THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[SEVENTH SERIES.]

OCTOBER 1937.

LII. *The "Converging Factor" in Asymptotic Series and the Calculation of Bessel, Laguerre and other Functions.* By JOHN R. AIREY, *D.Sc., Sc.D.**

IN the construction of tables of functions occurring in physical and mathematical problems, *e.g.*, Bessel functions, the series in ascending powers of the variable, such as

$$J_0(z)=1-\frac{z^2}{2^2}+\frac{z^4}{2^2.4^2}-\frac{z^6}{2^2.4^2.6^2}+\ldots,$$

may be conveniently employed for small values of z. Although convergent for all values of z, it is not suitable when z is large, since the calculations become very difficult and laborious owing to the large number of terms and high powers necessary to be included. In finding the value [1] of e^{-10} to 24 places of decimals the calculation must be carried as far as the term z^{62}, the sum of the positive and negative powers being respectively

11013 . 232 920 103 323 139 721 376 087

and

11013 . 232 874 703 393 377 236 524 556.

* Communicated by the Author.

An earlier example occurs in the calculation of Airy's Integral [2],

$$\int_0^{\infty} \cos\frac{\pi}{2}(\omega^3 - m\omega)\,d\omega$$

for $m=-5{\cdot}6$, the largest value of the argument for which this particular integral was computed. It is remarked that it is impossible to make the calculations for larger values of m, even with ten-figure logarithms, on account of the divergence of the first terms of the series.

$$\int_0^{\infty} \cos\frac{\pi}{2}(\omega^3 - m\omega)\,d\omega$$

$$= C_1\left\{1 - \frac{1}{3}\cdot\frac{m^3}{3\,!} + \frac{4}{3}\cdot\frac{1}{3}\cdot\frac{m^6}{6\,!} - \frac{7}{3}\cdot\frac{4}{3}\cdot\frac{1}{3}\cdot\frac{m^9}{9\,!} + \cdots\right\}$$

$$+ C_2\left\{m - \frac{2}{3}\cdot\frac{m^4}{4\,!} + \frac{5}{3}\cdot\frac{2}{3}\cdot\frac{m^7}{7\,!} - \frac{8}{3}\cdot\frac{5}{3}\cdot\frac{2}{3}\cdot\frac{m^{10}}{10\,!} + \cdots\right\}.$$

For $m=\pm 5{\cdot}6$ the largest term in the series is $169{\cdot}044826$, and it is necessary to proceed as far as the 45th power of m. The result $0{\cdot}000114$ for $m=-5{\cdot}6$ is obtained by combining the sum of the positive terms, $614{\cdot}149962$ with the sum of the negative terms $614{\cdot}149848$. A very striking example of the application of the ascending series for large values of the argument is that given by Glaisher [3] in his paper on "Tables of the numerical values of the sine-integral, cosine-integral and the exponential integral." In calculating these functions to twelve places of decimals when the argument is 20, the first term rejected was the one containing z^{76}, and to show how extremely unmanageable the formulæ had become it is stated that these values required the formation of about twenty-two thousand figures exclusive of verifications. For larger values of the argument it is much more convenient to use the asymptotic series, *e. g.*,

$$Ei(z) = \frac{e^z}{z}\left(1 + \frac{1\,!}{z} + \frac{2\,!}{z^2} + \frac{3\,!}{z^3} + \cdots\right),$$

$$si(z) = -\frac{\cos z}{z} 1 - \frac{2\,!}{z^2} + \frac{4\,!}{z^4} - \frac{6\,!}{z^6} + \cdots\Big)$$

$$-\frac{\sin z}{z}\left(\frac{1\,!}{z} - \frac{3\,!}{z^3} + \frac{5\,!}{z^5} - \frac{7\,!}{z^7} + \cdots\right),$$

and
$$ci(z) = \frac{\sin z}{z}\left(1 - \frac{2\,!}{z^2} + \frac{4\,!}{z^4} - \frac{6\,!}{z^6} + \ldots\right)$$
$$- \frac{\cos z}{z}\left(\frac{1\,!}{z} - \frac{3\,!}{z^3} + \frac{5\,!}{z^5} - \frac{7\,!}{z^7} + \ldots\right).$$

These series, as far as the least term, were used by Glaisher for values of z greater than 17, the results being correct to about seven or eight decimal places. "To remove every shade of doubt that might attach to the use of these divergent series, the functions for $z=20$ were found from both formulæ—ascending and descending powers of the variable—and the agreement was perfect to the eighth place which was as far as the semi-convergent series could give correct results for the value of z."

Much greater accuracy can be obtained from the use of these asymptotic series than is here claimed. By a simple transformation of the divergent part of the asymptotic series, other series are obtained which are easily evaluated even when the argument is complex; and in the case of the Bessel functions, when the order of the function is imaginary or complex.

Although these asymptotic series begin by converging, they eventually become divergent. If the remaining terms after the smallest be omitted the sum of the terms already found will give the value of the function with an error less than the last included term: and the view generally held was that the degree of approximation could not be carried beyond this point. Lord Rayleigh [(4)], referring to the asymptotic expansions of Bessel functions, remarks that "series of this kind are strictly speaking not convergent at all, for when carried sufficiently far the sum of the series may be made to exceed any assignable quantity. But though ultimately divergent, they begin by converging, and when a certain point is reached the terms become very small. It can be proved that if we stop here the sum of the terms already obtained represents the required value of the function subject to an error which, in general, cannot exceed the last included term. Calculations founded on these series are therefore only approximate, and the degree of approximation cannot be carried beyond a certain point. In numerical calculations, therefore, we are to include only the convergent part."

In the case of asymptotic series of the first kind [5], *i.e.*, series in which the signs of the terms are alternately positive and negative, a much closer degree of accuracy than that represented by the least term can be secured by breaking up the divergent part of the asymptotic series into more tractable series, as many as eight, ten or more decimal places can be added to increase the accuracy of the result. To take a simple example, that of the exponential integral $Ei(-10)$, the value of the asymptotic series

$$1-\frac{1!}{x}+\frac{2!}{x^2}-\frac{3!}{x^3}+\frac{4!}{x^4}-\ldots$$

can be found to about four places of decimals when the calculations stop at the least term. However, by expressing the divergent part of the series as the product of a particular term—usually the least term—and a factor, for convenience called the "converging factor," it is possible to reduce the error to a unit in the seventeenth place of decimals and $Ei(-10)$ found to about 21 or 22 places, the accuracy of the result being increased by 13 places. To secure this degree of accuracy from the series in ascending powers it would be necessary to carry the calculations to sixty terms and for $Ei(-20)$ to nearly double that number of terms. If the argument z is complex, $\nu e^{i\theta}$ and β written for $e^{-i\theta}$, the various series derived from the divergent part of the asymptotic series can be readily computed by the method of differences and detached coefficients. For example, the series

$$6\beta^4-50\beta^5+225\beta^6-735\beta^7+\ldots,$$

which occurs in the term $\frac{1}{\nu^3}$ of the "converging factor" of the exponential integral [6],

6	50	225	735	1960	...
6	44	175	510	1225	
6	38	131	335	715	
6	32	93	204	380	
6	26	61	111	176	
6	20	35	50	65	
6	14	15	15	15	
6	8	1	0	0	

The sum of the series is

$$\frac{6\beta^4-8\beta^5+\beta^6}{(1+\beta)^7}:$$

this gives the simple result $-\dfrac{1}{128}$ when the argument is real. $(\theta=0,\ \beta=1)$.

Exponential, Sine and Cosine Integrals.

For negative values of the argument the asymptotic series of the exponential integral is

$$1-\frac{1\,!}{z}+\frac{2\,!}{z^2}-\frac{3}{z^3}+\ldots,$$

the rth term, T_r, being $\dfrac{(r-1)\,!}{z^{r-1}}$. Put $z=\nu e^{i\theta}$, $\nu=r+h$,

r an integer and h a fraction between -1 and $+1$, $\beta=e^{-i\theta}$, then the rth term and all the following terms are equivalent to

$$T_r\left[1-\frac{r}{z}+\frac{r(r+1)}{z^2}-\frac{r(r+1)(r+2)}{z^3}+\ldots\right]$$

$$=T_r\left[1-\frac{(\nu-h)\beta}{\nu}+\frac{(\nu-h)(\nu+1-h)\,.\,\beta^2}{\nu^2}\right.$$

$$\left.-\frac{(\nu-h)(\nu+1-h)(\nu+2-h)\,.\,\beta^3}{\nu^3}+\ldots\right]$$

$$=T_r\left[1-\left(1-\frac{h}{\nu}\right)\beta+\left(1-\frac{h}{\nu}\right)\left(1+\frac{1-h}{\nu}\right)\beta^2\right.$$

$$\left.-\left(1-\frac{h}{\nu}\right)\left(1+\frac{1-h}{\nu}\right)\left(1+\frac{2-h}{\nu}\right)\beta^3+\ldots\right].$$

The part of the series in the bracket independent of ν is, of course,

$$1-\beta+\beta^2-\beta^3+\ldots=\frac{1}{1+\beta}.$$

The other series are easily evaluated, *e. g.*, the series containing $\frac{1}{\nu^2}$ is

$$-2\beta^3+11\beta^4-35\beta^5+85\beta^6-\ldots,$$

the sum of which is

$$\frac{-2\beta^3+\beta^4}{(1+\beta)^5}.$$

The "converging factor," the factor by which T_r must be multiplied to evaluate this term and the following terms, becomes

$$\begin{aligned}
&\frac{1}{1+\beta}+\frac{1}{\nu}\left[\frac{\beta^2}{(1+\beta)^3}+\frac{\beta}{(1+\beta)^2}\cdot h\right]\\
&\qquad+\frac{1}{\nu^2}\left[\frac{-2\beta^3+\beta^4}{(1+\beta)^5}-\frac{\beta^2-2\beta^3}{(1+\beta)^4}h+\frac{\beta^2}{(1+\beta)^3}h^2\right]\\
&\quad+\frac{1}{\nu^3}\left[\frac{6\beta^4-8\beta^5+\beta^6}{(1+\beta)^7}+\frac{2\beta^3-10\beta^4+3\beta^5}{(1+\beta)^6}h\right.\\
&\qquad\left.+\frac{-3\beta^3+3\beta^4}{(1+\beta)^5}h^2+\frac{\beta^3}{(1+\beta)^4}h^3\right]\\
&\quad+\frac{1}{\nu^4}\left[\frac{-24\beta^5+58\beta^6-22\beta^7+\beta^8}{(1+\beta)^9}\right.\\
&\qquad\qquad-\frac{6\beta^4-52\beta^5+43\beta^6-4\beta^7}{(1+\beta)^8}h\\
&\quad\left.+\frac{11\beta^4-28\beta^5-6\beta^6}{(1+\beta)^7}h^2-\frac{6\beta^4-4\beta^5}{(1+\beta)^6}\cdot h^3+\frac{\beta^4}{(1+\beta)^5}h^4\right]. \qquad (1)
\end{aligned}$$

This form of the "converging factor" is not suitable for numerical computation. However, if we write $\theta=2\phi$, $\beta=e^{-i2\phi}$, $\sigma=\sec\phi$, and put $\alpha=e^{-i\phi}$, then $\frac{1}{1+\beta}=\frac{\sigma}{2\alpha}$ and (1) becomes

$$\begin{aligned}
&\frac{\sigma}{2\alpha}+\frac{\sigma^2}{4\nu}\left[\frac{\sigma}{2}\alpha+h\right]+\frac{\sigma^3}{8\nu^2}\left[\frac{\sigma^2}{4}(-2\alpha+\alpha^3)-\frac{\sigma}{2}(1-2\alpha^2)h+\alpha h^2\right]\\
&\quad+\frac{\sigma^4}{16\nu^3}\left[\frac{\sigma^3}{8}(6\alpha-8\alpha^3+\alpha^5)+\frac{\sigma^2}{4}(2-10\alpha^2+3\alpha^4)h\right.\\
&\qquad\qquad\left.+\frac{\sigma}{2}(-3\alpha+3\alpha^3)h^2+\alpha^2h^3\right]
\end{aligned}$$

$$+\frac{\sigma^5}{32\nu^4}\Big[\frac{\sigma^4}{16}(-24\alpha+58\alpha^3-22\alpha^5+\alpha^7)$$

$$+\frac{\sigma^3}{8}(6-52\alpha^2+43\alpha^4-4\alpha^6)h$$

$$+\frac{\sigma^2}{4}(11\alpha-28\alpha^3+6\alpha^5)h^2$$

$$+\frac{\sigma}{2}(6\alpha^2-4\alpha^4)h^3+\alpha^3\,.\,h^4\Big]+\ldots \qquad (2)$$

If ν is an integer and s written for $\sigma/2$ the "converging factor" as far as the term containing $\frac{1}{\nu^{10}}$ becomes

$$s\,.\,\alpha+\frac{1}{\nu}s^3\,.\,\alpha+\frac{1}{\nu^2}[s^5(-2\alpha+\alpha^3]+\frac{1}{\nu^3}[s^7(6\alpha-8\alpha^3+\alpha^5)]$$

$$+\frac{1}{\nu^4}[s^9(-24\alpha+58\alpha^3-22\alpha^5+\alpha^7)]$$

$$+\frac{1}{\nu^5}[s^{11}(120\alpha-444\alpha^3+328\alpha^5-52\alpha^7+\alpha^9)]$$

$$+\frac{1}{\nu^6}[s^{13}(-720\alpha+3708\alpha^3-4400\alpha^5$$
$$+1452\alpha^7-114\alpha^9+\alpha^{11})]$$

$$+\frac{1}{\nu^7}[s^{15}(5040\alpha-33984\alpha^3+58140\alpha^5-32120\alpha^7$$
$$+5610\alpha^9-240\alpha^{11}+\alpha^{13})]$$

$$+\frac{1}{\nu^8}[s^{17}(-40320\alpha+341136\alpha^3-785304\alpha^5+644020\alpha^7$$
$$-195800\alpha^9+19950\alpha^{11}-494\alpha^{13}+\alpha^{15})]$$

$$+\frac{1}{\nu^9}[s^{19}(362880\alpha-3733920\alpha^3+11026296\alpha^5$$
$$-12440064\alpha^7+5765500\alpha^9-1062500\alpha^{11}$$
$$+67260\alpha^{13}-1004\alpha^{15}+\alpha^{17})]$$

$$+\frac{1}{\nu^{10}}[s^{21}(-3628800\alpha+44339040\alpha^3-162186912\alpha^5$$
$$+238904904\alpha^7-155357384\alpha^9+44765000\alpha^{11}$$
$$-5326160\alpha^{13}+218848\alpha^{15}-2026^{17}+\alpha^{19})]+\ldots$$

$$\ldots \qquad 3)$$

The numerical coefficients in the expressions of two consecutive powers of ν are simply related: thus, the coefficients in the term containing $\frac{1}{\nu^6}$ are

$$720=6.120\,;\quad 3708=5.120+7.444\,;\quad 4400=4.444+8.328\,;$$

$$1452=3.328+9.52\,;\quad 114=2.52+10.1.$$

If ν is real and equal to $n+h$, the C.F. is

$$\begin{aligned}&\frac{1}{2}+\frac{1}{4n}\left(\frac{1}{2}+h\right)-\frac{1}{8n^2}\left(\frac{1}{4}+\frac{h}{2}+h^2\right)-\frac{1}{16n^3}\left(\frac{1}{8}+\frac{h}{4}-h^3\right)\\&+\frac{1}{32n^4}\left(\frac{13}{16}+\frac{13h}{16}+\frac{7h^2}{4}+h^3-h^4\right)\\&-\frac{1}{64n^5}\left(\frac{47}{32}+\frac{47h}{16}+\frac{15h^2}{4}+\frac{15h^3}{4}+\frac{5h^4}{2}-h^5\right)\\&-\frac{1}{128n^6}\left(\frac{73}{64}+\frac{73h}{32}+\frac{13h^2}{16}-\frac{5h^3}{2}-5h^4-\frac{9h^5}{2}+h^6\right)+\ldots\end{aligned} \qquad (4)$$

This formula was used in extending the tables of the exponential integral as far as $\nu=20$ by intervals of 0·2, which were needed in constructing a table of the radiation integral [7]

$$\int_x^\infty \frac{dx}{x(e^x-1)}.$$

When ν is real, h zero, $\alpha=1$, $\sigma=1$, the calculations have been carried as far as the term $\frac{1}{\nu^{22}}$. If the "converging factor" is

$$\frac{1}{2}\left[1+\sum_{n=1}\frac{a_n}{(4\nu)^n}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the values of a_n are given in the table (p. 529).

The numbers in the table are derived from the numerical coefficients in (3) in a simple manner. For example,

$$a_6=-73=-720+3708-4400+1452-114+1,$$

and so on.

n.		a_n.				
1	+					1
2	−					1
3	−					1
4	+					13
5	−					47
6	−					73
7	+					2447
8	−					16811
9	−					15551
10	+				17	26511
11	−				189	94849
12	+				109	79677
13	+				29834	09137
14	−			4	84211	03257
15	+			13	50023	66063
16	+			1012	53200	47141
17	−			23203	31477	79359
18	+		1	30595	20092	04319
19	+		58	74028	26601	73759
20	−		1862	05713	25553	80307
21	+		16905	21942	11969	07793
22	+	5	27257	18724	48118	05207

(*a*) $Ei(-5)$. The exponential integral for argument $z=-5$ is given by

$$-\frac{e^{-5}}{5}\left[1-\frac{1}{5}+\frac{2\,!}{5^2}-\frac{3\,!}{5^3}+\frac{4\,!}{5^4}\times\text{C.F.}\right].$$

From (5) the C.F. $=0{\cdot}52372\ 087$: , $\dfrac{4\,!}{5^4}=0{\cdot}0384$,

their product $=0{\cdot}02011\ 08816$,

the bracket $=0{\cdot}85211\ 08816$,

and, finally,

$$Ei(-5)=-\frac{1}{5}\times 0{\cdot}00574\ 14779\ 57:$$

$$=-0{\cdot}00114\ 82955\ 91:$$

the correct value being $-0{\cdot}00114\ 82955\ 913$. There is thus an improvement in the accuracy of the calculation to the extent of seven or eight decimals even for this comparatively small value of the argument.

(*b*) $Ei\,(-10)$. The term, T_r in this case, is

$$-0{\cdot}00036\ 28800$$

and the C.F. is

$$0{\cdot}51218\ 19943\ 760:$$

thus one can add about thirteen places of decimals to the result of the calculation stopping at the least term

$$Ei\,(-10) = -0{\cdot}0^5\ 4156\ 9689\ 2968\ 5324,$$

the value given by Bauschinger [8], computed from the ascending series to twenty places,

$$-0{\cdot}0^5\ 4156\ 9689\ 2968\ 532.$$

(*c*) $Ei\,(-4{\cdot}5i)$.

Here $\quad \theta = \frac{\pi}{2}, \quad \alpha = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{1-i}{\sqrt{2}}.$

$$\alpha^2 = -i, \quad \alpha^3 = \frac{-1-i}{\sqrt{2}}, \text{ etc.} \quad \sigma = \sqrt{2}, \quad h = 0{\cdot}5.$$

The series representing this integral is

$$\frac{\sin\nu + i\cos\nu}{\nu}\left[1 + \frac{i}{\nu} - \frac{2\,!}{\nu^2} - \frac{3\,!\,i}{\nu^3}\,.\times\text{C.F.}\right].$$

Making the above substitutions in (2), it is found that the C.F. is

$$\left(\frac{1}{2} + \frac{1}{2\nu} - \frac{7}{16\nu^2} + \frac{19}{32\nu^3} - \frac{311}{128\nu^4} + \ldots\right)$$

$$+\left(\frac{1}{2} - \frac{1}{4\nu} - \frac{3}{16\nu^2} + \frac{3}{4\nu^3} + \frac{5}{128\nu^4} - \ldots\right)i.$$

Putting $\nu = 4{\cdot}5$, this becomes $0{\cdot}59 : +0{\cdot}44i$, the colon : representing approximately a half-unit in the last place of decimals; the product of this factor and the term $-\frac{3\,!\,i}{\nu^3}$ is equal to $0{\cdot}02897 - 0{\cdot}03917i$; finally, the bracket has the value

$$0{\cdot}93020 : +0{\cdot}18305i,$$

which, multiplied by

$$\frac{\sin\nu + i\cos\nu}{\nu} = \frac{2}{9}(-0{\cdot}977530 - 0{\cdot}210796i)$$

gives

$$Ei(-4{\cdot}5i) = -0{\cdot}19349 : -0{\cdot}08334i.$$

Since

$$Ei(-ix)=ci(x)-i\,si\,(x),$$

this result can be compared with ci (4·5) and si (4·5) from Glaisher's Tables [3].

In the first volume of mathematical tables issued by the Committee of the British Association the cosine integral for argument less than 5 is not given, but the difference between the integral and the natural logarithm of the argument, an arrangement which is both inconvenient and unnecessary.

(*d*) $Ei(-10i)$. The asymptotic expansion is given by

$$\frac{\sin\nu+i\cos\nu}{\nu}\left[1+\frac{i}{\nu}-\frac{2\,!}{\nu^2}-\frac{3\,!\,i}{\nu^3}+\frac{4\,!}{\nu^4}+\frac{5\,!\,i}{\nu^5}-\frac{6\,!}{\nu^6}\right.$$
$$\left.-\frac{7\,!\,i}{\nu^7}+\frac{8\,!}{\nu^8}+\frac{9\,!\,i}{\nu^9}\times\text{C.F.}\right].$$

In this case where ν is an integer, (3) can be used to calculate the C.F. as far as the term containing $\frac{1}{\nu^{10}}$: as in the previous example

$$\theta=\frac{\pi}{2},\quad \phi=\frac{\pi}{4},\quad \alpha=\cos\frac{\pi}{4}-i\sin\frac{\pi}{4};$$

with these substitutions the real and imaginary parts of the C.F. are

Real part :

$$\frac{1}{2}+\frac{1}{2^2\nu}-\frac{3}{2^3\nu^2}+\frac{13}{2^4\nu^3}-\frac{59}{2^5\nu^4}+\frac{185}{2^6\nu^5}+\frac{1309}{2^7\nu^6}-\frac{45387}{2^8\nu^7}$$
$$+\frac{832613}{2^9\nu^8}-\frac{12609823}{2^{10}\nu^9}+\frac{158544573}{2^{11}\nu^{10}}\cdots,$$

Imag. part :

$$\frac{1}{2}-\frac{1}{2^2\nu}+\frac{1}{2^3\nu^2}+\frac{3}{2^4\nu^3}-\frac{55}{2^5\nu^4}+\frac{599}{2^6\nu^5}-\frac{5823}{2^7\nu^6}+\frac{49595}{2^8\nu^7}$$
$$-\frac{266743}{2^9\nu^8}-\frac{2679473}{2^{10}\nu^9}+\frac{141494849}{2^{11}\nu^{10}}\cdots,$$

which give the C.F. to five places of decimals, viz.,

$$0{\cdot}52191+0{\cdot}47633i.$$

The product of this factor and $\frac{9\,!\,i}{\nu^9}$, which is $0{\cdot}0003\ 6288i$, is

$$-0{\cdot}0001\ 7285+0{\cdot}0001\ 8939i,$$

and the complete bracket is

$$0{\cdot}9819\ 1035+0{\cdot}0948\ 8539i$$

$$\sin 10=-0{\cdot}5440\ 2111,\ \cos 10=-0{\cdot}8390\ 7153,$$

and

$$Ei(-10i)=-0{\cdot}04545\ 64329-0{\cdot}08755\ 12675 : i,$$

which may be compared with [9]

$$ci(10)=-0{\cdot}04545\ 64330 \quad \text{and} \quad si(10)=-0{\cdot}08755\ 12674.$$

(*e*) $Ei\left(-5e^{i\frac{\pi}{6}}\right)$. Since

$$Ei(-z)=-\frac{e^{-z}}{z}\left(1-\frac{1}{z}+\frac{2\,!}{z^2}-\frac{3\,!}{z^3}+\frac{4\,!}{z^4}\times\text{C.F.}\right)$$

and

$$z=5\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=\frac{5(\sqrt{3}+i)}{2}, \quad \nu=5, \quad \theta=\frac{\pi}{6},$$

the bracket

$$=\left(1-\frac{\sqrt{3}}{2\nu}+\frac{1}{\nu^2}\right)+i\left(\frac{1}{2\nu}-\frac{\sqrt{3}}{\nu^2}+\frac{6}{\nu^3}\right)$$

$$-\frac{12}{\nu^4}(1+i\sqrt{3})\times\text{C.F.}$$

Now

$$e^{-z}=e^{\nu\cos\theta}\left[\cos(\nu\sin\theta)-i\sin(\nu\sin\theta)\right],$$

$$\log_{10} e^{\nu\cos\theta}=\bar{2}{\cdot}1194497 : , \quad \frac{1}{z}=\frac{1}{5}\cdot\frac{\sqrt{3}-i}{2},$$

$$\cos(\nu\sin\theta)=-0{\cdot}8011436, \quad \sin(\nu\sin\theta)=-0{\cdot}5984721 :$$

and

$$\frac{e^{-z}}{z}=-0{\cdot}00261\ 4867-0{\cdot}00030\ 9975i.$$

The C.F. calculated from (3) as far as the term $\frac{1}{\nu^4}$, by substituting

$$\phi=15^\circ,\ \sigma=\sec 15^\circ,\ \alpha=e^{-i\phi},$$

gives

$$0{\cdot}52505 : +0{\cdot}12671 : i,$$

and the bracket is

$$0{\cdot}8609\ 278+0{\cdot}0588\ 242i,$$

giving the result

$$Ei\,(-5e^{\frac{i\pi}{6}})=0{\cdot}0022\ 3298+0{\cdot}0004\ 2068i,$$

compared with the value from the ascending series

$$0{\cdot}0022\ 3299+0{\cdot}0004\ 2066i.$$

Without the use of the "converging factor" the result is

$$0{\cdot}0022\ 022+0{\cdot}0003\ 816,$$

with errors of three or four units in the fifth place of decimals.

Logarithmic Γ Function.

The series in ascending powers, viz.,

$$\log_e \Gamma(1+z)=\frac{1}{2}\log_e\left(\frac{\pi z}{\sin \pi z}\right)-\frac{1}{2}\log_e\left(\frac{1+z}{1-z}\right)+C_1 z\,.$$
$$-C_3 x^3-C_5 x^5\ldots,$$

where

$$C_1=1-\gamma=0{\cdot}42278\ 43351,$$

$$C_3=\frac{1}{3}\sum_2^\infty n^{-3}=0{\cdot}06735\ 23010:$$

$$C_5=\frac{1}{5}\sum_2^\infty n^{-5}=0{\cdot}00738\ 55510: \quad (10) \text{ and so on,}$$

is suitable for computing this function when $|z| \leqslant 1$. Thus, when $z=i$, the first term is real and the remaining terms imaginary,

$$\log_e \Gamma(1+i)=-0{\cdot}65092\ 31993-0{\cdot}30164\ 03205i.$$

The asymptotic expansion (Stirling's Series), which may be used for larger values of the argument z, is

$$\log \Gamma(z)=(z-\tfrac{1}{2})\log_e z-z+\tfrac{1}{2}\log_e 2\pi$$
$$+\frac{B_1}{2z}-\frac{B_2}{12z^3}+\frac{B_3}{30z^5}-\ldots$$

The term containing the rth Bernoullian number is

$$\frac{B_r}{(2r-1)\,2r\,.\,z^{2r-1}} \text{ and } B_r=\frac{(2r)\,!}{2^{2r-1}\,.\,\pi^{2r}}\sum_{n=1}^{\infty}\frac{1}{n^{2r}}.$$

If $z=\nu e^{i\theta}$, then this term of the Bernoullian series and all the following terms can be represented by

$$\frac{(2r-2)\,!}{2^{2r-1}\pi^{2r}z^{2r-1}}\left[\Sigma\frac{1}{n^{2r}}-\frac{(2r-1)2r}{2^2\pi^2\nu^2}e^{-2i\theta}\,\Sigma\frac{1}{n^{2r+2}}\right.$$
$$\left.+\frac{(2r-1)\,2r\,(2r+1)(2r+2)}{2^4\pi^4\nu^4}e^{-4i\theta}\,\Sigma\frac{1}{n^{2r+4}}-\ldots\right] \quad . \quad . \quad . \quad (1)$$

Put $t=2\pi\nu=(2r-1)-\eta$, where r is an integer fixed by the chosen value of ν and η lies between -1 and $+1$, $\beta=e^{-i\theta}$, and A is the coefficient outside the bracket. Thus (1) becomes

$$A\left[\Sigma\frac{1}{n^{2r}}-\frac{(t+\eta)(t+1+\eta)}{t^2}\beta^2\,\Sigma\frac{1}{n^{2r+2}}\right.$$
$$\left.+\frac{(t+\eta)(t+1+\eta)(t+2+\eta)(t+3+\eta)}{t^4}\beta^4\Sigma\frac{1}{n^{2r+4}}-\ldots\right],$$

and proceeding as in the case of the exponential integral we find the "converging factor" as far as the t^{-2} term.

$$\frac{1}{1+\beta^2}+\left[\frac{-\beta^2+3\beta^4}{(1+\beta^2)^3}+\frac{-2\beta^2}{(1+\beta^2)^2}\eta\right]\frac{1}{t}$$
$$+\left[\frac{11\beta^4-30\beta^6+7\beta^8}{(1+\beta^2)^5}+\frac{-\beta^2+14\beta^4-9\beta^6}{(1+\beta^2)^4}\eta\right.$$
$$\left.+\frac{-\beta^2+3\beta^4}{(1+\beta^2)^3}\eta^2\right]\frac{1}{t^2}\ldots$$

$$+\frac{1}{2^{2r}}\left\{\frac{1}{1+\gamma^2}+\left[\frac{-\gamma^2+3\gamma^4}{(1+\gamma^2)^3}+\frac{-2\gamma^2}{(1+\gamma^2)^2}\eta\right]\frac{1}{t}+\ldots\right\}$$

$$+\frac{1}{3^{2r}}\left\{\frac{1}{1+\delta^2}+\left[\frac{-\delta^2+3\delta^4}{(1+\delta^2)^3}+\frac{-2\delta^2}{(1+\delta^2)^2}\eta\right]\frac{1}{t}+\ldots\right\}+\ldots,$$

where $\gamma=\frac{\beta}{2}$, $\delta=\frac{\beta}{3}$, etc.

If $\nu=3$, $2r$ is approximately equal to 20 and $\frac{1}{2^{2r}}$ is about 10^{-6}, *i. e.*, the term of which the coefficient is $\frac{1}{2^{2r}}$ only affects the converging factor by less than a unit in the sixth place of decimals and may therefore be omitted.

Limiting the calculation to the first term, writing

$$s=\frac{\sec\theta}{2}, \text{ and } \frac{1}{1+\beta^2}=\frac{s}{\beta},$$

the converging factor becomes

$$\frac{s}{\beta}+\left[s^3\left(-\frac{1}{\beta}+3\beta\right)-2s^2\cdot\eta\right]\frac{1}{t}$$

$$+\left[s^5\left(\frac{11}{\beta}-30\beta+7\beta^3\right)+s^4\left(-\frac{1}{\beta^2}+14-9\beta^2\right)\eta+s^3\left(-\frac{1}{\beta}+3\beta\right)\eta^2\right]\frac{1}{t^2}$$

$$+\left[s^7\left(\frac{6}{\beta^3}-\frac{183}{\beta}+511\beta-245\beta^3+15\beta^5\right)+s^6\left(\frac{22}{\beta^2}-208+222\beta^2-28\beta^4\right)\eta+s^5\left(\frac{18}{\beta}-60\beta+18\beta^3\right)\eta^2+s^4(4-4\beta^2)\eta^3\right]\frac{1}{t^3}$$

$$+\left[s^9\left(-\frac{274}{\beta^3}+\frac{4303}{\beta}-12216\beta+8634\beta^3-1422\beta^5+31\beta^7\right)\right.$$

$$+s^8\left(\frac{6}{\beta^4}-\frac{627}{\beta^2}+4568-6314\beta^2+1850\beta^4-75\beta^6\right)\eta$$

$$+s^7\left(\frac{11}{\beta^3}-\frac{433}{\beta}+1491\beta-875\beta^3+70\beta^5\right)\eta^2$$

$$+s^6\left(\frac{6}{\beta^2}-114+170\beta^2-30\beta^4\right)\eta^3$$

$$+s^5\left(\frac{1}{\beta}-10\beta+5\beta^3\right)\eta^4\Big]\frac{1}{t^4}+\ldots\ldots \qquad (2)$$

If the argument is taken along the "semi-imaginary" axis, *i. e.*, $\theta=\frac{\pi}{4}$,

$$s=\frac{1}{\sqrt{2}} \text{ and } \beta=\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}=\frac{1-i}{\sqrt{2}}.$$

To take an extreme case, viz., to find $\log_e\Gamma(1+i)$ from the asymptotic series, $z=1+i$, $\nu=\sqrt{2}$,

$$\log_e z=\frac{1}{2}\log_e 2+i\frac{\pi}{4}, \quad t=2\pi\sqrt{2};$$

whence $r=5$ and $\eta=0{\cdot}114$. Substituting these values in (2), the converging factor is found to be approximately $0{\cdot}52+0{\cdot}50i$, and the product of this and the term $\frac{B_5}{90z^9}$, $[=0{\cdot}0^4263(1-i)]$

is $$0{\cdot}0^427-0{\cdot}0^400 : i.$$

The sum of the series up to the B_4 term is

$$-0{\cdot}650948-0{\cdot}301637 : i.$$

Adding the above product, we get the value of $\log_e\Gamma(1+i)$, about two units in error in the sixth place of decimals, viz.,

$$\log_e\Gamma(1+i)=-0{\cdot}650921-0{\cdot}301638i.$$

Naturally, with this very small value of z, only one or two decimal places can be added to the result.

Gauss [11] calculated $\log_e \Gamma(1+i)$ to seven places of decimals from $\log_e\Gamma(11+i)$, using, apparently, Stirling's Series as far as the term $\frac{B_3}{30z^5}$, B_3 being the least

Bernoullian number. The difference equation gave the value of

$$\log_{10} \Pi(i) = \bar{1}\cdot 7173075 - 17^\circ\ 16'\ 57''\ \cdot 693 i,$$

or

$$\log_e \Gamma(1+i) = -0\cdot 6509235 : -0\cdot 3016399 i.$$

When $z=10+i$ the asymptotic series may be used as far as $\dfrac{B_{32}}{63.64 z^{63}}$, which is approximately equal to $0\cdot 0^{28}4$, *i. e.*, with an accuracy of some 27 decimal places.

Confluent Hypergeometric Function.

This function $M(\alpha . \gamma . x)$ is closely related to the exponential integral, and is defined by the series

$$M(\alpha . \gamma . x) = 1 + \frac{\alpha}{\gamma} x + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \cdot \frac{x^2}{2\,!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)} \cdot \frac{x^3}{3\,!} + \ldots,$$

satisfying the differential equation

$$x . \frac{d^2y}{dx^2} + (\gamma - x) . \frac{dy}{dx} - \alpha . y = 0.$$

Attention has been drawn to the importance of this function (12) in the solution of differential equations of the second order, *e. g.*,

$$\frac{d^2y}{dx^2} + (px+q)\frac{dy}{dx} + (lx^2+mx+n)y = 0,$$

$$\frac{d^2y}{dx^2} + \left(p + \frac{q}{x}\right)\frac{dy}{dx} + \left(l + \frac{m}{x} + \frac{n}{x^2}\right)y = 0,$$

and Laplace's equation

$$(a_2+b_2x)\frac{d^2y}{dx^2} + (a_1+b_1x)\frac{dy}{dx} + (a_0+b_0x)y = 0.$$

The function plays an important rôle in the solution of many physical problems, such diverse problems as the deflexion of electrons, collision of proton and neutron, hyperfine structure, and the positron theory, and it has been stated that for the further development of the quantum theory the main gaps are those which seem likely to be filled by more accurate numerical computations.

The asymptotic expansion of M $(\alpha.\gamma.x)$ is

$$\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}.(-x)^{-\alpha}\left\{1-\frac{\alpha(\alpha-\gamma+1)}{x}+\frac{\alpha(\alpha+1)(\alpha-\gamma+1)(\alpha-\gamma+2)}{2!\,x^2}-\ldots\right\}$$

$$+\frac{\Gamma(\gamma)}{\Gamma(\alpha)}e^x x^{\alpha-\gamma}\left\{1+\frac{(1-\alpha)(\gamma-\alpha)}{x}+\frac{(1-\alpha)(2-\alpha)(\gamma-\alpha)(\gamma-\alpha+1)}{2!\,x^2}+\ldots\right\}.$$

Among the functions which can be expressed in terms of the confluent hypergeometric function there are :

Incomplete gamma function

$$\gamma(n,x)=\int_0^x e^{-t}t^{n-1}dt=\frac{1}{n}.e^{-x}.x^n\mathrm{M}(1,n+1,x).$$

Integrals of the type

$$\int_0^x e^{-a^m x^m}dx=x\mathrm{M}\left(\frac{1}{m},\frac{m+1}{m},-a^m x^m\right).$$

Laguerre function

$$\mathrm{L}_p^q(x)=\frac{\Gamma(q+p+1)}{\Gamma(p+1)\,\Gamma(q+1)}.\mathrm{M}(-p,q+1,x).$$

Bessel functions, imaginary argument

$$\Gamma\left(\frac{\gamma+1}{2}\right).\mathrm{I}_{\frac{\gamma-1}{2}}\left(\frac{x}{2}\right)=2^{1-\gamma}.e^{-\frac{x}{2}}.x^{\frac{\gamma-1}{2}}.\mathrm{M}(\tfrac{1}{2}\gamma,\gamma,x).$$

Bessel functions, real argument

$$\mathrm{J}_\nu(x)+i\mathrm{Y}(x)=\left(\frac{x}{2}\right)^\nu\frac{e^{-ix}}{\Gamma(\nu+1)}.\mathrm{M}(\nu+\tfrac{1}{2},2\nu+1,2ix),$$

and the ω function of Cunningham [13], an important generalization of the Hermite functions. The differential equation of $\omega_{n,m}(\xi)$ is

$$\xi\frac{d^2y}{d\xi^2}+(\xi+1+m)\frac{dy}{d\xi}+\left(n+1+\frac{m}{2}\right)y=0,$$

the solution of which is

$$\mathrm{M}\left(n+1+\frac{m}{2},m+1,-\xi\right)\text{ or }e^{-x}\mathrm{M}\left(\frac{m}{2}-n,m+1,\xi\right).$$

Incomplete Gamma Function.

The asymptotic expansion of this function

$$\int_z^\infty t^{r-1}e^{-t}dt$$

is

$$z^{r-1}e^{-z}\left\{1-\frac{1-r}{z}+\frac{(1-r)(2-r)}{z^2}\ldots\right.$$
$$\left.(-)^{n-1}\frac{(1-r)(2-r)\ldots(n-1-r)}{z^{n-1}}\times\text{C.F.}\right\}.$$

The variable z, r, and n (an integer) being known, if we put $n-r=m$ and $z=m+h$, both m and h can be found; h is a small quantity between -1 and $+1$. Thus if $z=4$, and $r=\frac{3}{4}$, then, if $n=5$, $m=\frac{17}{4}$ and $h=-\frac{1}{4}$. The converging factor in this particular case is found by the foregoing method to be of the same form as that of the exponential integral for real values of the variable, viz.,

$$\frac{1}{2}+\frac{1}{4m}\left(\frac{1}{2}+h\right)-\frac{1}{8m^2}\left(\frac{1}{4}+\frac{h}{2}+h^2\right)-\frac{1}{16m^3}\left(\frac{1}{8}+\frac{h}{4}-h^3\right)$$
$$+\frac{1}{32m^4}\left(\frac{13}{16}+\frac{13h}{8}+\frac{7h^2}{4}+h^3-h^4\right)-\ldots$$

The converging factor in the three cases where $h=-\frac{1}{4},-\frac{1}{2},-\frac{3}{4}$ are

$$h=-\frac{1}{4};\ \frac{1}{2}+\frac{1}{4.4m}-\frac{3}{16.8m^2}-\frac{5}{64.16m^3}+\frac{127}{256.32m^4}$$
$$-\frac{943}{1024.64m^5}-\frac{2643}{4096.128m^6}+\frac{190907}{16384.256m^7}\ldots,$$

$$h=-\frac{1}{2};\ \frac{1}{2}-\frac{1}{4.8m^2}-\frac{1}{8.16m^3}+\frac{1}{4.32m^4}-\frac{21}{32.64m^5}$$
$$-\frac{23}{64.128m^6}+\frac{229}{32.256m^7}\ldots;$$

$$h=-\frac{3}{4};\ \frac{1}{2}-\frac{1}{4.4m}-\frac{7}{16.8m^2}-\frac{23}{64.16m^3}-\frac{41}{256.32m^4}$$
$$-\frac{841}{1024.64m^5}-\frac{2479}{4096.128m^6}+\frac{64225}{16384.256m^7}\ldots$$

Use was made of these formulæ in computing the integrals

$$\int_x^\infty x^{-\alpha}e^{-x}\,dx, \quad \alpha=\frac{1}{4},\ \frac{1}{2},\ \frac{3}{4}$$

required in the construction of tables of the radiation integrals $\int_x^\infty \frac{dx}{x^\alpha(e^x-1)}$.

For $\int_4^\infty x^{-\frac{1}{4}}e^{-x}\,dx$ the converging factor is 0·51338, thus increasing the accuracy of the result by five places of decimals; the value of the integral so found is 0·0123 1169, with a possible error of a unit in the eighth place of decimals. From Pearson's table [14] of the incomplete gamma function this value is approximately 0·0123 1167 :

In the example just given ν is real, but the converging factor can also be applied in the case where ν is imaginary, *e. g.*, in calculating $M(i,\ 1+i,\ -x)$, of which the expansion is

$$\frac{\Gamma(1+i)}{x^i}-\frac{i}{xe^x}\left[1-\frac{1-i}{x}+\frac{(1-i)(2-i)}{x^2}\right.$$
$$\left.-\frac{(1-i)(2-i)(3-1)}{x^3}+\ldots\right].$$

If $x=4$, $r=i$, and $n=4$, $m=4-i$, $h=i$, and $\frac{1}{m}=\frac{4+i}{17}$.

Making these substitutions, the converging factor is

$$\frac{1}{2}+\frac{2+9i}{136}+\frac{61-6i}{9248}+\frac{3454+2555i}{314432}-\frac{7393+5830i}{42672752}\ldots$$
$$=0{\cdot}5321+0{\cdot}0735i.$$

Therefore

$$1-\frac{1-i}{4}+\frac{(1-i)(2-i)}{16}-\frac{(1-i)(2-i)(3-i)}{64}\times\text{C.F.}$$
$$=(0{\cdot}8125-0{\cdot}0115)+(0{\cdot}0625+0{\cdot}0831\ :)\,i$$
$$=0{\cdot}8010+0{\cdot}1456\ :\ i.$$

Also $\Gamma(1+i)=0{\cdot}49801\ :\ -0{\cdot}15495i,$

and $4^{-i}=0{\cdot}183457-0{\cdot}983028i$ [15].

Whence

$$M(i,\ 1+i,\ -4)=-0{\cdot}06029-0{\cdot}52166i.$$

From the series in ascending powers of x,

$$M(i,\ 1+i,\ -4) = -0{\cdot}060296 - 0{\cdot}521665i.$$

Integrals $\int_0^x e^{-a^m x^m}\,dx$.

Integrals of the above type include the Error Function, which has been extensively tabulated [16], and the integral $\int_0^x e^{-x^4}\,dx$, which appears in electrical conductivity problems in gases [17]. In connexion with this integral the asymptotic series is given in the form $(\lambda = x^4)$,

$$\frac{\Gamma\left(\frac{5}{4}\right)}{x} - \frac{1}{4\lambda}\,.\,e^{-\lambda}\left[1 - \frac{3}{(2x^2)^2} + \frac{3{\cdot}7}{(2x^2)^4} - \ldots\ldots \pm \frac{3\,.\,7\,.\,11\,\ldots\,(4n-1)}{(2x^2)^{2n}}(1-\theta)\right],$$

with the remark that the ratio of two consecutive terms is $\frac{4n-1}{2x^2}$, which for sufficiently large values of n increases without limit. When x is large $(n < x^4)$ the resulting error is less than the last calculated term. Since the general integral is equal to

$$xM\left(\frac{1}{m},\ \frac{m+1}{m},\ -a^m x^m\right),$$

$$\gamma - \alpha = 1 \text{ and } \alpha - \gamma + 1 = 0 \text{ in } M(\alpha\,.\,\gamma\,.\,x),$$

with the result that only the first term, unity, remains in the first asymptotic series. When

$$x = 1{\cdot}5,\quad x^4 = \rho + h = 5{\cdot}0625,\quad r = \tfrac{1}{4},$$

the integral may be computed with an accuracy of 8 or 9 places of decimals. If

$$n = 5,\quad \rho = 4{\cdot}75 \text{ and } h = 0{\cdot}3125,$$

or

$$n = 6,\quad \rho = 5{\cdot}75 \text{ and } h = -0{\cdot}6875,$$

and the converging factor can be computed from the series in the preceding section

$$\frac{1}{2} + \frac{1}{4\rho}\left(\frac{1}{2} + h\right) - \frac{1}{8\rho^2}\left(\frac{1}{4} + \frac{h}{2} + h^2\right) - \ldots$$

to five or six decimals, giving for the value of the integral

$$0{\cdot}9995\ 4149 :$$

Laguerre Function $L_p^q(z)$.

Among the numerous physical problems, into the solution of which the Laguerre function enters, may be mentioned the theory of scattering of protons by protons, photoelectric absorption for X-rays, cosmic ray absorption, vibrational isotope effect, and Coulomb wave functions in repulsive fields [18].

In the last case the Laguerre function expressed in terms of the confluent hypergeometric function is proportional to

$$M(L+1-i\eta,\ 2L+2,\ 2i\rho)\ ;$$

the regular solution of the differential equation

$$\frac{d^2M}{d\rho^2}+\left\{1-\frac{2\eta}{\rho}-\frac{L(L+1)}{\rho^2}\right\}M=0$$

is $\qquad e^{-ix}.x^{L+1}.M(L+1-i\eta,\ 2L+2,\ 2i\rho).$

For the regions of small energy, η large, by change of variables $M=\zeta f$ and $\zeta=i(8\rho\eta)^{\frac{1}{2}}$, the differential equation becomes

$$\left[\frac{d^2}{d\zeta^2}+\frac{1}{\zeta}\frac{d}{d\zeta}+1-\frac{(2L+1)^2}{\zeta^2}+\frac{\zeta^2}{16\eta^2}\right]f=0.$$

If the term $\dfrac{\zeta^2}{16\eta^2}$ is neglected the differential equation is that of the ordinary Bessel function.

For the Laguerre function $L_n(u)$ Sommerfeld [19] has given the following expression :

$$L_n(u)=e^{i\pi n}\left[\frac{u^n}{\Gamma(1+n)}-\frac{n}{u}\cdot\frac{e^u(-u)^{-n}}{\Gamma(1-n)}\right],$$

which is equivalent to taking only the first terms, unity, in the two asymptotic series of $M(-n,\ 1,\ x)$. The numerical results obtained from this formula are considerably in error and of little practical value. Sexl [20] had previously included three or four terms in the asymptotic series, viz.:

$$\mathrm{L}_n(u) = \frac{e^{i\pi n}u^n}{\Gamma(n+1)}\left[1 - \frac{n^2}{u} + \frac{n^2(n-1)^2}{2!\,u^2} - \ldots\right]$$

$$-\frac{n}{u}e^{i\pi n}\frac{e^u(-u)^{-n}}{\Gamma(1-n)}\left[1 + \frac{(n+1)^2}{u} + \frac{(n+1)^2(n+2)^2}{u^2} + \ldots\right].$$

This function may also be expressed in terms of an integral involving the Bessel function [21], $\mathrm{I}_0(x\sqrt{u})$, and also in terms of the Hankel Cylinder Functions [22] of order $\frac{1}{3}$ and $\frac{2}{3}$.

As an example of the application of the converging factor method calculate the value of $\mathrm{M}(1-i,\ 2,\ 3i)$.

Now

$$\mathrm{M}(1-i,\ 2,\ i\rho)$$

$$= \frac{\Gamma(2)}{\Gamma(1+i)}(-i\rho)^{-1+i}\left\{1 - \frac{(1-i)i}{i\rho} + \frac{(1-i)(2-i)(-i)(1-i)}{2!\,(i\rho)^2} - \ldots\right\}$$

$$+ \frac{\Gamma(2)}{\Gamma(1-i)}e^{i\rho}(i\rho)^{-1-i}\left\{1 + \frac{i(1+i)}{i\rho} + \frac{i(1+i)(1+i)(2+i)}{2!\,(i\rho)^2} + \ldots\right\},$$

and $$(-i\rho)^{-1+i} = \frac{i}{\rho}e^{\frac{\pi}{2}}(\cos\log_e\rho + i\sin\log_e\rho)$$

$$e^{\frac{\pi}{2}} = 4{\cdot}8104\ 7738$$

$$\frac{1}{\Gamma(1+i)} = 1{\cdot}8307444 + 0{\cdot}5696\ 076 : i,$$

and $$3^i = \cos\log_e 3 + i\sin\log_e 3$$

$$= 0{\cdot}4548\ 3242 + 0{\cdot}8905\ 7704i.$$

For the first asymptotic series of $\mathrm{M}(1-i,\ 2,\ ir)$, if T_r is the term

$$\frac{(1-i)(2-i)\ldots(r-i)(-i)(1-i)\ldots(r-1-i)}{r!\,(ir)^r},$$

the C.F. in this case is, as far as the $\frac{1}{r^3}$ term,

$$\left(\frac{1}{2}+\frac{1}{4r}+\frac{1}{8r^2}+\frac{1}{16r^3}-\ldots\right)+\left(\frac{1}{2}+\frac{3}{4r}+\frac{1}{8r^2}-\frac{5}{16r^3}+\ldots\right)i,$$

and for the second asymptotic series the conjugate of this expression.

When $\rho=r=3$, $T_r=0{\cdot}18518:+0{\cdot}06173i$,

and the C.F. $=0{\cdot}592+0{\cdot}758i$ approximately.

Their product is $0{\cdot}1769+0{\cdot}0628i$, and the two asymptotic series are, respectively, $1{\cdot}618:-0{\cdot}267:i$, $1{\cdot}618:+0{\cdot}267:i$. Finally, $M(1-i,\ 2,\ 3i)=0{\cdot}186+2{\cdot}621i$, compared with the value from the ascending series $0{\cdot}1859+2{\cdot}6217i$.

The value of the function calculated from the asymptotic series as far as the least term is $0{\cdot}260+3{\cdot}662:i$, and using the first term only of the two series, $0{\cdot}043+0{\cdot}611i$, which bears no relation to the true value of the function.

It may be of interest to set down the values for $L=0$ as in the above example, $L=1$, $L=2$.

$$M(1-i,\ 2,\ 3i)=0{\cdot}1859\ 1843:+2{\cdot}6217\ 1393i,$$

$$M(2-i,\ 4,\ 3i)=0{\cdot}1168\ 0548:+1{\cdot}6471\ 2320i,$$

$$M(3-i,\ 6,\ 3i)=0{\cdot}0986\ 1954+1{\cdot}3906\ 7558i$$ [23].

Bessel Functions, $K_\nu(z)$, $J_\nu(z)$.

(*a*) The asymptotic expansion of $K_\nu(z)$ may be written in the form

$$\sqrt{\frac{\pi}{2z}}\cdot e^{-z}\left\{1+\frac{4\nu^2-1}{8z}+\ldots\right.$$
$$\left.\frac{(4\nu^2-1)(4\nu^2-9)\ldots[4\nu^2-(4n-1)^2]}{(2n)!\,(8z)^{2n}}\times\text{C.F.}\right\},$$

and the converging factor then becomes

$$1+\frac{[4\nu^2-(4n+1)^2]}{(2n+1)8z}+\frac{[4\nu^2-(4n+1)^2][4\nu^2-(4n+3)^2]}{(2n+1)(2n+2)(8z)^2}+\ldots$$

If z is real and equal to $n+h$, writing $\gamma=\dfrac{1-4\nu^2}{4}$, and carrying out the summations as in the previous examples, the factor simplifies to

$$\frac{1}{2}+\frac{1}{4n}\left(\frac{1}{4}+h\right)-\frac{1}{16n^2}\left(\frac{1}{8}+\frac{h}{2}+2h^2+\gamma\right)$$
$$-\frac{1}{64n^3}\left(\frac{1}{16}+\frac{h}{4}-4h^3-\frac{\gamma}{2}\right)\ldots$$

Similar but more complicated factors occur when z is complex. If $z=(n+h)e^{2i\theta}$, the first three terms of the real and imaginary parts of the converging factor, if $\sigma=\sec\theta$, $\tau=\tan\theta$, and γ as before is $\frac{1-4\nu^2}{4}$,

$$\frac{1}{2}+\frac{\sigma^2}{16n}(1+4h)-\frac{\sigma^2}{128n^2}\{2\sigma^2(1+4h)$$
$$+(8\gamma-1+3\tau^2-4h+4h\tau^2+16h^2)\}\ldots$$
$$+i\tau\left[\frac{1}{2}-\frac{\sigma^2}{16n}+\frac{\sigma^2}{128n^2}\{2\sigma^2-(3-\tau+8h+16h^2)\}\ldots\right].$$

For $z=5e^{i\frac{\pi}{4}}$, the argument taken along the semi-imaginary axis, $\theta=\frac{\pi}{8}$ and

$$\mathrm{Ker}\,(5)+i\,\mathrm{Kei}\,(5)=-0{\cdot}0115117+0{\cdot}0111876i.$$

(*b*) The two Bessel functions of fractional order $J_{\pm\frac{1}{4}}(x)$ are given by

$$J_{\frac{1}{4}}(x)=\sqrt{\frac{2}{\pi x}}\left[P_{\frac{1}{4}}(x).\sin\left(x+\frac{\pi}{8}\right)+Q_{\frac{1}{4}}(x).\cos\left(x+\frac{\pi}{8}\right)\right],$$

$$J_{-\frac{1}{4}}(x)=\sqrt{\frac{2}{\pi x}}\left[P_{-\frac{1}{4}}(x).\cos\left(x-\frac{\pi}{8}\right)-Q_{-\frac{1}{4}}(x)\sin\left(x-\frac{\pi}{8}\right)\right]$$

When x is an integer, n and $\gamma=\frac{1-4\nu^2}{4}$,

$$\mathrm{CF}_P=\frac{1}{2}+\frac{1}{8n}-\frac{1}{4n^2}\left(\frac{3}{8}+\frac{\gamma}{2}\right)+\frac{1}{8n^3}\left(\frac{13}{16}+\frac{\gamma}{2}\right)$$
$$-\frac{1}{16n^4}\left(\frac{59}{32}-\frac{7\gamma}{8}-\frac{\gamma^2}{4}\right)+\frac{1}{32n^5}\left(\frac{185}{64}-\frac{57\gamma}{8}-\frac{3\gamma^2}{2}\right)\ldots,$$

and

$$\mathrm{CF_Q}=\frac{1}{2}+\frac{3}{8n}-\frac{1}{4n^2}\left(\frac{3}{8}+\frac{\gamma}{2}\right)-\frac{1}{8n^3}\left(\frac{1}{16}+\frac{\gamma}{2}\right)$$

$$+\frac{1}{16n^4}\left(\frac{61}{32}+\frac{23\gamma}{8}+\frac{\gamma^2}{4}\right)-\frac{1}{32n^5}\left(\frac{709}{64}+\frac{41\gamma}{8}+\gamma^2\right)\ldots,$$

when

$$n=5,\quad \mathrm{CF_P}=0\cdot52107,\quad \mathrm{CF}\ =0\cdot57028,$$

$$\mathrm{P}_{\pm\frac{1}{4}}(5)=0\cdot99805\ 88187:,\qquad \mathrm{Q}_{\pm\frac{1}{4}}(5)=-0\cdot01836\ 83506:$$

$$\text{Bracket }(\mathrm{J}_{\frac{1}{4}})=-0\cdot78742\ 29801.$$

$$\text{Bracket }(\mathrm{J}_{-\frac{1}{4}})=-0\cdot12295\ 81443,$$

and

$$\mathrm{J}_{\frac{1}{4}}(5)=-0\cdot28097\ 20667,$$

$$\mathrm{J}_{-\frac{1}{4}}(5)=-0\cdot04387\ 45181.$$

(*c*) The calculation of the ber, bei functions proceeds on similar lines.

$$\mathrm{I}_0(x\sqrt{i})=\mathrm{ber}\,x+i\,\mathrm{bei}\,x$$

$$=\frac{e^{x/\sqrt{2}}}{\sqrt{2\pi x}}\left\{\cos\left(\frac{x}{\sqrt{2}}-\frac{\pi}{8}\right)+i\sin\left(\frac{x}{\sqrt{2}}-\frac{\pi}{8}\right)\right\}$$

$$\times\left\{\mathrm{P}_0\left(xe^{-\frac{i\pi}{4}}\right)+i\mathrm{Q}_0\left(xe^{-\frac{i\pi}{4}}\right)\right\}$$

$$+\frac{ie^{-x/\sqrt{2}}}{\sqrt{2\pi x}}\left\{\cos\left(\frac{x}{\sqrt{2}}+\frac{\pi}{8}\right)-i\sin\left(\frac{x}{\sqrt{2}}+\frac{\pi}{8}\right)\right\}$$

$$\times\left\{\mathrm{P}_0\left(xe^{-\frac{i\pi}{4}}\right)-i\mathrm{Q}_0\left(xe^{-\frac{i\pi}{4}}\right)\right\}.$$

The second term is no other than $\frac{i}{\pi}$ (Ker $x+i$ Kei x), which was, by most mathematicians, including Kummer and Kirchhoff, regarded as a negligible series. This term must be retained, not on account of any question of symmetry, but because it forms an essential part of the function. No amount of manipulation of the remainder of the first series will compensate for the neglect of the second series. Through the omission of the so-called negligible series the first zero of ber x is erroneously given as 2·835, instead of the value, correct to three places, 2·849. Tables of sin, cos, sinh and cosh $\frac{x}{\sqrt{2}}$ to twelve places have been constructed over the range 0·0 to 20·0 by 0·1 intervals (24).

The series $P_\nu(z)$ may be written as

$$1-\frac{(4\nu^2-1)(4\nu^2-9)}{2!\,(8z)^2}+\frac{(4\nu^2-1)(4\nu^2-9)(4\nu^2-25)(4\nu^2-49)}{4!\,(8z)^4}$$

$$+(-)^n\frac{(4\nu^2-1)(4\nu^2-9)\ldots[4\nu^2-(4n-3)^2][4\nu^2-(4n-1)^2]}{(2n)!\,(8z)^{2n}}$$

$$\times \text{C.F.}_{\text{P}}.$$

Put $z=(n+h)e^{i\theta}$, $\frac{1-4\nu^2}{4}=\gamma$, $\sigma=\sec\theta$, and $\alpha=e^{i\theta}$, the converging factor C.F._{P} becomes

$$\frac{\sigma}{2}\alpha-\frac{\sigma}{2n}\left[\frac{\sigma^2}{8}(\alpha-3\alpha^{-1})-h\sigma\right]$$

$$-\frac{\sigma^2}{4n^2}\left[\frac{\gamma}{2}-\frac{\sigma^3}{32}(11\alpha-30\alpha^{-1}+7\alpha^{-3})\right.$$

$$\left.-\frac{h\sigma^2}{4}(\alpha^2-8+3\alpha^{-2})+\frac{h^2\sigma}{2}(3\alpha-\alpha^{-1})\right]$$

$$+\frac{\sigma^3}{8n^3}\left[\frac{\sigma\gamma}{8}(\alpha^2+8-5\alpha^{-2})+h\gamma(\alpha-\alpha^{-1})\right.$$

$$+\frac{\sigma^4}{128}(6\alpha^3-183\alpha+511\alpha^{-1}-245\alpha^{-3}+15\alpha^{-5})$$

$$-\frac{h\sigma^3}{16}(22\alpha^2-123+88\alpha^{-2}-7\alpha^{-4})$$

$$\left.-\frac{3h^2\sigma^3}{8}(\alpha^3-15\alpha+15\alpha^{-1}-\alpha^{-3})+2h^2\sigma(\alpha^2-1)\right]+\ldots.$$

Similarly, if $Q_\nu(z)$ is written in the form

$$\frac{4\nu^2-1}{8z}-\frac{(4\nu^2-1)(4\nu^2-9)(4\nu^2-25)}{3!\,(8z)^3}+\ldots$$

$$+(-)^{n-1}\frac{(\nu^2-1)(4\nu^2-9)\ldots[4\nu^2-(4n-5)^2][4\nu^2-(4n-3)^2]}{(2n-1)!\,(8z)^{2n-1}}$$

$$\times \text{C.F.}_{\text{Q}},$$

with the same notation, the converging factor $C.F._{Q}$ is

$$\frac{\sigma}{2}\alpha+\frac{\sigma}{2n}\left[\frac{\sigma^2}{8}(\alpha+5\alpha^{-1})+h\sigma\right]$$

$$-\frac{\sigma^2}{4n^2}\left[\frac{\gamma}{2}+\frac{\sigma^3}{32}(\alpha+30\alpha^{-1}-19\alpha^{-3})+\frac{h\sigma^2}{4}(\alpha^2+8-5\alpha^{-2})+\frac{h^2\sigma}{2}(3\alpha-\alpha^{-1})\right]$$

$$+\frac{\sigma^3}{8n^3}\left[\frac{\sigma\gamma}{8}(\alpha^2+4-9\alpha^{-2})+h\gamma(\alpha-\alpha^{-1})\right.$$

$$-\frac{\sigma^4}{128}(2\alpha^3+29\alpha-413\alpha^{-1}+455\alpha^{-3}-65\alpha^{-5})$$

$$-\frac{h\sigma^3}{16}(2\alpha^4+87\alpha^2-136+19\alpha^{-2})$$

$$\left.+\frac{h^2\sigma^2}{8}(3\alpha^3+35\alpha-59\alpha^{-1}+5\alpha^{-3})+2h^3\sigma(\alpha^2-1)\right]+\ldots$$

For the ber, bei functions,

$$\theta=-\frac{\pi}{4},\quad \sigma=\sqrt{2},\quad \alpha=e^{i\theta},\quad \gamma=\frac{1}{4},$$

the converging factors become

$$C.F._{P}=\left(\frac{1}{2}+\frac{1}{4n}-\frac{7}{8n^2}+\frac{71}{16n^3}-\ldots\right)$$

$$+\left(-\frac{1}{2}+\frac{1}{2n}-\frac{17}{16n^2}+\frac{13}{4n^3}-\ldots\right)i,$$

$$C.F._{Q}=\left(\frac{1}{2}+\frac{3}{4n}-\frac{13}{8n^2}+\frac{49}{8n^3}-\ldots\right)$$

$$+\left(-\frac{1}{2}+\frac{1}{2n}-\frac{5}{16n^2}-\frac{3}{4n^3}+\ldots\right)i.$$

W th the help of these converging factors the first term of $I_0(5\sqrt{i})$ can be calculated to about six places of decimals, viz.,

$$-6{\cdot}226519+0{\cdot}119697i,$$

to which must be added the product of $\frac{i}{\pi}$ and the "negligible" series,

$$-0{\cdot}0035611-0{\cdot}0036643i,$$

giving the result

$$I_0(5\sqrt{i})=\text{ber }5+i\text{ bei }5=-6{\cdot}230080+0{\cdot}116033i,$$

the error being one or two units in the last place.

(*d*) Bessel functions, imaginary order, occur, of course, much less frequently in physical problems than those of real order. Böcher [25] found the potential within a solid bounded by two coaxial cylinders and four planes, two through the axis of the cylinders and two perpendicular to this axis, involving, under special conditions, Bessel functions whose order or index is imaginary, whilst functions whose order is complex and argument real are required in the investigation of the De Sitter universe.

The Bessel function of imaginary order $J_i(x)$ for $x=4$ has been computed from the series in ascending powers of x and from the asymptotic series.

$$J_i(x)=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left[P_i(x)\cos\left(x-\frac{2i+1}{4}\pi\right)\right.$$
$$\left.-Q_i(x)\sin\left(x-\frac{2i+1}{4}\right)\pi\right],$$

where

$$P_i(x)=1-\frac{5.13}{2!\,(8x)^2}+\frac{5.13.29.53}{4!\,(8x)^4}-$$

and

$$Q_i(x)=-\frac{5}{8x}+\frac{5.13.29}{3!\,(8x)^3}-\frac{5.13.29.53.85}{5!\,(8x)^5}+\ldots.$$

When x is an integer, as in this example, the converging factor for $Q_i(n)$ is

$$\frac{1}{2}+\frac{3}{8n}-\frac{1}{4n^2}\left(\frac{3}{8}+\frac{\gamma}{2}\right)-\frac{1}{8n^3}\left(\frac{1}{16}+\frac{\gamma}{2}\right)$$
$$+\frac{1}{16n^4}\left(\frac{61}{32}+\frac{23\gamma}{8}+\frac{\gamma^2}{4}\right)-\frac{1}{32n^5}\left(\frac{709}{64}+\frac{41\gamma}{8}+\gamma^2\right)\ldots,$$

and for $P_i(n)$

$$\frac{1}{2}+\frac{1}{8n}-\frac{1}{4n^2}\left(\frac{3}{8}+\frac{\gamma}{2}\right)+\frac{1}{8n^3}\left(\frac{13}{16}+\frac{\gamma}{2}\right)$$

$$-\frac{1}{16n^4}\left(\frac{59}{32}-\frac{7\gamma}{8}-\frac{\gamma^2}{4}\right)+\frac{1}{32n^5}\left(\frac{185}{64}-\frac{57\gamma}{8}-\frac{3\gamma^2}{2}\right)\ldots,$$

and $\gamma=\dfrac{1-4\nu^2}{4}=\dfrac{5}{4}$, when $\nu=i$ and for $n=4$,

$$\text{C.F}_Q=0\cdot5776 \quad\text{and}\quad \text{C.F}_P=0\cdot5182.$$

The last term of the Q series is

$$+0\cdot0010\ 6043,$$

and of the P series

$$+0\cdot0009\ 4859:$$

$$P_i(4)=+0\cdot9713501 \text{ and } Q_i(4)=-0\cdot1481589.$$

Finally

$$J_i(4)=-0\cdot9805664+0\cdot070697i,$$

the value $-0\cdot9805664+0\cdot070695i$ being obtained from the ascending series. If the calculation is carried as far as the least term the result is

$$J_i(4)=-0\cdot9801+0\cdot0712i.$$

The Bessel function of imaginary order and complex argument $J_i\left(4e^{\frac{i\pi}{4}}\right)$ has also been computed by both methods.

From the ascending series

$$J_i\left(4e^{\frac{i\pi}{4}}\right)=-0\cdot51967-0\cdot52763i.$$

For the asymptotic series the converging factors are

$$\text{P series}: \left(\frac{1}{2}+\frac{1}{4n}-\frac{9}{8n^2}+\frac{79}{16n^3}\ldots\right)$$

$$+\left(\frac{1}{2}-\frac{1}{2n}+\frac{17}{16n^2}-\frac{23}{8n^3}\ldots\right)i,$$

$$\text{Q series}: \left(\frac{1}{2}+\frac{3}{4n}-\frac{15}{8n^2}+\frac{107}{32n^3}\ldots\right)$$

$$+\left(\frac{1}{2}-\frac{1}{2n}+\frac{5}{16n^2}+\frac{11}{8n^3}\ldots\right)i.$$

The last term of the P series is real and equal to

$$0{\cdot}0009486 \text{ with converging factor } 0{\cdot}57+0{\cdot}40i,$$

the last term of the Q series is complex and equal to $0{\cdot}0007498\,(1+i)$ with converging factor $0{\cdot}62+0{\cdot}41i$.

The final result gives

$$J_i\left(4e^{\frac{i\pi}{4}}\right)=-0{\cdot}51970-0{\cdot}52764 : i,$$

and without the converging factor

$$-0{\cdot}51909-0{\cdot}52690i.$$

The " converging factor " method has been extensively employed in the construction of tables of Bessel, Neumann and other functions for the Mathematical Tables Committee of the British Association.

Neumann functions [26] or Bessel functions of the second kind, of zero and unit orders: $G_0(x)$ and $G_1(x)$.

Bessel functions, $Y_0(x)$ and $Y_1(x)$ according to Neumann's definition [27].

Lommel-Weber functions [28] $\Omega_0(x)$, $\Omega_1(x)$, $\Omega_{\frac{1}{2}}(x)$, and $\Omega_{-\frac{1}{2}}(x)$ where asymptotic series such as

$$B_0(x)=\frac{1}{x}-\frac{1}{x^3}+\frac{3^2}{x^5}-\frac{3^25^2}{x^7}+\ldots,$$

$$B_{\frac{1}{2}}(x)=\frac{1}{x}\left\{1-\frac{1\cdot3}{(2x)^2}+\frac{1.3.5.7}{(2x)^4}-\frac{1.3.5.7.9.11}{(2x)^6}+\ldots\right\},$$

have to be calculated.

Confluent hypergeometric function [29], $M(\alpha.\gamma.x)$ for various values of the parameters, α and γ.

Exponential, sine and cosine integrals [30].

Probability integral and its integrals [31].

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