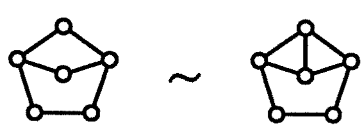
#76 **222233/2** B e H P
4:2 2 2
3 2

#81 **223333/2** H P
3:2 2 3
3 2 $Z_2 \times Z_2$

#77 **222233/3** e H P
3:1 2 3
3 2

#80 **223333/3** H P
3:1 2 3
2 2 Z_2

#78 **222233/4** e h P
4:1 2 3
2 2



#79 **223333/4** H P
3:2 2 3
2 2 $Z_2 \times Z_2$