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A METHOD FOR THE DETERMINATION OF CONVERGING FACTORS, APPLIED TO THE ASYMPTOTIC EXPANSIONS FOR THE PARABOLIC CYLINDER FUNCTIONS

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1. The method of converging factors, for hastening the convergence of slowly convergent series and improving the accuracy of asymptotic expansions, was introduced by J. R. Airey and is well known to computers (see Airey (1) and Rosser (2)). The principle is as follows. It is required to compute a quantity which is expressed as an infinite series

$$S = u_0 + u_1 + u_2 + \dots.$$

The series may be either convergent or asymptotic and divergent. If it is written as a finite sum

$$S = u_0 + u_1 + \dots + u_{n-1} + R_n,$$

this sum may be computed either if the remainder term R_n is known to be negligible or if R_n can be estimated sufficiently accurately. To establish that R_n is negligible it is usual to determine an upper bound to $|R_n|$; often, however, this bound turns out not to be negligible. Airey therefore directed his attention to the estimation of R_n. He first chose u_n to be one of the smaller terms of the series, and then, writing

$$R_n = \pm C_n u_n,$$

he concentrated on evaluating C_n, which he called a *converging factor*. In practice the range of variation of C_n is much less than that of R_n or of u_n; this usually simplifies the problem of obtaining a useful expansion for C_n.

By somewhat empirical methods Airey obtained converging factors for several asymptotic expansions of terms with alternating signs, testing his results numerically by comparing them with those obtained from other expansions. The method was highly successful, and results were obtained to many more decimal places than are usually considered possible when, for example, an asymptotic expansion is available with an error bound of the form $|R_n| < |u_n|$.

When the asymptotic series has terms which are not strictly alternating in sign, at least for $n > n_0$, Airey's methods still work, with more or less effect, so long as the signs change infinitely often. Series with a fixed sign for terms with $n > n_0$ are, however, of an entirely different character, and Airey obtained converging factors only in two or three cases where an explicit expression for R_n was known as an integral, and where this had been redeveloped, e.g. by Stieltjes (3), in a form immediately useful for Airey's purposes.

In the present paper it is shown how a convenient expansion for C_n may be developed whenever S is a function of x satisfying a linear differential equation, so that C_n also satisfies such an equation in x, while at the same time u_n, and thence C_n, satisfies a linear difference equation in n. If the number of changes of sign in the sequence u_n is infinite, C_n may be determined *either* from the differential equation (denoted here by D.E.) *or* from the difference equation (Δ.E.). If, however, the terms u_n are ultimately all one-signed, there is difficulty with constants of integration and summation, and *both* D.E. and Δ.E. are needed for the determination of C_n.

The method is outlined by its application to the asymptotic expansions for the solutions to Weber's equation

$$\frac{d^2y}{dx^2} = (a+\tfrac{1}{4}x^2)\,y,$$

giving results which have not been previously obtained except for certain particular values of a.

The methods have been completely successful in the present case. A warning must, however, be given, since asymptotic representations, satisfactory in the Poincaré sense, are not unique. In theoretical developments of such expansions, results are often obtained in different ways having the same 'major' series—as they must—but having different multiples of an accompanying subsidiary series, whose ratio to the major series is a negative exponential in the independent (complex) variable z, and so treated as negligible. These differing expansions have overlapping z-regions of validity and are such that in the common region the parts of the expansion which differ are relatively negligible, though the roles of major and subsidiary series may well be interchanged outside this common region. The well-known Stokes phenomenon refers to a change in the multiple of the subsidiary series somewhere in such a common region.

If, however, a converging factor is used, the increase of precision may be such that the subsidiary series is not negligible numerically for values of z where the expansion is useful. In this case, it is clear that the correct multiple of the subsidiary series must be used—in other words, that regions of validity must have strictly non-overlapping ranges for the phase (or 'argument') of z. See, for example, Airey (1), p. 546, where the asymptotic expansion for $I_0(x\sqrt{i}) = \operatorname{ber} x + i \operatorname{bei} x$ is considered, the main part being a series with factor $e^{x/\sqrt{2}}$, but where it is shown that a term $(i/\pi)(\operatorname{ker} x + i \operatorname{kei} x)$, with expansion having a factor $e^{-x/\sqrt{2}}$, remains numerically significant. Such refinements indicate that the change referred to above in connexion with the Stokes phenomenon is abrupt and not diffused over a range of phases.

2. The *Weber functions* or *parabolic cylinder functions* with which this paper is concerned satisfy the D.E.

$$\frac{d^2y}{dx^2} - (a+\tfrac{1}{4}x^2)\,y = 0. \tag{2·1}$$

The asymptotic expansions for solutions to this equation involve two particular independent solutions $S_1(a,x)$ and $S_2(a,x)$ such that

$$S_1 \equiv S_1(a,x) \sim e^{-\frac{1}{4}x^2}x^{-a-\frac{1}{2}}\left\{1 - \frac{(a+\frac{1}{2})(a+\frac{3}{2})}{2\,.\,x^2} + \frac{(a+\frac{1}{2})(a+\frac{3}{2})(a+\frac{5}{2})(a+\frac{7}{2})}{2\,.\,4\,.\,x^4} - \dots\right\}$$
$$\equiv v_0 - v_1 + v_2 - \dots, \tag{2·2}$$

$$S_2 \equiv S_2(a,x) \sim e^{+\frac{1}{4}x^2}x^{+a-\frac{1}{2}}\left\{1 + \frac{(a-\frac{1}{2})(a-\frac{3}{2})}{2\,.\,x^2} + \frac{(a-\frac{1}{2})(a-\frac{3}{2})(a-\frac{5}{2})(a-\frac{7}{2})}{2\,.\,4\,.\,x^4} + \dots\right\}$$
$$\equiv u_0 + u_1 + u_2 + \dots. \tag{2·3}$$

In these v_r and u_r satisfy the Δ.E.'s

$$\left.\begin{aligned} 2rx^2v_r &= (a+2r-\tfrac{3}{2})(a+2r-\tfrac{1}{2})v_{r-1}, \\ 2rx^2u_r &= (a-2r+\tfrac{3}{2})(a-2r+\tfrac{1}{2})u_{r-1}. \end{aligned}\right\} \qquad (2\cdot4)$$

Write, now,

$$S_{1,r} = \sum_0^{r-1} (-1)^s v_s, \quad R_{1,r} = S_1 - S_{1,r}, \qquad S_{2,r} = \sum_0^{r-1} u_s, \quad R_{2,r} = S_2 - S_{2,r}, \qquad (2\cdot5)$$

whence $\qquad R_{1,r} - R_{1,r+1} = (-1)^r v_r, \qquad R_{2,r} - R_{2,r+1} = u_r. \qquad (2\cdot6)$

Then it is desired to find expressions from which $R_{1,r}$ and $R_{2,r}$ may be calculated. This is done by writing, say,

$$R_{1,r} = (-1)^r \Gamma_r v_r \quad \text{or} \quad R_{2,r} = C_r u_r. \qquad (2\cdot7)$$

Then the converging factors Γ_r and C_r each satisfy a D.E. and a Δ.E., and from these equations suitable expansions may be derived.

3. Consider first the series for $S_1(a, x)$, which is easier to deal with; it will be apparent later that the reason for this is the alternation of sign in the successive terms of the series.

From (2·4), (2·6) and (2·7) it is readily verified that Γ_r satisfies the Δ.E.

$$2rx^2(\Gamma_{r-1} - 1) + (a+2r-\tfrac{3}{2})(a+2r-\tfrac{1}{2})\Gamma_r = 0. \qquad (3\cdot1)$$

Again, substitution of (2·2) in (2·1), using $R_{1,r}$ to curtail the series, gives a D.E. for $R_{1,r}$, namely,

$$\frac{d^2R_{1,r}}{dx^2} - (a+\tfrac{1}{4}x^2)R_{1,r} = (-1)^r\, 2rv_r. \qquad (3\cdot2)$$

The substitution (2·7) then gives

$$x^2\frac{d^2\Gamma_r}{dx^2} - x(x^2+2a+4r+1)\frac{d\Gamma_r}{dx} + (a+2r+\tfrac{1}{2})(a+2r+\tfrac{3}{2})\Gamma_r + 2rx^2(\Gamma_r - 1) = 0. \qquad (3\cdot3)$$

Suppose now that v_r is the smallest term, or nearly so, then v_r and v_{r-1} are nearly of the same size so that, from (2·4),

$$2r = x^2 - 2(a-1) - k, \qquad (3\cdot4)$$

in which k is small, of the order unity. To simplify the expressions, write

$$b = a-2, \quad \mu = (a-1)^2 - \tfrac{1}{4} = (a-\tfrac{1}{2})(a-\tfrac{3}{2}). \qquad (3\cdot5)$$

Then the Δ.E. (3·1), keeping x constant, so that r and k vary together, becomes

$$x^4(\Gamma_r + \Gamma_{r-1} - 1) - x^2[(k+2b+2)(\Gamma_r + \Gamma_{r-1} - 1) + k\Gamma_r] + [k^2 + 2(b+1)k + \mu]\Gamma_r = 0, \qquad (3\cdot6)$$

whilst the D.E. (3·3), keeping r constant, so that x and k vary simultaneously, becomes

$$x^4(4\Gamma_r'' - 6\Gamma_r' + 2\Gamma_r - 1) + x^2[4(k+b)\Gamma_r' - (3k+4b)\Gamma_r + (k+2b+2)] + [k^2 + 2(b-1)k + \mu - 4b]\Gamma_r = 0, \qquad (3\cdot7)$$

where accents denote differentiation with respect to k.

A solution to these is sought in the form

$$\Gamma_r = \frac{\beta_0(k)}{2} + \frac{\beta_1(k)}{2^2 . x^2} + \frac{\beta_2(k)}{2^3 . x^4} + \ldots + \frac{\beta_r(k)}{2^{r+1} x^{2r}} + \ldots \tag{3·8}$$

suitable for numerical work when x is fairly large. Either equation, D.E. or Δ.E., may be solved, giving identical results for a particular integral. The accompanying complementary function will be assumed absent, not being expressible as a series in descending powers of x; its neglect will then be justified in §6.

4. To obtain a solution to (3·7), substitute (3·8) therein, and equate to zero successive descending powers of x^2. The resulting relations are

$$\left.\begin{aligned}
2\beta_0'' - 3\beta_0' + \beta_0 &= 1,\\
2\beta_1'' - 3\beta_1' + \beta_1 &= -4(k+b)\beta_0' + (3k+4b)\beta_0 - (2k+4b+4),\\
2\beta_2'' - 3\beta_2' + \beta_2 &= -4(k+b-2)\beta_1' + (3k+4b-6)\beta_1 - 2\{k^2+2(b-1)k+\mu-4b\}\beta_0,\\
2\beta_{r+1}'' - 3\beta_{r+1}' + \beta_{r+1} &= -4(k+b-2r)\beta_r' + (3k+4b-6r)\beta_r + 8(r-1)(k+b-r)\beta_{r-1}'\\
&\qquad - 2\{k^2+2(b-1)k+\mu-4b\}\beta_{r-1}.
\end{aligned}\right\} \tag{4·1}$$

These may be solved in succession, the right-hand side being known at each stage. The operator $2D^2 - 3D + 1$ on the left sides of (4·1) may be removed in the usual way by application of the operator

$$\frac{1}{1-3D+2D^2} = \frac{2}{1-2D} - \frac{1}{1-D} = 1 + 3D + 7D^2 + 15D^3 + \ldots + (2^n - 1)D^{n-1} + \ldots. \tag{4·2}$$

The result obtained is

$$\begin{aligned}
\Gamma_r \sim \tfrac{1}{2} &+ \frac{1}{2^2x^2}\{k-1\} + \frac{1}{2^3x^4}\{k^2-3k-(2\mu-1)\} + \frac{1}{2^4x^6}\{k^3-6k^2-(8\mu-7)k-(8\mu b-2\mu-1)\}\\
&+ \frac{1}{2^5x^8}\{k^4-10k^3-(22\mu-25)k^2-(48\mu b-16\mu+5)k-(28\mu^2-56\mu b-26\mu+13)\}\\
&+ \frac{1}{2^6x^{10}}\{k^5-15k^4-(52\mu-65)k^3-(184\mu b-68\mu+60)k^2\\
&\quad -(228\mu^2-480\mu b-184\mu+83)k-(96\mu^2 b-144\mu^2+152\mu b+146\mu-47)\}+\ldots.
\end{aligned} \tag{4·3}$$

Concerning the coefficients in { }, it may be noted that (i) none contains a term in b alone, and (ii) their values when $k = 0$, $\mu = 0$, namely $-1, +1, +1, -13, +47, \ldots$, are
A1662 the constants $-a_n$ tabulated by Airey ((1), p. 529) for his converging factors for the exponential integral; they also occur when dealing with the ascending series for the exponential function. This is to be expected, since $a = -\tfrac{1}{2}$ gives a D.E. satisfied by $y = \exp(-\tfrac{1}{4}x^2)$.

5. Consider next the substitution of (3·8) in the Δ.E. (3·6) and the relations obtained by equating to zero the coefficients of successive powers of $1/x^2$. The resulting relations, in which $\beta_r^*(k)$ is written for $\beta_r(k+2)$—for k becomes $k+2$ when r becomes $r-1$, so that

$$\Gamma_{r-1} = \frac{\beta_0^*(k)}{2} + \frac{\beta_1^*(k)}{2^2 . x^2} + \frac{\beta_2^*(k)}{2^3 . x^4} + \ldots \tag{5·1}$$

—are

$$\left.\begin{aligned}
\tfrac{1}{2}(\beta_0+\beta_0^*) &= 1,\\
\tfrac{1}{2}(\beta_1+\beta_1^*) &= (k+2b+2)(\beta_0+\beta_0^*-2)+k\beta_0,\\
\tfrac{1}{2}(\beta_{r+1}+\beta_{r+1}^*) &= (k+2b+2)(\beta_r+\beta_r^*)+k\beta_r-2\{k^2+2k(b+1)+\mu\}\beta_{r-1}.
\end{aligned}\right\} \tag{5·2}$$

The last relation may be rewritten

$$\tfrac{1}{2}(\beta_{r+1}+\beta^*_{r+1}-2k\beta_r) = (k+2b+2)(\beta_r+\beta^*_r-2k\beta_{r-1})-2\mu\beta_{r-1}, \tag{5·3}$$

whence may be derived the relations

$$\left.\begin{aligned}
\tfrac{1}{2}(\beta_0+\beta^*_0) &= 1,\\
\tfrac{1}{2}(\beta_1+\beta^*_1) &= k\beta_0,\\
\tfrac{1}{2}(\beta_2+\beta^*_2) &= k\beta_1-2\mu\beta_0,\\
\tfrac{1}{2}(\beta_3+\beta^*_3) &= k\beta_2-2\mu\beta_1-4(k+2b+2)\mu\beta_0,\\
\tfrac{1}{2}(\beta_4+\beta^*_4) &= k\beta_3-2\mu\beta_2-4(k+2b+2)\mu\beta_1-8(k+2b+2)^2\mu\beta_0,
\end{aligned}\right\} \tag{5·4}$$

and so on. These are useful when seeking early coefficients.

The details of methods of solving these equations are not uninteresting, but are omitted. The coefficients in (4·3) result as before.

6. Lastly, the complementary functions that may arise during the solution of (3·6) and (3·7) remain to be considered. It is convenient to consider the D.E. and Δ.E. for $R_{1,r}$ itself for this purpose, and to omit those terms which give rise to the right-hand sides of (3·6) and (3·7). The Δ.E. for $R_{1,r}$ is given in (2·6), so that, omitting the term $(-)^r v_r$, which leads to the term $-x^4+(k+2b+2)x^2 = -2rx^2$ in (3·6), the complementary function is given by

$$\bar{R}_{1,r} = \text{constant} \quad \text{or} \quad \bar{\Gamma}_r = (-1)^r \times \text{constant}/v_r, \tag{6·1}$$

where the constant may, of course, be a function of x.

The D.E. for $R_{1,r}$ is (3·2), whence

$$\frac{d^2}{dx^2}\bar{R}_{1,r}-(a+\tfrac{1}{4}x^2)\bar{R}_{1,r} = 0, \tag{6·2}$$

so that

$$\bar{R}_{1,r} = A_r S_1(a,x)+B_r S_2(a,x), \tag{6·3}$$

in which A_r, B_r may depend on r.

The desired complementary function, which must satisfy simultaneously both (6·1) and (6·3), is thus of the form

$$\bar{R}_{1,r} = (-1)^r\bar{\Gamma}_r v_r = AS_1(a,x)+BS_2(a,x), \tag{6·4}$$

where A and B are constants independent both of x and of r.

Now, as $x\to\infty$, $S_1\to 0$ while $S_2\to\infty$. Also $R_{1,r}\to 0$ for fixed r. Hence $B=0$. Likewise, if $A \neq 0$, $S_{1,r}+(-1)^r R^*_{1,r} = (1+A)S_1$, in which $R^*_{1,r} = R_{1,r}+\bar{R}_{1,r}$, indicating an incorrect identification of the series with $S_1(a,x)$; if this is ruled out, clearly $A=0$. Hence $R_{1,r}=0$ and so $\bar{\Gamma}_r = 0$.

7. Next the series $S_2(a,x)$ must be considered. In a manner similar to that described in §3, it may be shown that $R_{2,r}$ satisfies the Δ.E. (2·6) and the D.E.

$$\frac{d^2R}{dx^2} = (a+\tfrac{1}{4}x^2)R-2ru_r, \tag{7·1}$$

whence the converging factor C_r satisfies the Δ.E.

$$2rx^2C_{r-1}-(a-2r+\tfrac{3}{2})(a-2r+\tfrac{1}{2})C_r = 2rx^2 \tag{7·2}$$

and the D.E.

$$x^4C_r''+x^2(x^2+2a-4r)\,C_r'+\{(a-2r-\tfrac{1}{2})\,(a-2r-\tfrac{3}{2})-2rx^2\}\,C_r+2rx^2=0, \qquad (7\cdot3)$$

in which accents denote differentiation with respect to h, where

$$2r=x^2+2(a+1)-2h. \qquad (7\cdot4)$$

By elimination of r and rearrangement, these take the forms

$$x^4(C_r-C_{r-1}+1)+x^2[-(4h-2c)\,C_r+(2h-2c)\,(C_{r-1}-1)]+(4h^2-4hc+\lambda)\,C_r=0 \quad (7\cdot5)$$

and

$$x^4(C_r''-C_r'+1)+x^2[(4h-2c-2)\,C_r'-(2h-4)\,C_r-(2h-2c)]$$
$$+[4h^2-4h(c+2)+\lambda+4(c+1)]\,C_r=0, \quad (7\cdot6)$$

in which $c=a+1,\quad \lambda=c^2-\tfrac{1}{4}=(a+\tfrac{1}{2})\,(a+\tfrac{3}{2}).$ (7·7)

8. A solution is sought in the form

$$C_r=\alpha_0(h)+\frac{\alpha_1(h)}{x^2}+\frac{\alpha_2(h)}{x^4}+\frac{\alpha_3(h)}{x^6}+\dots, \qquad (8\cdot1)$$

which, on substitution in (7·6) and on equating to zero the coefficients of successive powers of $1/x^2$, gives the relations

$$\left.\begin{aligned}\alpha_0'-\alpha_0''&=1,\\ a_1'-\alpha_1''&=2(2h-c-1)\,\alpha_0'-(2h-4)\,\alpha_0-2(h-c),\\ \alpha_2'-\alpha_2''&=2(2h-c-3)\,\alpha_1'-(2h-6)\,\alpha_1+\{4h^2-4h(c+2)+\lambda+4(c+1)\}\,\alpha_0,\\ \alpha_{r+1}'-\alpha_{r+1}''&=2(2h-c-2r-1)\,\alpha_r'-(2h-2r-4)\,\alpha_r-4(r-1)\,(2h-c-r-1)\,\alpha_{r-1}\\ &\qquad+\{4h^2-4h(c+2)+\lambda+4(c+1)\}\,\alpha_{r-1}.\end{aligned}\right\} \quad (8\cdot2)$$

Likewise, noting that

$$C_{r-1}=\alpha_0^*(h)+\frac{\alpha_1^*(h)}{x^2}+\frac{\alpha_2^*(h)}{x^4}+\dots, \qquad (8\cdot3)$$

in which $\alpha_r^*(h)=\alpha_r(1+h),$ (8·4)

substitution in (7·5) yields the relations

$$\left.\begin{aligned}\alpha_0^*-\alpha_0&=1,\\ \alpha_1^*-\alpha_1&=-2h\alpha_0-2(h-c)\,(\alpha_0-\alpha_0^*+1),\\ \alpha_2^*-\alpha_2&=-2h\alpha_1-2(h-c)\,(\alpha_1-\alpha_1^*)+(4h^2-4hc+\lambda)\,\alpha_0,\\ \alpha_{r+1}^*-\alpha_{r+1}&=-2h\alpha_r-2(h-c)\,(\alpha_r-\alpha_r^*)+(4h^2-4hc+\lambda)\,\alpha_{r-1}.\end{aligned}\right\} \quad (8\cdot5)$$

9. This time the complementary function includes an additive constant *which cannot be assumed zero*. The constant arises from the fact that the operator D or $E-1$ occurs as a factor on the left of the equations (8·2) or (8·5); these factors in turn are connected with the non-alternation of sign in the terms of Σu_n. This connexion is of considerable interest, as it exhibits a reason for the special behaviour of asymptotic series with all terms ultimately positive, as evinced here and in the Stokes phenomenon. The remainder term R_r in an asymptotic sum such as those of § 2, but allowing complex argument z in place of x, can usually be written in the form $R_r=F_r\exp(i\chi_r)$ in which F_r is a function which varies steadily with r, without oscillations, and which is eventually one-signed, whilst usually $\chi_r=\lambda r$ (exactly or approximately) where λ is ϕz^p or

$\phi z^p + \pi$, ϕt indicating the phase of the complex number t. In this z^p is the increase in the power of z in the denominator when passing from one term of the series to the next. Thus for S_1 and S_2 of § 2, λ is respectively π or 0.

The difference $R_{r+1} - R_r$ of (2·6) may thus be written

$$e^{i(\lambda+1)r}(E - e^{-i\lambda})F_r,$$

so that the operator $E-1$ is a factor if and only if $\lambda = 0$. Similar arguments show that D is then a factor of the corresponding D.E. Only in such cases can an additive constant arise as part of the complementary function.

The other part of the complementary function will be ignored until it is discussed in § 10.

Both α_0 equations yield immediately

$$\alpha_0 = h + \epsilon_0. \tag{9·1}$$

$$\left.\begin{aligned} &\text{Then, from (8·2)} \quad \alpha_1' - \alpha_1'' = -2h^2 + (6 - 2\epsilon_0)h - (2 - 4\epsilon_0), \\ &\text{and from (8·5)} \quad \alpha_1^* - \alpha_1 = -2h^2 - 2h\epsilon_0 = -2h(h-1) - 2h(1+\epsilon_0). \end{aligned}\right\} \tag{9·2}$$

On solution, these yield respectively

$$\left.\begin{aligned} \alpha_1 &= -\tfrac{2}{3}h^3 + (1-\epsilon_0)h^2 + 2\epsilon_0 h + \epsilon_1, \\ \alpha_1 &= -\tfrac{2}{3}h^3 + (1-\epsilon_0)h^2 + (\epsilon_0 - \tfrac{1}{3})h + \epsilon_1, \end{aligned}\right\} \tag{9·3}$$

whence

$$\epsilon_0 = -\tfrac{1}{3}. \tag{9·4}$$

The next pair of equations gives

$$\left.\begin{aligned} \alpha_2 &= \tfrac{4}{15}h^5 - \tfrac{4}{3}h^4 + \tfrac{20}{9}h^3 + (\tfrac{1}{2}\lambda - \tfrac{4}{3} - \epsilon_1)h^2 + (\tfrac{2}{3}\lambda + 4\epsilon_1)h + \epsilon_2, \\ \text{and} \quad \alpha_2 &= \tfrac{4}{15}h^5 - \tfrac{4}{3}h^4 + \tfrac{20}{9}h^3 + (\tfrac{1}{2}\lambda - \tfrac{4}{3} - \epsilon_1)h^2 + (\tfrac{8}{45} - \tfrac{5}{6}\lambda + \epsilon_1)h + \epsilon_2, \end{aligned}\right\} \tag{9·5}$$

whence

$$\epsilon_1 = \tfrac{8}{135} - \tfrac{1}{2}\lambda. \tag{9·6}$$

Proceeding in this way, the value finally obtained is

$$\begin{aligned} C_r = (h - \tfrac{1}{3}) &- \frac{1}{x^2}(\tfrac{2}{3}h^3 - \tfrac{4}{3}h^2 + \tfrac{2}{3}h + \tfrac{1}{2}\lambda - \tfrac{8}{135}) \\ &+ \frac{1}{x^4}[\tfrac{4}{15}h^5 - \tfrac{4}{3}h^4 + \tfrac{20}{9}h^3 + (\lambda - \tfrac{188}{135})h^2 - (\tfrac{4}{3}\lambda - \tfrac{32}{135})h + (\lambda c + \lambda + \tfrac{32}{2835})] \\ &- \frac{1}{x^6}[\tfrac{8}{105}h^7 - \tfrac{32}{45}h^6 + \tfrac{112}{45}h^5 + (\tfrac{2}{3}\lambda - \tfrac{544}{135})h^4 - (\tfrac{10}{3}\lambda - \tfrac{1192}{405})h^3 + (2\lambda c + \tfrac{16}{3}\lambda - \tfrac{1984}{2835})h^2 \\ &\qquad + (\tfrac{1}{2}\lambda^2 - \tfrac{8}{3}\lambda c - \tfrac{368}{135}\lambda - \tfrac{64}{945})h + (\tfrac{13}{6}\lambda^2 + \tfrac{14}{3}\lambda c + \tfrac{19}{6}\lambda + \tfrac{128}{8505})] \\ &+ \frac{1}{x^8}[\tfrac{16}{945}h^9 - \tfrac{16}{63}h^8 + \tfrac{32}{21}h^7 + (\tfrac{4}{15}\lambda - \tfrac{1888}{405})h^6 - (\tfrac{8}{3}\lambda - \tfrac{3088}{405})h^5 + (\tfrac{4}{3}\lambda c + \tfrac{92}{9}\lambda - \tfrac{3536}{567})h^4 \\ &\qquad + (\tfrac{2}{3}\lambda^2 - 8\lambda c - \tfrac{2168}{135}\lambda + \tfrac{15488}{8505})h^3 + (2\lambda^2 + 16\lambda c + \tfrac{601}{45}\lambda + \tfrac{2432}{8505})h^2 \\ &\qquad + (2\lambda^2 c - 3\lambda^2 - \tfrac{1276}{135}\lambda c - \tfrac{15424}{2835}\lambda - \tfrac{1024}{8505})h + \epsilon_4] + \ldots. \end{aligned} \tag{9·7}$$

To evaluate ϵ_4 involves the almost complete evaluation of α_5, which has not been carried out.

10. Precisely as in §6, the complementary function may be found from (7·1) and (2·6). The latter shows that $\bar{R}_{2,r}$ is independent of r, whence

$$\bar{C}_r = f(x)/u_r, \tag{10·1}$$

whilst (7·1) shows that $$\bar{R}_{2,r} = A_r S_1(a,x) + B_r S_2(a,x), \tag{10·2}$$

whence, combining the results,

$$\bar{C}_r = \{AS_1(u,x) + BS_2(a,x)\}/u_r, \tag{10·3}$$

in which A and B are constants.

As before, B must be zero, for $B \neq 0$ would imply identification of $S_2(a,x)$ with the wrong multiple of the series (2·3).

The term $AS_1(a,x)$ in $\bar{R}_{2,r}$ is less easy to dispose of, for it becomes relatively insignificant as $x \to \infty$. However, the constant A has a definite value which may be investigated *numerically* to whatever accuracy is attainable by any method which may be available. Its presence or absence does not, therefore, affect the numerical use of the converging factor; it is necessary only to evaluate A at both ends of the range of x for which C_r is desired, by alternative calculation (for example by means of the ascending series), and to use the better of the two values so calculated—they should agree to within the estimated error of the less accurate—at intermediate points in the range.

As an illustration of this point, consider the case $a = -\frac{1}{2}$. A solution of this equation (2·1) in this case is

$$e^{-\frac{1}{4}x^2} \int_K^x e^{\frac{1}{4}t^2}\, dt$$

for any constant K. Change of the constant K adds a constant multiple of the other solution of (2·1), namely $e^{-\frac{1}{4}x^2}$, corresponding to a change in the constant A of (10·2) or (10·3). It happens here, as often, that the most natural choice of K, namely $K = 0$, corresponds to $A = 0$, as will be partially verified in §13.

11. The converging factors (4·3) and (9·7) may be expressed in other forms. In particular, expansion in powers of $1/r$, rather than of $1/x^2$, is useful, since r is integral. The re-expansion may be done by expressing $(x^2)^{-p} = \{2r + 2(b+1) + k\}^{-p}$ or $(x^2)^{-p} = \{2r - 2c + 2h\}^{-p}$ in powers of $1/r$ and rearranging the terms. Alternatively, the expansions may be obtained by rewriting (3·6), (3·7), (7·5) and (7·6) in terms of r and h or k instead of x and h or k, and equating to zero successive powers of $1/r$ after substituting

$$\Gamma_r = \frac{\delta_0(k)}{2} + \frac{\delta_1(k)}{2^3 r} + \frac{\delta_2(k)}{2^5 r^2} + \ldots$$

or

$$C_r = \gamma_0(h) + \frac{\gamma_1(h)}{r} + \frac{\gamma_2(h)}{r^2} + \ldots.$$

Other expansion in terms of inverse factorials may also be developed. It is possible that some of these may be convergent.

12. *Numerical illustrations.* The following numerical illustrations all refer to cases with $x = 4$—rather a small value for effective use of asymptotic expansions. The values of a used are $a = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ for $S_1(a,x)$ and $a = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}$ for $S_2(a,x)$.

For these cases it is readily verified that

$$S_1(\tfrac{1}{2}, x) = e^{\frac{1}{2}x^2}\int_x^\infty e^{-\frac{1}{2}t^2}\,dt, \qquad S_1(\tfrac{1}{2}, 4) = 0{\cdot}00433\;44395\;876,$$

$$-S_1(\tfrac{3}{2}, x) = xS_1(\tfrac{1}{2}, x) - e^{-\frac{1}{2}x^2}, \qquad S_1(\tfrac{3}{2}, 4) = 0{\cdot}00097\;78805\;38,$$

$$-2S_1(\tfrac{5}{2}, x) = xS_1(\tfrac{3}{2}, x) - S_1(\tfrac{1}{2}, x), \qquad S_1(\tfrac{5}{2}, 4) = 0{\cdot}00021\;14587\;17.$$

Also

$$S_2(-\tfrac{1}{2}, x) = e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}t^2}\,dt, \qquad S_2(-\tfrac{1}{2}, 4) = 14{\cdot}76313\;75272,$$

$$S_2(-\tfrac{3}{2}, x) = xS_2(-\tfrac{1}{2}, x) - e^{\frac{1}{2}x^2}, \qquad S_2(-\tfrac{3}{2}, 4) = 4{\cdot}45440\;00758,$$

$$2S_3(-\tfrac{5}{2}, x) = xS_2(-\tfrac{3}{2}, x) - S_2(-\tfrac{1}{2}, x), \qquad S_2(-\tfrac{5}{2}, 4) = 1{\cdot}52723\;13879.$$

The value of $S_1(\frac{1}{2}, x)$ was extracted from the NBSCL tables of the probability integral, and the last digit is doubtful for this value and for the two quantities derived from it. The value of $S_2(-\frac{1}{2}, x)$ was obtained from the ascending series

$$S_2(-\tfrac{1}{2}, x) = e^{-\frac{1}{2}x^2}\left(x + \frac{x^3}{2.3} + \frac{x^5}{2.4.5} + \frac{x^7}{2.4.6.7} + \dots\right).$$

Values of k, h, μ, λ, b, c (defined in (3·4), (3·5), (7·4) and (7·7)) are as follows:

$S_1(a, x)$	a	b	μ	k	$S_2(a, x)$	a	c	λ	h
	$\frac{1}{2}$	$-\frac{3}{2}$	0	$17-2r$		$-\frac{1}{2}$	$+\frac{1}{2}$	0	$\frac{1}{2}(17-2r)$
	$\frac{3}{2}$	$-\frac{1}{2}$	0	$15-2r$		$-\frac{3}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}(15-2r)$
	$\frac{5}{2}$	$+\frac{1}{2}$	2	$13-2r$		$-\frac{5}{2}$	$-\frac{3}{2}$	2	$\frac{1}{2}(13-2r)$.

The converging factors Γ_1 are the same for $a = \frac{1}{2}$ and $a = \frac{3}{2}$, while C_1 is the same for $a = -\frac{1}{2}$ and $a = -\frac{3}{2}$. The effect of μ when $a = \frac{5}{2}$ is not great, but that of λ when $a = -\frac{5}{2}$ will be seen to be very pronounced.

13. *The alternating series $S_1(a, x)$.*

$S_1(\frac{1}{2}, 4)$:

r	v_r	$S_{1,r+1}$	k	Γ_r	$R_{1,r}$	$S_1(\frac{1}{2}, 4)$
		0·00				0·00
0	0·00457 89097 2218	457 89097 222				
1	28 61818 5764	429 27278 645	+15	0·8534	$-0{\cdot}0^3 24\;42$	433 47
2	5 36590 9831	434 63869 628	13	0·77716	$+0{\cdot}0^4\;4\;1702$	433 4430
3	1 67684 6822	432 96184 946	11	0·71246 9	$-0{\cdot}0^4\;1\;19470\;1$	433 44399 5
4	73362 0485	433 69546 995	9	0·65716 69	$+0{\cdot}0^5\;48211\;11$	433 44396 06
5	41266 1523	433 28280 842	7	0·60948 558	$-0{\cdot}0^5\;25151\;125$	433 44395 870
6	28370 4797	433 56651 322	5	0·56802 092	$+0{\cdot}0^5\;16115\;026$	433 44395 868
7	23051 0147	433 33600 308	3	0·53166 641	$-0{\cdot}0^5\;12255\;450$	433 44395 872
8	21610 3263	433 55210 634	+1	0·49955 587	$+0{\cdot}0^5\;10795\;565$	433 44395 873
9	22960 9717	433 32249 662	−1	0·47100 623	$-0{\cdot}0^5\;10814\;761$	433 44395 873
10	27266 1539	433 59515 816	−3	0·44546 03	$+0{\cdot}0^5\;12145\;99$	433 44395 65

The column $S_{1,r+1}$ indicates how closely the result is given—to about $5\frac{1}{2}$ decimals—by the unaltered asymptotic series, and how partial sums are alternately in excess and in defect. The final column indicates a good range of k, from about -1 to $+7$, where the converging factor is very effective; in fact, with 13 decimals, the study of accuracy is not quite complete—the figures suggest that the check value 0·00433 44395 876 given in §12 is itself inadequate. Fewer details are given below:

$S_1(\frac{3}{2}, 4)$:

r	v_r	k	$R_{1,r}$	$S_1(\frac{3}{2}, 4)$
0	0·00114 47274 3054			
1	21 46363 9323			
2	6 70738 7288			
3	2 93448 1939			
4	1 65064 6090			
5	1 13481 9187			
6	92204 0590	+3	$+0{\cdot}0^5$ 49021 801	$0{\cdot}0^3$ 97 78805 400
7	86441 3053	+1	$-0{\cdot}0^5$ 43182 262	97 78805 396
8	91843 8869	−1	$+0{\cdot}0^5$ 43259 043	97 78805 395
9	1 09064 6156	−3	$-0{\cdot}0^5$ 48583 96	97 78805 28

$S_1(\frac{5}{2}, 4)$:

r	v_r	k	Γ_r	$R_{1,r}$	$S_1(\frac{5}{2}, 4)$
0	0·00028 61818 576				
1	10 73181 966				
2	5 03054 047				
3	2 93448 194				
4	2 06330 761				
5	1 70222 878	+3	0·52865 4	$-0{\cdot}0^5$ 89989 0	$0{\cdot}0^3$ 21 14584 2
6	1 61357 103	+1	0·49726 294	$+0{\cdot}0^5$ 80236 907	21 14587 253
7	1 72882 610	−1	0·46922 185	$-0{\cdot}0^5$ 81120 297	21 14587 152

It will be seen that Γ_r is determined to about 8 or 9 decimals, at $r = 6$ or 7 for $a = \frac{1}{2}$ and $\frac{3}{2}$, with perhaps slightly less accuracy for $a = \frac{5}{2}$. In fact, for $k = -1$

$$a = \tfrac{1}{2}, \quad \Gamma_r = \tfrac{1}{2} - \frac{2}{2^2 . x^2} + \frac{5}{2^3 . x^4} - \frac{13}{2^4 . x^6} + \frac{28}{2^5 . x^8} - \frac{11}{2^6 . x^{10}},$$

$$a = \tfrac{5}{2}, \quad \Gamma_r = \tfrac{1}{2} - \frac{2}{2^2 . x^2} + \frac{1}{2^3 . x^4} - \frac{1}{2^4 . x^6} - \frac{4}{2^5 . x^8} + \frac{49}{2^6 . x^{10}}.$$

The accuracy falls off as k gets away from the optimum range; it also falls off as $|a|$ increases, while the optimum k also changes. Thus, for $a = -4\frac{1}{2}$, $k = 6$ is about the best, and this gives

$$a = -\tfrac{9}{2}, \quad \Gamma_r = \tfrac{1}{2} + \frac{5}{2^2 . x^2} - \frac{41}{2^3 . x^4} + \frac{223}{2^4 . x^6} - \frac{67}{2^5 . x^8} - \frac{15295}{2^6 . x^{10}}.$$

In this the numerator 67 is abnormally small (compared with about 1500 to 2000 for this neighbourhood in k); the last coefficient is normal—for the range $k = 5$ to $k = 12$

its modulus does not exceed about 25000, although increasing rapidly outside the limits of this range.

For effective use of these formulae a fuller study is desirable.

14. *The single-sign series* $S_2(a, x)$.

r	$S_2(-\frac{1}{2}, 4)$ u_r	$S_{2,r+1}$	$S_2(-\frac{3}{2}, 4)$ u_r	$S_2(-\frac{5}{2}, 4)$ u_r
0	13·64953 75083	13·64953 75083	3·41238 43771	0·85309 60943
1	85309 60943	14·50263 36026	63982 20707	31991 10354
2	15995 55177	14·66258 91203	19994 43971	14995 82978
3	4998 60993	14·71257 52196	8747 56737	8747 56737
4	2186 89184	14·73444 41380	4920 50665	6150 63331
5	1230 12666	14·74674 54046	3382 84832	5074 27248
6	845 71208	14·75520 25254	2748 56426	4809 98745
7	687 14106	14·76207 39360	2576 77899	
8	644 19475	14·76851 58835	2737 82768	
9	684 45692	14·77536 04257	3251 17037	
10	812 79259	14·78348 83786		

The partial sums $S_{2,r+1}$ are given, for $S_2(-\frac{1}{4}, 4)$ only, in order to exhibit the way in which $S_{2,r}$ increases up to and, at the least term, through the value 14·76313 75272 given in §12.

Take in each case the least term, respectively the 8th, 7th or 6th, so that $h = \frac{1}{2}$ for each. Thus

$$\text{for} \qquad a = -\tfrac{1}{2}, -\tfrac{3}{2}, \quad C_r = \frac{1}{6} - \frac{13}{540}\frac{1}{x^2} - \frac{353}{22680}\frac{1}{x^4} + \frac{1423}{136080}\frac{1}{x^6} + \ldots,$$

$$\text{for} \qquad a = -\tfrac{5}{2}, \quad C_r = \frac{1}{6} - \frac{553}{540}\frac{1}{x^2} - \frac{41933}{22680}\frac{1}{x^4} - \frac{3\cdot 7}{x^6} - \ldots,$$

whence, for $x = 4$, $\quad a = -\frac{1}{2}, -\frac{3}{2}, \quad C_r = 0{\cdot}16510\,379$,

$\qquad a = -\frac{5}{2}, \quad C_r = 0{\cdot}0945$,

the former to 7 or 8 decimals, the latter to about 3—the effect of non-zero λ is very pronounced. Thence

$a = -\frac{1}{2}$	$a = -\frac{3}{2}$	$a = -\frac{5}{2}$
$S_{2,8} =$ 14·76207 39360	$S_{2,7} =$ 4·45014 57109	$S_{2,6} =$ 1·52279 0
$R_{2,8} =$ 106 35900	$R_{2,7} =$ 425 43598	$R_{2,6} =$ 454 7
$S_2 =$ 14·76313 75260	$S_2 =$ 4·45440 00707	$S_2 =$ 1·52733 7

These are correct to $8\frac{1}{2}$, 8 and 4 decimals respectively.

No serious attempt has been made to search for the best range of values of h for convergence, but small ones seem likely to be best.

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