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Enumeration of Stochastic Matrices with Integer Elements

by

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ENUMERATION OF STOCHASTIC MATRICES
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ABSTRACT

The problem here treated is that of enumerating the number of nonnegative integer $n \times n$ matrices with common line sum r. The case in which the entries are restricted to be (0,1) is also treated. Although a general formula (for both cases) is given, it is not useful except for small r; the explicit expressions are written out for r = 2, 3, 4. Numerical enumeration is also considered, and an efficient branching process for obtaining the enumerating numbers is described. Finally, several formulae of the Gupta type (fixed n, variable r) are given in the nonnegative integer case. The cases n = 2, 3, 4 are also treated theoretically using an approach based on properties of the so-called Schur function coefficients of Kostka. Several numerical tables are included.

I. INTRODUCTION

Consider the class of $n \times n$ matrices all of whose elements are nonnegative integers less than or equal to some prescribed integer r; the cardinality of this class is clearly $(r+1)^{n^2}$. Let us now restrict our interest to that subclass characterized by the condition that all the row and column sums of each matrix have precisely the value r. How many such matrices are there? We may call these matrices "r-stochastic" by analogy with the familiar nonintegral case in which the elements are real numbers between 0 and 1 and the common line sum is unity. The enumeration of r-stochastic matrices is one of those fundamental combinatorial problems which, in a sense, can be considered "solved" but which, in fact, does not yet possess any really satisfactory solution. A similar—possibly more interesting—problem arises if we impose the further restriction that all the matrix elements be either 0 or 1. This report treats these two problems from both a theoretical and a practical (i.e., calculational) point of view without, however, by any means exhausting the subject.

In Sec. II we reformulate the prescription of MacMahon[1] and derive several explicit enumeration formulae. In Sec. III we further exploit MacMahon's idea to develop an efficient calculational scheme; numerical tables are included in the hope that they will be of value to others who may wish to pursue the investigation. Finally, in Sec. IV we give another formulation of the enumeration problem employing Kostka's[2] connection coefficients—here called Schur function coefficients. This approach yields the result that the number H_r^n of r-stochastic matrices is, for fixed n, a polynomial of degree $(n-1)^2$ in r, thus verifying in part a conjecture of Gupta et al.[3] The exact form of Gupta's conjecture, however, does not emerge from our treatment. *

II. EXPLICIT ENUMERATION FORMULAE

Throughout this report we shall adopt the customary notation for the three most common types of symmetric functions: h_j is the homogeneous product sum (of degree j); a_j is the elementary symmetric function; and s_j is the power sum (for further details, see Ref. 1). We do not specify the number of indeterminants, which is always assumed to be sufficiently large so that all possible types of products appear.

* Stanley says this proof is wrong

Many years ago MacMahon[1,4] pointed out that the number of $k \times m$ nonnegative integer arrays with preassigned row sums $r_1, r_2, \ldots, r_k$ and column sums $c_1, c_2, \ldots, c_m$ is given by the coefficient of the monomial symmetric function $(c_1 c_2 \ldots c_m)$ in the expansion of the product $h_{r_1} h_{r_2} \ldots h_{r_k}$. (The c's and r's may, of course, be interchanged in this theorem, a fact which leads to a well-known symmetry law.) If we wish to enumerate the subclass of arrays in which all the elements are 0 or 1, it is only necessary to replace the product $h_{r_1} \ldots h_{r_k}$ by the product $a_{r_1} \ldots a_{r_k}$. In the present case (r-stochastic matrices) MacMahon's theorem asserts that our enumerating number H_r^n is the coefficient of the monomial symmetric function (r^n) in the expansion of $(hr)^n$. Correspondingly, in the (0,1) case, the enumerating number A_r^n is the coefficient of (r^n) in the expansion of $(a_r)^n$. In this form, MacMahon's "solution," however, amounts to little more than a restatement of the original problem; for how is one to obtain these coefficients? MacMahon's own prescription is a branching process (making use of Hammond's operators) which can, in fact, be adapted to give an efficient scheme (see Sec. III), but it does not lead to explicit enumeration formulae.

Let us express h_r and a_r in terms of the power sums s_i, $i = 1, 2, \ldots, r$ by means of the well-known relations:[5]

$$\left.\begin{aligned} h_r &= \frac{1}{r!} \Sigma\, C_\rho s_\rho \\ a_r &= \frac{1}{r!} \Sigma\, C'_\rho s_\rho \end{aligned}\right\} . \qquad (1)$$

Here

$$\left.\begin{aligned} s_\rho &\equiv s_1^{\rho_1} s_2^{\rho_2} \ldots \\ \Sigma i\rho_i &= r \\ C_\rho &= \frac{r!}{1^{\rho_1}\rho_1!\, 2^{\rho_2}\rho_2! \ldots}, \quad C'_\rho = (-1)^{r+\nu(\rho)} C_\rho \end{aligned}\right\} , \qquad (2)$$

where $\nu(\rho)$ is the number of parts of the partition ρ. The sums run over all partitions ρ of r (where the partitions are written in "signature" form, as indicated by the condition $\Sigma i\rho_i = r$). Equations (1) are sometimes known as "Newton's relations," although the latter term is more properly applied to the recursive formulae connecting the h's, a's, and s's.

It follows that

$$(h_r)^n = \frac{1}{(r!)^n} (\Sigma\, C_\rho s_\rho)^n = \frac{1}{(r!)^n} \Sigma\, \bar{C}_\mu s_\mu , \qquad (3)$$

where the sum includes those partitions μ of nr that actually arise in the expansion of $(\Sigma\, C_\rho s_\rho)^n$, and the $\bar{C}_\mu$ stand for the corresponding products of the $C\rho$'s, including multinomial factors. (For the number of distinct partitions μ which appear in this sum, see Ref. 6.) An analogous expression holds for $(a_r)^n$, the only difference being that some of the terms appear with a minus sign. Now, as shown by Littlewood,[5] the coefficient of (r^n) in the product $s_\mu = s_1^{r_1} s_2^{r_2} \ldots$, $\Sigma i\mu_i = nr$, is given by the compound character $\phi_\mu^{(r^n)}$ of the symmetric group on nr letters. Explicitly,

$$\phi_\mu^{(r^n)} = \sum \frac{\mu_1!}{\mu_{11}!\, \mu_{12}! \ldots} \; \frac{\mu_2!}{\mu_{21}!\, \mu_{22}! \ldots} \ldots , \qquad (4)$$

where the sum runs over all solutions of the system

$$\left.\begin{aligned} \sum_{j=1}^{r} j\, \mu_{ji} &= r, \quad i = 1, 2, \ldots \\ \sum_{i=1}^{r} \mu_{ji} &= \mu_j, \quad j = 1, 2, \ldots \end{aligned}\right\} . \qquad (5)$$

In other words, Eq. (4) is summed over all "separations" of μ such that the "separates" are themselves partitions of r.

The solution to our problem is therefore given by the expression

$$H_r^n = \frac{1}{(r!)^n} \Sigma\, \bar{C}_\mu \phi_\mu^{(r^n)} . \qquad (6)$$

A_r^n is, of course, given by the same formal expression; in this case some of the $\bar{C}_\mu$ carry a minus sign.

Equation (6) is a more or less explicit solution of the enumeration problem, albeit not a very useful one; one would like to replace the sum over separations of μ by something more straightforward. Although this does not seem possible in the general case, the sum does simplify greatly for small values of r.

Let us denote by $[s_\mu;\ r^n]$ the coefficient of $n!(r^n)$ in the expansion of the product s_μ. Then from Eq. (3),

$$H^n_r = \frac{n!}{(r!)^n} \Sigma\, \bar{C}_\mu\, [s_\mu;\ r^n]\ . \qquad (7)$$

Now the product s_μ is of the form $s_1^{\mu_1} s_2^{\mu_2} \ldots s_r^{\mu_r}$, with $\Sigma i\mu_i = nr$. A moment's reflection makes it clear that the coefficient $[s_\mu;\ r^n]$ is just the number of ways of combining μ_1 1's, μ_2 2's, ..., μ_r r's by addition to form the partition r^n. Thus, for example, if $r = 3$, $n = 4$, we find that

$$\left[s_3 s_2^2 s_1^5;\ 3^4\right] = 20\ ,$$

$$\left[s_2^3 s_1^6;\ 3^4\right] = 120, \text{ etc.}$$

For $r = 2$ this coefficient is very easy to evaluate; the resulting expressions are

$$A^n_2 = \frac{n!}{2^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (2n-2j-1)!!\ , \qquad (8)$$

$$H^n_2 = \frac{n!}{2^n} \sum_{j=0}^{n} \binom{n}{j} (2n-2j-1)!!\ . \qquad (9)$$

It may be verified that A^n_2 has the recurrence

$$A^n_2 = \frac{n(n-1)^2}{2} \left\{(2n-3)\, A^{n-2}_2 + (n-2)^2\, A^{n-3}_2\right\}\ . \qquad (10\text{-a})$$

Similarly:

$$H^n_2 = n^2\, H^{n-1}_2 - \frac{n(n-1)^2}{2}\, H^{n-2}_2\ . \qquad (10\text{-b})$$

These were previously derived by Gupta et al.[3] in a different manner.

For $r = 3$ we get a more complicated formula:

$$A^n_3 = \frac{n!}{6^n} \Sigma(-1)^{n_2} \binom{n}{n_1 n_2 n_3} 2^{n_1}\, 3^{n_2} \frac{(3n_3 + n_2)!}{n_3!\, 6^{n_3}}, \qquad (11)$$

where the sum runs over all $\binom{n+2}{2}$ compositions of n into three parts (including zero): $n_1 + n_2 + n_3 = n$, $n_i \geq 0$. H^n_3 is, of course, given by precisely the same expression with the factor $(-1)^{n_2}$ suppressed.

In the interest of completeness we give the result for $r = 4$; if $r > 4$ the enumerating expressions become too complicated to be worth writing down.

Let

$$\left.\begin{aligned} \sigma(k) &= \frac{(4k)!}{k!\,(24)^k} \\ \sigma(1) &= \sigma(0) = 1,\ \sigma(-k) = 0 \end{aligned}\right\}\ . \qquad (12)$$

We now introduce two auxiliary products:

$$\left.\begin{aligned} P^k_e(m,j) &= \frac{\binom{2m}{2} \cdots \binom{2j+2}{2}}{(m-j)!} \binom{4k-4m}{2} \cdots \binom{4k-4m-4j+2}{2} \\ &\text{with} \\ P^k_e(m,0) &= \frac{\binom{2m}{2} \cdots \binom{2}{2}}{m!}\ , \\ P^k_e(m,m) &= \binom{4k-4m}{2} \cdots \binom{4k-4m+2}{2} \end{aligned}\right\} \qquad (13)$$

and

$$\left.\begin{aligned} P^k_O(m,j) &= \frac{\binom{2m+1}{2} \cdots \binom{2j+1}{2}}{(m+1-j)!} \binom{4k-4m-2}{2} \cdots \binom{4k-4m-4j+2}{2} \\ &\text{with} \\ P^k_O(m,1) &= \frac{\binom{2m+1}{2} \cdots \binom{3}{2}}{m!} \binom{4k-4m-2}{2} \\ P^k_O(m,m+1) &= \binom{4k-4m-2}{2} \cdots \binom{4k-8m-2}{2} \end{aligned}\right\}\ . \qquad (14)$$

The subscripts e and O stand, respectively, for even and odd. The products defined above arise in evaluating the following coefficients:

$$\left[s_2^{2m}\ s_1^{4k-4m}\right] = \sum_{j=0}^{m} P^k_e(m,j)\ \sigma(k-m-j)\ , \qquad (15)$$

$$\left[s_2^{2m+1}\ s_1^{4k-4m-2}\right] = \sum_{j=1}^{m+1} P^k_O(m,j)\ \sigma(k-m-j)\ . \qquad (16)$$

The enumerating expression for A^n_4 can then be written in the form:

$$A^n_4 = \frac{n!}{(24)^n} \sum (-1)^{n_1+n_4} \binom{n}{n_1 n_2 \ldots n_5} 2^{n_1+3n_2+n_4}$$

$$\times\ 3^{n_1+n_3+n_4} \prod_{i=0}^{n_2-1} (2n_4+4n_5+n_2-i) \qquad (17)$$

$$\left[s_2^{2n_3+n_4}\ s_1^{2n_4+4n_5}\right].$$

The sum runs over all $\binom{n+4}{4}$ compositions of n into five parts (including zero): $\sum_{i=1}^{5} n_i = n$, $n_i \geq 0$. The bracket is to be evaluated using Eq. (15) or Eq. (16), depending on whether the exponent of s_2 is even or odd. As before, H_4^n is given by the same expression with the minus sign omitted.

Despite its forbidding appearance, Eq. (17) is well suited to evaluation on an electronic computer. Equations (8), (11), and (17) (and their counterparts with the minus sign suppressed) were, in fact, used to generate part of the numerical tables given in Appendix B.

III. THE BRANCHING TREE

In this section we describe a scheme for the efficient calculation of H_r^n and A_r^n. It is in essence a branching process, but with the special feature that all elements (or "nodes") of the branching tree are known in advance. For this reason both H_r^n and A_r^n can be written as a scalar product involving rectangular matrices, all of whose matrix elements can be calculated separately (rather than recursively).

Consider the partition r^n or r,r,...,r (n terms). Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be any partition of r into not more than r parts. If we subtract the parts of λ from the parts of r^n and order the result, we obtain a partition of (n-1)r of the form $r, r, r, \ldots, r-\lambda_k, r-\lambda_{k-1}, \ldots, r-\lambda_1$. If we do this in all possible ways, we generate the same partition but with a numerical factor $C_{r^n,\lambda}$ which counts the number of ways of making the subtraction. Writing λ in signature form, $\Sigma i\alpha_i = \lambda$, we clearly have

$$C_{r^n,\lambda} = \binom{n}{\alpha_1}\binom{n-\alpha_1}{\alpha_2} \cdots \binom{n-\alpha_1-\alpha_2 \cdots -\alpha_{r-1}}{\alpha_r}$$

$$= \frac{n!}{\alpha_1!\, \alpha_2! \ldots \alpha_r!}\,. \tag{18}$$

Let the 0^{th} level of the branching tree consist of the single partition r^n. To generate the first level of the tree we subtract from r^n all p(r,n) partitions λ' of r with $\nu(\lambda') \leq n$, each subtraction producing a new partition λ'' of (n-1)r with coefficient $C_{r^n,\lambda''}$. The general branching step, i.e., the generation of the $j+1^{st}$ level from the j^{th} level, consists in passing from a set of partitions $\left|\lambda': \Sigma_{\lambda_i'} = (n-j)r\right|$ to a set $\left|\lambda'': \Sigma_{\lambda_i''} = (n-j-1)r\right|$ by subtracting all partitions λ of r with $\nu(\lambda) \leq n$ in all possible ways from the first (j^{th} level) set. For any particular λ' it may, of course, be impossible to carry out the subtraction of one or more of the $\dot{\lambda}$. This is taken care of by the following

<u>Definition I.</u> Let λ be a partition of (n-j)r and let λ' be a partition of (n-j-1)r. Let $\lambda'_{i_1}, \lambda'_{i_2}, \ldots, \lambda'_{i_k}$ be an arbitrary permutation of the parts of λ'. The "coupling coefficient" $C_{\lambda,\lambda'}$ is then the number of such permutations satisfying the condition

$$\lambda_j - \lambda'_{i_j} \geq 0,\ 1 \leq j \leq k\,. \tag{19}$$

According to this definition, we shall have $C_{\lambda,\lambda'} = 0$ if the subtraction is impossible.

From the method of constructing any level of the branching tree from the previous level, it is clear that at the j^{th} level we shall have as partition labels <u>all</u> partitions λ such that, simultaneously

(a) λ is a partition of (n-j)r,

(b) $\nu(\lambda) \leq n$, (20)

(c) the largest part of λ is less than or equal to r, i.e., $\lambda_1 \leq r$.

We observe that the branching tree is spindle-shaped, since the partitions on the j^{th} level are in 1-1 correspondence with those on the $n-j^{th}$ level, the correspondence being complementation with respect to the initial partition r^n (λ' is said to be the <u>complement</u> of λ with respect to r^n if $\lambda' = (r,r,\ldots r,r-\lambda_k,r-\lambda_{k-1},\ldots,r-\lambda_1)$).

It is a simple matter to prepare in advance the list of partition labels for all levels of the branching tree. At any step, of course, the corresponding coefficient labels will be sums of products of coefficients from the previous levels. The final step of the process yields a set of p(r,n) coefficients, each attached to some partition of r into not more than n parts, and the sum of these coefficients is then the required number, i.e., the coefficient of (r^n) in the product $(h_r)^n$. This process of summing over products of coefficients from successive levels is conveniently represented as the scalar product of—generally rectangular—matrices; that it is actually a scalar product, that is, a number, follows from the fact that the branching tree is spindle-shaped. The k^{th} matrix will have its rows labelled by k-1^{st} level partitions and its columns labelled by k^{th} level partitions; the matrix elements are just the coupling coefficients $C_{\lambda,\lambda'}$.

As an illustration we exhibit the scheme for the case $n=4$, $r=3$. The coefficient H_3^4 of (3^4) in $(h_3)^4$ is given by the matrix scalar product $H_3^4 = M_1\ M_2\ M_3\ M_4$, where:

$M_1 =$

	3^3	$3^2 21$	32^3
3^4	4	12	4

,

$M_2 =$

	3^2	321	31^3	2^3	$2^2 1^2$
3^3	3	6	0	1	0
$3^2 21$	1	8	2	1	3
32^3	0	6	1	4	6

,

$M_3 =$

	3	21	1^3
3^2	2	2	0
321	1	4	1
31^3	1	3	4
2^3	0	6	1
$2^2 1^2$	0	6	4

,

$M_4 =$

	0
3	1
21	1
1^3	1

.

(Note that M_n is always a column matrix with $p(r,n)$ rows consisting entirely of 1's.)

In the present case, the product yields $H_3^4 = 2008$; this may easily be verified by independent calculation, e.g., by use of Eq. (11) with the minus sign omitted.

The calculation of A_r^n is analogous, but there are some noteworthy differences; for example, $A_r^n = 0$ for $r > n$, since there can be at most n 1's in any given line. Further,

$$A_r^n = A_{n-r}^n \quad . \tag{21}$$

This follows on interchanging 0's and 1's in the matrices of the set. Therefore, for given n we need only calculate A_r^n for the range $1 \le r \le \left[\frac{n}{2}\right]$. Equation (21) applied to the analogue of Eq. (6) could lead to some interesting identities; these have not yet been investigated.

In calculating A_r^n we pass from level j to level j+1 by subtracting 1^r in all possible ways from each j^{th}-level partition of $(n-j)r$.

Naturally, the set of partition labels is more restricted than that occurring in the calculation of H_r^n.

The set of "legal" partitions in the (0,1) case can be defined as follows: (22)

(a) For $j = n-1, n-2, \ldots, n-\left[\frac{n}{2}\right]$
  1. λ is a partition of $(n-j)r$
  2. $r \le \nu(\lambda) \le n$
  3. the largest part of λ satisfies $\lambda_1 \le \min(r, n-j)$

(b) For $j > n-\left[\frac{n}{2}\right]$ the corresponding partitions are the complements with respect to r^n of those at level n-j (see the definition of complement following Eq. (20)).

The definition of the coupling coefficient $C_{\lambda,\lambda'}$ undergoes an obvious modification.

Definition II. Let λ, λ' be two partitions of $(n-j)r$ and $(n-j-1)r$, respectively, satisfying the conditions (22). Let $\nu(\lambda) = m$, $\nu(\lambda') = k$, where, by (22) $m \ge k$. We then write λ' as an m-part partition by appending m-k zeros. Now let $\left(\lambda'_{i_1}, \lambda'_{i_2}, \ldots, \lambda'_{i_m}\right)$ be an arbitrary permutation of the parts of λ' (including the m-k zero parts). The coupling coefficient $C_{\lambda,\lambda'}$ is the number of permutations such that the m numbers $\lambda_j - \lambda'_{i_j}$, $1 \le j \le m$ contain precisely r 1's and m-r 0's.

The expression for A_r^n as a matrix product is analogous to that for H_r^n. Here, however, there is only one partition belonging to the first level, namely, $r^{n-r}(r-1)^r$, with coupling coefficient $\binom{n}{r}$. We therefore have

$$A_r^n = \binom{n}{r} \prod_{i=2}^{n-1} M_i \ ; \tag{23}$$

The row and column labels of M_i are partitions from the i-1st and ith levels respectively. An an example we take n=6, r=3. Then:

M_2 =

	3^31^3	$3^22^21^2$	32^41	2^6
3^32^3	1	9	9	1

,

M_3 =

	3^3	3^221	3^21^3	32^3	32^21^2	321^4	2^41	2^31^3
3^31^3	1	9	0	0	9	0	0	1
$3^22^21^2$	0	2	2	2	8	2	2	2
32^41	0	0	0	0	6	4	4	6
2^6	0	0	0	0	0	0	0	20

,

M_4 =

	2^3	2^21^2	21^4	1^6
3^3	1	0	0	0
3^221	1	1	0	0
3^21^3	0	3	0	0
32^3	0	3	0	0
32^21^2	1	4	1	0
321^4	0	6	4	0
2^41	0	6	4	0
2^31^3	1	9	9	1

,

M_5 =

	1^3
2^3	1
2^21^2	2
21^4	6
1^6	20

.

Since $\binom{n}{r} = \binom{6}{3} = 20$, we get

$A_3^6 = 20 \prod_{i=2}^{5} M_i = 297200$, a result which may be verified independently, e.g., by use of Eq. (11).

The branching process described above for the calculation of H_r^n, A_r^n is trivially justified by reference to the algebraic meaning of the products $(h_r)^n$ and $(a_r)^n$. Of course, if one wanted the full expansion of these products in terms of monomial symmetric functions, one would proceed in the opposite direction; the inverse process is used because only a single coefficient in this expansion is required. From a diagrammatic point of view, the calculation at the j^{th} level of the branching tree can be described as follows. We enumerate, with respect to all possible column sums, the number of $j \times n$ arrays with all row sums equal to r. For each column sum vector, we then calculate the number of $(n-j) \times n$ arrays, also with all row sums equal to r, such that their column sum vectors are the complements with respect to r^n of those of the first set. For the nonnegative integral case, i.e., for the product $(h_r)^n$, all partitions satisfying the obvious conditions (20) are possible column vectors; in other words, any rectangular array with all row sums equal to r can be completed in at least one way to form an r-stochastic matrix. This is not true in the (0,1) case, so that we must impose the added restrictions given in (22) to ensure that we deal only with "legal" column sum vectors. (From the calculational point of view this is perhaps an unnecessary refinement, since any illegal column sum vector will contribute zero to the final answer.)

It seems difficult to extract useful enumeration formulae from the above branching scheme. That such formulae exist has been shown in Sec. II. There are, however, others, for example:

$$H_r^3 = \binom{r+2}{2} + 3\binom{r+3}{4} , \tag{24}$$

a result first obtained by Gupta et al.[3] A simple proof of Eq. (24) will be given in the next section, where we adopt a different approach to the enumeration problem.

IV. ENUMERATION BY MEANS OF SCHUR FUNCTION COEFFICIENTS

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$, be a partition of n into k parts. The Ferrers-Sylvester graph of λ defines a "shape" of k rows, the i^{th} row consisting of λ_i nodes. For the purposes of enumeration it is convenient to consider the nodes of the graph as boxes. Now let $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ be another partition of n. We associate with this second partition an "object of specification μ," namely, a set of n variables x_i, μ_1 of them equal to x_1, μ_2 equal to x_2, etc. The variables will be taken as ordered: $x_1 < x_2 < \dots < x_m$. We now ask the question: in how many ways can we distribute these n quantities (only m of which are distinct) in the n boxes of the λ-graph subject to the two restrictions:

1. the variables in each row are in nondecreasing order, and
2. the variables in each column are in strictly increasing order.

In the sequel we shall denote the number of arrangements in question by the symbol $y_{\lambda\mu}$. There does not seem to be any convenient closed expression for $y_{\lambda\mu}$ in the general case. In various special cases, however, useful formulae may be derived. The most familiar of these applies to the case $\mu = 1^n$ (i.e., all the x_i are distinct), when the problem reduces to that of enumerating standard Young diagrams.[7] If we let λ,μ run through all partitions of n in lexicographical order, we can define a matrix $Y(n) = (y_{\lambda\mu})$ with rows and columns indexed by the partitions. It is easily seen that $Y(n)$ is upper triangular, with 1's on the diagonal.

The following development takes as its starting point the definition of Schur functions given in Littlewood's book (Ref. 5, Chap. VI) as well as certain formulae relating the Schur functions to other symmetric functions. The coefficients $y_{\lambda\mu}$ enter our problems through the formula

$$h_\mu = \Sigma y_{\lambda\mu}\{\lambda\} \; ; \tag{25}$$

here $\{\lambda\}$ is yet another type of symmetric function, usually called a Schur function (or, by Littlewood, an S-function), and h_μ stands as usual for the product $h_{\mu_1} h_{\mu_2} \dots$. As first noted by Kostka,[2] the same coefficients appear in the formula relating the Schur functions $\{\lambda\}$ to the monomial symmetric functions (ρ):

$$\{\lambda\} = \Sigma y_{\lambda\rho}(\rho) \; . \tag{26}$$

The analogue of Eq. (25) for a product $a_\mu = a_{\mu_1} a_{\mu_2} \dots$ ... of elementary symmetric function is

$$a_\mu = \Sigma y_{\lambda\mu}\{\tilde{\lambda}\} \; , \tag{27}$$

where $\tilde{\lambda}$ is the partition conjugate to λ.

In view of Eqs. (25) through (27) it is not unreasonable to call the $y_{\lambda\mu}$ "Schur function coefficients," and we shall adopt this nomenclature in what follows.

Although we shall not make use of the fact, it is interesting to note that there is a close connection between the Schur function coefficients and the characters of the symmetric group. From Littlewood[5] we have the relation

$$s_\rho = \Sigma \chi^\lambda_\rho \{\lambda\} \; , \tag{28}$$

where χ^λ_ρ is the value of the character χ^λ for the class ρ. Alternatively, we can express the power sum product s_ρ in terms of monomial symmetric functions

$$s_\rho = \Sigma f_{\rho\mu}(\mu) \; ; \tag{29}$$

the $f_{\rho\mu}$ are, except for a factor, the same coefficients we encountered in Sec. II. For lexicographically partitions, the matrix $F(n) = (f_{\rho\mu})$ is lower triangular with 1's on the diagonal. Substituting Eq. (26) into Eq. (28) and using Eq. (29) we derive the matrix identity:

$$\tilde{X} = F\, Y^{-1} \; , \tag{30}$$

$\tilde{X}$ being the transpose of the character table. F and Y are easy to compute recursively, and since Y is triangular, its inversion is trivial. Therefore, Eq. (30) is actually a practical method of calculating the symmetric group characters, provided, of course, that the full table is required; if only certain particular classes are needed, the usual branching methods[8] are clearly superior. (NOTE: This method of computing the character table requires more storage than do branching methods. On a medium-sized machine such as Maniac II at Los Alamos, the calculation is practicable through n = 15, but might become cumbersome for larger n).

Returning to the problem at hand, we observe that Littlewood's formulae immediately yield

$$\left.\begin{aligned} h_\mu &= \Sigma g_{\mu\rho}(\rho) \\ g_{\mu\rho} &= \Sigma y_{\lambda\mu} y_{\lambda\rho} \end{aligned}\right\} \; , \tag{31}$$

and

$$\left.\begin{aligned} a_\mu &= \Sigma \bar{g}_{\mu\rho}(\rho) \\ \bar{g}_{\lambda\rho} &= \Sigma y_{\lambda\mu} y_{\tilde{\lambda}\rho} \end{aligned}\right\} \quad , \tag{32}$$

where, as usual, $\tilde{\lambda}$ denotes the partition conjugate to λ. Clearly, $g_{\mu\rho} > 0$; this is why any rectangular array with all row sums equal to r can be completed to an r-stochastic matrix. $\bar{G} = (\bar{g}_{\mu\rho})$, on the other hand, is not a strictly positive matrix; this reflects the more restrictive nature of the (0,1) r-stochastic matrix problem.

Setting $\mu = \rho = r^n$ in Eqs. (31) and (32) we find

$$H^n_r = \Sigma y^2_{\lambda,r^n} \quad , \tag{33}$$

$$A^n_r = \Sigma y_{\lambda,r^n}\, y_{\tilde{\lambda},r^n} \quad . \tag{34}$$

Equation (34) does not seem to lead to anything useful, although it does provide an alternative method of calculation; Eq. (33), on the other hand, is capable of significant development.

We begin by noting two elementary properties of y_{λ,r^n}.

1. $y_{\lambda,r^n} = 0$ if $\nu(\lambda) > n$ (35)

   This follows from the requirement that the column ordering be strictly increasing.
2. Let $\nu(\lambda) = n$. Then

$$\left.\begin{aligned} &y_{\lambda,r^n} = y_{\lambda',(r-\lambda_n)^n} \, , \\ &\text{where } \lambda' = (\lambda_1-\lambda_n, \lambda_2-\lambda_n, \ldots, \lambda_{n-1}-\lambda_n) \, . \end{aligned}\right. \tag{36}$$

This can be seen as follows. Given a λ' with $\nu(\lambda') \le n$ and an r', we wish to pass to $r = r' + 1$ and a λ such that $\nu(\lambda) = n$. We must find a new shape that incorporates one more of each of the variables $x_1, x_2, \ldots, x_n$. But these can be appended to the original shape λ' in only one way, namely, as a column of length n tacked on to the left of λ'; all other methods of incorporation violate the ordering rules. Iterating this argument, we arrive at Eq. (36).

Turning to Eq. (33), we first consider the trivial case n = 2. Then, clearly, $y_{\lambda,r^2} = 1$ for all $\lambda \ge r^2$ (lexicographical ordering of partitions is assumed) which are partitions of 2r with $\nu(\lambda) \le 2$. Therefore,

$$H^2_r = \sum_{\lambda \ge r^2} 1 = r + 1 \quad , \tag{37}$$

a result which is obvious from consideration of the 2 x 2 array.

Next we treat the case n = 3. Let $\lambda = (\lambda_1,\lambda_2,\lambda_3)$ be a partition of 3r. We now decompose λ into a sum of partitions of 2r by removing partitions of r in all possible ways consistent with the ordering rules. This process of "disjoint subtraction" is inverse to the process of building up partitions of 2r by adding r copies of the new variable x_3 in all possible ways such that the ordering relations are satisfied. From Eq. (36) we know that we need only consider $\nu(\lambda) \le 2$. Two cases arise:

1. $\lambda_2 \le r$, $y_{\lambda,r^3} = \sum_0^{\lambda_2} 1 = \lambda_2 + 1$,

2. $\lambda_2 > r$, $y_{\lambda,r^3} = \sum_{\lambda_2-r}^{\lambda_1-r} 1 = \lambda_1 - \lambda_2 + 1$.

Now if we have a general partition of 3r, $\lambda = (\lambda_1,\lambda_2,\lambda_3)$, we may reduce it to the case just treated by means of the substitution $\lambda_1 \to \lambda_1 - \lambda_3$, $\lambda_2 \to \lambda_2 - \lambda_3$, $r \to r - \lambda_3$, but since r does not appear in the formulae, we may ignore this last substitution. The general result is therefore

$$y_{\lambda,r^3} \left.\begin{aligned} &= \lambda_2 - \lambda_3 + 1 \text{ if } \lambda_2 \le r \\ &= \lambda_1 - \lambda_2 + 1 \text{ if } \lambda_2 > r \end{aligned}\right\} \, . \tag{38}$$

We are now in a position to derive Eq. (24). If $\lambda_2 \le r$, $y^2_{\lambda,r^3} = (\lambda_2-\lambda_3+1)^2 = 2\binom{\lambda_2-\lambda_3+1}{2} + \binom{\lambda_2-\lambda_3}{1}$ + 1, whence

$$H^3_{\lambda_2\le r} = \sum_{\lambda_2=0}^{r} \sum_{\lambda_3=0}^{r} y^2_{\lambda,r^3} = 2\binom{r+3}{4} + \binom{r+2}{3} + \binom{r+2}{2} . \tag{39}$$

For $\lambda_2 > r$, the variables have the range $\lambda_2 \le \lambda_1 \le 3r-\lambda_2$, $r + 1 \le \lambda_2 \le \left[\frac{3r}{2}\right]$.

Since $y^2_{\lambda,r^3} = 2\binom{\lambda_1 - \lambda_2 + 1}{2} + \binom{\lambda_1 - \lambda_2}{1} + 1$, we have

$$H^3_{\lambda_2 > r} = \sum_{\lambda_2 = r+1}^{\left[\frac{3r}{2}\right]} \sum_{\lambda_1 = \lambda_2}^{3r-\lambda_2} y^2_{\lambda,r^3} =$$

$$= \sum_{j=r+1}^{\left[\frac{3r}{2}\right]} \left\{ \binom{3r+2-2j}{3} + \binom{3r+3-2j}{3} \right\}, \tag{40}$$

the intermediate steps being elementary. At first sight the sum looks complicated owing to the appearance of the "integer part" function in the upper limit. If we write out the terms, however, we see that the sum is a complete sequence of binomials with no gap; in fact,

$$H^3_{\lambda_2 > r} = \sum_{i=0}^{r+1} \binom{i}{3} = \binom{r+2}{4}. \tag{41}$$

Since $\binom{r+2}{4} = \binom{r+3}{4} - \binom{r+2}{3}$, we have $H^3_r = H^3_{\lambda_2 \le r} + H^2_{\lambda_2 > r} = \binom{r+2}{3} + 3\binom{r+3}{4}$, which is just Eq. (24).

Gupta et al. derived this result by summing compositions of $3r$ over a 3×3 array—a much more complicated procedure.

In a similar manner, we may derive the general expression for y_{λ,r^4}, namely by summing Eq. (38) over λ_2, λ_3. Owing to the necessity of subdividing the range of summation in a relatively complicated manner, the carrying out of this summation is tedious; the details are relegated to App. A. It is clear, however, that y_{λ,r^4} will be a polynomial in λ_1, λ_2, λ_3, and r (we make use, of course, of Eq. (36)) of maximum order 3; this follows on noting that the summation runs over the full range of all the variables. In principle, we could evaluate $H^4_r = \Sigma y^2_{\lambda,r^4}$, which would then emerge as a polynomial in r of highest degree 9 (since we only have to sum over 3 of the λ_i by virtue of the relation $\sum_{i=1}^{4} \lambda_i = 4r$). This sort of argument does not apply to Eq. (34), which clearly involves only a restricted set of λ's. In any event, A^n_r cannot be a polynomial in r for fixed n in view of (21): $A^n_r = A^n_{n-r}$.

The process of constructing formulae for y_{λ,r^n} can be continued, at least conceptually. For $n = 5$ we first consider only those λ with $\nu(\lambda) \le 4$ because of Eq. (36). We then have to sum y_{λ',r^4} over the three variables λ'_2, λ'_3, λ'_4, the result being a polynomial of degree 6. Squaring this and summing over the four variables λ_2, λ_3, λ_4, λ_5 (after restoring the general case by means of Eq. (36)) we obtain a polynomial in r of degree 16. In general:

1. y_{λ,r^n} is a polynomial in $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, r$ of degree $\binom{n-1}{2}$,
2. H^n_r is a polynomial in r of degree $(n-1)^2$. (42)

This method of obtaining y_{λ,r^n} and H^n_r from the corresponding expressions for $n-1$ will always give results satisfying (42) because we always sum over the full range of λ_i. It is true that the limits of summation in any given case may be extremely complicated (see App. A for the simplest nontrivial case), but the composite formula for y_{λ,r^n} is of course "continuous," i.e., the polynomial expressions which hold in the various multidimensional λ-regions coincide on all common boundaries.

The Gupta Conjecture. Gupta et al.[3] conjectured the following formula for H^n_r:

$$H^n_r = \sum_{i=0}^{\binom{n-1}{2}} C^{(n)}_i \binom{r+n-1+i}{n-1+2i}, \tag{43}$$

where the coefficients $C^{(n)}_i$ are independent of r. Note that this says that H^n_r is a polynomial of degree $(n-1)^2$ in r, in agreement with (42). After calculating H^4_r, $r = 0, 1, 2, 3$, these authors were able to write down the explicit formula

$$H^4_r = \binom{r+3}{3} + 20\binom{r+4}{5} + 152\binom{r+5}{7} + 352\binom{r+6}{9}. \tag{44}$$

Although (42) says nothing about the precise form of H^n_r, it does imply that the fitting process is valid. Therefore, Eq. (44) will hold for all r. We have verified this through $r = 8$—which seems far enough—by calculating H^4_r using the branching method. A short list of values is given in Table I:

TABLE I

A1496

r	H^4_r
0	1
1	24
2	282
3	2008
4	10147
5	40176
6	132724
7	381424
8	981541

By use of our tables (App. B), we have derived two additional Gupta-type formulae; the coefficients are given in Table II below:

TABLE II

A5466 A5467

i	$c_i^{(5)}$	$c_i^{(6)}$
0	1	1
1	115	714
2	5390	196677
3	101275	18941310
4	858650	809451144
5	3309025	17914693608
6	4718075	223688514048
7		1633645276848
8		6907466271384
9		15642484909560
10		14666561365176

Each of these formulae has been checked by calculating one more value of H^n_r than is necessary to determine the $c_i^{(n)}$ and comparing the value with that predicted by Eq. (43).

It is unfortunate that the method of Schur function coefficients does not serve to establish the Gupta formula; clearly, some new ideas are needed. Perhaps the following remark will provide a clue. The inverse of Eq. (43) is[9]

$$c_r^{(n)} = \sum_{i=0} (-1)^{r+i} \left\{\binom{2r+n-1}{r-i} - \binom{2r+n-1}{r-i-1}\right\} H^n_i \,. \quad (45)$$

Since $c_r^{(n)} = 0$, $r > \binom{n-1}{2}$, Eq. (45) has the form of an inclusion-exclusion expression. This suggests that the terms may have a direct combinatorial meaning. (A similar situation arises in the enumeration of the number of distinct terms in Eq. (3); see Ref. 6, end of Sec. IV.)

APPENDIX A

EXPLICIT FORMULAE FOR y_{λ,r^4}

Let $(\lambda_1,\lambda_2,\lambda_3)$ be a partition of $4r$ with the parts in the usual nonincreasing order. Similarly, let $(\lambda_1',\lambda_2',\lambda_3')$ be a partition $3r$ where the λ_i' are restricted as follows:

$$\left.\begin{aligned} &\lambda_1' \ge \lambda_2' \ge \lambda_3' \\ &\lambda_2 \le \lambda_1' \le \lambda_1, \lambda_3 \le \lambda_2' \le \lambda_2,\ 0 \le \lambda_3' \le \lambda_3 \\ &\lambda_1 - \lambda_1' + \lambda_2 - \lambda_2' + \lambda_3 - \lambda_3' = r \end{aligned}\right\} \,. \quad (A.1)$$

The quantity we wish to evaluate is

$$y_\lambda = \sum_{\lambda_2'} \sum_{\lambda_3'} \bar{y}_\lambda \,, \quad (A.2)$$

$$\text{where } \left.\begin{aligned} \bar{y}_\lambda &= \lambda_2' - \lambda_3' + 1, \quad \lambda_2' \le r \\ \bar{y}_\lambda &= \lambda_1' - \lambda_2' + 1, \quad \lambda_2' > r \end{aligned}\right\} \,. \quad (A.3)$$

If $\lambda_2 \le r$, then also $\lambda_2' \le r$, and the third condition in Eq. (A.1) imposes no extra restriction. Consequently, we have

$$y_\lambda(\lambda_2 \le r) = \sum_{\lambda_2'=\lambda_3}^{\lambda_2} \sum_{\lambda_3'=0}^{\lambda_3} (\lambda_2' - \lambda_3' + 1) =$$

$$(\lambda_3+1)\left\{\binom{\lambda_2+2}{2} - \binom{\lambda_3+1}{2}\right\} - (\lambda_2 - \lambda_3 + 1)\binom{\lambda_3+1}{2} \,. \quad (A.4)$$

In the sequel we may then restrict ourselves to the case

$$r \le \lambda_2 \le 2r. \quad (A.5)$$

The general expression for the sum, Eq. (A.2), may be written

$$y_\lambda = \sum_{j=\text{Min}2}^{\text{Max}2} \sum_{k=\text{Min}3}^{\text{Max}3} (j+1-k) - 3 \sum_{j=r+1}^{\text{Max}2} \sum_{k=\text{Min}3}^{\text{Max}3} (j-r), \quad (A.6)$$

where the various limits will be specified below; we have also used the fact that $\lambda_1' - \lambda_2' + 1 = 3r + 1 - 2\lambda_2' - \lambda_3'$ (i.e., $\lambda_1' + \lambda_2' + \lambda_3' = 3r$). From Eq. (A.1) we see that

$$\left.\begin{aligned} \text{Min2} &= \text{Max}\left\{\lambda_2 - r, \lambda_3\right\} \\ \text{Max2} &= \text{Min}\left\{\lambda_2, \lambda_1 + \lambda_3 - r\right\} \end{aligned}\right\} \quad . \tag{A.7}$$

Similarly, if the value of j in Eq. (A.6) is fixed, we have

$$\left.\begin{aligned} \text{Min3} &= \text{Max}\left\{0, \lambda_2 + \lambda_3 - r - j\right\} \\ \text{Max3} &= \text{Min}\left\{\lambda_3, 3r - \lambda_2 - j\right\} \end{aligned}\right\} \quad . \tag{A.8}$$

Let us now introduce the auxiliary quantities

$$\left.\begin{aligned} \ell &\equiv \lambda_2 + \lambda_3 - r \\ u &\equiv 2r - \ell = \lambda_1 - r \end{aligned}\right\} \quad . \tag{A.9}$$

Clearly, $\ell + u = 2r$. Conditions (A.8) then take the form

$$\left.\begin{aligned} &\text{If } j \le \ell, \ \text{Min}\,3 = \ell - j \\ &\text{If } j > \ell, \ \text{Min}\,3 = 0 \\ &\text{If } j \le u, \ \text{Max}\,3 = \lambda_3 \\ &\text{If } j > u, \ \text{Max}\,3 = u + \lambda_3 - j \end{aligned}\right\} \quad . \tag{A.10}$$

Elementary manipulation shows that the following five situations can occur:

$$\left.\begin{aligned} &\text{T.1:} \quad u \le \text{Min2} \le \text{Max2} \le \ell \\ &\text{T.2:} \quad u \le \text{Min2} \le \ell \le \text{Max2} \\ &\text{T.3:} \quad \text{Min2} \le u \le \ell \le \text{Max2} \\ &\text{T.4:} \quad \text{Min2} \le \ell \le u \le \text{Max2} \\ &\text{T.5:} \quad \text{Min2} \le \ell \le \text{Max2} \le u \end{aligned}\right\} \quad . \tag{A.11}$$

All other orderings are excluded by the basic inequalities (recall that we are assuming $\lambda_2 > r$).

If we carry out the inner sum in Eq. (A.6) we obtain

$$\left.\begin{aligned} y_\lambda &= \Sigma_1 + \Sigma_2 \\ \Sigma_1 &= \sum_{j=\text{Min2}}^{\text{Max2}} \left\{(\text{Max3}-\text{Min3}+1)(j+1) - \binom{\text{Max3}+1}{2} + \binom{\text{Min3}}{2}\right\} \\ \Sigma_2 &= -3\sum_{j=r+1}^{\text{Max2}} (j-r)\left\{\text{Max3}-\text{Min3}+1\right\} \end{aligned}\right\} \tag{A.12}$$

The rest is a matter of straightforward summation, using conditions (A.10) and (A.11). To make the formulae easier to print, we introduce the additional notations:

$$\begin{aligned} M &\equiv \text{Max2} \\ m &\equiv \text{Min2} \\ L &= \lambda_3 + 1 \end{aligned} \tag{A.13}$$

The results are then as follows. As usual, $\binom{a}{b} = 0$ if $a < b$.

<u>Case T.1</u>: $(u \le m \le M \le \ell)$

$$\left.\begin{aligned} \Sigma_1 &= (u+L-\ell)\left\{\binom{M+2}{2} - \binom{m+1}{2}\right\} - \binom{u+L+1-m}{3} + \binom{u+L-M}{3} \\ &\quad + \binom{\ell+1-m}{3} - \binom{\ell-M}{3} \\ \Sigma_2 &= -3(u+L-\ell)\left\{\binom{M+1-r}{2} - \binom{m-r}{2}\right\} \end{aligned}\right\} \tag{A.14}$$

Case T.2: $(u \le m \le \ell \le M)$

$$\left.\begin{aligned}
\Sigma_1 &= (u+L-\ell)\left\{\binom{\ell+2}{2} - \binom{m+1}{2}\right\} + (u+L)\left\{\binom{M+2}{2} - \binom{\ell+2}{2}\right\} \\
&\quad - 2\left\{\binom{M+2}{3} - \binom{\ell+2}{3}\right\} + \binom{\ell+1-m}{3} - \binom{u+L+1-m}{3} + \binom{u+L-M}{3} \\
\Sigma_2 &= -3(u+L-\ell)\binom{\ell+1-r}{2} - 3(u+L+1-r)\left\{\binom{M+1-r}{2} - \binom{\ell+1-r}{2}\right\} \\
&\quad + 6\left\{\binom{M+2-r}{3} - \binom{\ell+2-r}{3}\right\}
\end{aligned}\right\} \tag{A.15}$$

Case T.3: $(m \le u \le \ell \le M)$

$$\left.\begin{aligned}
\Sigma_1 &= (L-\ell)\left\{\binom{u+2}{2} - \binom{m+1}{2}\right\} + 2\left\{\binom{u+2}{3} - \binom{m+1}{3}\right\} \\
&\quad + (u+L-\ell)\left\{\binom{\ell+2}{2} - \binom{u+2}{2}\right\} + (u+L)\left\{\binom{M+2}{2} - \binom{\ell+2}{2}\right\} \\
&\quad - 2\left\{\binom{M+2}{3} - \binom{\ell+2}{3}\right\} + \binom{\ell+1-m}{3} - (u+1-m)\binom{L}{2} \\
&\quad - \binom{L}{3} + \binom{u+L-M}{3} \\
\Sigma_2 &= -3(u+L-\ell)\binom{\ell+1-r}{2} - 3(u+L+1-r)\left\{\binom{M+1-r}{2} - \binom{\ell+1-r}{2}\right\} \\
&\quad + 6\left\{\binom{M+2-r}{3} - \binom{\ell+2-r}{3}\right\}
\end{aligned}\right\} \tag{A.16}$$

Case T.4: $(m \le \ell \le u \le M)$

$$\left.\begin{aligned}
\Sigma_1 &= (L-\ell)\left\{\binom{\ell+2}{2} - \binom{m+1}{2}\right\} + 2\left\{\binom{\ell+2}{3} - \binom{m+1}{3}\right\} + L\left\{\binom{u+2}{2} - \binom{\ell+2}{2}\right\} \\
&\quad + (u+L)\left\{\binom{M+2}{2} - \binom{u+2}{2}\right\} - 2\left\{\binom{M+2}{3} - \binom{u+2}{3}\right\} + \binom{\ell+1-m}{3} \\
&\quad - (u+1-m)\binom{L}{2} - \binom{L}{3} + \binom{u+L-M}{3} \\
\Sigma_2 &= -3L\binom{u+1-r}{2} - 3(u+L+1-r)\left\{\binom{M+1-r}{2} - \binom{u+1-r}{2}\right\} \\
&\quad + 6\left\{\binom{M+2-r}{3} - \binom{u+2-r}{3}\right\}
\end{aligned}\right\} \tag{A.17}$$

Case T.5: $(m \le \ell \le M \le u)$

$$\left.\begin{aligned} \Sigma_1 &= (L-\ell)\left\{\binom{\ell+2}{2} - \binom{m+1}{2}\right\} + 2\left\{\binom{\ell+2}{3} - \binom{m+1}{3}\right\} \\ &\quad + L\left\{\binom{M+2}{2} - \binom{\ell+2}{2}\right\} + \binom{\ell+1-m}{3} - (M-m+1)\binom{L}{2} \\ \Sigma_2 &= -3L\binom{M+1-r}{2} \end{aligned}\right\} \qquad (A.18)$$

Equation (A.4) together with Eqs. (A.14) through (A.18) constitute the complete expression for y_{λ,r^n} if $\nu(\lambda) \le 3$; when $\nu(\lambda) = 4$ we use Eq. (36) to reduce λ to three or fewer parts.

APPENDIX B
NUMERICAL RESULTS

In this Appendix we present several numerical tables relevant to the enumeration problem discussed in this report.

Table B.1 gives values of H_r^n for r = 1, 2, ..., 11 and n = 4, 5, 6. These values, calculated by the branching method, are more than sufficient to fit and check the corresponding Gupta formulae (cf. Eq. (43)) for the three n values considered. The coefficients for n = 4 are displayed in Eq. (44), while those for n = 5, 6 are given in Table I of the text. The Gupta formulae for n = 2, 3 are proved in the text (cf. Eqs. (24) and (37)) and require no numerical data to determine them.

Tables B.2 through B.5 give both H_r^n, A_r^n for the range n = 1, 2, ..., 15 and r = 2, 3, 4, 5. Tables B.2, B.3, and B.4 (for r = 2, 3, 4 respectively) were calculated by means of Eqs. (8), (11), and (17) (and their analogues with the minus sign suppressed); Table B.5 was obtained by the branching process. The first three tables could be easily extended to large values of n, but there seems little point in doing this (see, however, Tables B.7 and B.8).

Table B.6 consists of only three entries, namely A_6^{12}, A_6^{13}, and A_6^{14}. These were obtained by the branching method and are included because, except for the single missing value A_7^{14}, they serve to complete the full table of A_r^n for all $n \le 14$. (Recall that $A_r^n = A_{n-r}^n$.) The missing value would take several hours to obtain, and it has not been thought worth while to invest the computer time.

In his thesis (unpublished), O'Neil[10] derived asymptotic expressions for the number of distinct n x m arrays of 0's and 1's with prescribed row and column sums. A result of his which is relevant to our problem is

$$A_r^n = \frac{(rn)!}{(r!)^{2n}}\, e^{-\frac{(r-1)^2}{2}} \left[1 + O\left(n^{-1+\delta}\right)\right]. \qquad (B.1)$$

This holds for sufficiently large n and arbitrary $\delta > 0$, provided that

$$r < (\ln^n)^{1/4} \quad . \qquad (B.2)$$

For the case r = 2, this result can be substantially improved. As Everett has shown,* we can replace O by o in (B.1):

$$A_2^n = \frac{(2n)!}{2^{2n}}\, e^{-1/2} \left\{1 + o\left(n^{-1+\delta}\right)\right\} \quad . \qquad (B.3)$$

Further,

$$H_2^n = \frac{(2n)!}{2^{2n}}\, e^{1/2} \left\{1 + o\left(n^{-1+\delta}\right)\right\} \quad . \qquad (B.4)$$

Thus the limit of the ratio H_2^n/A_2^n is e. The best result to date is as follows*

$$\text{Let} \quad I_a = \frac{(2n)!}{2^{2n}}\, e^{-1/2} \quad , \qquad (B.5)$$

$$I_b = \frac{(2n)!}{2^{2n}}\, e^{1/2} \quad . \qquad (B.6)$$

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*C. J. Everett, private communication

Then

$$1 - e^{1/2}\left\{\frac{1}{\frac{32n}{5}\left(1-\frac{1}{2n}\right)} + \frac{1}{2^{n+1}(n+1)!}\right\} < \frac{A_2^n}{I_a} < 1 - e^{1/2}\left\{\frac{1-\frac{9}{4n}}{8n} - \frac{1}{2^{n+1}(n+1)!}\right\} \quad (B.7)$$

$$1 - e^{-1/2}\left\{\frac{1}{\frac{8}{3}n} - \frac{1}{2^{n}(n+1)!}\right\} < \frac{H_2^n}{I_b} < 1 + e^{-1/2}\left\{\frac{1}{2n\left(1-\frac{1}{2n}\right)}\right\} . \quad (B.8)$$

These sharp results were obtained utilizing the exact formulae, Eqs. (8) and (9) of the text. In Tables B.7 and B.8 we list A_2^n and H_2^n for n = 10, 20, 30, ..., 250. In each table the first and third columns are, respectively, the Everett limits of (B.7) and (B.8)—here labelled L_e and U_e—multiplied by I_a for $\left(A_2^n\right)$ or I_b for $\left(H_2^n\right)$.

TABLE B.1

r	H_r^4
1	24
2	282
3	2008
4	10147
5	40176
6	132724
7	381424
8	981541
9	2309384
10	5045326
11	10356424

r	H_r^5
1	120
2	6210
3	153040
4	2224955
5	22069251
6	164176640
7	976395820
8	4855258305
9	20856798285
10	79315936751
11	272095118010

r	H_r^6
1	720
2	202410
3	20933840
4	1047649905
5	30767936616
6	602351808741
7	8575979362560
8	94459713879600
9	842286559093240
10	6292583664553881
11	40447642842118656

TABLE B.2

n	H_2^n
1	1
2	3
3	21
4	282
5	6210
6	202410
7	9135630
8	545007960
9	41511583320
10	3930730106200
11	452785322266200
12	62347376347779600
13	10112899541133589200
14	1908371363642760216400
15	414517594539154672566000

n	A_2^n
1	0
2	1
3	6
4	90
5	2040
6	67950
7	3110940
8	187530640
9	14396171200
10	1371785398200
11	158615367962000
12	21959547410077200
13	3574340599104475200
14	676508133623135814000
15	147320988741542099484000

Table B.1

A1496

r	H_r^4
1	24
2	282
3	2008
4	10147
5	40176
6	132724
7	381424
8	981541
9	2309384
10	5045326
11	10356424

A3438

r	H_r^5
1	120
2	6210
3	153040
4	2224955
5	22069251
6	164176640
7	976395820
8	4855258305
9	20856798285
10	79315936751
11	272095118010

A3439

r	H_r^6
1	720
2	202410
3	20933840
4	1047649905
5	30767936616
6	602351808741
7	8575979362560
8	94459713879600
9	842286559093240
10	6292583664553881
11	40447642842118656

Table B.2

A681

n	H_2^n
1	1
2	3
3	21
4	282
5	6210
6	202410
7	9135630
8	545007960
9	41514583320
10	3930730108200
11	452785322266200
12	62347376347779600
13	10112899541133589200
14	1908371363842760216400
15	414517594539154672566000

A1499

n	A_2^n
1	0
2	1
3	6
4	90
5	2040
6	67950
7	3110940
8	187530840
9	14398171200
10	1371785398200
11	158815387962000
12	21959547410077200
13	3574340599104475200
14	676508133623135814000
15	147320988741542099484000

Table B.3

A1500

n	H_3^n
1	1
2	4
3	55
4	2008
5	153040
6	20933840
7	4662857360
8	1579060246400
9	772200774683520
10	523853880779443200
11	477360556805016931200
12	569060910292172349004800
13	868071731152923490921728000
14	1663043727673392444887284377600
15	3937477620391471128913917360384000

A1501

n	A_3^n
1	0
2	0
3	1
4	24
5	2040
6	297200
7	68938800
8	24046189440
9	12025780892160
10	8302816499443200
11	7673688777463632000
12	9254768770160124288000
13	14255616537578735986867200
14	27537152449960680597739468800
15	65662040698002721810659005184000

Table B.4 A172806 ~~1502~~

n	H_4^n
1	1
2	5
3	120
4	10147
5	2224955
6	1047649905
7	936670590450
8	1455918295922650
9	3680232136895819610
10	14356628851597700179050
11	82857993930808028192521800
12	683327637694741065563262206250
13	7821620120684573354895941635688250
14	121226756408657335034315697817193707350
15	2490562784819660349490404693413463514984500

A58528 ~~1503~~

n	A_4^n
1	0
2	0
3	0
4	1
5	120
6	67950
7	68938800
8	116963796250
9	315031400802720
10	1289144584143523800
11	7722015017013984456000
12	65599839591251908982712750
13	769237071909157579108571190000
14	12163525741347497524178307740904300
15	2541436678226866358505906615550954680000

Table B.5

~~1504~~

A172862

n	H_5^n
1	1
2	6
3	231
4	40176
5	22069251
6	30767936616
7	94161778046406
8	56930469099440025 6
9	6274236760589024662176
10	1182858301266601234748447 52
11	36234402121984614113810725755 12
12	1728504524983984203103700973452421 12
13	123902303941900710288672672324875489997 12
14	1294628556532107050462104023408654483047089152
15	192083384534505302178334348591748127157718391470256

~~1505~~

A75754

n	A_5^n
1	0
2	0
3	0
4	0
5	1
6	720
7	3110940
8	24046189440
9	315031400802720
10	6736218287430460752
11	22688523170021571353568 0
12	11649337108041078980732943360
13	8852827762101207150867156197241 60
14	96986285294151066094112970262797953280
15	1496262881677497094077277774008499852173825 6

TABLE B.5

n	H_5^n
1	1
2	6
3	231
4	40176
5	22069251
6	30767936616
7	94161778046406
8	569304690994400256
9	6274236760589024662176
10	116285830126660123474844752
11	3623440212198461411381072575512
12	172850452498398420310370097345242112
13	1239023039419007102886726723248754899712
14	1294628556532107050462104023408654483047089152
15	192083384534505302176334348591748127157718391470256

n	A_5^n
1	0
2	0
3	0
4	0
5	1
6	720
7	3110940
8	24046189440
9	315031400802720
10	6736218287430460752
11	226885231700215713535680
12	11649337108041078980732943360
13	885282776210120715086715619724160
14	96986285294151066094112970262797953280
15	1496262881677497094077277740084998521738256

TABLE B.6

$$A_6^{12} = 640513758899273800355498043336$$

$$A_6^{13} = 28278447452854953938096018206821120$$

$$A_6^{14} = 1904041926627879976663103246184913901304$$

Table B.7

n	L_e	A_2^n	U_e
10	1.36911×10^{12}	1.37179×10^{12}	1.38117×10^{12}
20	4.44143×10^{35}	4.44432×10^{35}	4.45683×10^{35}
30	4.33929×10^{63}	4.34090×10^{63}	4.34845×10^{63}
40	3.56730×10^{94}	3.56821×10^{94}	3.57268×10^{94}
50	4.44212×10^{127}	4.44298×10^{127}	4.44733×10^{127}
60	3.03922×10^{162}	3.03970×10^{162}	3.04213×10^{162}
70	5.83647×10^{198}	5.83723×10^{198}	5.84119×10^{198}
80	1.95029×10^{236}	1.95051×10^{236}	1.95165×10^{236}
90	7.92817×10^{274}	7.92894×10^{274}	7.93305×10^{274}
100	2.96904×10^{314}	2.96930×10^{314}	2.97068×10^{314}
110	8.20163×10^{354}	8.20227×10^{354}	8.20571×10^{354}
120	1.39343×10^{396}	1.39353×10^{396}	1.39406×10^{396}
130	1.25144×10^{438}	1.25152×10^{438}	1.25196×10^{438}
140	5.22689×10^{480}	5.22720×10^{480}	5.22891×10^{480}
150	9.09720×10^{523}	9.09771×10^{523}	9.10047×10^{523}
160	5.99914×10^{567}	5.99945×10^{567}	6.00115×10^{567}
170	1.37923×10^{612}	1.37930×10^{612}	1.37967×10^{612}
180	1.02721×10^{657}	1.02726×10^{657}	1.02752×10^{657}
190	2.32169×10^{702}	2.32179×10^{702}	2.32234×10^{702}
200	1.50213×10^{748}	1.50219×10^{748}	1.50253×10^{748}
210	2.63962×10^{794}	2.63972×10^{794}	2.64029×10^{794}
220	1.20129×10^{841}	1.20134×10^{841}	1.20158×10^{841}
230	1.35599×10^{888}	1.35603×10^{888}	1.35630×10^{888}
240	3.64936×10^{935}	3.64948×10^{935}	3.65016×10^{935}
250	2.25847×10^{983}	2.25854×10^{983}	2.25895×10^{983}

Table B.8

n	L_e	H_2^n	U_e
10	3.91237×10^{12}	3.93073×10^{12}	3.96185×10^{12}
20	1.23738×10^{36}	1.23948×10^{36}	1.24352×10^{36}
30	1.19895×10^{64}	1.20015×10^{64}	1.20260×10^{64}
40	9.81608×10^{94}	9.82296×10^{94}	9.83746×10^{94}
50	1.21933×10^{128}	1.21999×10^{128}	1.22140×10^{128}
60	8.32884×10^{162}	8.33248×10^{162}	8.34040×10^{162}
70	1.59759×10^{199}	1.59818×10^{199}	1.59947×10^{199}
80	5.33380×10^{236}	5.33549×10^{236}	5.33922×10^{236}
90	2.16679×10^{275}	2.16738×10^{275}	2.16873×10^{275}
100	8.11005×10^{314}	8.11205×10^{314}	8.11656×10^{314}
110	2.23931×10^{355}	2.23981×10^{355}	2.24094×10^{355}
120	3.80311×10^{396}	3.80389×10^{396}	3.80563×10^{396}
130	3.41451×10^{438}	3.41515×10^{438}	3.41659×10^{438}
140	1.42576×10^{481}	1.42600×10^{481}	1.42656×10^{481}
150	2.48089×10^{524}	2.48130×10^{524}	2.48220×10^{524}
160	1.63570×10^{568}	1.63594×10^{568}	1.63650×10^{568}
170	3.75987×10^{612}	3.76040×10^{612}	3.76161×10^{612}
180	2.79980×10^{657}	2.80016×10^{657}	2.80101×10^{657}
190	6.32716×10^{702}	6.32795×10^{702}	6.32977×10^{702}
200	4.09315×10^{748}	4.09363×10^{748}	4.09475×10^{748}
210	7.19184×10^{794}	7.19264×10^{794}	7.19450×10^{794}
220	3.27266×10^{841}	3.27302×10^{841}	3.27382×10^{841}
230	3.69375×10^{888}	3.69412×10^{888}	3.69500×10^{888}
240	9.94007×10^{935}	9.94104×10^{935}	9.94329×10^{935}
250	6.15110×10^{983}	6.15168×10^{983}	6.15301×10^{983}

REFERENCES

1. P. A. MacMahon, Combinatory Analysis, 2 vols, Cambridge University Press, 1915 (reprinted as 2 vols in 1 by Chelsea, 1960).

2. C. Kostka, "Ueber den Zusammenhang zwischen einigen Formen von symmetrischen Functionen," Crelle's Jour., xciii, 89-123 (1881).

3. H. Anand, V. C. Dumir, and H. Gupta, "A combinatorial distribution problem," Duke Math. J. 33, 757-770 (1966).

4. D. E. Barton, F. N. David, and M. G. Kendall, Symmetric Function and Allied Tables, Cambridge University Press, pp. 10-11 (1966).

5. D. E. Littlewood, The Theory of Group Characters, 2nd Ed., Oxford, 1950.

6. N. Metropolis and P. R. Stein, "An elementary solution to a problem in restricted partitions," J. Comb. Theory (to appear).

7. G. de B. Robinson, Representation Theory of the Symmetric Group, Univ. of Toronto Press, Chap. 2, 1961.

8. R. Bivins, N. Metropolis, P. R. Stein, and M. B. Wells, "Characters of the symmetric group of degree 15 and 16," MTAC, Vol. 8, No. 48 (1954).

9. J. Riordan, Combinatorial Identities, John Wiley, New York, p. 68, 1968.

10. Patrick O'Neil, unpublished Ph.D. Thesis, The Rockefeller University, Chap. 2, 1968.