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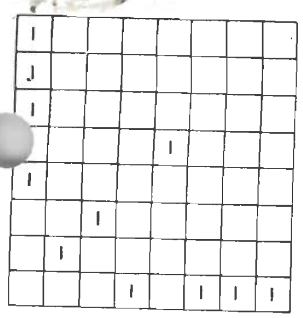
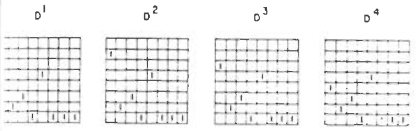
he discriminant D of this system is



o that $S=4$ indicating four different solu-
ions defined by



D^1	D^2	D^3	D^4
$\cdot\ x_2 + x_3$	$X_1 = \bar{x}_1 + x_2 + x_3$	$X_1 = x_2 + x_3$	$X_1 = x_2 + x_3$
$_2 = x_3 + x_1$	$X_2 = x_3 + x_1$	$X_2 = \bar{x}_2 + x_3 + x_1$	$X_2 = x_3 + x_1$
$_3 = x_1 + x_2$	$X_3 = x_1 + x_2$	$X_3 = x_1 + x_2$	$X_3 = \bar{x}_3 + x_1 + x_2$

SINGULAR CASES

When the discriminant of W has $[D]=0$ here is no solution of the form (5). But here is a solution of the form

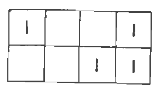
$$X_k = W^{s}{}_k(x_j) \qquad k = 1, 2, \cdots, R$$
$$j = 1, 2, \cdots, Q$$
$$f(x_j) = 1 \qquad s = 1, 2, \cdots, S \qquad (6)$$

where $f=1$ (or an equivalent condition between the independent variables) represents restr[illegible]n of the domain of definition of he fu[illegible]ns X_k. The restriction rules out ll columns in D which do not have any nonero elements. Thanks to this restriction the unctions $W_k{}^s(x_j)$ in (6) can take on any alues. This fact can be used to make the inal results algebraically simpler.

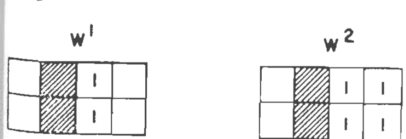
Example 3: The system given by the ingle equation

$$X_1 + x_1 = x_2$$

as the discriminant

$D =$ 

with zero value. There is no solution of the orm (5). To find it for the form (6) we rule ut the column $q=1$ containing only zeros. The condition $\bar{x}_1+x_2=1$ represents the deired restriction. The maps for W^1, W^2 beome



where the shaded area can be used to simlify the results (don't care conditions). Here e have a case with no simplification possile. The[illegible]re two solutions,

$$W^1: \quad X_1 = \bar{x}_1 x_2 \qquad W^2: \quad X_1 = x_2$$
$$\bar{x}_1 + x_2 = 1 \qquad \bar{x}_1 + x_2 = 1,$$

which cover all conditions included in the given system (see Remark 3). The solution $W^3FX_1 = x_1 + x_2$, $\bar{x}_1 + x_2 = 1$ uses the don't care condition of W^2 but is less simple.

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The Number of Equivalence Classes of Boolean Functions Under Groups Containing Negation*

INTRODUCTION

The purpose of this communication is to point out a new derivation of the number of self-complementary symmetry types and to discuss a conjecture of Elspas.[1]

In the synthesis of digital systems, it is sometimes convenient to have both a signal and its negation available. This idea suggests a modification of the conventional notion of equivalence under a group. (See Harrison[2] for a discussion of equivalence classes induced by a group.) If $\mathfrak{G}$ is a permutation group on the domain D of Boolean functions, then one usually says that a function f is *equivalent* to a function g if there exists an $\alpha \in \mathfrak{G}$ such that $f(d) = g(\alpha(d))$ for every $d \in D$. The concept of equivalence can be broadended if one allows the condition to be that $f(d) = g(\alpha(d))$ or $f(d) = \bar{g}(\alpha(d))$ for every $d \in D$. Under the proposed definition of equivalence, the Boolean functions are again resolved into equivalence classes. Ninomiya[3] has called an equivalence class of the latter type a *genus* while the conventional class is called a *symmetry type*.

In the present communication, a new method for obtaining the number of genera is presented. The problem can be recast in terms of group theory by considering negation to be obtained by the action of a negation group to be denoted by $\mathfrak{N}$. $\mathfrak{N}$ has order two; one element is the identity mapping and the other element denoted by η has the property

$$\eta: \quad f \to \bar{f}$$

for any Boolean function f.

Note that $\mathfrak{N}$ is really a permutation group on the range $\{0, 1\}$ of the Boolean functions. The situation can be summed up by saying that we are asking for the number of classes of functions under a group $\mathfrak{g}$ on the domain and $\mathfrak{N}$ on the range.

DE BRUIJN'S THEOREM

A recent theorem of DeBruijn[4] is stated which indicates an immediate solution to the problem of counting the number of classes. Let F be the class of functions from D into R, and suppose that D has s elements and R has r elements. Let $\mathfrak{G}$ be a permutation group of order g and degree s acting on D while $\mathfrak{H}$ denotes a group of order h and degree r on R.[5] Two functions f and g are said to be *equivalent* if and only if there exist elements $\alpha \in \mathfrak{G}$, $\beta \in \mathfrak{H}$ such that $f(d) = \beta(g(\alpha(d)))$ for every $d \in D$. Since this is a genuine equivalence relation, the family of functions is decomposed into equivalence classes and we desire the number of these classes. The pertinent theorem of DeBruijn is given in terms of cycle index polynomials. The cycle index polynomial is a multivariate generating function for the cycle structure of $\mathfrak{G}$ acting on D. Let $f_1, \cdots, f_s$ be s indeterminates and let $g_{j_1 j_2 \cdots j_s}$ be the number of permutations of $\mathfrak{G}$ having j_k cycles of length k for $k = 1, 2, \cdots, s$. Naturally

$$\sum_{i=1}^{s} i j_i = s. \tag{1}$$

Then define the cycle index of $\mathfrak{G}$ acting on D as

$$Z_{\mathfrak{G}}(f_1, \cdots, f_s) = \frac{1}{g} \sum_{(1)} g_{j_1, j_2, \cdots, j_s} f_1^{j_1} f_2^{j_2} \cdots f_s^{j_s}$$

where the sum is taken over all partitions of s, *i.e.* the non-negative integer solutions of (1).

We can now state DeBruijn's[4] theorem.

Theorem 1: Let

$$h_t = \exp\left\{ t \sum_{k=1}^{\infty} z_{kt} \right\} \qquad \textit{for } t = 1, \cdots, r$$

then the number of equivalence classes is given by

* Received July 2, 1962; revised manuscript received May 22, 1963. This research was supported by the United States Air Force under Contract No. AF 33(657)-7811.

[1] B. Elspas, "Self-complementary symmetry types of Boolean functions," IRE TRANS. ON ELECTRONIC COMPUTERS, vol. EC-9, pp. 264–266; June, 1960.

[2] M. A. Harrison, "Combinatorial Problems in Boolean Algebras and Applications to the Theory of Switching," Ph.D. dissertation, The University of Michigan, Ann Arbor; 1963.

[3] I. Ninomiya, "On the number of genera of Boolean functions of n variables," *Memoirs of the Faculty of Engineering of Nagoya University*, vol. 11, pp. 54–58; November, 1959.

[4] N. G. De Bruijn, "Generalization of Pólya's fundamental theorem in enumerative combinatorial analysis," *Koninklijke Nederlandse Akademe Van Wetenschappen*, vol. LXII, pp. 56–59; 1959.

[5] The *order* of a group is the number of elements in the group. The *degree* of a permutation group is the cardinality of the object set.

$$Z_{\mathfrak{G}}\left(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_s}\right) Z_{\mathfrak{H}}(h_1, \cdots, h_r)$$

evaluated at $z_1 = z_2 = \cdots = z_s = 0$.

Thus, the counting problem is solved once we know both cycle indexes. It is to be understood that the variables $(z_1, z_2, \cdots, z_s)$ in these polynomials are indeterminates. Therefore, we can differentiate formally and no questions of existence or convergence need ever arise.

Lemma 2: A term $h_1^{j_1} \cdots h_r^{j_r}$ *in* $Z_{\mathfrak{H}}$ *gives rise to*

$$Z_{\mathfrak{G}}\left(\sum_{t|1} tj_t, \cdots, \sum_{t|s} tj_t\right).$$

Proof

We compute

$$\frac{\partial}{\partial z_i}(h_1^{j_1} \cdots h_r^{j_r}).$$

This yields

$$\begin{aligned}\frac{\partial}{\partial z_i}(h_1^{j_1} \cdots h_r^{j_r}) &= \frac{\partial}{\partial z_i}\left(\prod_{t=1}^{r} \exp\left(tj_t \sum_{k=1}^{\infty} z_{kt}\right)\right) \\ &= \frac{\partial}{\partial z_i} \exp\left(\sum_{t=1}^{r}\sum_{k=1}^{\infty} tj_t z_{kt}\right) \\ &= \left(\exp \sum_{t=1}^{r}\sum_{k=1}^{\infty} tj_t z_{kt}\right) \sum_{t=1}^{r}\sum_{k=1}^{\infty} tj_t \delta_{i,kt}\end{aligned}$$

where $\delta_{i,kt}$ is the Kronecker delta function, *i.e.*,

$$\delta_{i,kt} = \begin{cases} 1 & \text{if } i = kt \\ 0 & \text{otherwise.} \end{cases}$$

Taking all the z's equal to zero gives

$$\frac{\partial}{\partial z_i}(h_1^{j_1} \cdots h_r^{j_r}) = \sum_{t|i} tj_t.$$

Applications

The specialization to Boolean functions is immediate, since the domain D becomes $\{0, 1\}^n$ and the range R is $\{0, 1\}$. The effect of complementation upon the range is to permute 0 and 1. Thus, the group $\mathfrak{H}$ on the range is $\mathfrak{N}$, the complementing group. It is immediately apparent that $\mathfrak{N}$ is the symmetric group of degree (and order) two; hence

$$Z_{\mathfrak{H}} = \tfrac{1}{2}(f_1^2 + f_2).$$

Using DeBruijn's theorem and the lemma gives

Theorem 3: If $\mathfrak{G}$ *is any permutation group on the domain of the Boolean functions, then the number of equivalence classes of Boolean functions under* $\mathfrak{G}$ *allowing complementation of the functions is*

$$\tfrac{1}{2}[Z_{\mathfrak{G}}(2, 2, \cdots, 2) + Z_{\mathfrak{G}}(0, 2, \cdots, 0, 2)].$$

We note that $Z_{\mathfrak{G}}(2, 2, \cdots, 2)$ is the total number of classes of functions under the group (cf. Harrison[2,6]). Once the cycle indexes $Z_{\mathfrak{G}}$ are constructed in order to count the classes under $\mathfrak{G}$, no additional work is required to count the number of genera.

[6] M. A. Harrison, "The Number of Transitivity Sets of Boolean Functions," *J. SIAM*; September, 1963.

The groups to be considered in the present discussion are listed below.

1) $\mathfrak{C}_2^n$ is the group of all 2^n complementations of variables. This group was first studied as a group on Boolean functions by Ashenhurst.[7]
2) $\mathfrak{S}_n$ denotes the symmetric group on the n variables, *i.e.*, the group of all permutations of input letters. The order of $\mathfrak{S}_n$ is $n!$.
3) $\mathfrak{G}_n$ is the smallest group containing $\mathfrak{C}_2^n$ and $\mathfrak{S}_n$ and is very well known.
4) $\mathfrak{GL}_n(Z_2)$ is the general linear group on the variables; the group has been studied by Slepian,[8] Harrison,[2,9] and Lechner.[10]

TABLE I

THE NUMBER OF CLASSES UNDER THE ENLARGED GROUPS

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n	$\mathfrak{C}_2^n$	$\mathfrak{S}_n$	$\mathfrak{G}_n$
1	2	2	2
2	5	6	4
3	30	40	14
4	2,288	1,992	222
5	67,172,352	18,666,624	616,126
6	144,115,192,303,714,304	12,813,206,169,137,152	200,253,952,527,184

613 614

n	$\mathfrak{GL}_n(Z_2)$	$\mathfrak{A}_n(Z_2)$
1	2	2
2	4	3
3	10	6
4	46	18
5	1,372	206
6	475,499,108	7,888,299

5) $\mathfrak{A}_n(Z_2)$ denotes the least group containing $\mathfrak{T}_2^n$ and $\mathfrak{GL}_n(Z_2)$; this group is the affine group on the variables and has been studied by Nechiporuk,[11] Harrison,[2,9] and Lechner.[10]

The cycle indexes of all these groups are derived in Harrison.[2]

Now we shall apply Theorem 3 to count the appropriate numbers for five groups.

Theorem 4: The number of classes of functions with $\mathfrak{C}_n^2$ *on the domain and* $\mathfrak{N}$ *on the range is*

$$\frac{1}{2^{n+1}}\left[2^{2^n} + (2^n - 1)2^{2^{n-1}+1}\right].$$

Proof

The result follows directly from the fact[7] that

$$Z_{\mathfrak{C}_2^n} = \frac{1}{2^n}\left[f_1^{2^n} + (2^n - 1)f_2^{2^{n-1}}\right].$$

Theorem 5: The number of classes of functions with $\mathfrak{S}_n$ *on the domain and* $\mathfrak{N}$ *on the range is exactly one half the number of classes with just* $\mathfrak{S}_n$ *on the domain.*

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[10] R. Lechner, "Affine Equivalence of Switching Functions," Ph.D. thesis Abstract, Harvard University, Cambridge, Mass.; 1963.

[11] E. I. Nechiporuk, "On the synthesis of networks using linear transformations," *Doklady Akad. Nauk*, vol. 123 December, 1958; pp. 610–612, available in English in *Automation Express*, vol. xx, pp. 12–13; April, 1959.

Proof

Since two domain elements are symmetric, every term of the cycle contains a factor f_1^k for $k \geq 2$.

Theorem 6: The number of classes of functions with $\mathfrak{GL}_n(Z_2)$ *on the domain and* $\mathfrak{N}$ *on the range is exactly one half the number of classes with just* $\mathfrak{GL}_n(Z_2)$ *on the domain.*

Proof

Every linear transformation leaves the origin invariant so every term of the cycle index contains a factor f_1^k for $k \geq 1$.

The calculated numerical results follow in Table I.

It is interesting to note the ease with which these results were obtained and to compare these methods with those of Elspas[1] and Ninomyia.[3] Both these men computed the same results but with considerable effort and only for the group $\mathfrak{G}_n$.

Self-Complementary Classes

We shall use the results of the previous section to obtain some results concerning groups without negation on the range. Suppose $\mathfrak{G}$ is a group on the domain and we desire the number of classes of $\mathfrak{G}$ closed under complementation of the functions. Clearly only classes of neutral functions can have this property. *Neutral functions* are those functions with as many ones as zeroes in their graphs, *i.e.* functions of weight 2^{n-1}.

Theorem 7: The number of classes of functions under $\mathfrak{G}$ *which are equivalent to their complements, i.e., self-complementary, is*

$$Z_{\mathfrak{G}}(0, 2, 0, 2, \cdots, 0, 2).$$

Proof

Let T_n be the number of classes of functions under $\mathfrak{G}$ alone. Note that $Z_{\mathfrak{G}}(2, \cdots, 2) = T_n$. Let N_{sc} be the number of self-complementary classes. Then

$$\tfrac{1}{2}[Z_{\mathfrak{G}}(2, \cdots, 2) + Z_{\mathfrak{G}}(0, 2, \cdots, 0, 2] = \tfrac{1}{2}(T_n - N_{sc}) + N_{sc}$$

which implies

$$N_{sc} = Z_{\mathfrak{G}}(0, 2, \cdots, 0, 2).$$

Theorem 8: The number of self-complementary classes under $\mathfrak{C}_2^n$ *is*

$$(2^n - 1)2^{2^{n-1}-n}.$$

Proof

See the proof of Theorem 4.

Theorem 9: There exists no self-complementary classes under $\mathfrak{S}_n$.

Proof

See Theorem 5.

Theorem 10: There exist no self-complementary classes under $\mathfrak{G}\mathfrak{L}_n(Z_2)$.

Proof

See Theorem 6.

Numerical calculations are reported in Table II.

TABLE II
THE NUMBER OF SELF-COMPLEMENTARY CLASSES

n	$\mathfrak{C}_2{}^n$	$\mathfrak{S}_n$	$\mathfrak{G}_n$
1	1	0	1
2	3	0	2
3	14	0	6
4	240	0	42
5	63,488	0	4,094
6	4,227,858,432	0	98,210,640

n	$\mathfrak{G}\mathfrak{L}_n(Z_2)$	$\mathfrak{A}_n(Z_2)$
1	0	1
2	0	1
3	0	2
4	0	4
5	0	30
6	0	7,679

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Elspas[1] made an interesting conjecture in relation to his solution of this problem for $\mathfrak{G}_n$ and suggested that the ratio of the number of self-complementary classes to the number of neutral classes went to zero for increasing n under the group $\mathfrak{G}_n$. This implies that the phenomenon of the self-complementary class is rather rare.

Elspas conjecture was proved true by Lorens[12] for the group $\mathfrak{G}_n$. We now derive a theorem which implies the result of Lorens and gives a condition on $\mathfrak{G}$ under which the conjecture of Elspas is true.

In order to obtain the result, a lower bound on the number of neutral classes and an upper bound on the number of self-complementary classes are needed. These bounds are obtained in the following lemma.

Lemma 11: The number of neutral classes under a group $\mathfrak{G}$ *having order g is greater than or equal to*

$$\frac{1}{g}\binom{2^n}{2^{n-1}}.$$

The number of self-complementary functions is less than or equal to

$$2^{2^{n-1}}.$$

Proof

The first part is obtained by using the well-known[12] lower bound s/g, where s is the number of objects on which a permutation group of order g acts. The result follows upon noting that there are

$$\binom{2^n}{2^{n-1}}$$

neutral Boolean functions. The second part of the theorem follows from noting that the largest term in $Z\mathfrak{G}(0, 2, \cdots, 0, 2)$ is $2^{2^{n-1}}/g$. An upper bound is certainly

$$\frac{1}{g}\sum_{\theta\in\mathfrak{G}} 2^{2^{n-1}} = 2^{2^{n-1}}.$$

The ratio we desire can now be computed, but it is convenient to have an estimate for the binomial coefficient. Using Stirling's formula, we get

$$\binom{2^n}{2^{n-1}} \sim \sqrt{\frac{2}{\pi}}\, 2^{2^n - n/2}.$$

Finally, the pertinent theorem can be proven.

Theorem 12: Let $\mathfrak{G}$ *be any group defined on the domain of the Boolean functions. If the order of* $\mathfrak{G}$ *does not exceed*

$$\sqrt{\frac{2}{\pi}}\, 2^{2^{n-1}} - \frac{n}{2} - \epsilon \log_2 n$$

for any $\epsilon > 0$, *then the number of self-complementary classes of functions tends to zero with the number of classes of neutral functions for increasing* n.

Proof

Compute the ratio for the upper bound.

As an immediate corollary to Theorem 12, we note that self-complementary classes are rather rare for the five groups that we have discussed.

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Bounds on Threshold Gate Realizability*

SUMMARY

A well-known bound estimates the number of switching functions realizable by a single threshold gate. In this communication this bound is generalized to apply to incompletely specified functions. Application is made to prove analytically an experimental result of Koford: the number of patterns discriminable by a threshold gate is twice the number of inputs (roughly). Also, Cameron's lower bound on the number of threshold gates needed in a network to realize an arbitrary function is improved. Finally, a lower bound on the number of gates needed in a two-level network is found; it is substantially lower than Koford's experimental results.

* Received February 21, 1963; revised manuscript received May 27, 1963. Most of these results were announced at the AIEE Winter General Meeting in conjunction with a delivered paper (Winder[1]). The research reported in this paper was sponsored by the Electronics Research Directorate, Air Force Cambridge Research Laboratories, Office of Aerospace Research, under Contract AF19(604)-8423.

[1] R. O. Winder, "Threshold logic in artificial intelligence," in *Artificial Intelligence*, IEEE Publication S-142; January, 1963.

It is well known[2-4] that the number of threshold functions of n arguments is less than

$$B_n \equiv 2\sum_{i=0}^{n}\binom{2^n-1}{i}.$$

Using this bound, Cameron[3] shows that the number of n-argument functions realizable by a network of at most k-threshold gates is (asymptotically) less than 2^{k^3}. As Cameron pointed out, this implies that at least one switching function of n arguments (and probably most of them) requires more than $2^{n/3}$ threshold gates for realization. The purpose of this communication is to generalize these results to functions incompletely specified; an improvement in the asymptotic bound $2^{n/3}$ will also be obtained. Applications in character recognition and self-organizing systems are discussed.

The basic bound B_n is derived from the following basic lemma (see Cameron[3] for a good discussion of its proof; the proof in Winder[5] is virtually identical, but less well explained):

Lemma: If m hyperplanes are passed through the origin of an $(n+1)$-dimensional Euclidean space, the space is divided into a number of regions—at most,

$$B_n{}^m \equiv 2\sum_{i=0}^{n}\binom{m-1}{i}.$$

The bound B_n is then obtained by considering an $(n+1)$-dimensional "realization space"—the space consisting of points $a=(a_0, a_1, \cdots, a_n)$, each of which represents the realization of some threshold function (a bias and n weights). (We assume a ± 1 logic.) By taking all possible choices of sign, we consider 2^n hyperplanes,

$$a_0 \pm a_1 \pm a_2 \pm \cdots \pm a_n = 0.$$

Two points in the realization space represent the same function if and only if they are not separated by any of these hyperplanes. Thus the regions, with boundaries defined by these hyperplanes, correspond one-to-one with threshold functions. Thus setting $m = 2^n$ in the lemma gives B_n.

Now, suppose we select exactly m out of the 2^n possible input combinations. How many switching functions, no two of which agree in value on all m of these points, can be realized by a single threshold gate? Clearly, by the same argument, there are at most $B_n{}^m$. (Because the m points correspond to m of the hyperplanes, and again, we're asking how many regions the realization space is divided into by m hyperplanes.)

[2] D. T. Perkins, D. G. Willis, and E. A. Whitmore, unpublished work at Lockheed Aircraft Corp., Missiles and Space Div., Sunnyvale, Calif.

[3] S. H. Cameron, "An Estimate of the Complexity Requisite in a Universal Decision Network," *Bionics Symp.*, Dayton, Ohio, December, 1960, pp. 197–212, WADD Rept. 60-600.

[4] R. O. Winder, "Single stage threshold logic," in *Switching Circuit Theory and Logical Design*, AIEE Special Publication S-134; September, 1961.

[5] R. O. Winder "Threshold Logic," Ph.D. dissertation, Princeton University, N. J.; May, 1962.