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# ON THE WEIGHTED FINITE LINEAR SPACES

by Pierre ROBILLARD

*Résumé*. — La notion d'espace linéaire pondéré (e.l.p.) est une extension de la notion d'espace linéaire fini (e.l.) introduite récemment par P. Libois. Cette nouvelle notion apparaît naturellement dans un problème d'énumération des schémas confondus de classe ($s^n$, $s^2$).

La première partie de cette étude concerne les e.l. où nous obtenons une formule donnant le nombre d'espaces linéaires comme fonction des espaces linéaires connexes. Nous dérivons aussi une borne inférieure et supérieure pour le nombre d'espaces linéaires non isomorphes de $n$ points.

Dans une seconde partie nous définissons la notion de e.l.p. et de e.l.p. connexes. Nous obtenons une formule donnant le nombre de e.l.p. non isomorphes comme fonction du nombre de e.l.p. connexes.

Utilisant ces résultats et ceux concernant les e.l.p. nous calculons une borne inférieure et supérieure pour le nombre de e.l.p. On trouvera en appendice l'énumération des 177 e.l.p. connexes dont la somme totale des poids est inférieure à 9.

## Introduction

We want to consider an extension of the notion of finite linear space defined by P. Libois [2], which occurs naturally in the combinatorial problem of the enumeration of confounded factorial designs of class ($s^n$, $s^2$) see [4]. We recall first the notion of finite linear space.

A linear space (*l.s.*) is any set of objects called points some subsets of which are called lines. We require that these points and lines satisfy the following axioms.

A1: Any two points belong to one and only one line.

A2: Any line contains at least two points.

From this definition, it is obvious that two distinct lines contain at most one point in common.

An isomorphism of a linear space L on a linear space L′ is any bijection of L on L′ which maps any line of L on a line of L′. Two linear spaces are isomorphic if there exists an isomorphism from one on the other. An automorphism of a linear space L is on isomorphism of L on itself.

We define a linear variety V of a linear space L as a subset of L in which any line containing two distinct points of V is entirely in V. It is easily seen that a linear variety is itself a linear space and that any intersection of linear varieties is a linear variety.

A linear space is connected if it is not the union of two disjoint non null linear varieties. A connected linear variety maximal with respect to inclusion is called a connected component. We have the results

— In a linear space L any point belongs to one and only one connected component.

— If V is a linear variety in a linear space L, then L − V is a variety if and only if any line with a point in V and a point in L − V has no other points in it.

**The enumeration problem**

From now on we shall consider the linear spaces with a finite number $n$ of points. The problem we want to consider is to count the number of non isomorphic linear spaces of $n$ points; in other words if we define $N(n)$ the number of non isomorphic *l.s.* of $n$ points and by $N_c(n)$ the number of non isomorphic connected *l.s.* of $n$ points, one wants to express $N(n)$ and $N_c(n)$ as a function of $n$ only. The question is far from being solved, however some partial results were found by Jean Doyen [1]. As we need these results later we shall now summarize them.

Jean Doyen has enumerated the linear spaces of $n$ points for $n \leqslant 9$ and the values of $N(n)$ and $N_c(n)$ are given in table 1 below:

TABLE 1

| $n$ | $N(n)$ | $N_c(n)$ |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1 | 1 |
| 2 | 1 | 0 |
| 3 | 2 | 1 |
| 4 | 3 | 1 |
| 5 | 5 | 2 |
| 6 | 10 | 4 |
| 7 | 24 | 13 |
| 8 | 69 | 42 |
| 9 | 384 | 308 |

For higher values of $n$ he has the following theorems:

THEOREM 1 (Doyen)

The function $f\colon n \to \mathrm{N}(n)$ is convex i.e.

$$\mathrm{N}(n) \leqslant \frac{1}{2}[\mathrm{N}(n-1) + \mathrm{N}(n+1)] \qquad n > 0 \tag{1}$$

THEOREM 2 (Doyen)

$$\mathrm{N}_c(n) > 2^{n-1} \qquad \text{for all} \qquad n \geqslant 9 \tag{2}$$

THEOREM 3 (Doyen)

$$\mathrm{N}(n) > 2^n \qquad \text{for all} \qquad n \geqslant 10 \tag{3}$$

THEOREM 4 (Doyen)

$$\mathrm{N}(n) < 2^{\binom{n}{3}} = \mathrm{N}_u^*(n) \qquad \text{for} \qquad n \geqslant 4 \tag{4}$$

Before proceeding to the generalizalition of the notion of linear space we want to give the following results

THEOREM 5

$$\mathrm{N}(n) = \sum_{\substack{n = y_1 + 2y_2 + \ldots + ny_n \\ y_i \geqslant 0}} \left[ \prod_{i=1}^{n} \binom{\mathrm{N}_c(i) + y_i - 1}{\mathrm{N}_c(i) - 1} \right] \tag{5}$$

The summation is over the set of non negative integers $\{y_i\}_{i=1 \ldots n}$ solutions of $n = y_1 + 2y_2 + \ldots + ny_n$.

We use the convention that $\binom{n}{-1} = 0$.

PROOF. — We realize easily that any linear space of $n$ points can be expressed in a unique way as the union of disjoint connected components. More precisely if the connected components of L are $\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_p$ then

$$\mathrm{C}_i \cap \mathrm{C}_j = \emptyset \quad \text{if} \quad i \neq j$$

$$\sum_{i=1}^{p} |\mathrm{C}_i| = n,$$

where $|\mathrm{C}_i|$ is the number of points in $\mathrm{C}_i$, and we can write

$$\mathrm{L} = \sum_{i=1}^{p} \mathrm{C}_i.$$

If L′ is another *l.s.* of $n$ points whose decomposition in connected components is written as

$$L' = \sum_{i=1}^{p'} C'_i$$

then a necessary condition for L and L′ to be isomorphic is that

— $p = p'$ and

— $\{|C_i| \mid i = 1, 2, \ldots, p\}$ and $\{|C'_i| \mid i = 1, 2, \ldots, p\}$

represent the same partition of $n$ in $p$ positive integers.

Thus for a given partition $y_1 + 2y_2 + \ldots + ny_n = n$ ($y_k$ being the number of times $k$ appears in the partition) we obtain the number of *l.s.* of $n$ points as equal to

$$\prod_{i=1}^{n} \binom{N_c(i) + y_i - 1}{N_c(i) - 1} \tag{6}$$

where

$$\binom{N_c(i) + y_i - 1}{N_c(i) - 1}$$

is the number of ways to choose $y_i$ connected *l.s.* of $i$ elements each.

Summing (6) over all possible permutations gives (5).

The last theorem shows that in order to solve the problem of enumeration for the linear spaces it is sufficient to enumerate the connected linear spaces and conversely.

We can now utilize the theorems (2) and (5) together and obtain a new lower bound for N($n$) which is

$$L(n) = \sum_{\substack{n = y_1 + 2y_2 + \ldots + ny_n \\ y_i \geqslant 0}} \left[ \prod_{i=1}^{n} \binom{N^*_c(i) + y_i - 1}{N^*_c(i) - 1} \right]$$

where

$$N^*_c(i) = N_c(i) \quad \text{if} \quad i \leqslant 9$$

$$2^{i-1} \quad \text{if} \quad i > 9$$

In table 2, we compare L($n$) with the value $2^n$ for $10 \leqslant n \leqslant 20$

TABLE 2

| $n$ | $2^n$ | L($n$) | L($n$)/$2^n$ |
|---|---|---|---|
| 10 | 1024 | 917 | |
| 11 | 2048 | 2007 | |
| 12 | 4096 | 4449 | 1.08 |
| 13 | 8192 | 9524 | 1.16 |
| 14 | 16384 | 20200 | 1.23 |
| 15 | 32768 | 42877 | 1.30 |
| 16 | 65536 | 91862 | 1.40 |
| 17 | 131072 | 200459 | 1.53 |
| 18 | 262144 | 462830 | 1.76 |
| 19 | 524288 | 1055862 | 2.01 |
| 20 | 1048576 | 2390058 | 2.30 |

We realise that for $n \geqslant 12$ the values of L($n$) are greater than $2^n$ and in general this inequality is true; if we write the first few terms of L($n$), we have that:

$$\begin{aligned} L(n) &= N_c^*(n) + N_c^*(n-1) + N_c^*(n-2) \\ &\quad + 2N_c^*(n-3) + 3N_c^*(n-4) + \dots \end{aligned}$$

and for $n > 12$

$$\begin{aligned} L(n) &= 2^{n-1} + 2^{n-2} + 2^{n-3} + 2 \cdot 2^{n-4} + 3N_c^*(n-4) + \dots \\ &= 2^n + 3N_c^*(n-4) + \dots > 2^n \end{aligned}$$

THEOREM 6

$$N(n) \leqslant N(n-1)\left[1 + \sum_{K=0} \frac{1}{(K+1)!} \prod_{j=0}^{K} \binom{n-1-2j}{2}\right] \tag{7}$$

$$\text{where } \binom{s}{t} = 0 \quad \text{for} \quad s < t$$

PROOF. — Suppose that we delete a point of a *l.s.* and also all the lines of two points passing through it. We obtain then a *l.s.* L* of $n - 1$ points. Let us count the number of ways one can add a point to L* and obtain a *l.s.* with $n$ points. When we talk of adding a point P to L* we suppose that we also add all the lines of two points joining P to any point of L* such that {L*} + P is a *l.s.*

One can place the new point outside any existing lines of L*; there is one way to do that. One can also place the new point on an existing line and not at the intersection of any other lines. There are a maximum of $\binom{n-1}{2}$ lines in L* which give $\binom{n-1}{2}$ possibilities. In general, one can place the new point as the intersection of exactly K non intersecting lines of L* which give

$$\frac{1}{K!}\prod_{j=0}^{K-1}\binom{n-1-2d}{2}$$

possibilities. The sum of these expressions gives the maximum number of *l.s.* of $n$ points one can obtain from one *l.s.* of $n-1$ points. The equation (7) follows.

We use (7) to calculate recursively an upper bound for N($n$). For example

$$N(10) \leqslant N(9)\cdot\left[1+\binom{9}{2}+\binom{9}{2}\binom{7}{2}\Big/2+\binom{9}{2}\binom{7}{2}\binom{5}{2}\Big/6\right.$$
$$\left.+\binom{9}{2}\binom{7}{2}\binom{5}{2}\binom{3}{2}\Big/24\right]$$

i.e.

$$N(10) \leqslant 1{,}006{,}080 = U(10)$$

In general

$$N(n) \leqslant U(n-1)\cdot\left[1+\sum_{K=0}\frac{1}{(K+1)!}\prod_{j=0}^{K}\binom{n-1-2j}{2}\right] = U(n)$$

We now compare this bound U($n$) with N*($n$) given in (4).

We have that

$$Q(n)=\frac{U(n)}{U(n-1)}=1+\sum_{K=0}\frac{1}{(K+1)!}\prod_{j=0}^{K}\binom{n-1-2j}{2}<\frac{n-1}{2}\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}$$

and $\log_2 Q(n) < n\log_2 n$

On the other hand

$$Q^*(n)=\log_2\frac{N_u^*(n)}{N_u^*(n-1)}=\frac{n^2-3n+2}{2}$$

We realize that

$$n\log_2^2 n < \frac{n^2-3n+2}{2} \quad \text{for} \quad n \geqslant 10$$

The bound U($n$) is thus smaller than the bound $N_u^*(n)$. However, the difference between U($n$) and L($n$) is very large as we can see for the case $n = 10$ and increases with $n$.

**Weighted linear spaces** (w.l.s.)

We define a weighted linear space (*w.l.s.*) as a set of elementary objects called weighted points; there are the 1-points or points with weight one, the 2-points of points with weight 2, in general the $w$-points or points with weight $w$. Some subsets of these objects will be distinguished and called lines. A line may be composed of any kind of points. In the following the word point without specification of weight will designate any kind of $w$-points. On this set of objects we specify two axioms

W1 — two different points belong to one and only one line

W2 — any line contains at least two points.

The notion of *w.l.s.* is an extension of the notion of *l.s.* defined below where we considered uniquely the 1-points.

We call an isomorphism from a *w.l.s.* W on a *w.l.s.* W′ any bijection of W on W′ which apply

— a $w$- point of W on a $w$- point of W′

— any line of W on a line of W′

Two *w.l.s.* will be called *isomorphic* if there exists an isomorphism from one to the other.

The notion of linear variety, connexity and connected component defined in the case of *l.s.* can be extended in a very obvious and natural way for the case of *w.l.s.*

We shall consider now the finite weighted linear spaces, that is the spaces whose sum of point's weights equals $n$. We shall say that they have total weight $n$. The problem of interest for us is the enumeration of the non-isomorphic *w.l.s.* of total weight $n$. More precisely we say that there are $M(n)$ non-isomorphic *w.l.s.* of total weight $n$ and $M_c(n)$ connected ones. Obviously $M_c(n) \leqslant M(n)$.

For $n < 9$ we have enumerated all the non-isomorphic *w.l.s.* We give the values of $M(n)$ and $M_c(n)$ in table 3 below.

This problem of enumerating the *w.l.s.* is related as we have said earlier to the problem of enumerating the confounded factorial designs of class $(s^n, s^2)$ which will be treated elsewhere. We shall consider only the geometrical structure here.

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TABLE 3

| $n$ | $M(n)$ | $M_c(n)$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 2 | 1 |
| 3 | 4 | 2 |
| 4 | 8 | 3 |
| 5 | 16 | 6 |
| 6 | 36 | 13 |
| 7 | 85 | 35 |
| 8 | 239 | 116 |

Using the following theorem, the problem of enumerating the *w.l.s.* of total weight $n$ can be reduced to the problem of enumerating the connected *w.l.s.* :

THEOREM 7

$$M(n) = \sum_{\substack{n=y_1+2y_2+\ldots+ny_n \\ y_i \geqslant 0}} \left[ \prod_{i=1}^{n} \binom{M_c(i)+y_i-1}{M_c(i)-1} \right] \tag{9}$$

This is analog to theorem 5 and is proved in a similar fashion.

In appendix 1 we give the complete enumeration of the connected *w.l.s.* of weight sum $\leqslant 8$. For $n > 8$ the enumeration is very fastidious, thus we shall from now on derive only some bounds for $M(n)$ and $M_c(n)$

THEOREM 8

$$M(n) \geqslant \sum_{k=1}^{n} p_k(n)\, N(k) = B^1(n) \tag{10}$$

where $N(k)$ is the number of *l.s.* of $k$ points and $p_k(n)$ is the number of partitions of $n$ in exactly $k$ parts.

PROOF. — If one replaces the points of an *l.s.* L of $k$ points by *w*-points in such a way that the sum of weights equals $n$, one obtains this way a *w.l.s.* of total weight $n$. The set of $k$ weights distribùted among the $k$ points of L represents a partition of $n$ in $k$ parts. It is easily seen that if we had started with another partition, the *w.l.s.* obtained would have been different. Also if we had started with a different *l.s.* than L, we would have obtained a different result. Enumerating all these possibilities leads to (10).

THEOREM 9

$$M_c(n) \geqslant \sum_{k=1}^{n} p_k(n) N_c(k) = B_c^1(n)$$

where $N_c(k)$ is the number of different connected *l.s.* of $k$ points.

The proof is analogous to the proof of the theorem 8. Obviously we do not know $N(k)$ or $N_c(k)$, but we have by theorem 2 and theorem 3 a lower bound for each of these values; this gives us:

THEOREM 10

$$M(n) \geqslant \sum_{k=1}^{n} p_k(n) N^*(k) = B^2(n)$$

$$M_c(n) \geqslant \sum_{k=1}^{n} p_k(n) N_c^*(k) = B_c^2(n)$$

where

$$N^*(k) = \begin{cases} N(k) & \text{for} \quad k \leqslant 9 \\ 2^k & \text{for} \quad k > 9 \end{cases}$$

and

$$N_c^*(k) = \begin{cases} N_c(k) & \text{for} \quad k \leqslant 9 \\ 2^{k-1} & \text{for} \quad k > 9 \end{cases}$$

An other way of obtaining a lower bound for $M(n)$, is to replace in (9) the values $M_c(k)$ by $B_c^2(k)$. We obtain then

THEOREM 11

$$M(n) \geqslant \sum_{\substack{n = y_1 + 2y_2 + \dots + ny_n \\ y_i \geqslant 0}} \left[ \prod_{i=1}^{n} \binom{B_c^2(i) + y_i - 1}{y_i} \right] = B^3(n)$$

$B^3(n)$ is a much more complicated expression than $B^2(n)$ and it becomes very difficult to state which one is better.

I whish to express my appreciation to Jean Doyen for numerous suggestions.

## APPENDIX I

**Enumeration of the connected weighted linear spaces**

We shall give the graphical representations as follows: Each $w$-point is represented by a dot and the value $w$ is written near to it if $w > 1$. If all the $w$-points are collinear we write only the value of $w$, each separated by a coma.

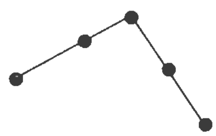
$n = 1$ 1

$n = 2$ 2

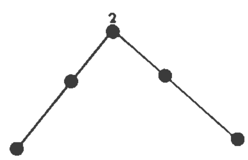
$n = 3$ 1 - 3 2 - 1, 1, 1,

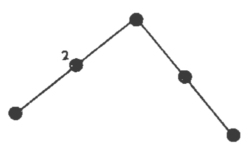
$n = 4$ 1 - 4
2 - 2, 1 , 1,
3 - 1, 1, 1, 1,

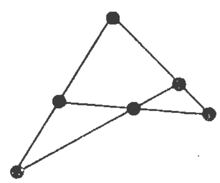
$n = 5$ 1 - 5
2 - 3, 1, 1,
3 - 2, 2, 1
4 - 2, 1, 1, 1
5 - 1, 1, 1, 1, 1,

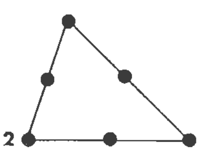
6 

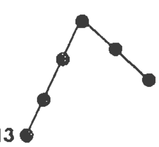
$n = 6$ 1 - 6
2 - 4, 1, 1
3 - 3, 2, 1
4 - 3, 1, 1, 1
5 - 2, 2, 2
6 - 2, 2, 1, 1
7 - 2, 1, 1, 1, 1
8 - 1, 1, 1, 1, 1, 1,

9 

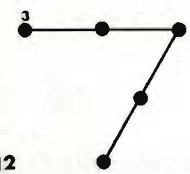

10 


11 

12 

13 

$n = 7$ 1 - 7
2 - 5, 1, 1
3 - 4, 2, 1
4 - 4, 1, 1, 1
5 - 3, 3, 1
6 - 3, 2, 1 ,1
6 - 3, 2, 2
8 - 3, 1, 1, 1, 1
9 - 2, 2, 2, 1
10 - 2, 2, 1, 1, 1
11 - 2, 1, 1, 1, 1, 1



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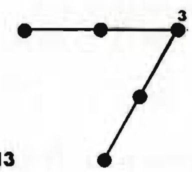

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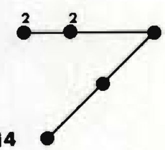

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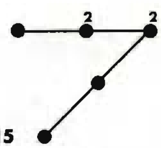

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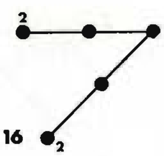

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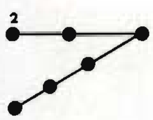

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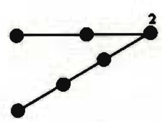

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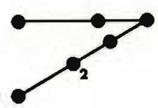

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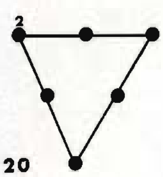

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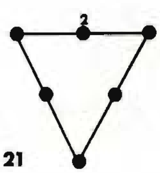

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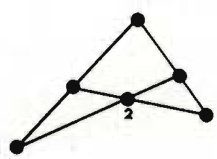


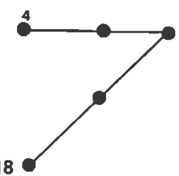
22

the *w.l.s.* 23 to 35 consist of the 13 connected *l.s.* described by Doyen.

$n = 8$

1 - 8
2 - 6, 1, 1
3 - 5, 2, 1
4 - 5, 1, 1, 1
5 - 4, 3, 1
6 - 4, 2, 2
7 - 4, 2, 1, 1
8 - 4, 1, 1, 1, 1
9 - 3, 3, 2
10 - 3, 3, 1, 1
11 - 3, 2, 2, 1
12 - 3, 2, 1 ,1, 1
13 - 3, 1, 1, 1, 1, 1
14 - 2, 2, 2, 2
15 - 2, 2, 2, 1, 1
16 - 2, 2, 1, 1, 1, 1
17 - 2, 1, 1, 1, 1, 1, 1

the connected *w.l.s.* 75 to 116 consist of the 42 connected *l.s.* described by Doyen.



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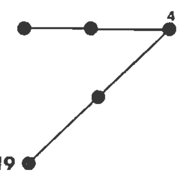

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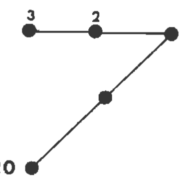

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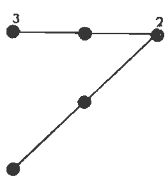

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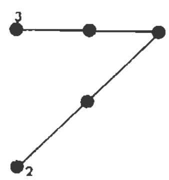

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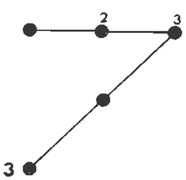

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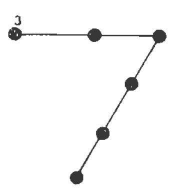

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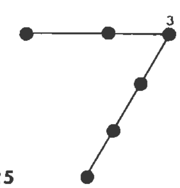

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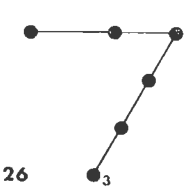

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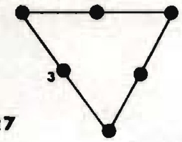

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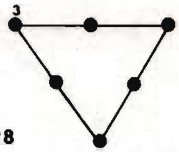

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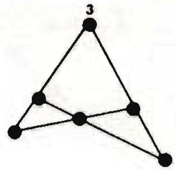

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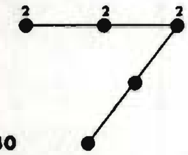

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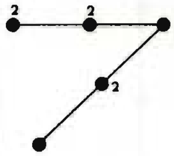

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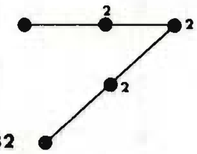

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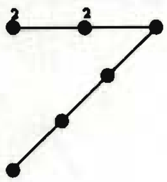

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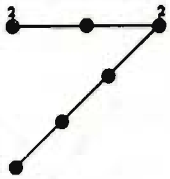

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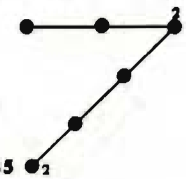

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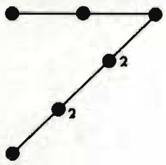

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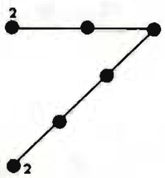

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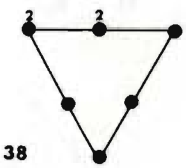

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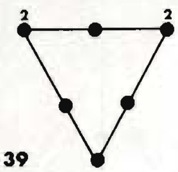

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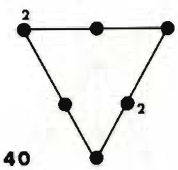

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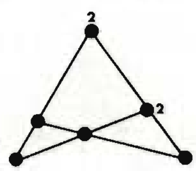

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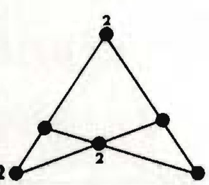

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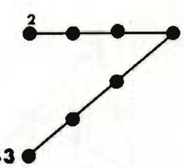

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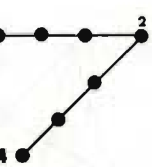

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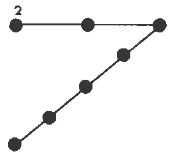


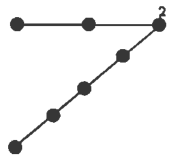

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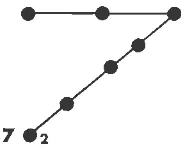

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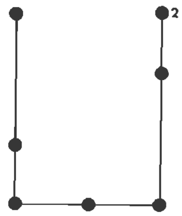

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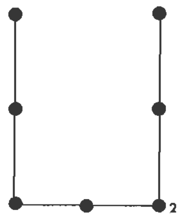

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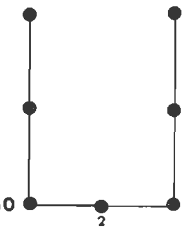

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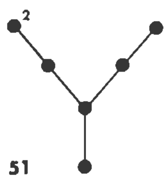

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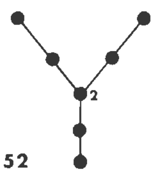

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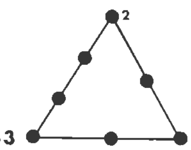

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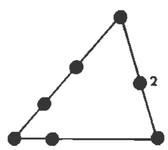

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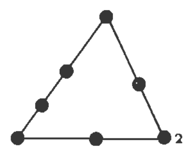

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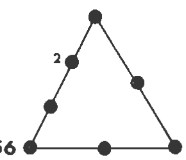

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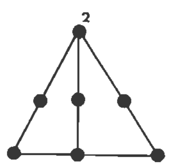

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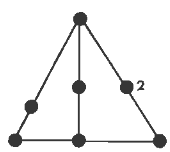

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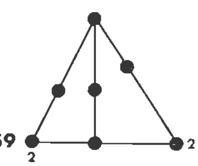

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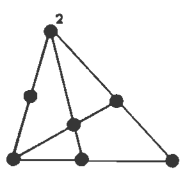

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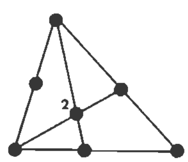

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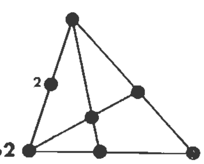

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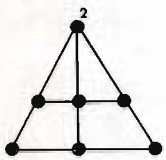


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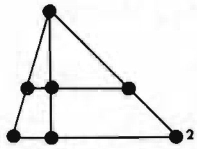


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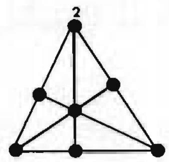


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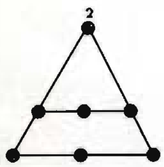


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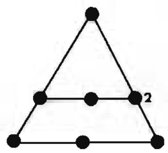


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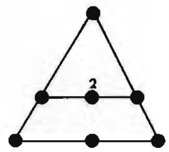


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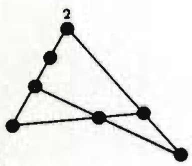


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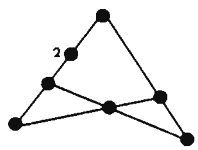


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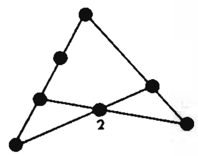


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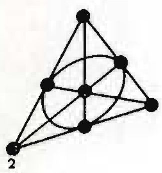


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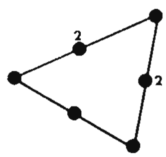


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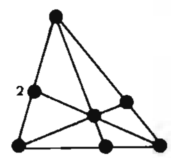


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