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Mr. Neil J. A. Sloane
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Dear Neil,

I have just had my first real "success" using your index of sequences, finding a sequence treated by Cayley that turns out to be identical to another (a priori quite different) sequence that came up in connection with computer sorting.

The copy of your index that I have was a relatively early version; for example it doesn't contain the sequence of prime powers. What is the current status of the index? Do you have definite publication plans yet?

Cordially,

Don Knuth

Donald E. Knuth
Professor

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April 28, 1970

Prof. John Riordan
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Dear John:

The following problem must be well known but I was unsuccessful in locating any recent references; perhaps you can help me. Let T_n be the number of outcomes of a tournament with ties allowed; for example, $T_3 = 13$ because the outcomes for players 1 , 2 , 3 are:

$1 = 2 = 3$, $1 = 2 < 3$, $3 < 1 = 2$, $1 = 3 < 2$, $2 < 1 = 3$,

$2 = 3 < 1$, $1 < 2 = 3$, $1 < 2 < 3$, $1 < 3 < 2$, $2 < 1 < 3$,

$2 < 3 < 1$, $3 < 1 < 2$, $3 < 2 < 1$.

It is not hard to prove that $\Sigma\binom{n}{k}T_k = 2T_n$, so that

n	0	1	2	3	4	5	
T_n	1	1	3	13	75	541	...

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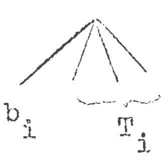
I looked up this sequence in Neil Sloane's table and found that T_n was given by Cayley (Papers, vol. 4, p. 113) as the number of ordered (i.e., planar) trees with n+1 terminal nodes all at the same level; other levels contain no terminal nodes and at least one branch. There is an interesting one-to-one correspondence between the outcome of an n-player contest and an (n+1)-terminal tree in Cayley's sense: Start with one node labeled 0 . Let the result of the contest be $S_1 < S_2 < \ldots < S_m$ where the S's are nonempty disjoint sets whose union is $\{1,2,\ldots,n\}$, representing tied players. Then do the following for $k = 1,2,\ldots,m$: If the labeled terminal nodes now present are $0 = b_1 < b_2 < \ldots < b_j$, split S_k into sets $T_1,T_2,\ldots,T_j$ so that $b_1 < T_1 < b_2 < T_2 < \ldots < b_j < T_j$; and replace node b_i by



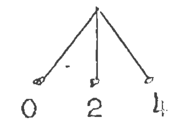
, for each $i = 0\ 1 \ldots j$.

For example, suppose the outcome of the contest is

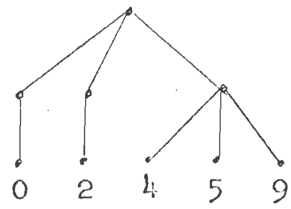
$$2 = 4 < 5 = 9 < 3 < 1 = 7 = 8 < 6$$

then we have

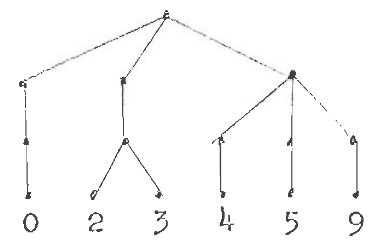
k = 1



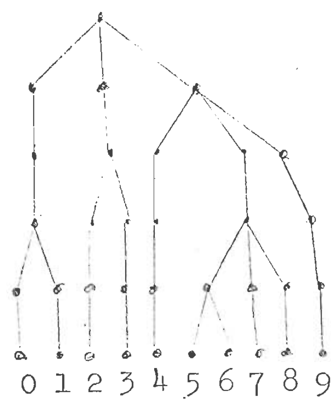
k = 2



k = 3



etc., and the final tree is



Cayley states that the result can be expressed in terms of Stirling numbers of the second kind, because of the generating function

$$\Sigma T_n x^n/n! = 1/(2 - e^x)$$

which is an immediate consequence of the recurrence. In fact,

$$\Sigma_{k \geq 0}(e^x - 1)^k = \Sigma T_n x^n/n!$$

is equivalent to

$$T_n = \Sigma_m \left\{ {n \atop m} \right\} m! \quad ,$$

a relation which can be obtained immediately from the original statement of the problem since $\left\{{n \atop m}\right\}$ is the number of ways to partition n players into m nonempty classes. The generating function also gives an interesting series not mentioned by Cayley,

$$T_n = \frac{1}{2} n!\left((\ln 2)^{-n-1} + 2 \sum_{k \geq 1} \mathrm{Re}\left((2\pi k i + \ln 2)^{-n-1}\right)\right) ,$$

which shows the asymptotic behavior.

Have you seen anything published on T_n since Cayley?

Best regards

Donald E. Knuth
Professor

DEK/pw