Signum Equations and Extremal Coefficients

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Let $a(n)$ denote the number of sign choices + and − such that

$$\pm 1 \pm 2 \pm 3 \pm \cdots \pm n = 0$$

and $b(n)$ denote the number of solutions of

$$\varepsilon_1 \cdot 1 + \varepsilon_2 \cdot 2 + \varepsilon_3 \cdot 3 + \cdots + \varepsilon_n \cdot n = 0$$

where each $\varepsilon_j \in \{-1, 0, 1\}$. It can be proved that [1, 2]

$$a(n) \text{ is the coefficient of } x^{n(n+1)/2} \text{ in the polynomial } \prod_{k=1}^{n} \left(1 + x^{2k}\right),$$

$$b(n) \text{ is the coefficient of } x^{n(n+1)/2} \text{ in the polynomial } \prod_{k=1}^{n} \left(1 + x^{k} + x^{2k}\right).$$

Clearly $a(n) = 0$ when $n \equiv 1, 2 \bmod 4$. If we think of sign choices as independent random variables with equal weight on $\{-1, 1\}$, then

$$\mathrm{E}\left(\sum_{k=1}^{n} \pm k\right) = 0, \qquad \mathrm{Var}\left(\sum_{k=1}^{n} \pm k\right) = \frac{n(n+1)(2n+1)}{6} \sim \frac{n^3}{3}$$

as $n \to \infty$. By the Central Limit Theorem,

$$\mathrm{P}\left(\sqrt{3} n^{-3/2} \sum_{k=1}^{n} \pm k \leq x\right) \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt$$

which implies that [3, 4]

$$\mathrm{P}\left(\sum_{k=1}^{n} \pm k = 0\right) \sim s \sqrt{\frac{3}{2\pi}} n^{-3/2} \exp\left(-\frac{x^2}{2}\right) \Bigg|_{x=0}$$

where $s = 1 - (-1) = 2$ is the span of the distribution of $\pm$; hence [5, 6]

$$a(n) \sim \sqrt{\frac{6}{\pi}} n^{-3/2} 2^n.$$

In the same way,

$$b(n) \sim \frac{1}{2\sqrt{\pi}} n^{-3/2} 3^{n+1}.$$

Let $c(n)$ denote the number of sign choices such that

$$\pm 1 \pm 2 \pm 3 \pm \cdots \pm n = \pm 1 \pm 2 \pm 3 \pm \cdots \pm n.$$

Here [7]

$$c(n) \text{ is the coefficient of } x^{n(n+1)/2} \text{ in the polynomial } \prod_{k=1}^{n} \left(1 + x^k\right)^2$$

and [8, 9, 10, 11]

$$c(n) \sim \sqrt{\frac{3}{\pi}} n^{-3/2} 2^{2n}.$$

Define [12]

$$\alpha(n) \text{ to be the maximal coefficient in the polynomial } \prod_{k=1}^{n} \left(1 + x^{2k}\right),$$

$$\beta(n) \text{ to be the maximal coefficient in the polynomial } \prod_{k=1}^{n} \left(1 + x^k + x^{2k}\right),$$

$$\gamma(n) \text{ to be the maximal coefficient in the polynomial } \prod_{k=1}^{n} \left(1 + x^k\right)^2.$$

The first of these has an immediate combinatorial interpretation: $\alpha(n)$ is the number of sign choices such that

$$\pm 1 \pm 2 \pm 3 \pm \cdots \pm n \text{ is } 0 \text{ or } 1.$$

While $\beta(n)$ seems not to have such a representation, the last sequence satisfies trivially $\gamma(n) = c(n)$ always.

We look at several more examples. Define [13]

$$\lambda_{\max}(n) \text{ to be the maximal coefficient in } \prod_{k=1}^{n} \left(1 - x^{2k}\right)$$

and $-\lambda_{\min}(n)$ to be the corresponding minimal coefficient;

$$\mu_{\max}(n) \text{ to be the maximal coefficient in } (-1)^n \prod_{k=1}^{n} \left(1-x^k\right)^2$$
and $-\mu_{\min}(n)$ to be the corresponding minimal coefficient.

Only the third of these possesses a clear simplification:

$$\mu_{\max}(n) \text{ is the coefficient of } x^{n(n+1)/2} \text{ in } (-1)^n \prod_{k=1}^{n} \left(1-x^k\right)^2$$

and the asymptotics

$$\mu_{\max}(n)^{1/n} \sim 1.48... \sim 2\,e^{-0.29...}$$

are of interest [14, 15]. Greater understanding of the other sequences is desired.

0.1. Number Partitioning. What is the number of ways to partition the set $\{1, 2, \ldots, n\}$ into two subsets whose sums are as nearly equal as possible? If $n \equiv 0, 3 \bmod 4$, the answer is $\alpha(n)$; if $n \equiv 1, 2 \bmod 4$, the answer is $\alpha(n)/2$. In the former case, the subsets have the same sum; in the latter, the subsets have sums that differ by 1 [16, 17]. Partitioning arbitrary sets of n integers, each typically of order 2^m, is an NP-complete problem. The ratio m/n characterizes the difficulty in searching for a perfect partition (one in which subset sums differ by at most 1). A phase transition exists for this problem (at $m/n = 1$, in fact) and perhaps similarly for all NP problems [17, 18, 19].

As an aside, we observe that

$$\lambda_{\max}(n) \text{ is the coefficient of } x^{n(n+1)/2} \text{ in the polynomial } \prod_{k=1}^{n} \left(1-x^{2k}\right)$$

for $n \equiv 0 \bmod 4$, but this fails elsewhere (a conjectural relation involving $x^{(n+1)^2/2}$ coefficients for $n \equiv 3 \bmod 4$ falls apart when $n = 27$). It seems to be true that

$$\lambda_{\max}(n)^{1/n} \sim 1.21... \sim 2\,e^{-0.50...}$$

as $n \to \infty$ via multiples of 4.

As another aside, if $d(n)$ is the number of solutions of

$$\varepsilon_1 \cdot 1 + \varepsilon_2 \cdot 2 + \varepsilon_3 \cdot 3 + \cdots + \varepsilon_n \cdot n = \varepsilon_{-1} \cdot 1 + \varepsilon_{-2} \cdot 2 + \varepsilon_{-3} \cdot 3 + \cdots + \varepsilon_{-n} \cdot n,$$

then [20]

$$d(n) \text{ is the coefficient of } x^{n(n+1)} \text{ in the polynomial } \prod_{k=1}^{n} \left(1+x^k+x^{2k}\right)^2$$
(in fact, it is the maximal such coefficient)

and

$$d(n) \sim \frac{1}{2\sqrt{2\pi}} n^{-3/2} 3^{2n+1}.$$

This grows more quickly than $b(n)$, of course. We wonder what else can be said in both cases. For example, what is the mean percentage of $0s$ in $\{\varepsilon_j\}$ taken over all solutions, as $n \to \infty$? It may well be $1/3$ for both, but it may be $> 1/3$ for one or the other.

0.2. Addendum. Define a function $G : (0,1) \to \mathbb{R}$ by

$$G(x) = \int_0^1 \ln(\sin(\pi x t))\, dt.$$

There is a unique point $x_0 = 0.7912265710...$ at which G attains its maximum value $G(x_0) = -0.4945295653....$ Let

$$r = \exp(2G(x_0)) = 0.3719264606... = \frac{1}{4}(1.4877058426...),$$

$$C = \frac{4\sin(\pi x_0)}{x_0}\sqrt{\frac{\pi}{-G''(x_0)}} = 2.4057458393...$$

then [21]

$$\mu_{\max}(n) \sim C\frac{(4r)^n}{\sqrt{n}}$$

as $n \to \infty$, making impressively precise our earlier conjecture. An analogous formula for $\lambda_{\max}(n)$ for $n \equiv 0 \bmod 4$ remains open.

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