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39 1947 1569

ASYMPTOTIC SOLUTION OF THE "PROBLÈME DES MÉNAGES"

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The "problème des ménages" consists in finding the number of ways in which n married couples may be seated at a round table, men alternating with women, so that no wife sits next to her own husband.

Let the number of ways be set equal to $2(n!)a(n)$. Several solutions have been given before by expressing $a(n)$ in various ways. Mac Mahon (1915), for instance, gives the following operational formula for $a(n)$:

$$a(n) = (3!/n!)^n [D_{a+b+c+} \ldots (cde\ldots)(ade\ldots)(abe\ldots)\ldots]_{a=1} \tag{1}$$

Moreau (Lucas, 1891) gives a difference equation for $a(n)$ equivalent to

$$(n-2)a(n) = n(n-2)a(n-1) + na(n-2) - 4(-1)^n, \tag{2}$$

and then calculates $a(n)$ for $n \leq 20$.

Recently, however, more explicit formulæ have been obtained for $a(n)$. Touchard (1934) expresses $a(n)$ in terms of definite integrals, and Schöbe (1943) shows that

$$a(n+1) = 2(-1)^{n+1} + \frac{(n+1)(-1)^n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} h_k^2, \tag{3}$$

where h_k is Whitworth's "sub-factorial" k or

$$h_k = k! \sum_{r=0}^{k} (-1)^r/(r!).$$

Schöbe also establishes that

$$a(n)/n! \sim 1/e^2 \tag{4}$$

Very recently, Kaplansky (1945) by an interesting application of Cayley and Sylvester's principle of "cross-classification" showed that

$$a(n) = 2(-1)^n + (-1)^{n-1} n \sum_{r=0}^{n-1} (-1)^r \binom{n+r}{2r+1} r! \tag{5}$$

It is intended here to establish an asymptotic series for $a(n)$, assuming (2) and (4).

I

To investigate a particular solution of the difference equation (2), I set

$$a(n) = \sum_r c_r n^{-r},$$

assuming convergence for all $n > n_0$, and find

$$c_0 = 0, \quad c_1 = c_3 = c_5 = c_7 = c_9 = c_{11} = 0,$$

$$c_2 = 4(-1)^n, \quad c_4 = -5c_2, \quad c_6 = 31c_2, \quad c_8 = -125c_2, \quad c_{10} = -1769c_2, \text{ etc.}$$

Hence the particular solution runs

$$4(-1)^n\left[\frac{1}{n^2} - \frac{5}{n^4} + \frac{31}{n^6} - \frac{125}{n^8} - \frac{1769}{n^{10}} \cdots + \cdots\right].$$

Next, I seek a solution of the homogeneous equation

$$(n-2)a(n) - n(n-2)a(n-1) - na(n-2) = 0 \tag{6}$$

such that

Writing $a(n)/(n!) \sim e^{-2}$.

$$a(n) = \frac{n!}{e^2}\left[1 + \sum_{r=1}^{\infty} d_r n^{-r}\right]$$

in (6) and assuming the infinite series to converge for $n > n_1$, I find

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#1569

$1!\, d_1 = -1,$	$6!\, d_6 = 1501,$
$2!\, d_2 = -1,$	$7!\, d_7 = -31354,$
$3!\, d_3 = 2,$	$8!\, d_8 = -1451967,$
$4!\, d_4 = 37,$	$9!\, d_9 = -39284461,$
$5!\, d_5 = 329,$	$10!\, d_{10} = -737652869,$ etc.

Hence the asymptotic value of $a(n)$ is given by

$$a(n) = \frac{n!}{e^2}\left[1 - \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \frac{37}{24n^4} + \frac{329}{120n^5} + \frac{1501}{720n^6} - \frac{15677}{2520n^7} - \frac{1451967}{40320n^8} - \frac{39284461}{362880n^9} - \frac{737652869}{3628800n^{10}} - \cdots + \cdots\right]$$

$$+4(-1)^n\left[\frac{1}{n^2} - \frac{5}{n^4} + \frac{31}{n^6} - \frac{125}{n^8} - \frac{1769}{n^{10}} - \cdots + \cdots\right] \tag{7}$$

I believe that for large values of n, the above provides a more explicit solution of the "problème des ménages" than any previously given.

II

As Lucas 1891 remarks, $a(n)$ is the number of permutations of 1, 2, . . . n discordant with each of the two permutations

$$\begin{matrix} 1, & 2, & 3, & 4, & \ldots & (n-1), & n \\ 2, & 3, & 4, & 5, & \ldots & n, & 1. \end{matrix}$$

If the two lines be replaced by any two discordant permutations of 1, 2, . . . n, we get the problem of completing the third line in a three-deep Latin rectangle, of which the first two lines are fixed. The asymptotic number of solutions in this case was recently found by Erdös and Kaplansky (1946) to be

$$\frac{n!}{e^2}\left[1-\frac{1}{n}-\frac{1}{2n^2}+\cdots\right],$$

but the series has been extended by me (1947) as follows:

$$\frac{n!}{e^2}\left[1-\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+\frac{49}{24n^4}+\frac{629}{120n^5}+\frac{6961}{720n^6}+\frac{19183}{2520n^7}\right.$$

$$\left.-\frac{633229}{13440n^8}-\frac{133065253}{362880n^9}-\frac{6482111309}{3628800n^{10}}-\cdots+\cdots\right].$$

It is instructive to compare this last result with (7). The two asymptotic expansions are identical as far as the fourth term, but differ from the fifth term onwards. This throws interesting light on some conjectures of Erdös and Kaplansky about the asymptotic number of the general n by k Latin rectangle. I hope to examine the implications in a subsequent paper.

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