#91

March 24, 1976

Suggests ~~new~~ changing names of various sequences

— the rest is probably a "Bart" letter

3-24-76 (1)

John Norris Tangen

Phone 1-617-729-3822

16 Winslow Road

Winchester, Mass. 02138

666
88
595
273

1930

Dear N J A Sloane

In regard to the opening pages

of your 1973 book:

A Handbook of Integer Sequences

I think a serious terminological problem has

arrisen. First of all this is, of course, my own

opinion. But also please note that

Frank Harary and Edgar Palmer

in their 1973 book Graphical Enumeration

have (wisely) described digraphs not as

"reflexive relations"

but (because no loops are allowed) as irreflexive

relatxtions. Surely they are correct. The concept

of a loop is properly identified with xRx, the

relation which is reflexive

Thus one includes all of the "connected" digraphs,)

plus all of the completely trivial digraphs,

consisting only of n points. But there is one

overlap, o is counted as "connected"

o o is not counted as connected but

we count it in our set of 3 because it

is trivial. Of the 16 digraphs for

3 points, 13 are connected, and one is

trivial { o ⁰ o } = completely trivial

so we are counting 3 of 3

14 of 16

1 + 199 of 218 = 200 of 218

200 + 14 + 3 = 217, unfortunately the

digraph o is counted twice, once as

trivial, once as connected, thus the confusion,

1 + 3 + 14 + 200 = 218

2/

I propose a revision of terminology which would make use of the concept of a loop.

You describe symmetric relations as "graphs with loops of length one allowed". I think such relations, with loops of length one allowed, should be called symmetric reflexive relations, hence, by your terminology, "graphs".

N646   2, 6, 20, 90, 544 ...; $2^{\frac{n^2+n}{2}}$

(666) The relations usually described as graphical would hereafter be called symmetric graphs

N479   1, 2, 4, 11, 34, 156, 1044, 12346, ... ; $2^{\frac{n^2-n}{2}}$

(88) When we refer to "digraphs with loops of length one allowed" we should say "reflexive relations", in deference to the loops, (x R x), which are allowed

595   2, 10, 104, 3044, — ; 2^{n^2}

Mere "digraphs" then are truly unrestricted, yet less numerous, ..

15

In two-dimensional space we have 7 strip patterns for which the dimension is, strictly speaking, (per Coxeter) dimension $\frac{3}{2} = 1\frac{1}{2}$, rather than 2; and we have 17 discrete groups of direct isometries for 2-d space. Thus each of the 7 strip patterns can multiply the 17 group (full) patterns to give us 119 imaginary unity elements, each of which has a conjugate (projection), inasmuch as the space is 3-dimensional, with digraphs for 4 or ~~fewer~~ points. The numbers of (Connected) digraphs for 1, 2, 3, 4 points are $1+2+13+199 = 215$, so that

$$1+1+1+2+1+13+1+199 = 219$$

where 219 is Coxeter's Number of "Purely Geometric" GROUPS

3./

(273)

1, 3, 16, 218, 9608, — — ; 2^{n^2-n}

I am now in a position to set out the proposed new terminology

symmetric graphs

symmetric only (no loop)

N479 (88) 1, 2, 4, 11, 34, 156, 1044, 12346 ; $2^{\frac{n^2-n}{2}}$

graphs

symmetric reflexive (loops allowed)

(666) 2, 6, 20, 90, 544, ... ; $2^{\frac{n^2+n}{2}}$

looped digraphs

reflexive relations (loop allowed)

(595) 2, 10, 104, 3044, ... ; 2^{n^2}

digraphs

unrestricted relations (no loops)

(273) 1, 3, 16, 218, ... ; 2^{n^2-n}

change some names

19

We now have $2 + 6 + 6 = 14$

There are 6 remaining 3 point digraphs, which are complicatedly connected but not maximally connected

~~[illegible]~~

$$\begin{array}{l} \frac{1+j+k+ie}{2}, \frac{1-j-k-ie}{2} \\[2ex] \frac{1+j-k+ie}{2}, \frac{1-j+k-ie}{2} \\[2ex] \frac{1+j+k-ie}{2}, \frac{1-j-k+ie}{2} \end{array} \left(\begin{array}{ll} \frac{1+j+k+i(e)}{2}, & \frac{1-j-k-i(e)}{2} \\[2ex] \frac{1+j-k+i(e)}{2}, & \frac{1-j+k-i(e)}{2} \\[2ex] \frac{1+j+k-i(e)}{2}, & \frac{1-j-k+i(e)}{2} \end{array} \right)$$

These 6, plus the 218 digraphs for 4 points give us the 224, hence we have

$2+6+6+224 = 238$. Since the 238 imaginary unity elements exist in 119 pairs, we note that

$119 = 7(17)$

4/

These digraphs 1, 3, 16, 218 are extremely useful for work with algebras because if we assume an abstract entity $(+1) = 1$ then $1 = +1$ and $0 = 111 = -1$ gives us the second unity element which is necessary for real algebra

$$1 + 3 = 4 = \left| \{\pm 1, \pm \sqrt{-1}\} \right|$$

so we have the necessary units for complex (Gaussian) algebra

$$1 + 3 + 16 = 20$$

so we add the units $(\pm j)$ and $(\pm k) = \pm (ij)$

Thus the 16 digraphs for 3 points correspond to the 16 quaternion unity elements

$$\tfrac{1}{2}(\pm 1 \pm \sqrt{-1} \pm j \pm k)$$

Thus $\pm j, \pm k, +1 + 3 + 16 = 24$

Thus we have the 24 unity elements of quaternion algebra

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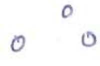
Some progress seems possible

o and o o are trivial

$\circ \to \circ$ and $\circ \rightleftarrows \circ$ can be grouped

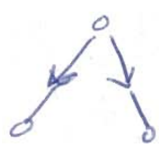
with  and  and 

and 

Of the 16 digraphs for 3 points, we find that 3 are not connected 

whereas one is maximally connected 

So 12 remain, in 6 pairs, to complete our set of 6 we select

, , , ,

, and 

the latter is cyclic (clockwise)

5/

When we come to Octave algebra it is clear that

$1+3+16+218 = 238$

We can draw 238 digraphs for 4 or fewer points, so we only need add the abstract real units, ± 1, in order to have the 240 unity elements necessary for the octave algebra. It has always been argued that Octave algebra is Distributive, (Reflexive), because the Norm of Products is equal to the Product of Norms

but now we have a form of Octave algebra which is nondistributive, since we can draw each unity element in "graphical" form as digraphs with no loops allowed, and points unlabelled.

Thus an imaginary unity element not only corresponds to a digraph, it __is__ a digraph,

for 4 or fewer points

$$1 = \sqrt{-1} = i$$

$$3 = j, k, e$$

$$16 = -i, -j, -k, -e$$

$$\pm i(e), \pm j(e), \pm ke$$

$$\left\{\begin{array}{cc} & 4 \\ +6 & 10 \\ +6 & 16 \end{array}\right\}$$

plus 6 of the 224

described by Coxeter in his classical (1946) paper on octaves, The other 218, of the 224, require 4 points for digraphical representation.

Ideally we would wish for a sum

$$2 + 6 + 6 + 224 = 238$$ instead of

$$1 + 3 + 16 + 218$$

$$2 = \pm\sqrt{-1}\ ;\ 6 = \begin{matrix}\pm j \\ \pm k \\ \pm e\end{matrix}\ ;\ 6 = \left\{\begin{matrix}\pm i(e) \\ \pm j(e) \\ \pm k(e)\end{matrix}\right\}$$

By restricting the number of points to
powers of 2, here 1, 2, and 4, we get
1+3+218 = 222, which is the number
of crystal classes or molecular "point groups"
in 4-d space
corresponding to the 32 such known for 3-d space,
I would predict that in 4-d space the number
of atomic space groups would be the number of
digraphs (no loops) which are possible
for 5 (or perhaps 8) points:
For 5 points we have 9608 digraphs
For 8 points we have 1,793,359,192, 848 digraphs
The analogous number for 3-d space is 218,
which is close enough to the 219 sum computed by
Coxeter.

$$\circ = \sqrt{-1}$$

$$\circ \quad \circ = -\sqrt{-1}$$

$$\circ \rightarrow \circ = \frac{1+\sqrt{-1}}{\sqrt{2}} = \frac{1+\circ}{\sqrt{2}}$$

$$\circ \rightleftarrows \circ = \frac{1-\sqrt{-1}}{\sqrt{2}} = \frac{1+\circ\circ}{\sqrt{2}} = \frac{1-\circ}{\sqrt{2}}$$

It is only in the case of the quaternion algebra that we need 4 abstract units, plus the 20 digraphs for 3 or fewer points. In the case of the octave algebra we need only the usual two abstract units, ± 1, (of real algebra) plus the 238 digraphs for 4 or fewer points. Thus each of the 238 digraphs is an imaginary unity element of octave algebra

It is customary to say that we have

$65 + 165 = 230$ atomic space groups

in 3-d space. But, as Coxeter has pointed out, since 22 of the 65 exist in 11 enantiomorphic pairs the proper sum from the standpoint of pure geometry is $54 + 165 = 219$.

Thus if we have digraphs for any (n^2) number of points, $n = 1, 2$, then we have $1 + 218 = 219$, as required

We should look at the 218 digraphs (They are displayed in the 1969 Harary book on Graph Theory)

and see if we can select out 53,

$1 + 53 + 165 = 1 + 218 = 219$

Thus the "digraph" for a single point corresponds to a rotation or a translation or a "reflection" but not to a "reflexive" relation —

Since 6 circles (2-d spheres) can be lattice packed about a central equal circle in 2-d space ("6 pennies on a table top")

I have always argued that

$$\frac{1+\sqrt{-1}}{\sqrt{2}} \quad \text{and} \quad \frac{1-\sqrt{-1}}{\sqrt{2}}$$

are "units" of Gaussian (Complex) algebra

$$\frac{-1+\sqrt{-1}}{\sqrt{2}} \qquad \frac{-1-\sqrt{-1}}{\sqrt{2}}$$

$$\frac{+1+\sqrt{-1}}{\sqrt{2}} \qquad \frac{+1-\sqrt{-1}}{\sqrt{2}}$$

along with ± 1 and $\pm\sqrt{-1}$

This would be ± 1 plus 4 digraphs
$$\left\{\begin{matrix} \circ \\ \circ \quad \circ \\ \circ \rightarrow \circ \\ \circ \rightleftarrows \circ \end{matrix}\right\} \quad \begin{matrix} \sqrt{-1} \\ -\sqrt{-1} \\ \frac{1+\sqrt{-1}}{\sqrt{2}} \\ \frac{1-\sqrt{-1}}{\sqrt{2}} \end{matrix}$$

8/

No doubt clarification is needed in regard to not only "digraphs" but also in regard to reflections as operations and to reflexivity as an (abstract) relation

Despite numerous **books** on reflections, including **high school texts on the geometry of reflections**, the concept of a reflection, in relation to the concept of reflexivity, remains less clear than it should be

" $x R x$ " implies a loop or a rotation.

$x R x$ implies, algebraically:

$$x\,(x^{-1}x) = x = x\,(x\,x^{-1}) = (x\,x^{-1})\,x = (x^{-1}x)\,x$$

$$X + (x + (-x)) = X + 0 = X$$

$$X \text{ times } (X \text{ times } X^{-1}) = X \text{ times } 1 = X$$

$x + yz$; $\{x\,(y+z) - (xy+xz)\} + x = x$

$$x + \{x\,(y+z) - (xy+xz)\} = x$$

over

9

$$x + \{(x + yz) - (x+y)(x+z)\} = x$$

if $x = 1$ and $y = \pm\sqrt{-1}$, $z = \mp\sqrt{-1}$

$$x \text{ times } \left\{\frac{x + yz}{(x+y)(x+z)}\right\} = x \quad \left\{\begin{array}{l} x = 1 \\ y = \pm\sqrt{-1} \\ z = \mp\sqrt{-1} \end{array}\right\}$$

$$x \text{ times } \left\{\frac{x(y+z)}{xy + x\cancel{z}}\right\} = x$$

$$\left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^2 = +\sqrt{-1}$$

$$\left(\frac{1-\sqrt{-1}}{\sqrt{2}}\right)^2 = -\sqrt{-1}$$

$\frac{2i}{\sqrt{2}} = \sqrt{2}i$

$$\left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)\left(\frac{1-\sqrt{-1}}{\sqrt{2}}\right) = 1$$

$$\left(\frac{1+\sqrt{-1}}{\sqrt{-2}}\right)\left(\frac{1-\sqrt{-1}}{\sqrt{-2}}\right) = -1$$

The counting $1+3+14+200$

includes all trivial digraphs $\circ$, $\circ\circ$, $\overset{\circ}{\circ\ \circ}$, $\overset{\circ\ \circ}{\circ\ \circ}$,

plus all connected digraphs. Probably the list

of connected digraphs should not include $\{\circ\}$.

So $\quad 0+2+13+199 = 214$

$\quad\quad 1+1+1+1 = \underline{\ \ 4\ \ }$

$\quad\quad\quad\quad\quad\quad\quad\quad 218$

In that case we can match

218 digraphs ~~with~~ which are trivial or connected

with 218 digraphs (connected or not) for 4 points

Note that if we have an even number of points, the

connected digraphs are $2+199 = 31+90+65+15$

$$= \sum_{b=1}^{4} \left(\sum_{n=0}^{b+1} \frac{(-1)^n\,(b+1-n)^6}{(n)!\,(b+1-n)!} \right)$$

$$= S(6,2)+S(6,3)+S(6,4)+S(6,5)$$

Clearly we have $2(2 + 10 + 104) = 232$.

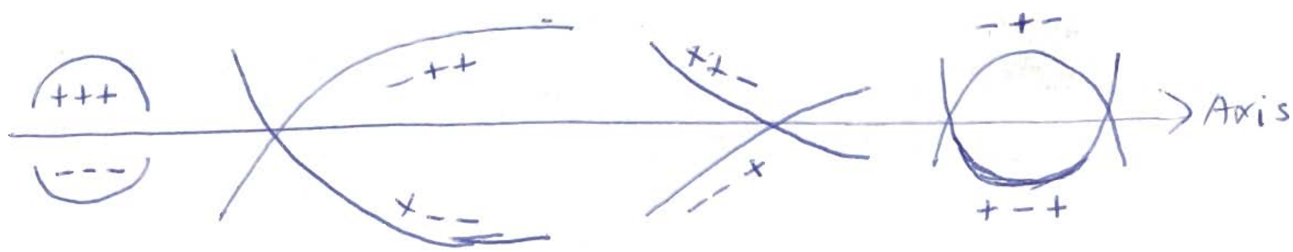
The other eight are abstract, such as

~~± ± √ ±~~

$+1$, $+\sqrt{-1}$, $+j$, $+k$, $+e$, $+i(e)$, $+j(e)$, $+k(e)$

The eight permutations two things (+ and −) taken 3 at a time

Thus with 3 moments or points minimally required to establish oscillation or curvature we have 8 forms (abstract) in 4 pairs



$+1 = +++$
$+\sqrt{-1} = ---$
$j = -++$
$k = +--$

$e = ++-$
$i(e) = --+$
$j(e) = -+-$
$k(e) = +-+$

Cheers, John Targer

These are the non-trivial, nonvoid disjoint subsets for 6 = 3! labelled elements (points) distributed into unlabelled (symmetric, transitive, reflexive) subsets

digraphs are not symmetric
not transitive
not reflexive

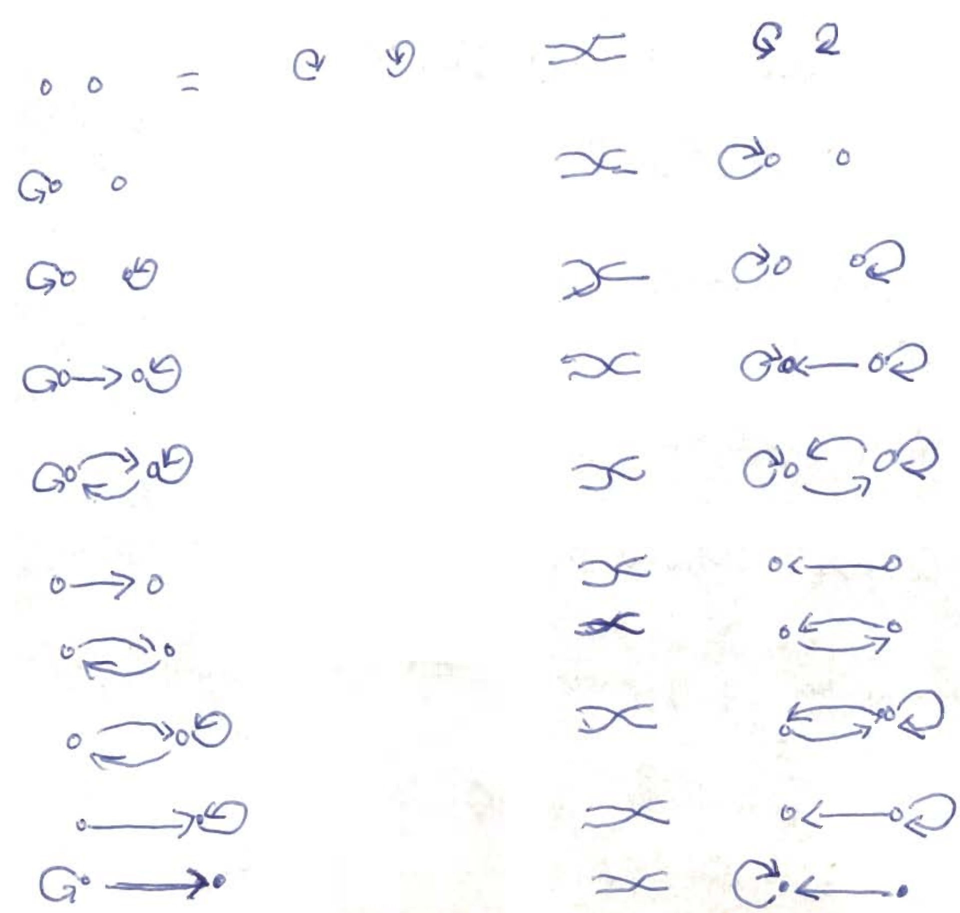
So we see that, subject to conditions specified above, we can remove the labels from all 6 of a set of 3! = !(3) = 6 points and remove 2 of the 4 points and then permit digraphical (connected) RELATIONS, which turn out to be UNRESTRICTED, neither symmetric nor transitive nor reflexive — (no loops allowed). — for 4 or fewer (even) numbers of points, if the number is odd, we get the 13 lattice packed spheres, in digraphical form all 13 digraphs are connected.

$$2^3 + 2(2+10+104) = 2(4) + 2(2+10+104)$$

Each such looped digraph represents a "pair" (a conjugate pair) of algebraic elements, because every left loop (counterclockwise) is similar to a right loop (clockwise)

For 2 points the 10 relations have

10 enantiomorphs



$2 + 13 = 15$ equals 3-d simplex (tetrahedron)
including the center –

Now if we think of just 232 of the 238
digraphs or unity elements, and if we think
of them as existing in 116 pairs, then
each such pair is represented by the
reflexive relations (loops of length one allowed)
for 1, 2, 3 unlabelled points

$$2 + 10 + 104 = 116$$

The 8 extra units $\quad 8 + 232 = 240 = 120$ pairs

exist in 4 pairs, as shown above

$$\pm 1 \qquad \pm\sqrt{-1}$$

$$\frac{+1 \pm \sqrt{-1}}{\sqrt{2}}, \qquad \frac{-1 \pm \sqrt{-1}}{2}$$

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So we have seen how a theory of digraphs, with or without loops, can be added to a theory of abstract algebraic "elements"

such as ± 1,

to give us the 240 unity elements of the (Degen-Graves-Cayley) "Octave" Algebra,

whilst the number of unlabelled points decreases from 4 to 3,

$$1 + 2 + 10 = 13$$

$$2 + 4 + 20 = 26$$

$$2 + 2 + 10 + 2(1 + 2 + 10 + 104) = 248$$

$$(10 - 2) + 2(2 + 10 + 104) = 240$$

The idea that a unity element in an algebra should be a digraph (looped or not), on a few points, adds great geometric clarity to algebra. over

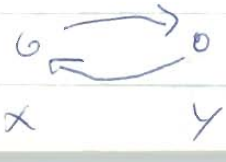
April 13, 1975

4-13-75

Dr NJA Sloane,

The terminology used in your *Handbook of Integer Sequences* (1973) is not only in conflict with such texts on graph theory as those of Harary but also is in conflict with common sense. For example Harary suggests that digraphs are irreflexive rather than reflexive insofar as we refer to digraphs which do not include loops i.e. to your series 1, 3, 16, 218, 9608, ...

After all, since a reflexive relation relates x to x : i.e. xRx, it is a loop which most directly relates x to x.

The figure 

allows X to indirectly relate to X, but only via Y.

29

The series

1	3	9	33	139	718	4535
1	3	7	18	52	208	1252
		2	15	87	510	$\boxed{3283}$

suggest that the graphs for 5 or fewer points form a subsets, i.e. $s \geq 3$, of the possible topologies for 5 points

1	4	13	46	185	903	5438
1	3	7	18	52	208	1252
	1	6	28	133	695	4186
						863
		7	35	168	863	5049

34	154	874	$7! + 7 + 2$
$\sum_{s=0}^{4} s!$	$\sum_{s=0}^{5} s!$	$= \sum_{s=0}^{6} s!$	$1! + 2! + 3! + 7!$

$28 + 133 + 695 + 4186 = 2! + 7! = 5042$

161   856 $\frac{856}{5042}$

2/

Thus the proper word and concept to describe

$\circ \rightleftarrows \circ$ is $xRy \Rightarrow yRx$ which is, as you say,

and as all seem to agree, the symmetric relation

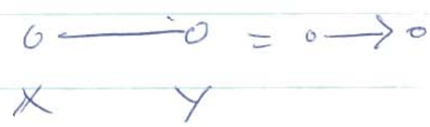
I have no quarrel with the word and concept of

transitivity to describe

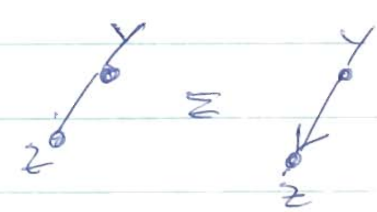
$$xRy \text{ and } yRz \Longrightarrow xRz$$

but all this implies to me, graphically,

is that if there is a line from x to y



and a line from Y to Z



that this implies a line from x to z

23

After much work I have concluded that we can identify series with RELATIONS implied in a rigorous and unambiguous manner, for unlabelled (undistinguished) points

595 2, 10, 104, 3044, 291,968 ; symmetric reflexive (looped digraphs)

1930 1 3 9 33 139 ; symmetric transitive / transitive reflexive

666 2 6 20 90 540 ; reflexive; looped graph

273 1 3 16 218 9608 ; symmetric; digraphs

88 1 2 4 11 34 ; unrestricted (or restricted

1	4	13	46	185	903	5438
1	3	9	33	139	718	4535
1	2	4	11	34	156	1044
	1	5	22	105	562	3491
	1	6	28	133	695	4186
	3−1=2	9−6 =3	33−28 =5	139−133 =6	718−695 =23	4535−[illegible]6 =349

$= \frac{1+3+9+33}{2}$

$349 < \frac{718}{2}$

$349 > 695/2$

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[dl] \\ z & \end{tikzcd}$$

$+\sqrt{-1}$ $-\sqrt{-1}$

-1

inversion is the operation which relates x and y

because $\pm\sqrt{-1}$ are not only reverse but also inverse in respect to each other

when this transitivity exists, it is exemplified

by associativity (metric transitivity), and

$$xy + z = (x+z)(y+z) = \frac{(xy+yx)}{x}$$

$$(x+y)z = (xz) + (yz) = \frac{xy-yx}{x}$$

Suppose $x = 1$; $y = \sqrt{-1}$; $z = \sqrt{0}$, such that $z^2 = 0$

Then $xy = yx$, $yz = zy$, $xz = zx$

$$x(y+z) = xy + xz = y + z$$

$\cancel{x(xy)}$ $(xy)z = y$; $x(yz) = y = (yz)x$

$(xy)z = (yz)x$

22

$4 + 6 + 21 + 165 = 196 = 14^2$

$8 \quad 12 + 42 + 165 = 227$

62

$3 + 8 + 12 + 42 + 165 = 5(2)\{3+8+12\}$

Consider 5 operations, 3 or fewer points

$5(4 + 42) = 230$

$4 + 42 + 165$

46

$$\begin{array}{r} 165 \\ 46 \\ \hline 211 \end{array}$$

translations, rotations, reflections

plus 2 types of screw displacements

5 { zero points + 2 points }

4

$\varepsilon = \sqrt{0}$

$i = \sqrt{-1}$

$1 = \sqrt{+1}$

$$(\varepsilon + i)(\varepsilon + i) = (i + \varepsilon)(i + \varepsilon) = 0 + 2i(\varepsilon) - 1$$

$$= x(2) - 1$$

$$= -1 + i(2) = (i+\varepsilon)(i+\varepsilon)$$

$$(\varepsilon + 1)(\varepsilon + 1) = (1 + \varepsilon)(1 + \varepsilon) = 0 + 2 + 1 = 3$$

$$(1 + \varepsilon)(1 + \varepsilon) = (\varepsilon + 1)(\varepsilon + 1) = 1 + 2 + 0 = 3$$

$$(1 + i)(1 + i) = 2i$$

$$(i + 1)(i + 1)$$

$$(i + \varepsilon)(i + \varepsilon) = -1 + 2i + 0$$

$$(1 + i + \varepsilon)(1 + i + \varepsilon) = (\varepsilon + i + 1)(\varepsilon + i + 1) = \begin{array}{cccc} 1 & + i & + \varepsilon & \\ -1 & i & \cancel{} & i\varepsilon \\ & & \varepsilon & \varepsilon i + 0 \\ \hline 0 & 2i & + 2\varepsilon & + 2i\varepsilon \end{array}$$

$$2 + i(4) = 4i + 2$$

21

So that only the case of two points,

dimension (1) is excluded, whereas

dimensions (+1), 0 and +2 are included

The 4 sets of rules are

I graphical set —— line ---- broken line
II digraphical set ⟶ directed line = dual ⟷
III looped graph ⊙ circle (neutral rotation)
IV looped digraph ⟳ directed circumference (directed rotation)

It is possible to get 154 relations, thus,

for zero, one and three points

And 165 relations for two and three points

$$165 + 6 + 4 = 175 = \{p(15) - 1\} = \{p(10+3+2)\} - 1$$

$$154 + (6+3+2) = 165 = 4 + 6 + 144 + 2+3+6$$

The 10 maximal relations on 2 points can be exploded

5.

$$(2+i4) = \sqrt{(2+i4)(2+i4)}$$

$$= \sqrt{4+16i-16}$$

$$= \sqrt{-12+16i}$$

$$(R+iR+\varepsilon R)(S+iS+\varepsilon S)$$

$RS + i\cancel{RS} + \varepsilon RS$

$-RS + iRS$

$i\varepsilon RS$

$+0 \quad \varepsilon i RS \quad \varepsilon RS$

$$i(4RS) + RS = RS + i(4RS)$$

$$(A+iB+\varepsilon C)(X+iY+\varepsilon Z) =$$

$AX + iAY + \varepsilon AZ$

$-BY \quad iBX$

$i\varepsilon BZ \quad \varepsilon CX$

$+0 \quad \varepsilon i CY$

$$i(AY+BX+BZ+CY) + \varepsilon(AZ+CX) = \sqrt{(-)(X)} + \sqrt{0}$$

20

The arrow indicates ~~wh~~whether the limit (being zero) is being approach from a positive ~~counter~~ (clockwise) or negative (counterclockwise) direction. Thus the theory of discrete groups of isometries in 3-d space is considerably simplified when we draw precise zeroes or limit points for each group.

In the simplest case 11 (14) = 154 we see that if we have relations defined (abstractly or concretely) for zero points and 4 sets of rules, then we get

$$4 + 6 + 144 = 154 = 11\,(14)$$

Clearly

$$\{(A+iB+\varepsilon C)(x+iy+\varepsilon z)\}^2 = -\{(Ay+By+Bz+Cy)^2\}$$

$$= -\{(Ay+B(y+z)+Cy)^2\}$$

$$= -[\{y(A+C)+B(y\pm z)\}^2]$$

We don't really have transitivity unless we have 3 units which behave in 3 distinct trajectories. A unit $i(B)$ is the square root of any negative number of the form

$$-N = -\{B^2\}$$

and a unit $\varepsilon\beta$ is the square root of any form of zero, such as

$$o(N) = o(B^2)$$

where 384 is the maximum number of linear spaces in reference to a single (planar) figure in the set given on p 16 of your book. Coxeter's ideas can thus be refined still further, we can argue that we have just 27 + 165 = 192 pure groups (geometrically) when we speak of discrete groups of isometries in 3-d space

$$27 = 3 + 24$$
$$165 = 3 + 162 = 3 + 3(54)$$
$$192 = 6 + 186 = 3 + 27(7) = 6 + 6(31)$$

These can be regarded, also, as 384 discrete groups in 192 enantiomorphic pairs. When a point is represented by an arrow, it is a limit point,

7.

A dual unit $A + iB$

is a square root of $(A+iB)(A+iB) = A^2 - B^2 + i(2AB)$

$= C + i(2AB)$

$C = A^2 - B^2$

$(A+B)(A-B)$

$= (A+B)(A-B) + i(2AB)$

That is $A + i\beta$ can be a square root of any

Real Number $R = C = (A+B)(A-B)$

Plus $i\sqrt{2AB}$

$(5 + i10)^2 = -75 + i(100)$

$(10 + i5)^2 = +75 + i(100)$

The real part of $(A+iB)^2$ is negative or positive

depending on whether $B > A$ or $A > B$

In the case of the 10 pairs of looped digraph relations I did not draw all of the details, but I am sure it is clear as to how the symbolism and structure is reversed,

giving $(2^2)^2 + (2^2)$

Thus 54 groups in just 2 points ~~or~~ 1 point attained in this manner.

When we consider the 165 groups in 2 or 3 points, the 21 referring to 2 points are written with an extra (isolated) point on each background.

If we paired the 165 we would get

$165 + 165 + 54 = 384 = 2(4)6(8)$

$= 3(2^7)$

8.

$$(A+\varepsilon B)^2 = (A+\varepsilon B)(A+\varepsilon B) = A^2 + 2AB$$

$$A + \varepsilon\{(AB)^2\}$$

Finally

$(A + iB + \varepsilon C)$ is the square root of

[illegible]

$$(A+iB+\varepsilon C)(A+iB+\varepsilon C) = A^2 + iAB + AC$$
$$+ iAB \qquad -B^2$$
$$+ iBC$$
$$iCB + \cancel{CA}$$

$$= A^2 - B^2 + 2AC + i(2AB) + i(2BC)$$

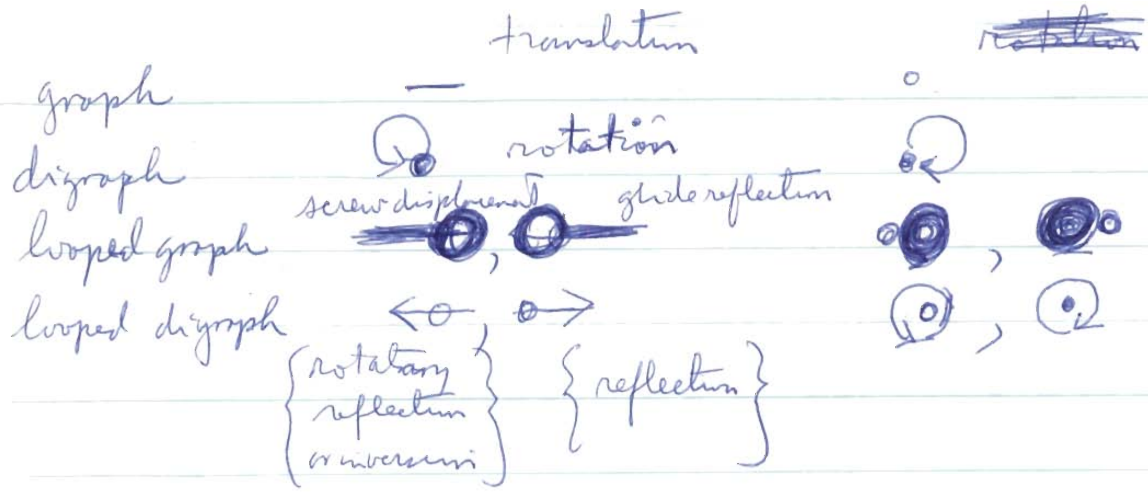
$$A + iB + \varepsilon C = \sqrt{A^2 - B^2 + 2AC + i\{(2AB) + (2BC)\}}$$

This is complicated but can be any real number plus $i\{(2AA + 2BC)\} = i\{2B(A+C)\}$

3(10) 10(5)

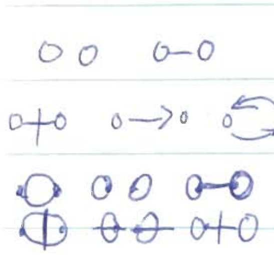
17

I will draw the 54 relations in 27 pairs



These are six pairs of one point relations

The 21 pairs of two point relations are

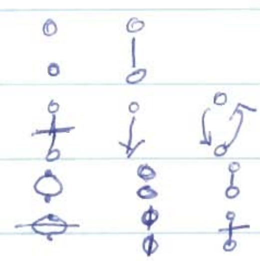


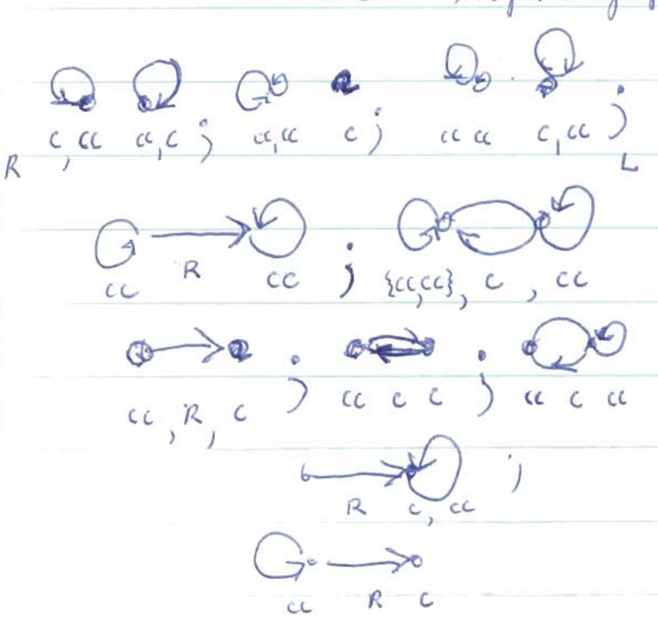
2 graph + 2

3 digraph + 3

6 looped graph + 6

10 looped digraph + 10





reverse all of the symbolism & structure

9

That is $(A + iB + \varepsilon C)$ is the square root

of a complex number, just as is $(A + iB)$

whereas $(A + \varepsilon B)$ is the square root of the

real number $A^2 + 2(AB)$

Note that $(A + iB + \varepsilon C)$

is the square root of a complex number

for which the imaginary part is even

<u>not odd</u> !!!!

Note that

$A^2 + AB$ is a narrow set of real numbers

whereas

$A^2 + B^2 + 2AC + i\{2B(A+C)\}$

reaches a wider class of reals

plus even imaginaries, hence a wider class of complex n.

we have 22 which are known to exist in 11 enantiomorphic pairs, so that Coxeter argues in favor of just 54 such groups. It is, thus, easy for the rest of us to see that these 54 groups exist in 27 pairs, since we allow each of the 6 relations on a single point to have an analog which is, however, a relation between 2 points. This leaves 15 relations between 2 points which have no analogs for relations ~~on~~ a single point

$6+6+6+6+15+15=54$.

That is, we simply allow each of 27 relations to have an enantiomorphic relation,

10

The maximum number of relations amongst

3 unlabelled (undistinguished points) is

$$104 = 2\{\beta(3)\} = 2^3 \mathcal{E}(3) = 2^3(1+2+10)$$

$$= 2^3\{U(0)+U(1)+U(2)\}$$

$$= C(3)\{U(0)+U(1)+U(2)\}$$

$U(0) = 1$

The maximum number of relations amongst

3 unlabelled points is 4

4, 16, 20, 104   4, 20, 16, 104

12   41   84

-4   88

11, 218, 90, 3044

11   90   218   3044

79   128   2826

3|1413

3|471

157

15

If we allow 2 or 3 points, but not one,

our total becomes 21 + 144 = 165 = 3(5)11

If we allow 1 or 2 points but not 3 our total becomes

27. It is customary in discussing the

theory of isometry groups in 3-d space to refer

to 230 discrete groups, of which 165

include not only direct isometries but also

opposite (reflection) isometries. Thus these 165

groups are called mixed isometry groups, by which

we mean that they involve relations between

2 or 3 points. There are 65 discrete groups

of direct isometries which are generally

recognized, but amongst these 65,

11

34 ; 544 ; 9608 ; 291,968

$$\begin{array}{r} 291,968 \\ -\ 9\,608 \\ \hline 282,360 \end{array}$$

510 9064 282,360

255 4532 141 180

15(17) 2266 47 060

1133 ~~23530~~ 235 ~~80~~

3(5)17 (2)^4 11(103) ~~1140~~ 11 ~~7~~ 70

~~5445~~ 58 ~~85~~ = 5(1177)

~~1119~~ ~~2885~~ = 5(577)

11(107)

$$\begin{array}{r} 16\,6 \\ 17 \overline{)\,282\,360} \\ 272 \\ \hline 103 \end{array}$$

$$\begin{array}{r} 1660 \\ 17\overline{)282\,360} \\ 17 \\ \hline 112 \\ 102 \\ \hline 103 \\ 102 \\ \hline 16 \end{array}$$

$$\begin{array}{r} 93 \\ 13\overline{)\cancel{1177}} \end{array}$$

47 060

23 530

5 | 11765

= 5(2353)

2 | 141180

70590

282,360

141 180

70590

3 | 35295

5 | 11765

13 | 2353

181

$(13^2+13-1)\ 3(2^3)5(13)$

$13^3+13^2-13\ (3)\ 2^3(5)$

2 3 6 10

2 6 3 10

} = 21

consequently we run up a total of 3363 relations, For other polytopes the analogous products would be

$59(62) = 3487$

$22(26) = \overset{52}{52} = 572$

$23(26) = 598$

$11(14) = 154 = 2p(12)$

note 6 21 144

27 171

26 -154

$= 17$

~~4~~

~~10~~

~~30~~

~~175~~

~~$7 \overline{)3367}$~~

~~$13 \overline{)481}$~~

~~37~~

A cube or icosahedron, with a center, is the sum of possible relations for 3 or fewer points

12

With just one point and 4 sets of rules

we have 6 possible relations

With two points and 4 sets of rules we have

21 possible relations

With three points and 4 sets of rules we have

144 possible relations

With 4 points and 4 sets of rules we have

3363 possible relations

6 + 21 + 144 + 3363
27 171 171
2 | 3534 = 2(3) 19 (31)
3 | 1767
19 | 589 3(4)·20 (32)
31 9~~6~~60

57 (62)

OVER

13/

Clearly this set of all possible relations amongst 4 or fewer points can be described as follows. Consider a dodecahedron, which has,

20 vertices + 30 edgelines + 12 faces = 62

i.e. 62 components. Since each face is a pentagon (having 5 edgelines) we think of the face being preserved but the edgelines not counted (hence remaining invariant). Varying thereafter we have 20 vertices, + 25 edgelines + 12 faces, hence 57 components varying. Each varying component is allowed to become a transformer over the entire set of 62,

25

$$5049 = 3(1683) = 9(561)$$
$$= 27(187)$$
$$= 3^3(11)17$$

$$5049 = (0+0!)! + (0+1!)! + (1+2!) + (1+3!)!$$

$$(1+0)! \quad (1+1)! + (1+2)! + (1+(0+1+2)!)!$$

$$(1+0)! \quad (1+(0+1)!)! + (1+(0+2)!)! + (1+(0+1+2)!)!$$

three things 0, 1, 2 and if we combine them 1 at a time we use only zero if we combine them 2 at a time we omit the combination 1+2 that is, ~~[illegible]~~, we use only zero pairing 0,1 and 0,2

When, however, we combine 3 at a time, we get 0+1+2 = 3 which is the same result as 1+2 as a pairing, the only omitted pairing

The requirement is that we are to use only the 4 of 8 possible combinations of 0, 1, and 2 which include zero ≡ 0,

950	101	26	9	4	2	1 1
996	143	31	10	4	2	1
46	42	5	1			1

46 − 42 = 4 5 − 1 = 4

1051	35	6
1139	41	6
88	6	6

950	101	26	9	4	2	1	1
853	112	21	6	2	1	1	1
97	−11	5	3	2	1	0	
97	0	11	6	3	1		

For 6 or fewer points the number of connected graphs (143) is precisely equal to the number of geometries for 7 or fewer points (143) provided the number of points is larger than one

$$* G_2(7) = C_{g_2}(6) = 143 = \sum_2^7 G(s) = \sum_1^6 Cg(s)$$

26

Saying it another way, we omit $1, 2, 1+2$, and the empty set

all of those combinations (all 4) which don't explicitly display a zero $= 0$

The sum must have 2 or more parts if we are to take the

factorial of it

$$(1+S_0)! = (1+0)! = 1!$$

$$(1+S_1)! = (1+(0+1)!) = (1+1!)! = 2!$$

$$(1+S_2)! = (1+(0+2)!) = (1+2!)! = 3!$$

$$(1+S_3)! = (1+(0+1+2)!) = (1+3!) = 7!$$

$$1+5049 = 5050 = \frac{100^2+100}{2} = \sum_{S=0}^{3}(1+S!)!$$

$$100 = 5! - 4! + 2! + 1! + 0!$$

$$= \{5! - 2!\} - \{4! + 2!\}$$

Some regular or semi-regular pattern exists

in this comparison of series

2, 8, 12, 47

It seems worthwhile to make a serious effort to match geometries for for S+1 points with graphs for S points; that is we should seek some compromise to make the totals come out precisely. In the case of connected graphs

$1 + 1 + 2 + 6 + 21 + 112 + 853$

1	2	4	10	31	143	996
	2	4	11	34	156	1044
			1	3	13	48
						= 3 (3+13)

The total number of connected graphs for S or fewer points is only slightly less than the number of graphs for just S points

2.7

1	6	28	133	695	4186
-1	-2	-3	-4	-5	-6
0	4	25	129	690	4180
6^2	$=2^2$	$=5^2$	$3(43)$	$5(138)$	$=\sum_{n=1}^{17} F_n$
				$10(69)$	

$\alpha = \frac{1+\sqrt{5}}{2}$; $\beta = \frac{1-\sqrt{5}}{2}$ $\quad 2(3)5(23)$ $\quad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$

$\alpha - \beta = \sqrt{5}$

$$\alpha = \frac{1 + \alpha - \beta}{1 + \alpha + \beta} \; ; \; \beta = \frac{1 - \alpha - \beta}{1 + \alpha + \beta}$$

$$\alpha = \frac{\alpha+\beta \;+\; \alpha - \beta}{\alpha+\beta \;+\; \alpha+\beta} \; ; \; \beta = \frac{\alpha+\beta - \alpha - \beta}{\alpha+\beta + \alpha + \beta}$$

1	6	28	133	695	4186
-0	-1	-2	-3	-4	-5
1	5	26	130	691	4181
			$=5(26)$	a prime	$=F(19)$

$=5^2(26)+5^2+4^2$

$=5^2(26)+26+5+5+5$

$$\begin{array}{ccccc} 5^2(26) + 26 & + 5 + & + 5 + & 0 & \\ +1 & +1 & +1 & +1 & +1 \\ \hline 651 & 678 & 684 & 690 & + 691 \end{array}$$

Also if $s = 5$ then the total number

of graphs plus points for 5 or fewer points is

$T(5)$ i.e. the number of topologies for 5 points

1	1	2	4	11	34	156	1044	12346
1	1	2	4	5	26	101	950	11051
				2	8	~~156~~	~~1044~~	
						55	94	$= F(11) + F(5)$

The number of graphs for s points is always

$g(s) \geq G(s+1)$, where $G(s+1)$ is the

number of Geometries for $(s+1)$ points

1	1	2	4	11	34	156	1044	12346
1	1	2	4	9	26		950	
1	1	1	2	6	21	112	853	11117
1	2	3	5	11	32	101	986	11051
						133	144 997	
	1	2	3	5		11	97	66

Geometries for $(s+1)$ points are similar to connected graphs for s points

For those who still think that the simplest graphs are symmetric reflexive, the pattern of comparison between transitive reflexive and symmetric reflexive will be considered as a pair RELATION pattern comparison; but I regard the simplest graphs as neither symmetric nor reflexive, they are simply topologies which have become not only irreflexive but also intransitive. In particular they have become intransitive. I have grave doubts as to topologies being reflexive (ever), because they don't have loops, whereas loops would imply $x R x$ which is reflexivity —

3#

2, 4, 9, 26, 101, 950 = 2(475)

Clearly the total number of ~~ork~~ n points graphs plus posets is similar to the

total number of possible geometries for (n+2) points if $n \leq 5$. If $n = 6$ then this sum is $\approx \frac{1}{2}$ G(8)

G	=	2	4	9	26	101	950
S+AS	=	2	4	9	27	97	2(474) + 2
					+1	−4	

8-d space, 950 geometries	474	A+AS for	≤ 6
7-d space, 101 " "	97	A+AS for	$n \leq 5$
6-d space, 26 " "	27	A+AS for	$n \leq 4$
5-d space, 9	9	"	≤ 3
4-d space, 4	4	A+AS	$n \leq 2$
3 d space, 2	2	"	$n \leq 1$

$$950 = 2 + 2\Sigma$$
$$101 = 4 + \Sigma$$
$$26 = -1 + \Sigma$$
$$\left\{ \begin{array}{l} 9 = \Sigma \\ 5 = \Sigma \end{array} \right\}$$

at most the topologies are transitive and a metric topology is possessed of metric transitivity, i.e. associativity — I cannot accept the suggestion that topologies are reflexive. If they are not merely transitive then they are symmetric transitive,

addition is, for example, symmetric transitive (because addition is not "distributive" i.e. reflexive with respect to multiplication — Topology is only non-metric in the sense of being based upon an addition, an accumulation, which is not distributive with respect to multiplication.

over

30/

The one thing which is clear is that graph theorists must meet again, soon, to agree upon definitions of the meanings of geometrical representations of such RELATIONS as symmetry, transitivity and reflexivity.

1	2	4	11	34	156	1044	12346
1	2	5	16	63	318	2045	
2	4	9	27	97	474	3089	
2	6	15	42	139	139 / 613	3702	
1	3	9	33		718	4535	
1	4	13	46	185	903	5438	

$1+3=4$

$4+9=13$

$2+4+27=33$

0	0	1	5	29	162	1001	
		1	6	35	197	1198	$=6^4-3^4-[illegible]^4\ [illegible]^4$
		4	−13	−46 = −11	−185 = 12	−1252 = (−54)	
		−3	−6	34	156	1044	−74
				1	41	154	+12
							=62

143 is the number of 7 th powers required to represent any number as a sum of 7 th powers

or to represent any number as a sum of

1	first power	1
4	squares	5
9	cubes	14
19	biquadrates	33
37	fifth powers	70
73	sixth powers	143

It is therefore fascinating that if we have more than 1 point but less than 8, the total number of possible geometries is 143, which is also the number of connected graphs if the number of points is 6 or fewer: 6, 5, 4, 3, 2, 1 —

We have plenty of questions to ask, still, about representations by figures of logical relations.

36/

Of 16 proposed digraphs, only 1 has the property that $xRy \Rightarrow yRx$

Can logical relations exist even when they cannot be rigorously represented in graphical form?

The various figures are simply partitions which build up to the full figure, with a number of lines $f(p) = l$ such that $l > p$, which represents all of the possibilities for logical relations on p points, according to 1 of 4 or 5 or 6 sets of rules —

Sincerely,

John Tangen
TANGEN
16 Winslow Road
Winchester, Mass. 01890
Phone: 729-3822

9-20-76

Bell Laboratories

600 Mountain Avenue
Murray Hill, New Jersey 07974
Phone (201) 582-3000

September 20, 1976

Mr. J. N. Tangen
16 Winslow Road
Winchester, Massachusetts 02138

Dear Mr. Tangen:

I am sorry it has taken me so long to reply to your two very interesting letters. However, for the last six months I have been working on different problems altogether, and have had very little time for the Handbook of Integer Sequences.

I greatly appreciate the time and effort you have put into those letters, and I shall try to reply to them as best I can.

Concerning the definitions of graphs and relations. I tried to follow standard terminology, which of course is not always logically correct, and you are right in pointing this out. But I do not altogether agree with your proposed changes. For example, you propose that sequence 646 be called graphs. This conflicts with the almost universal convention in graph theory, that a graph should contain no loops. When the time comes for a second printing (or edition) of the book, I shall re-examine these names more carefully in the light of your letter.

Your discussion of the relationship between the sequence 1,3,16,218,... and various algebras is extremely interesting, although I am not sure I completely understand what you are saying. Have you written to Coxeter about this?

(Later) I'm sorry, although I have tried once more to follow your argument, but I am afraid it is over my head.

It is clear, is it not, that the number of relations with any given property in which every point is reflexive

is the same as the member with the same property in which every node is irreflexive? For in the first case there is a loop at every point, while in the second case there is a loop at no point. So the numbers are equal. It is for this reason that I did not pay too much attention to the distinction between reflexive and irreflexive relations.

Thank you again for taking the trouble to write to me.

Yours sincerely,

MH-1216-NJAS-mv N. J. A. Sloane

P.S. Perhaps you will find the enclosed amusing.

Enc.
As above

10-1-76

Dear Sloane, due to the long delay it happens I am now back in Oxford again, having left your "Handbook" in our Cambridge, Mass "study-studio", my wife and I have now the necessity of maintaining not only this home in Oxford but also the studio apartment in Mass. I am very sure that I can correct these names or labellings for you; but it will take time away from my regular work and it will involve some expenses for me, This work is sufficiently important so that your company should be willing to authorize a modest outlay (a total of $100). When I shall hear from you or your company, &/or get a check for $100, I will feel fully obliged to drop what I am doing — my current work is underfunded, — and get on with the correction of your graph names, I refer only to corrections of the graph names, Other ideas which I have communicated to you are sufficiently speculative so that contract work is not justified. Whenever I wrote to you about #646 I may possibly have been wrong but that was many months ago. I have great admiration for Harold Scott Macdonald COXETER 1907 — but he is age 69. I won't be writing him often on these matters, nor would I expect him often to reply. The question is not one of authority but experience,

Hpu seqs fgr

2/ As O W Holmes Jr 1840-1935 once said about (law) — —
the life of (mathematics) has not been logic but experience. In recent
years, and in recent months, my work in the area where
Coxeter is experienced has gone far beyond the work of Coxeter or
anyone else. But I never would have gone back, this year, to the graph
names, if my wife had not decided to forward your letter to me.
I am not precisely looking for extra work -- as I am
overwhelmed with writing — but my integrity requires me to
follow up on this naming project for the graphs. Sincerely,
N. TANGEN
Oct 1, 1976

To open slit here

To open slit here

Sender's name and address (Please show your postcode)

John Norris Tangen
11 TACKLEY PLACE
Oxford England
OX 2 6AR

An air letter should not contain any enclosure

By air mail Air letter
Par avion Aerogramme

OXFORD
7 15PM
1 OCT

11P

N.J.A. Sloane
Bell Laboratories
600 Mountain Avenue
Murray Hill New Jersey
07974 USA