DMJ 6 (1940)

NUMERICAL ANALYSIS OF CERTAIN FREE DISTRIBUTIVE STRUCTURES

By Randolph Church

Consider the set Σ_n of all formal cross-cuts and unions[1] of n symbols $A_1, A_2, \cdots, A_n$. Disjoint classes which exhaust Σ_n can be formed with respect to an equivalence introduced according to the axioms of a distributive structure if a suitable axiom as to the independence of the A_i is assumed. A decision as to the equality of the classes containing arbitrary elements of Σ_n can be reached in a finite number of steps. These classes form the elements of a distributive structure, Δ_n, the *free distributive structure based on n elements.* Its elements can be represented by the unique cross-cut of unions of the A_i contained in each class. Δ_n contains a finite number of elements, $N(\Delta_n)$, its order.[2]

Dedekind[3] gave the order of Δ_n for $n \leqq 4$. The purpose of this paper is to present an analysis of $N(\Delta_n)$, $n \leqq 5$. The analysis depends on the notion of conjugate elements. Let X_1 and X_2 be two elements of Δ_n, written as the cross-cut of unions of the A_i; if there exists a permutation α of the A_i such that $\alpha X_1 = X_2$, we say that X_2 is conjugate to X_1. The relation of conjugacy is symmetric, reflexive and transitive, dividing Δ_n into disjoint sets $\{X\}$ of conjugate elements. The number of conjugates in a set $\{X\}$ is $h = n!/k$, where k is the order of $G_n(X)$, the group of degree n which leaves X unchanged. A conjugate belongs to a transformed group: $G_n(\alpha X) = \alpha G_n(X)\alpha^{-1}$. The rank of an element in Δ_n is invariant under permutations of the A_i so that the elements of a set of conjugates are of the same rank.

The facts thus sketched determine the arrangement of the following tables. The number of elements of rank r, denoted by N_r, is given in the right-hand column, so that the sum of the entries in this column is $N(\Delta_n)$. The entries in the body of a table give, for each value of r, the number of sets consisting of h conjugates. The data presented here was obtained by listing representatives

Received April 29, 1940; presented to the American Mathematical Society, April 26, 1940.

[1] For the axioms and essential properties of distributive structures, reference may be made to O. Ore, *On the foundations of abstract algebra*, I, Annals of Math., vol. 36(1935), pp. 406–437.

[2] The details of this existence proof were included in the writer's dissertation, Yale, 1935. The chains of Boolean structures (defective with respect to one unit) composing the free distributive structure, referred to at the end of this paper, were there considered in detail.

[3] R. Dedekind, *Über Zerlegungen von Zahlen durch ihre grössten gemeinsamen Teiler*, Werke II, Braunschweig, 1931, pp. 103–147; p. 147.

$n = 2$

$r \backslash h$	1	2	N_r
2	1		1
1		1	2
0	1		1
	2	1	4

$n = 3$

$r \backslash h$	1	3	N_r
6	1		1
5		1	3
4		1	3
3	1	1	4
2		1	3
1		1	3
0	1		1
	3	5	18

$n = 4$

$r \backslash h$	1	3	4	6	12	N_r
14	1					1
13			1			4
12				1		6
11			1	1		10
10	1				1	13
9				1	1	18
8		1	1		1	19
7			3		1	24
6		1	1		1	19
5				1	1	18
4	1				1	13
3			1	1		10
2				1		6
1			1			4
0	1					1
	4	2	9	6	7	166

$n = 5$

$r \backslash h$	1	5	10	12	15	20	30	60	120	N
30	1									
29		1								
28			1							
27			2							
26		1					1			
25	1						2			
24			2		1			1		
23			1		1	2	1	1		1
22			1		1	2	1	2		2
21		1	2		1		1	4		3
20		1	1	1			2	5		3
19		1	1		1	1	2	6		4
18			1			2	4	4	1	5
17					2	2	3	5	1	5
16		1	1			1	3	6	1	6
15	1	2	2			1	3	6	1	6
14		1	1			1	3	6	1	6
13					2	2	3	5	1	5
12			1			2	4	4	1	5
11		1	1		1	1	2	6		4
10		1	1	1			2	5		3
9		1	2		1		1	4		3
8			1		1	2	1	2		2
7			1		1	2	1	1		1
6			2		1			1		
5	1						2			
4		1					1			
3			2							
2			1							
1		1								
0	1									
	5	14	28	2	14	21	43	74	7	75

NUMERICAL ANALYSIS OF CERTAIN FREE DISTRIBUTIVE STRUCTURES

By Randolph Church

Consider the set Σ_n of all formal cross-cuts and unions[1] of n symbols $A_1, A_2, \cdots, A_n$. Disjoint classes which exhaust Σ_n can be formed with respect to an equivalence introduced according to the axioms of a distributive structure if a suitable axiom as to the independence of the A_i is assumed. A decision as to the equality of the classes containing arbitrary elements of Σ_n can be reached in a finite number of steps. These classes form the elements of a distributive structure, Δ_n, the *free distributive structure based on n elements.* Its elements can be represented by the unique cross-cut of unions of the A_i contained in each class. Δ_n contains a finite number of elements, $N(\Delta_n)$, its order.[2]

Dedekind[3] gave the order of Δ_n for $n \leqq 4$. The purpose of this paper is to present an analysis of $N(\Delta_n)$, $n \leqq 5$. The analysis depends on the notion of conjugate elements. Let X_1 and X_2 be two elements of Δ_n, written as the cross-cut of unions of the A_i; if there exists a permutation α of the A_i such that $\alpha X_1 = X_2$, we say that X_2 is conjugate to X_1. The relation of conjugacy is symmetric, reflexive and transitive, dividing Δ_n into disjoint sets $\{X\}$ of conjugate elements. The number of conjugates in a set $\{X\}$ is $h = n!/k$, where k is the order of $G_n(X)$, the group of degree n which leaves X unchanged. A conjugate belongs to a transformed group: $G_n(\alpha X) = \alpha G_n(X)\alpha^{-1}$. The rank of an element in Δ_n is invariant under permutations of the A_i so that the elements of a set of conjugates are of the same rank.

The facts thus sketched determine the arrangement of the following tables. The number of elements of rank r, denoted by N_r, is given in the right-hand column, so that the sum of the entries in this column is $N(\Delta_n)$. The entries in the body of a table give, for each value of r, the number of sets consisting of h conjugates. The data presented here was obtained by listing representatives

Received April 29, 1940; presented to the American Mathematical Society, April 26, 1940.

[1] For the axioms and essential properties of distributive structures, reference may be made to O. Ore, *On the foundations of abstract algebra*, 1, Annals of Math., vol. 36(1935), pp. 406–437.

[2] The details of this existence proof were included in the writer's dissertation, Yale, 1935. The chains of Boolean structures (defective with respect to one unit) composing the free distributive structure, referred to at the end of this paper, were there considered in detail.

[3] R. Dedekind, *Über Zerlegungen von Zahlen durch ihre grössten gemeinsamen Teiler*, Werke II, Braunschweig, 1931, pp. 103–147; p. 147.

$n = 2$

r \ h	1	2	N_r
2	1		1
1		1	2
0	1		1
	2	1	4

$n = 3$

r \ h	1	3	N_r
6	1		1
5		1	3
4		1	3
3	1	1	4
2		1	3
1		1	3
0	1		1
	3	5	18

$n = 4$

r \ h	1	3	4	6	12	N_r
14	1					1
13			1			4
12				1		6
11			1	1		10
10	1				1	13
9				1	1	18
8		1	1		1	19
7			3		1	24
6		1	1		1	19
5				1	1	18
4	1				1	13
3			1	1		10
2				1		6
1			1			4
0	1					1
	4	2	9	6	7	166

$n = 5$

r \ h	1	5	10	12	15	20	30	60	120	N_r
30	1									1
29		1								5
28			1							10
27			2							20
26		1					1			35
25	1						2			61
24			2		1			1		95
23			1		1	2	1	1		155
22			1		1	2	1	2		215
21		1	2		1		1	4		310
20		1	1	1			2	5		387
19		1	1		1	1	2	6		470
18			1			2	4	4	1	530
17					2	2	3	5	1	580
16		1	1			1	3	6	1	605
15	1	2	2			1	3	6	1	621
14		1	1			1	3	6	1	605
13					2	2	3	5	1	580
12			1			2	4	4	1	530
11		1	1		1	1	2	6		470
10		1	1	1			2	5		387
9		1	2		1		1	4		310
8			1		1	2	1	2		215
7			1		1	2	1	1		155
6			2		1			1		95
5	1						2			61
4		1					1			35
3			2							20
2			1							10
1		1								5
0	1									1
	5	14	28	2	14	21	43	74	7	7579

of the classes of conjugates together with the number of conjugates in each class and determining the corresponding ranks by obtaining the structure inclusion relations between the elements of the various classes. The sets of conjugates thus obtained were checked by converting the representative elements into their duals, and by arranging the elements of the structure in chains of Boolean structures according to general theory.

It is considered desirable to make the results here given available now instead of including them with related investigations as originally planned when the analysis of Δ_5 was completed in 1936.

United States Naval Academy.