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ON THE AVERAGES OF THE DIVISORS OF A NUMBER

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Let n be some integer and let

$$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$$

be its decomposition into prime factors. As usual we shall denote the *number* of divisors of n by $\nu(n)$ and the *sum* of the divisors by $\sigma(n)$ so that

$$\nu(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1),$$

and

$$\sigma(n) = \sum_{d/n} d = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdots \frac{p_r^{\alpha_r+1} - 1}{p_r - 1}.$$

The various means of the divisors may be computed easily from the formulas that follow. For the *arithmetic mean* one has

(1) $$A(n) = \frac{\sigma(n)}{\nu(n)}$$

and for the *geometric mean* the result is

(2) $$G(n) = \sqrt[\nu(n)]{\prod_{d/n} d} = \sqrt{n}.$$

Finally the harmonic mean is defined by

$$\frac{1}{H(n)} = \frac{1}{\nu(n)} \cdot \sum_{d/n} \frac{1}{d}$$

and since

$$n \sum_{d/n} \frac{1}{d} = \sum_{d/n} \frac{n}{d} = \sum_{d/n} d$$

it follows that

(3) $$H(n) = \frac{n\nu(n)}{\sigma(n)}.$$

The combination of the three formulas (1), (2), and (3) gives

$$H(n) \cdot A(n) = G(n)^2;$$

in other words, *the geometric mean of the divisors is the geometric mean of the arithmetic and harmonic means of the divisors.*

$$H(n) A(n) = G(n)^2 = n$$

One may now ask in which cases these means may be integers. For the geometric mean this is trivial: *The geometric mean of the divisors is an integer only for square numbers.*

For the other two means the problem is by no means simple. We may observe first that both the arithmetic and harmonic means have the multiplicative property

$$A(a\cdot b) = A(a)\cdot A(b), \qquad H(a\cdot b) = H(a)\cdot H(b)$$

provided a and b are relatively prime. Thus one may only look for the *primitive* integral means, that is, such numbers that they are not the product of relatively prime factors, each of which has an integral arithmetic or harmonic mean.

Let us consider the arithmetic mean in a few special cases. For an odd prime p one has

$$A(p) = \frac{p+1}{2}$$

and this is always integral, while $p=2$ is an exception since $A(2)=3/2$ is not integral. From the multiplicative property we conclude that every number which is the product of different odd primes has an integral arithmetic mean of divisors while in the case of different prime factors including 2 the mean is only integral if one of the primes is of the form $4k-1$.

One may consider also when the powers of a prime may have an integral arithmetic mean of divisors. Since

$$A(p^\alpha) = \frac{p^{\alpha+1}-1}{(p-1)(\alpha+1)},$$

this is a problem closely connected with the solution of the congruence

$$x^k \equiv 1 \pmod{k}.$$

We shall not go into details about this problem. In certain cases the arithmetic mean is integral; for instance,

$$A(5^3) = 39, \qquad A(5^5) = 651.$$

It can be shown that no power of 2 can have an integral arithmetic mean.

The corresponding problems for the harmonic mean seem more interesting. In certain simple cases it can be established easily that the harmonic mean of the divisors cannot be integral. We mention first:

For the power of a prime the harmonic mean is not integral.

If namely $n=p^\alpha$ then

$$H(p^\alpha) = \frac{p^\alpha(\alpha+1)}{p^\alpha + p^{\alpha-1} + \cdots + p + 1}.$$

Here p^α is relatively prime to the denominator and since $\alpha+1$ is a number less than the denominator the harmonic mean cannot be integral.

Another observation is:

When $n \neq 6$ is the product of different prime factors the harmonic mean cannot be integral.

We write

$$n = p_1 p_2 \cdots p_r$$

where the primes are arranged in increasing order, and one finds

$$H(n) = \frac{p_1 p_2 \cdots p_r}{(p_1 + 1) \cdots (p_r + 1)} \cdot 2^r.$$

Let us assume first that n is odd, and let us write

$$H(n) = \frac{p_1 \cdots p_r}{\dfrac{p_1 + 1}{2} \cdots \dfrac{p_r + 1}{2}}.$$

Here the denominator contains at least r prime factors, and since p_r is not among them the quotient cannot be integral. Next let $p_1 = 2$ so that

$$H(n) = \frac{2^{r+1} \cdot p_2 \cdots p_r}{3 \cdot (p_2 + 1) \cdots (p_r + 1)} = \frac{4 \cdot p_2 \cdots p_r}{3 \cdot \dfrac{p_2 + 1}{2} \cdots \dfrac{p_r + 1}{2}},$$

and this expression can only be integral if $p_2 = 3$ so that

$$H(n) = \frac{2 p_3 \cdots p_r}{\dfrac{p_3 + 1}{2} \cdots \dfrac{p_r + 1}{2}}.$$

Since none of the $r - 2$ factors in the denominator are equal to p_r we conclude that each of them must be equal to one of the other prime factors in the numerator. But the assumption

$$\frac{p_3 + 1}{2} = 2$$

or $p_3 = 3$ was already excluded.

A result of some interest is the following:

A perfect number has an integral harmonic mean of divisors.

Since a perfect number is defined by the property that

$$\sigma(n) = 2n$$

it follows that

$$H(n) = \frac{n\nu(n)}{\sigma(n)} = \frac{\nu(n)}{2}.$$

When n is an even perfect number it has the form $n=2^{\alpha}\cdot p$ so that $\nu(n)=(\alpha+1)\cdot 2$ and $H(n)=\alpha+1$ is integral. When we have an odd perfect number it is known that one of the exponents in the prime factor decomposition is odd so that $\nu(n)$ is even also in this case.

By means of the tables of J. W. L. Glaisher giving the values of $\sigma(n)$ and $\nu(n)$ one can determine rather easily the values of n below 10000 for which $H(n)$ is integral. One finds the following table:

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TABLE I

n	$H(n)$	$A(n)$
$6=2\cdot 3$	2	3
$28=2^2\cdot 7$	3	$9\frac{1}{3}$
$140=2^2\cdot 5\cdot 7$	5	28
$270=2\cdot 3^3\cdot 5$	6	45
$496=2^4\cdot 31$	5	$99\frac{1}{5}$
$672=2^5\cdot 3\cdot 7$	8	84
$1638=2\cdot 3^2\cdot 7\cdot 13$	9	182
$2970=2\cdot 3^3\cdot 5\cdot 11$	11	270
$6200=2^3\cdot 5^2\cdot 31$	10	620
$8128=2^6\cdot 127$	7	$1161\frac{1}{7}$
$8190=2\cdot 3^2\cdot 5\cdot 7\cdot 13$	15	546

It would be of interest to make a systematic examination of such numbers up to higher numerical limits. This requires an extension of Glaisher's Number-divisor tables, a project which would be desirable also in connection with other number theoretical investigations.

There are, however, various methods by means of which one can construct new numbers with integral harmonic means from given ones. Let us suppose that n has this property so that

$$n\cdot\nu(n) = a\cdot\sigma(n), \qquad a = H(n).$$

We multiply both sides by some number k relatively prime to n and write $n_1=k\cdot n$. Then one finds

$$H(n_1) = \frac{n_1\cdot\nu(n_1)}{\sigma(n_1)} = \frac{a\cdot k\cdot\nu(k)}{\sigma(k)}.$$

Thus if the right-hand expression is an integer the number n_1 will also have an integral harmonic mean. This may again be examined by means of Glaisher's tables. The values of a in the last column of the table given above lead successively to the following numbers:

TABLE II

n	$H(n)$	n	$H(n)$
$2\cdot 3^2\cdot 7\cdot 13\cdot 17$	17	$2^3\cdot 3^3\cdot 5^2\cdot 31\cdot 53$	53
$2\cdot 3^2\cdot 5\cdot 7\cdot 13\cdot 29$	29	$2^6\cdot 13\cdot 127$	13
$2^3\cdot 3\cdot 5^2\cdot 31$	15	$2^6\cdot 3^2\cdot 13\cdot 127$*	27
$2^3\cdot 3^3\cdot 5^2\cdot 31$	27	$2^6\cdot 3^2\cdot 5\cdot 13\cdot 127$	45
$2^3\cdot 3^3\cdot 5^2\cdot 17\cdot 31$	51	$2^6\cdot 3^2\cdot 13\cdot 17\cdot 127$	51
$2^3\cdot 5^2\cdot 19\cdot 31$	19	$2^6\cdot 3^2\cdot 5\cdot 13\cdot 17\cdot 127$	85
$2^3\cdot 5^2\cdot 7^2\cdot 19\cdot 31$	49	$2^6\cdot 3^2\cdot 5\cdot 13\cdot 29\cdot 127$	87
$2^2\cdot 5^2\cdot 7^2\cdot 13\cdot 19\cdot 31$	91	$2^6\cdot 3^2\cdot 13\cdot 53\cdot 127$	53
$2^3\cdot 3\cdot 5^2\cdot 29\cdot 31$	29	$2^6\cdot 3^2\cdot 5\cdot 13\cdot 89\cdot 127$	89
$2^3\cdot 5^2\cdot 19\cdot 31\cdot 37$	37		

Table I shows that there are eleven numbers below 10,000 with integral harmonic mean for the divisors. Among these are the four perfect numbers below this limit. One verifies simply that an even perfect number cannot have an integral arithmetic mean of the divisors. However, if these are excluded from Table I the remaining numbers have the property that also their arithmetic means of divisors are integral. One might be led to the conjecture that this would be a general property and the computation of Table II was executed with this in mind. However, the number marked by an asterisk proved to be a counter example. It is not perfect and its arithmetic mean of divisors is not integral. A much more interesting conjecture, however, appears from our numerical computations, as an extension of the famous conjecture for perfect numbers, namely, that *a number with integral harmonic mean of divisors must be even.*

POLYGONS WITH TWO EQUIANGULAR POINTS

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1. Introduction. In 1816 A. L. Crelle [1] discovered that in every triangle ABC there are two points G and G' such that $\measuredangle GAB=GBC=GCA=G'BA=G'CB=G'AC=\omega$. After H. Brocard [2] rediscovered this feature in 1875, he and other mathematicians found a great number of related facts [3]. Since then the points G, G' and angle ω have been called Brocard's points and Brocard's angle. Interest in the subject remained at a high level for many years, and in 1886 R. Tucker [4] found that Moebius' harmonic quadrangle [5] has a similar property. Although this type of geometry of the triangle fascinated quite a few, all efforts failed to construct or calculate polygons with more than four sides having two equiangular points [6] as defined above. In 1930 K. Hagge [7] pub-